Sparse Dirichlet optimal control problems*

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Abstract

In this paper, we analyze optimal control problems governed by an elliptic partial differential equation, in which the control acts as the Dirichlet data. Box constraints for the controls are imposed and the cost functional involves the state and possibly a sparsity-promoting term, but not a Tikhonov regularization term. Two different discretizations are investigated: the variational approach and a full discrete approach. For the latter, we use continuous piecewise linear elements to discretize the control space and numerical integration of the sparsity-promoting term. It turns out that the best way to discretize the state equation is to use the Carstensen quasi-interpolant of the boundary data, and a new discrete normal derivative of the adjoint state must be introduced to deal with this. Error estimates, optimization procedures and examples are provided.

Keywords: optimal control, boundary control, sparse controls, finite element approximation

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary Γ, $-\infty < \alpha < \beta < +\infty$, $y_d \in L^p(\Omega)$ for some $p > 2$, and $\mu \geq 0$. We will study the following control problem

$$\min_{u \in U_{\rm ad}} J(u) := F(u) + \mu j(u), \tag{P}$$

where

$$F(u) = \frac{1}{2}\|y_u - y_d\|^2_{L^2(\Omega)}, \qquad j(u) = \|u\|_{L^1(\Gamma)},$$

and

$$U_{\rm ad} = \{u \in L^\infty(\Gamma) : \alpha \leq u(x) \leq \beta \ \text{ for a.e. } x \in \Gamma\}.$$

Above y_u denotes the state associated to the control u related by the following state equation

$$Ay_u = f \text{ in } \Omega, \quad y_u = u \text{ on } \Gamma, \tag{1}$$

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where $f \in H^{-1}(\Omega)$. Here A denotes the operator

$$Ay = -\sum_{i,j=1}^{2} \partial_{x_j}(a_{i,j}\partial_{x_i} y) + a_0 y,$$

where $a_0 \in L^p(\Omega)$, $p > 2$, satisfies $a_0 \geq 0$, and the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$ satisfy the uniform ellipticity condition

$$\exists \lambda_A > 0 \text{ such that } \lambda_A |\xi|^2 \leq \sum_{i,j=1}^{2} a_{i,j}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^2 \text{ and a.a. } x \in \Omega.$$

The solution of equation (1) must be understood in the transposition sense; see Definition 2.2 below. This problem has two special distinguishing characteristics: the control acts as Dirichlet boundary data and there is a non-differentiable sparsity promoting term, namely $j(u)$. Moreover, we do not introduce a regularizing Tikhonov term.

Since the so-called seminal papers for these topics, [13] for Dirichlet control problems in 2006 and [23] in 2009 for sparse optimal control problems, there has been an increasing interest in both of them in the last years. Just to give a measure of this interest, the first one has 85 citations and the second one 100 citations, according to Web of Science, at the moment of writing these lines. Nevertheless, the author is not aware any work dealing with sparse Dirichlet optimal control problems.

The main difficulty in the study of boundary Dirichlet optimal control problems lies in the lack of regularity of the solution. Even for Tikhonov regularized problems the optimal control is usually not even in $H^1(\Gamma)$; see [1]. This leads to serious difficulties in both the analysis and the numerical approximation of this kind of problems. In the case at hand, one cannot improve the regularity of the optimal control via the adjoint state equation and the optimality system, so the situation is even worse than the one described in [1].

The plan of the paper is as follows. In Section 2 we provide first order optimality conditions and state the sparsity properties of the optimal control. One of the objectives of this work is to introduce a discretization that preserves these sparsity properties. In Section 3 we first recall two different ways to discretize the state equation: using the $L^2(\Gamma)$-projection or the Cartensen interpolant of the boundary data. Unlike the $L^2(\Gamma)$-projection, the Carstenen interpolant of a continuous piecewise linear function need not coincide with itself. This has been seen in the literature as a disadvantage for application in optimal control; see [2, Remark 2.13]. We show how to circumvent these difficulties: the computation of the derivative of the resulting discrete functional requires the definition of a new approximation of the normal derivative of the adjoint state, different from the discrete normal derivative defined in [13]. We devote the second part of Section 3 to discuss the properties of this new concept.

In Section 4 we discuss the variational discretization of the control problem. The control is not discretized. This approach is quite effective to solve problems with bang-bang or bang-off-bang solutions; see [14] for a discussion about bang-bang solutions in the distributed case. In Section 5 we discretize completely the problem. To preserve sparsity we use a numerical integration formula to compute the non-differentiable sparsity-promoting term as is done in [9] or [18]. Error estimates are obtained for the two kinds of approximation for both the control and the state variable. In Section 6 we explain how to solve the problems and in Section 7 we show with some examples the properties discussed along the paper. The orders of convergence obtained in sections 4 and 5 are by no means optimal. The experimental orders of convergence obtained for the examples in Section 7 are higher than the theoretical ones in all the studied cases.

In this work we deal with convex polygonal domains to simplify the exposition. Similar results can be achieved for nonconvex polygonal domains, taking into account the decomposition of the adjoint state into a singular part plus a regular part carried out in [1]. On the other hand, we deal with the general elliptic operator with Lipschitz coefficients A. Up to our best knowledge, all the references to the study of Dirichlet control problems have been written for the Laplace operator $-\Delta$. All of the necessary results about regularity and approximations of partial differential equations involving $-\Delta$ are valid in our context, and we have included justifications and even some detailed proofs where we thought that the "translation" was not immediate.

The other choice we do is that the equation is linear, and this is because it is not clear to us how to treat the problem in the case of a semilinear equation. To deal with a semilinear equation we should obtain a second order sufficient optimality condition for a strong local minimum involving some norm of the state, and then prove somehow convergence of the discrete optimal states to the optimal state in that norm. This technique is used for distributed control problems using the norm of $L^\infty(\Omega)$ in [10] and [11]. A simple inspection of a 3D graph of a solution should be enough to convince us that uniform convergence of the states is impossible in our case; see e.g. Figure 1 in Section 7. Some stronger concept of local minimum should be used, involving, for instance, the $L^2(\Omega)$ norm of the states.

Through all the work, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and $(\cdot,\cdot)_\Gamma$ the inner product in $L^2(\Gamma)$.

2 The continuous problem

Before introducing the definition of transposition solution, we recall a classical regularity result.

Lemma 2.1. *For all $g \in L^2(\Omega)$ there exists a unique $\phi_g \in H_0^1(\Omega) \cap H^2(\Omega)$, variational solution of*

$$A^*\phi = g \text{ in } \Omega,\ \phi = 0 \text{ on } \Gamma. \tag{2}$$

Further

$$\|\partial_{\nu_{A^*}}\phi_g\|_{H^{1/2}(\Gamma)} \le C\|g\|_{L^2(\Omega)}, \tag{3}$$

where $\partial_{\nu_{A^*}}\phi_g$ is the co-normal derivative of ϕ_g associated to A^*, the adjoint operator of A, given by

$$A^*y = -\sum_{i,j=1}^{2} \partial_{x_j}(a_{j,i}\partial_{x_i}y) + a_0 y,$$

Proof. Existence and uniqueness of $\phi_g \in H_0^1(\Omega)$ follows from Lax-Milgram theorem. $H^2(\Omega)$-regularity follows from the convexity of Ω; see [15, Theorem 3.1.3.2].

A proof of the regularity of $\partial_{\nu_{A^*}}\phi_g$ in $H^{1/2}(\Gamma)$ for the case $A = -\Delta$ can be found in [12, Lemma A.2]. In Appendix A we adapt that proof for the general operator considered here. □

Thanks to Lemma 2.1, the next definition is meaningful.

Definition 2.2. For every $u \in H^{-1/2}(\Gamma)$ and $f \in H^{-1}(\Omega)$, we say that $y_u \in L^2(\Omega)$ is the transposition solution of (1) if

$$(y, g) = \langle f, \phi_g \rangle_{H^{-1}(\Omega),H^1(\Omega)} - \langle u, \partial_{\nu_{A^*}} \phi_g \rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \ \forall g \in L^2(\Omega),$$

where $\phi_g \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique variational solution of (2).

Notice that, if we denote y_0 the solution of (1) for $u = 0$, which is a variational solution and belongs to $H^1(\Omega) \subset L^r(\Omega)$ if $r < +\infty$, and redefine $y_d := y_d - y_0$, our problem is equivalent to

$$\min_{u \in U_{\rm ad}} J(u) := \frac{1}{2}\|y_u - y_d\|^2_{L^2(\Omega)} + \mu j(u), \tag{P}$$

where

$$Ay_u = 0 \text{ in } \Omega, \quad y_u = u \text{ on } \Gamma. \tag{4}$$

Therefore, in the rest of the work, we assume without loss of generality that $f = 0$.

We will denote $Y = H^{1/2}(\Omega) \cap L^\infty(\Omega)$. This is a Banach space endowed with the norm

$$\|y\|_Y = \|y\|_{H^{1/2}(\Omega)} + \|y\|_{L^\infty(\Omega)}.$$

Theorem 2.3. *Set $0 \le s \le 1/2$. For every $u \in H^{-s}(\Gamma)$ there exists a unique $y_u \in H^{1/2-s}(\Omega)$ solution of (4), and*

$$\|y_u\|_{H^{1/2-s}(\Omega)} \le C\|u\|_{H^{-s}(\Gamma)}$$

If, further, $u \in L^\infty(\Gamma)$, then $y_u \in Y$ and

$$\|y_u\|_Y \le \|u\|_{L^\infty(\Gamma)}. \tag{5}$$

Proof. The first result can be found in the proof of [1, Theorem 2.5] and the second one in [1, Theorem 2.8]. They are proved $A = -\Delta$, but the proofs only makes use of the definition in the transposition sense and the maximum principle for variational solutions, so the results also apply to the general operator A considered in this work. □

Remark 2.4. Since Ω is convex polygonal, we know that there exists $p^* > 2$ such that, for $g \in L^r(\Omega)$ with $r < p^*$, $\phi_g \in W^{2,r}(\Omega)$, see [15], and $\partial_{\nu_{A^*}} \phi_g = W^{1-1/r,r}(\Gamma)$, see [12, Lemma A.2]; the proof can be adapted as we have done for Lemma 2.1. Taking also into account that $H^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$, regarding the regularity of the target state y_d and of a_0, we assume for simplicity that $p \le 4$ and $2 < p < p^*$, since a greater p would not lead to better regularity of the adjoint state; see Lemma 2.5 below.

In the next lemma we collect some basic facts for later reference.

Lemma 2.5. *The control-to-state operator $G(u) = y_u$, where y_u is the transposition solution of equation (4), is linear continuous from $L^2(\Gamma)$ to $H^{1/2}(\Omega)$ and its restriction to $L^\infty(\Gamma)$ is linear continuous into Y.*

The functional F is of class C^∞ in $L^2(\Gamma)$, for all $u, v \in L^2(\Gamma)$

$$F'(u)v = (-\partial_{\nu_A} \varphi_u, v)_\Gamma \text{ and } F''(u)v^2 = \|y_v\|^2_{L^2(\Omega)}, \tag{6}$$

where $\varphi_u \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ *is the unique variational solution of*

$$A^*\varphi = y_u - y_d \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Gamma, \tag{7}$$

and

$$F(u+v) = F(u) + F'(u)v + \frac{1}{2}\|y_v\|^2_{L^2(\Omega)}. \tag{8}$$

Proof. The statements about G follow straightforward from Theorem 2.3. The regularity of the adjoint state follows from the assumption done in Lemma 2.1.

Since G is linear, its derivative can be computed as $G'(u)v = y_v$ and the expression for the derivatives of F follow using the chain rule and integration by parts in a standard way. Equation (8) is simply the Taylor expansion of F and follows just taking into account that F is quadratic. □

The function j is not differentiable, but it is convex and Lipschitz. We denote $j'(u; v)$ its directional derivative in the direction v. The convexity of j implies that

$$j'(u_1; u_2 - u_1) \leq j(u_2) - j(u_1) \ \forall u_1, u_2 \in L^1(\Gamma).$$

We denote $\partial j(u)$ the convex subdifferential of j at u.

Existence of a solution $\bar{u} \in U_{\text{ad}}$ of (P) follows in a standard way. From (5), we have that the control-to-state mapping is injective, and therefore the functional is strictly convex, so the solution is unique. First order optimality conditions read as follows.

Theorem 2.6. *Let* $\bar{u} \in U_{\text{ad}}$ *be the solution of* (P)*. Then, there exists a unique triplet* $\bar{y} \in Y$, $\bar{\varphi} \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ *and* $\bar{\lambda} \in \partial j(\bar{u})$ *such that*

$$A\bar{y} = 0 \text{ in } \Omega, \quad \bar{y} = \bar{u} \text{ on } \Gamma, \tag{9a}$$

$$A^*\bar{\varphi} = \bar{y} - y_d \text{ in } \Omega, \quad \bar{\varphi} = 0 \text{ on } \Gamma, \tag{9b}$$

$$(-\partial_{\nu_{A^*}}\bar{\varphi} + \mu\bar{\lambda}, u - \bar{u})_\Gamma \geq 0 \ \forall u \in U_{\text{ad}}. \tag{9c}$$

Further,

$$(-\partial_{\nu_{A^*}}\bar{\varphi}, u - \bar{u})_\Gamma + \mu j(u) - \mu j(\bar{u}) \geq 0 \ \forall u \in U_{\text{ad}}. \tag{10}$$

This theorem is proved in the same way as [7, Theorem 3] with the obvious changes. From (9c) the following corollary can be deduced in a straightforward way using the technique of [6, Theorem 3.1].

Corollary 2.7. *Suppose that the assumptions of Theorem 2.6 hold.*

If $\partial_{\nu_{A^*}}\bar{\varphi}(x) < -\mu$ *then* $\bar{u}(x) = \alpha$*. If* $\partial_{\nu_{A^*}}\bar{\varphi}(x) > \mu$ *then* $\bar{u}(x) = \beta$*.*

If $\mu > 0$ *then* $\bar{\lambda} \in W^{1-1/p,p}(\Gamma)$ *and*

$$\bar{\lambda}(x) = \text{Proj}_{[-1,+1]}\left(\frac{1}{\mu}\partial_{\nu_{A^*}}\bar{\varphi}(x)\right).$$

If, further, $\alpha < 0 < \beta$ *and* $|\partial_{\nu_{A^*}}\bar{\varphi}(x)| < \mu$*, then* $\bar{u}(x) = 0$*.*

The regularity of $\bar{\lambda}$ follows from the regularity of $\partial_{\nu_{A^*}}\bar{\varphi}$; see Remark 2.4. In particular, since $p > 2$, $\bar{\lambda}$ is a continuous function.

Under a structure assumption, we can deduce further properties of the control and enhanced first order optimality conditions; see [19, Lemma 6.3].

Lemma 2.8. *Suppose that*

$$\text{there exists } K > 0 \text{ such that } \operatorname{meas}\Big\{x \in \Gamma : \Big||\partial_{\nu_{A^*}} \bar\varphi(x)| - \mu\Big| < \varepsilon\Big\} \le K\varepsilon \ \forall \varepsilon > 0. \tag{H}$$

Then $\bar u(x) \in \{\alpha, \beta\}$ *if* $\mu = 0$, $\bar u(x) \in \{\alpha, 0, \beta\}$ *if* $\mu > 0$ *and*

$$(-\partial_{\nu_{A^*}} \bar\varphi, u - \bar u)_\Gamma + \mu j(u) - \mu j(\bar u) \ge \frac{1}{4(\beta - \alpha)} \|u - \bar u\|^2_{L^1(\Gamma)} \ \forall u \in U_{\rm ad}. \tag{11}$$

Remark 2.9.

1. In [20] it is shown by an example that the assumption of Lemma 2.8 may not be reasonable for 2D controls, and they replace the bound $K\varepsilon$ by the more realistic $K\varepsilon^\gamma$ for some $0 < \gamma < 1$. In our case, since the controls are 1D, the assumption is fulfilled in normal situations, although it is possible to build academic 1D examples where it is nor fulfilled; see [11, Section 7].
2. If the assumptions of Lemma 2.8 do not hold, then the optimal control may have singular arcs and the adjoint state does not provide any information about them; see Example 7.2.
3. For Dirichlet optimal control problems posed over convex polygonal domains, it is known, see [1], that $\partial_{\nu_{A^*}} \bar\varphi$ is a continuous function that vanishes at the corners of the domain. This means that, if $\mu > 0$, then the optimal control will be zero in a neighbourhood of the corners. This can be clearly seen in all the examples in Section 7.

We finish this section with a property of the optimal state.

Lemma 2.10. *Let* $\bar u$ *be the solution of* (P). *For any* $u \in U_{\rm ad}$ *we have*

$$\frac{1}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \le J(u) - J(\bar u).$$

Proof. Using (8), (6) and (10), we have that

$$\begin{aligned} J(u) =& F(u) + \mu j(u) \\ =& F(\bar u) + \mu j(\bar u) + F'(\bar u)(u - \bar u) + \mu j(u) - \mu j(\bar u) + \frac{1}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \\ =& J(\bar u) + (-\partial_{\nu_{A^*}} \bar\varphi, u - \bar u)_\Gamma + \mu j(u) - \mu j(\bar u) + \frac{1}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \\ \ge& J(\bar u) + \frac{1}{2}\|y_u - \bar y\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore, we have that

$$\frac{1}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \le J(u) - J(\bar u).$$

□

3 Discretizations of the state and adjoint state

For all $y, z \in H^1(\Omega)$, we define

$$a(y, z) = \sum_{i,j=1}^{2} (a_{i,j} \partial_{x_i} y, \partial_{x_j} z) + (a_0 y, z).$$

Let $\{\mathcal{T}_h\}_h$ be a quasi-uniform family of triangulations of $\bar\Omega$. For the discretization of the state and the adjoint state we use the space of linear finite elements $Y_h \subset H^1(\Omega)$,

$$Y_h = \{y_h \in C(\bar\Omega)\colon y_h \in P^1(T)\ \forall T \in \mathcal{T}_h\}.$$

We abbreviate $Y_{h0} = Y_h \cap H^1_0(\Omega)$ and Y_h^Γ the space of functions that are the trace of some element of Y_h.

In Section 4, we will not discretize the control (variational discretization), but in Section 5 we will use functions in Y_h^Γ (continuous piecewise linear approximations). We will denote $U_{h,\mathrm{ad}} = Y_h^\Gamma \cap U_{\mathrm{ad}}$.

Following [2], we define two different approximations of the state equation. For every $u \in L^2(\Gamma)$, $y_h(u) \in Y_h$ is the unique solution of

$$a(y_h, \eta_h) = 0 \text{ for all } \eta_h \in Y_{h0},\ y_h(u) = \pi_h u \text{ on } \Gamma, \tag{12}$$

where either $\pi_h u = \mathcal{P}_h u$ is the projection in the $L^2(\Gamma)$ sense onto Y_h^Γ, as proposed by Berggren in [3], or $\pi_h u = \mathcal{I}_h u$ is Carstensen quasi-interpolant. In case we need to distinguish them, we denote $y_h^{\mathcal{P}_h}(u)$ and $y_h^{\mathcal{I}_h}(u)$, otherwise, we simply write $y_h(u)$. We also introduce

$$F_h^{\mathcal{P}_h}(u) = \frac{1}{2}\|y_h^{\mathcal{P}_h}(u) - y_d\|^2_{L^2(\Omega)} \text{ and } F_h^{\mathcal{I}_h}(u) = \frac{1}{2}\|y_h^{\mathcal{I}_h}(u) - y_d\|^2_{L^2(\Omega)},$$

and do the same convention for $F_h(u)$.

Let us briefly recall the properties of both approximations. Regarding the $L^2(\Gamma)$ projection, we have that $\mathcal{P}_h u$ is the unique element of Y_h^Γ that satisfies

$$(\mathcal{P}_h u, v_h)_\Gamma = (u, v_h)_\Gamma\ \forall u \in L^2(\Gamma) \text{ and } v_h \in Y_h^\Gamma.$$

Notice that if $u_h \in Y_h^\Gamma$, then $u_h = \mathcal{P}_h u_h$ and $y_h^{\mathcal{P}_h}(u_h) \equiv u_h$ on Γ. It is also remarkable that it is possible that $u \in U_{\mathrm{ad}}$, but $\mathcal{P}_h u \notin U_{\mathrm{ad}}$; see Figure 1a.

To define Carstensen interpolation, denote $N_\Gamma = \dim Y_h^\Gamma$ and $\{x_j\}_{j=1}^{N_\Gamma}$ the boundary nodes of the triangulation. For $1 \le i \le N_\Gamma$, define e_i as the unique function in Y_h^Γ such that $e_i(x_j) = \delta_{i,j}$. We define

$$\mathcal{I}_h u = \sum_{j=1}^{N_\Gamma} \frac{(u, e_j)_\Gamma}{(1, e_j)_\Gamma} e_j.$$

For the scalar product in Y_h^Γ

$$(u_h, v_h)_h = \sum_{j=1}^{N_\Gamma} u_j v_j (1, e_j)_\Gamma, \tag{13}$$

it is straightforward to deduce the identity

$$(\mathcal{I}_h u, v_h)_h = (u, v_h)_\Gamma\ \forall u \in L^2(\Gamma) \text{ and } v_h \in Y_h^\Gamma. \tag{14}$$

The Carstenen interpolant of a function $u_h \in Y_h^\Gamma$ need not coincide with itself, so in general $y_h^{\mathcal{I}_h}(u_h) \neq u_h$ on Γ. On the other hand, $u \in U_{\mathrm{ad}}$ implies $\mathcal{I}_h u \in U_{\mathrm{ad}}$.

In the following result, we recall the concept of variational discrete normal derivative introduced in [13], which we denote $\partial^{\mathcal{P}_h}_{\nu_{A^*}} \phi_h$ because it will match the state equation when we use $\mathcal{P}_h$ and introduce a new one, which we denote $\partial^{\mathcal{I}_h}_{\nu_{A^*}} \phi_h$ because it will match the state equation when we use $\mathcal{I}_h$. We follow the same convention as for $y_h(u)$ and $F_h(u)$ and write $\partial^h_{\nu_{A^*}} \phi_h$ to refer to common properties of both $\partial^{\mathcal{P}_h}_{\nu_{A^*}} \phi_h$ and $\partial^{\mathcal{I}_h}_{\nu_{A^*}} \phi_h$.

Theorem 3.1. *Consider $g \in L^2(\Omega)$, and let $\phi_h \in Y_{h0}$ be the solution of*

$$a(\eta_h, \phi_h) = (g, \eta_h) \ \forall \eta_h \in Y_{h0}.$$

Then, there exists a unique $\partial^{\mathcal{P}_h}_{\nu_{A^*}} \phi_h \in Y_h^\Gamma$ solution of

$$(\partial^{\mathcal{P}_h}_{\nu_{A^*}} \phi_h, \zeta_h)_\Gamma = a(\zeta_h, \phi_h) - (g, \zeta_h) \ \forall \zeta_h \in Y_h,$$

and a unique $\partial^{\mathcal{I}_h}_{\nu_{A^*}} \phi_h \in Y_h^\Gamma$ solution of

$$(\partial^{\mathcal{I}_h}_{\nu_{A^*}} \phi_h, \zeta_h)_h = a(\zeta_h, \phi_h) - (g, \zeta_h) \ \forall \zeta_h \in Y_h.$$

Proof. Existence and uniqueness of $\partial^{\mathcal{P}_h}_{\nu_{A^*}} \phi_h$ is proved in [13, Proposition 4.2].

For the second part, we decompose $Y_h = Y_{h0} \oplus Y_h^\partial$, where Y_h^∂ is isomorphic to Y_h^Γ. If $\zeta_h \in Y_{h0}$, then the equation is trivially $0 = 0$, so we are left with the functions in Y_h^∂.

Abusing notation, we denote for $j = 1, \ldots, N_\Gamma$, $e_j \in Y_h^\partial$ the function that coincides with $e_j \in Y_h^\Gamma$ and is zero in the interior nodes of the triangulation. The system defining the coefficients of $\partial^{\mathcal{I}_h}_{\nu_{A^*}} \phi_h$ in this basis is a diagonal system, that trivially has a unique solution. □

The utility of the new discrete normal derivative is shown in the following Lemma. It tells us how to compute exactly the derivative of the discrete functional F_h when we use the Carstensen interpolant to approximate the boundary data of the state equation.

Lemma 3.2. *For every $u, v \in L^2(\Gamma)$*

$$F_h'(u)v = (-\partial^h_{\nu_{A^*}} \varphi_h(u), v)_\Gamma,$$

where $\varphi_h(u)$ is the solution of

$$a(\eta_h, \varphi_h) = (y_h(u) - y_d, \eta_h) \ \forall \eta_h \in Y_{h0}. \tag{15}$$

Further, for every $u, v \in L^2(\Gamma)$ and $v_h \in Y_h^\Gamma$

$$F_h^{\mathcal{I}_h\prime}(u)v = (-\partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h(u), \mathcal{I}_h v)_h, \quad F_h^{\mathcal{I}_h\prime}(u)v_h = (-\mathcal{I}_h \partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h^{\mathcal{I}_h}(u), v_h)_h.$$

Proof. The property for $F_h^{\mathcal{P}_h}$ is well known; see [13, eq. (4.6)].

Let us prove the expressions related to the Carstensen interpolant approach. First we apply the chain rule; next, we use the definition of discrete normal derivative related to the Carstensen interpolant of the Dirichlet data given in Theorem 3.1 together with the fact that $y_h^{\mathcal{I}_h}(v) \equiv \mathcal{I}_h v$ on Γ; in a third step, we use the fact that $\varphi_h^{\mathcal{I}_h}(u) \in Y_{h0}$ together with the discrete state equation (12); and, finally, we apply (14) to obtain:

$$\begin{aligned} F_h^{\mathcal{I}_h\prime}(u)v =& (y_h^{\mathcal{I}_h}(u) - y_d, y_h^{\mathcal{I}_h}(v)) = (-\partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h^{\mathcal{I}_h}(u), \mathcal{I}_h v)_h + a(y_h^{\mathcal{I}_h}(v), \varphi_h^{\mathcal{I}_h}(u)) \\ =& (-\partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h^{\mathcal{I}_h}(u), \mathcal{I}_h v)_h = (-\partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h^{\mathcal{I}_h}(u), v)_\Gamma. \end{aligned}$$

If, further $v_h \in Y_h^\Gamma$, applying again (14),

$$F_h^{\mathcal{I}_h\prime}(u)v_h = (-\partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h^{\mathcal{I}_h}(u), v_h)_\Gamma = (-\mathcal{I}_h \partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi_h^{\mathcal{I}_h}(u), v_h)_h.$$

□

We collect in the following results the approximation properties of Carstensen interpolant, the finite element approximation of the state equation and the discrete variational normal derivative. Other error estimates for the Carstensen quasi interpolation in negative Sobolev norms can be found in [21]

Lemma 3.3. *There exists $C > 0$ such that*

$$\|u - \mathcal{I}_h u\|_{L^2(\Gamma)} \leq C h^{1/2} \|u\|_{H^{1/2}(\Gamma)} \ \forall u \in H^{1/2}(\Gamma), \tag{16}$$

$$\|u - \mathcal{I}_h u\|_{H^{-1/2}(\Gamma)} \leq C h^{1/2} \|u\|_{L^2(\Gamma)} \ \forall u \in L^2(\Gamma). \tag{17}$$

Proof. Using the convexity of s^2 together with $0 \leq e_j \leq 1$ and $\sum_{j=1}^{N_\Gamma} e_j = 1$, we have that for every $u_h = \sum_{j=1}^{N_\Gamma} u_j e_j \in Y_h^\Gamma$,

$$\|u_h\|^2_{L^2(\Gamma)} = \int_\Gamma \left(\sum_{j=1}^{N_\Gamma} u_j e_j(x) \right)^2 dx \leq \int_\Gamma \sum_{j=1}^{N_\Gamma} u_j^2 e_j(x) dx = (u_h, u_h)_h.$$

Using this, (14), and Cauchy-Schwarz inequality, we deduce that for all $u \in L^2(\Gamma)$,

$$\|\mathcal{I}_h u\|^2_{L^2(\Gamma)} \leq (\mathcal{I}_h u, \mathcal{I}_h u)_h = (u, \mathcal{I}_h u)_\Gamma \leq \|\mathcal{I}_h u\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)},$$

and hence

$$\|\mathcal{I}_h u\|_{L^2(\Gamma)} \leq \|u\|_{L^2(\Gamma)}. \tag{18}$$

So we also have trivially that

$$\|u - \mathcal{I}_h u\|_{L^2(\Gamma)} \leq 2\|u\|_{L^2(\Gamma)} \ \forall u \in L^2(\Gamma).$$

In [5, Theorem 3.1.2] it is proved that

$$\|u - \mathcal{I}_h u\|_{L^2(\Gamma)} \leq C h \|u\|_{H^1(\Gamma)} \ \forall u \in H^1(\Gamma).$$

Estimate (16) follows, therefore, by interpolation.

Estimate (17) can now be obtained by duality. Since we do not have orthogonality with respect to the inner product in $L^2(\Gamma)$, we write the argument in detail. Let $V = \{v \in H^{1/2}(\Gamma) : \|v\|_{H^{1/2}(\Gamma)} = 1.\}$. Using that $u \in L^2(\Gamma)$, (14) and (16), we have

$$\begin{aligned}
\|u - \mathcal{I}_h u\|_{H^{-1/2}(\Gamma)} &= \sup_{v \in V} \langle u - \mathcal{I}_h u, v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \sup_{v \in V} (u - \mathcal{I}_h u, v)_\Gamma \\
&= \sup_{v \in V} (u - \mathcal{I}_h u, v - \mathcal{I}_h v)_\Gamma + (u, \mathcal{I}_h v)_\Gamma - (\mathcal{I}_h u, \mathcal{I}_h v)_\Gamma \\
&= \sup_{v \in V} (u - \mathcal{I}_h u, v - \mathcal{I}_h v)_\Gamma + (\mathcal{I}_h u, \mathcal{I}_h v)_h - (\mathcal{I}_h u, \mathcal{I}_h v)_\Gamma \\
&= \sup_{v \in V} (u, v - \mathcal{I}_h v)_\Gamma - (\mathcal{I}_h u, v - \mathcal{I}_h v)_\Gamma + (\mathcal{I}_h u, \mathcal{I}_h v)_h - (\mathcal{I}_h u, \mathcal{I}_h v)_\Gamma \\
&= \sup_{v \in V} (u, v - \mathcal{I}_h v)_\Gamma \leq \sup_{v \in V} \|u\|_{L^2(\Gamma)} \|v - \mathcal{I}_h v\|_{L^2(\Gamma)} \\
&\leq \sup_{v \in V} \|u\|_{L^2(\Gamma)} h^{1/2} \|v\|_{H^{1/2}(\Gamma)} = C h^{1/2} \|u\|_{L^2(\Gamma)}.
\end{aligned}$$

□

Lemma 3.4. *Consider $u \in L^2(\Gamma)$ and let y_u and $y_h(u)$ be respectively the transposition solution of (4) and the solution of (12). Then, there exists $C > 0$ independent of u such that*

$$\|y_u - y_h(u)\|_{L^2(\Omega)} \leq Ch^{1/2}\|u\|_{L^2(\Gamma)}. \tag{19}$$

Proof. This estimate is proved in [2, Corollary 3.3] for both choices of $\pi_h u$. The proof in that reference relies on duality techniques, the regularity of the solution and an estimate in [17, Lemma 3.2] of the discrete harmonic operator. All the steps can be repeated successfully for a general operator A using the discrete operator $A_h \pi_h u = y_h(u)$ instead of the discrete harmonic operator. The estimate for this operator follows in the same way as in [17] because y_u satisfies the same regularity requirements as the harmonic extension of u. □

Theorem 3.5. *Let g be an element of $L^2(\Omega)$. Consider $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$ the unique solution of*

$$A^*\phi = g \text{ in } \Omega,\ \phi = 0 \text{ on } \Gamma$$

and $\phi_h \in Y_{h0}$ its finite element approximation satisfying

$$a(\eta_h, \phi_h) = (g, \eta_h)\ \forall \eta_h \in Y_{h0}.$$

Then, there exists $C > 0$ independent of g such that

$$\|\partial_{\nu_{A^*}}\phi - \partial^h_{\nu_{A^*}}\phi_h\|_{L^2(\Gamma)} \leq Ch^{1/2}\|g\|_{L^2(\Omega)}, \tag{20}$$

$$\|\partial^h_{\nu_{A^*}}\phi_h\|_{H^{1/2}(\Gamma)} \leq C\|g\|_{L^2(\Omega)}. \tag{21}$$

Proof. Estimates (20) and (21) for $\partial^{\mathcal{P}_h}_{\nu_{A^*}}\phi_h$ are obtained in [13, Theorem 5.7 and Corollary 5.8].

Let us prove both estimates for $\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h$. By the triangle inequality

$$\|\partial_{\nu_{A^*}}\phi - \partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h\|_{L^2(\Gamma)} \leq \|\partial_{\nu_{A^*}}\phi - \mathcal{I}_h\partial_{\nu_{A^*}}\phi\|_{L^2(\Gamma)} + \|\mathcal{I}_h\partial_{\nu_{A^*}}\phi - \partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h\|_{L^2(\Gamma)}. \tag{22}$$

From (16) and Lemma 2.1, we have that

$$\|\partial_{\nu_{A^*}}\phi - \mathcal{I}_h\partial_{\nu_{A^*}}\phi\|_{L^2(\Gamma)} \leq Ch^{1/2}\|\partial_{\nu_{A^*}}\phi\|_{H^{1/2}(\Gamma)} \leq Ch^{1/2}\|g\|_{L^2(\Omega)}. \tag{23}$$

Let us estimate the $L^2(\Gamma)$-norm of $e_h = \mathcal{I}_h\partial_{\nu_{A^*}}\phi - \partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h \in Y_h^\Gamma$. First we notice that

$$\begin{aligned}\|e_h\|^2_{L^2(\Gamma)} =&(e_h, e_h)_\Gamma = (e_h, \mathcal{I}_h e_h)_h = (\mathcal{I}_h\partial_{\nu_{A^*}}\phi, \mathcal{I}_h e_h)_h - (\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h, \mathcal{I}_h e_h)_h \\ =&(\partial_{\nu_{A^*}}\phi, \mathcal{I}_h e_h)_\Gamma - (\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h, \mathcal{I}_h e_h)_h.\end{aligned}$$

Now we are going to compute the two terms in the right hand side of this equality. For the first one, we multiply both terms of the equation $A\phi = g$ by $y_h^{\mathcal{I}_h}(e_h) \in Y_h \subset H^1(\Omega)$ and apply Green's formula, to obtain

$$(g, y_h^{\mathcal{I}_h}(e_h)) = a(y_h^{\mathcal{I}_h}(e_h), \phi) - (\partial_{\nu_{A^*}}\phi, \mathcal{I}_h e_h)_\Gamma.$$

For the second one, we use the definition of the discrete normal derivative $\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h$ given in Theorem 3.1:

$$(g, y_h^{\mathcal{I}_h}(e_h)) = a(y_h^{\mathcal{I}_h}(e_h), \phi_h) - (\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h, \mathcal{I}_h e_h)_h.$$

Gathering the last three equalities, we obtain that

$$\begin{aligned}\|e_h\|^2_{L^2(\Gamma)} =& a(y_h^{\mathcal{I}_h}(e_h), \phi-\phi_h) \leq \|\phi-\phi_h\|_{H^1(\Omega)}\|y_h^{\mathcal{I}_h}(e_h)\|_{H^1(\Omega)}\\ \leq& Ch\|g\|_{L^2(\Omega)}\|e_h\|_{H^{1/2}(\Gamma)} \leq Ch\|g\|_{L^2(\Omega)}h^{-1/2}\|e_h\|_{L^2(\Gamma)}.\end{aligned}$$

where we have used the $H^2(\Omega)$ regularity of ϕ (see Lemma 2.1), the classical finite element error estimate, that

$$\|y_h^{\mathcal{I}_h}(e_h)\|_{H^1(\Omega)} \leq C\|\mathcal{I}_h e_h\|_{H^{1/2}(\Gamma)} \leq C\|e_h\|_{H^{1/2}(\Gamma)}$$

(see [4, Lemma 3.2] for the first estimate and apply [5, Theorem 3.1] and interpolation for the second estimate), and an inverse inequality. So we have that

$$\|e_h\|_{L^2(\Gamma)} \leq Ch^{1/2}\|g\|_{L^2(\Omega)}. \tag{24}$$

Estimate (20) follows from (22), (23) and (24).

To prove (21) we write introduce the term $\mathcal{I}_h\partial_{\nu_{A^*}}\phi$, which belongs to $H^{1/2}(\Gamma)$, and apply an inverse inequality, the continuity of $\mathcal{I}_h$ in $H^{1/2}(\Gamma)$, and (24) to obtain

$$\begin{aligned}\|\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h\|_{H^{1/2}(\Gamma)} \leq& \|\mathcal{I}_h\partial_{\nu_{A^*}}\phi\|_{H^{1/2}(\Gamma)} + \|\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h - \mathcal{I}_h\partial_{\nu_{A^*}}\phi\|_{H^{1/2}(\Gamma)}\\ \leq& C\|\partial_{\nu_{A^*}}\phi\|_{H^{1/2}(\Gamma)} + Ch^{-1/2}\|\partial^{\mathcal{I}_h}_{\nu_{A^*}}\phi_h - \mathcal{I}_h\partial_{\nu_{A^*}}\phi\|_{L^2(\Gamma)} \leq C\|g\|_{L^2(\Omega)}.\end{aligned}$$

Here, the continuity of $\mathcal{I}_h$ in $H^{1/2}(\Gamma)$ follows by interpolation between the estimate for the continuity in $L^2(\Gamma)$ proved in (18) and the continuity in $H^1(\Gamma)$ proved in [5, Theorem 3.13]. □

To obtain error estimates for the control variable, we will use the following result.

Corollary 3.6. *Consider $u \in U_{\rm ad}$, and let $\varphi_u \in H^1_0(\Omega)\cap H^2(\Omega)$ and $\varphi_h(u) \in Y_{h0}$ be the solutions respectively of* (7) *and* (15). *Then, there exists $C > 0$ independent of u such that*

$$\|\partial_{\nu_{A^*}}\varphi_u - \partial^h_{\nu_{A^*}}\varphi_h(u)\|_{L^2(\Gamma)} \leq Ch^{1/2}, \tag{25}$$

$$\|\partial^h_{\nu_{A^*}}\varphi_h(u)\|_{H^{1/2}(\Gamma)} \leq C. \tag{26}$$

Proof. Define $\phi \in H^2(\Omega)\cap H^1_0(\Omega)$ the solution of

$$A^*\phi = y_h(u) - y_d \text{ in } \Omega,\ \phi = 0 \text{ on } \Gamma.$$

We have that $\varphi_h(u)$ is the finite element approximation of ϕ. Estimate (25) follows from the triangle inequality, Lemma 2.1, estimate (19), (20) and the fact that $U_{\rm ad}$ is bounded.

$$\begin{aligned}\|\partial_{\nu_{A^*}}\varphi_u - \partial^h_{\nu_{A^*}}\varphi_h(u)\|_{L^2(\Gamma)} \leq& \|\partial_{\nu_{A^*}}\varphi_u - \partial_{\nu_{A^*}}\phi\|_{L^2(\Gamma)} + \|\partial_{\nu_{A^*}}\phi - \partial^h_{\nu_{A^*}}\varphi_h(u)\|_{L^2(\Gamma)}\\ \leq& C\|y_u - y_h(u)\|_{L^2(\Omega)} + Ch^{1/2}\|y_h(u)-y_d\|_{L^2(\Omega)} \leq Ch^{1/2}.\end{aligned}$$

Estimate (26) is deduced directly from (21) and the fact that, taking into account that $U_{\rm ad}$ is bounded, we have $\|y_h(u)-y_d\|_{L^2(\Omega)} \leq \|y_h(u)\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}$. □

4 Variational approach

For the variational approach, we consider both techniques to impose the boundary data. We will label with a subscript v the quantities related to this approach.

Taking into account that F_h may stand both for $F_h^{\mathcal{I}_h}$ or $F_h^{\mathcal{P}_h}$, we select $\pi_h \in \{\mathcal{I}_h, \mathcal{P}_h\}$ and consider the semi-discretized problem

$$\min_{u\in U_{\rm ad}} J_{\rm v}(u) := F_h(u) + \mu j(u). \tag{P$_{\rm v}$}$$

It it standard to check that (P$_{\rm v}$) has a solution $\bar u_{\rm v} \in U_{\rm ad}$. First order necessary conditions read as follows.

Theorem 4.1. *Let $\bar u_{\rm v} \in U_{\rm ad}$ be a solution of* (P$_{\rm v}$)*. Then, there exists a unique triplet $\bar y_{h,{\rm v}} \in Y_h$, $\bar\varphi_{h,{\rm v}} \in Y_{h0}$ and $\bar\lambda_{\rm v} \in \partial j(\bar u_{\rm v})$ such that*

$$a(\bar y_{h,{\rm v}}, \eta_h) = 0 \text{ for all } \eta_h \in Y_{h0},\ \bar y_{h,{\rm v}} = \pi_h \bar u_{\rm v} \text{ on } \Gamma. \tag{27a}$$

$$a(\eta_h, \bar\varphi_{h,{\rm v}}) = (\bar y_{h,{\rm v}} - y_d, \eta_h) \text{ for all } \eta_h \in Y_{h0}. \tag{27b}$$

$$(-\partial^h_{\nu_{A^*}}\bar\varphi_{h,{\rm v}} + \mu\bar\lambda_{\rm v}, u - \bar u_{\rm v})_\Gamma \geq 0\ \forall u \in U_{\rm ad}, \tag{27c}$$

Further,

$$(-\partial^h_{\nu_{A^*}}\bar\varphi_{h,{\rm v}}, u - \bar u_{\rm v})_\Gamma + \mu j(u) - \mu j(\bar u_{\rm v}) \geq 0\ \forall u \in U_{\rm ad}. \tag{28}$$

The discrete version of Corollary 2.7 follows in the same pointwise way, just replacing $\bar u$ and $\partial_{\nu_{A^*}}\bar\varphi$ by $\bar u_{\rm v}$ and $\partial^h_{\nu_{A^*}}\bar\varphi_{h,{\rm v}}$.

Remark 4.2. Notice that the solution need not be unique. Neither the projection operator $\mathcal{P}_h$ nor the quasi-interpolation operator $\mathcal{I}_h$ are injective. Therefore, if $\bar u_{\rm v}$ is a solution, any other admissible $u_{\rm v}$ such that $\pi_h\bar u_{\rm v} = \pi_h u_{\rm v}$ will satisfy the discrete state equation. If $\mu = 0$, then it is immediate that $u_{\rm v}$ is also a solution of (P$_{\rm v}$). For $\mu > 0$, this will be the case if $u_{\rm v}$ also satisfies $j(u_{\rm v}) = j(\bar u_{\rm v})$.

Unlike other problems discretized variationally, e.g., [8, Theorem 3.2], here the discrete control $\pi_h\bar u_{\rm v}$ may not be a solution of (P$_{\rm v}$). If we choose the $L^2(\Gamma)$-projection, then $\mathcal{P}_h\bar u_{\rm v}$ may overshoot the control constraints and hence it would not be in $U_{\rm ad}$. On the other hand $\mathcal{I}_h\mathcal{I}_h\bar u_{\rm v}$ may be different from $\mathcal{I}_h\bar u_{\rm v}$, so $y_h^{\mathcal{I}_h}(\mathcal{I}_h\bar u_{\rm v})$ would be different from $\bar y_{h,{\rm v}}$.

We have the following error estimate for the state variable.

Theorem 4.3. *Let $\bar u \in U_{\rm ad}$ be the solution of* (P)*, $\bar u_{\rm v} \in U_{\rm ad}$ be a solution of* (P$_{\rm v}$)*, and $\bar y \in H^{1/2}(\Omega)$, $\bar y_{h,{\rm v}} \in Y_h$ be respectively the solutions of the state equations* (9a) *and* (27a)*. Then there exists $C > 0$ such that*

$$\|\bar y - \bar y_{h,{\rm v}}\|_{L^2(\Omega)} \leq Ch^{1/4}\ \forall h > 0.$$

Proof. By the triangle inequality

$$\|\bar y - \bar y_{h,{\rm v}}\|_{L^2(\Omega)} \leq \|\bar y - y_{\bar u_{\rm v}}\|_{L^2(\Omega)} + \|y_{\bar u_{\rm v}} - \bar y_{h,{\rm v}}\|_{L^2(\Omega)}.$$

Using that $\|\bar u_{\rm v}\|_{L^2(\Gamma)} \leq (\beta - \alpha)|\Gamma|^{1/2}$, we deduce from estimate (19) that the second summand is of order $h^{1/2}$. To estimate the first one, using Lemma 2.10 we obtain

$$\begin{aligned}\frac{1}{2}\|y_{\bar u_{\rm v}} - \bar y\|^2_{L^2(\Omega)} &\leq J(\bar u_{\rm v}) - J(\bar u)\\ &= \Big(J(\bar u_{\rm v}) - J_{\rm v}(\bar u_{\rm v})\Big) + \Big(J_{\rm v}(\bar u_{\rm v}) - J_{\rm v}(\bar u)\Big) + \Big(J_{\rm v}(\bar u) - J(\bar u)\Big) = I + II + III.\end{aligned}$$

The second term is negative due to the optimality of $\bar u_{\rm v}$. Estimates for the first and third terms follow in the same way. Let us do III. Using (19), we obtain

$$\begin{aligned}|J_{\rm v}(\bar u) - J(\bar u)| =&|F_h(\bar u) + \mu j(\bar u) - F(\bar u) + \mu j(\bar u)|\\ \le&\int_\Omega \left|(y_h(\bar u) - y_d)^2 - (\bar y - y_d)^2\right| dx\\ =&\int_\Omega |(y_h(\bar u) - \bar y)(y_h(\bar u) + \bar y - 2y_d)|\, dx\\ \le&\|y_h(\bar u) - \bar y\|_{L^2(\Omega)}\|y_h(\bar u) + \bar y - 2y_d\|_{L^2(\Omega)} \le Ch^{1/2}, \end{aligned} \tag{29}$$

and the estimate follows. □

For bang-bang or bang-off-bang solutions satisfying the structure assumption, we have also information on the behaviour of the control.

Theorem 4.4. *Let $\bar u \in U_{\rm ad}$ be the solution of* (P) *and $\bar u_{\rm v} \in U_{\rm ad}$ be a solution of* ($\mathrm{P_v}$). *Suppose that the structure assumption* (H) *in Lemma 2.8 is satisfied. Then, there exists $C > 0$ such that*

$$\|\bar u - \bar u_{\rm v}\|_{L^1(\Gamma)} \le Ch^{1/3}.$$

Proof. Taking $u = \bar u$ in (28), $u = \bar u_{\rm v}$ in (11), and adding up both inequalities, we have

$$\begin{aligned}\frac{1}{4(\beta-\alpha)}&\|\bar u_{\rm v} - \bar u\|^2_{L^1(\Gamma)} \le (\partial^h_{\nu_{A^*}}\bar\varphi_{h,\rm v} - \partial_{\nu_{A^*}}\bar\varphi, \bar u_{\rm v} - \bar u)_\Gamma\\ &=(\partial^h_{\nu_{A^*}}\bar\varphi_{h,\rm v} - \partial_{\nu_{A^*}}\varphi(\bar u_{\rm v}), \bar u_{\rm v} - \bar u)_\Gamma + (\partial_{\nu_{A^*}}\varphi(\bar u_{\rm v}) - \partial_{\nu_{A^*}}\bar\varphi, \bar u_{\rm v} - \bar u)_\Gamma\\ &\le(\partial^h_{\nu_{A^*}}\bar\varphi_{h,\rm v} - \partial_{\nu_{A^*}}\varphi(\bar u_{\rm v}), \bar u_{\rm v} - \bar u)_\Gamma\\ &\le\|\partial^h_{\nu_{A^*}}\bar\varphi_{h,\rm v} - \partial_{\nu_{A^*}}\varphi(\bar u_{\rm v})\|_{L^2(\Gamma)}\|\bar u_{\rm v} - \bar u\|_{L^2(\Gamma)}\\ &\le Ch^{1/2}(\beta-\alpha)^{1/2}\|\bar u_{\rm v} - \bar u\|^{1/2}_{L^1(\Gamma)}, \end{aligned} \tag{30}$$

where we have used estimate (25), that $\|\bar u_{\rm v} - \bar u\|_{L^\infty(\Gamma)} \le \beta - \alpha$ and

$$(\partial_{\nu_{A^*}}\varphi(\bar u_{\rm v}) - \partial_{\nu_{A^*}}\bar\varphi, \bar u_{\rm v} - \bar u) =(F'(\bar u) - F'(\bar u_{\rm v}))(\bar u - \bar u_{\rm v}) = -\|\bar y - y_{\bar u_{\rm v}}\|^2_{L^2(\Omega)} \le 0. \tag{31}$$

This inequality follows from the mean value theorem, the fact that F is quadratic, and (6).

From (30) we have that

$$\|\bar u_{\rm v} - \bar u\|^{3/2}_{L^1(\Gamma)} \le Ch^{1/2},$$

which concludes the proof. □

5 Full discretization

For the full discretization, we only consider the Carstensen interpolant of the boundary data. For $u_h = \sum_{j=1}^{N_\Gamma} u_j e_j \in Y_h^\Gamma$, we define

$$j_h(u_h) = \sum_{j=1}^{N_\Gamma} |u_j|(1, e_j)_\Gamma.$$

We will make use of the following properties:

Lemma 5.1. *For all $u_h \in Y_h^\Gamma$ and all $u \in L^1(\Gamma)$,*

$$j(u_h) \le j_h(u_h), \tag{32}$$
$$j_h(\mathcal{I}_h u) \le j(u). \tag{33}$$

Proof. Take $u_h = \sum_{j=1}^{N_\Gamma} u_j e_j$. Using the triangle inequality, that $e_j(x) \ge 0$ and the linearity of the integral we have:

$$j(u_h) = \int_\Gamma \left|\sum_{j=1}^{N_\Gamma} u_j e_j(x)\right| dx \le \sum_{j=1}^{N_\Gamma} \int_\Gamma |u_j| e_j(x) dx = \sum_{j=1}^{N_\Gamma} |u_j|(e_j,1)_\Gamma = j_h(u_h).$$

Take $u \in L^1(\Gamma)$. Using the definition of $\mathcal{I}_h u$, that $e_j(x) \ge 0$, the triangle inequality for integrals and that $\sum_{j=1}^{N_\Gamma} e_j(x) = 1$ we obtain:

$$\begin{aligned} j_h(\mathcal{I}_h u) &= \sum_{j=1}^{N_\Gamma} \left|\frac{(u,e_j)_\Gamma}{(1,e_j)_\Gamma}\right| (1,e_j)_\Gamma = \sum_{j=1}^{N_\Gamma} \left|\int_\Gamma u(x) e_j(x) dx\right| \le \sum_{j=1}^{N_\Gamma} \int_\Gamma |u(x)| e_j(x) dx \\ &= \sum_{j=1}^{N_\Gamma} \int_\Gamma |u(x)| \sum_{j=1}^{N_\Gamma} e_j(x) dx = j(u) \end{aligned}$$

□

Given a discrete control $u_h \in Y_h^\Gamma$, we denote $\partial_h j_h(u_h)$ the subdifferential of $j_h(u_h)$ with respect to the scalar product $(\cdot,\cdot)_h$ defined in (13). This means that

$$\lambda_h \in \partial_h j_h(u_h) \subset Y_h^\Gamma \text{ if and only if } (\lambda_h, v_h - u_h)_h + j(u_h) - j(v_h) \le 0 \ \forall v_h \in Y_h^\Gamma. \tag{34}$$

Lemma 5.2. *Consider $u_h = \sum_{j=1}^{N_\Gamma} u_j e_j \in Y_h^\Gamma$. Every $\lambda_h \in \partial_h j_h(u_h)$ satisfies*

$$|\lambda_j| \le 1 \text{ for all } j \text{ and } \lambda_j = \operatorname{sign}(u_j) \text{ if } u_j \ne 0. \tag{35}$$

Conversely, every $\lambda_h \in Y_h^\Gamma$ satisfying (35) belongs to $\partial_h j_h(u_h)$.

The proof is straightforward taking into account the diagonal structure of the scalar product $(\cdot,\cdot)_h$.

Consider the discrete problem

$$\min_{u_h \in U_{h,\mathrm{ad}}} J_h(u_h) := F_h^{\mathcal{I}_h}(u_h) + \mu j_h(u_h). \tag{P$_h$}$$

It it standard to check that (P$_h$) has a unique solution $\bar u_h \in U_{h,\mathrm{ad}}$. First order necessary conditions read as follows.

Theorem 5.3. *Let $\bar u_h \in U_{h,\mathrm{ad}}$ be the solution of (P$_h$). Then, there exists a unique triplet $\bar y_h \in Y_h$, $\bar\varphi_h \in Y_{h0}$ and $\bar\lambda_h \in \partial_h j_h(\bar u_h)$ such that*

$$a(\bar y_h, \eta_h) = 0 \text{ for all } \eta_h \in Y_{h0},\ \bar y_h = \mathcal{I}_h \bar u_h \text{ on } \Gamma. \tag{36a}$$

$$a(\eta_h, \bar\varphi_h) = (\bar y_h - y_d, \eta_h) \text{ for all } \eta_h \in Y_{h0}. \tag{36b}$$

$$(-\mathcal{I}_h \partial_{\nu_{A^*}}^{\mathcal{I}_h} \bar\varphi_h + \mu\bar\lambda_h, u_h - \bar u_h)_h \ge 0 \ \forall u \in U_{h,\mathrm{ad}}. \tag{36c}$$

Further,

$$(-\partial_{\nu_{A^*}}^{\mathcal{I}_h} \bar\varphi_h, u_h - \bar u_h)_\Gamma + \mu j_h(u_h) - \mu j_h(\bar u_h) \ge 0 \ \forall u_h \in U_{h,\mathrm{ad}}. \tag{37}$$

Proof. The proof follows the same lines as that of [7, Theorem 3]. Equation (36c) follows from the last expression for the derivative of $F_h^{\mathcal{I}_h}$ obtained in Lemma 3.2 and (34). □

Denote now $\bar u_h = \sum_{j=1}^{N_\Gamma} \bar u_j e_j$, $\bar\phi_h = \mathcal{I}_h \partial_{\nu_{A^*}}^{\mathcal{I}_h} \bar\varphi_h = \sum_{j=1}^{N_\Gamma} \bar\phi_j e_j$, and $\bar\lambda_h = \sum_{j=1}^{N_\Gamma} \bar\lambda_j e_j$. The sparsity properties of the components of the optimal control follow from (36c) and the "diagonal" structure of the scalar product $(\cdot,\cdot)_h$ that allows us to write (36c) as

$$(\bar\phi_j + \mu\bar\lambda_j)(t - \bar u_j) \geq 0 \ \forall t \in [\alpha, \beta]$$

for every $j \in \{1, \dots, N_\Gamma\}$. This is not possible using $\mathcal{P}_h$.

Corollary 5.4. *If $\bar\phi_j < -\mu$, then $\bar u_j = \alpha$. If $\bar\phi_j > \mu$, then $\bar u_j = \beta$.*
If $\mu > 0$, then

$$\bar\lambda_j = \mathrm{Proj}_{[-1,1]}\left(\frac{\bar\phi_j}{\mu}\right).$$

If, further, $\alpha < 0 < \beta$ and $|\bar\phi_j| < \mu$, then $\bar u_j = 0$.

The above result does not only show how the sparsity properties of the optimal control are inherited by its discrete approximation, but is also the key to build an efficient optimization algorithm.

Next we prove convergence of the solutions of the discrete problems to the solution of (P) and error estimates. We start with the optimal states.

Theorem 5.5. *Let $\bar u \in U_{\mathrm{ad}}$ be the solution of* (P)*, $\bar u_h \in U_{h,\mathrm{ad}}$ be the solution of* (P_h) *and let $\bar y \in H^{1/2}(\Omega)$ and $\bar y_h \in Y_h$ be respectively the solutions of the state equations* (9a) *and* (36a)*. Then there exists $C > 0$ such that*

$$\|\bar y - \bar y_h\|_{L^2(\Omega)} \leq C h^{1/4} \ \forall h > 0.$$

Proof. By the triangle inequality

$$\|\bar y - \bar y_h\|_{L^2(\Omega)} \leq \|\bar y - y_{\bar u_h}\|_{L^2(\Omega)} + \|y_{\bar u_h} - \bar y_h\|_{L^2(\Omega)}.$$

Using that $\|\bar u_h\|_{L^2(\Gamma)} \leq (\beta - \alpha)|\Gamma|^{1/2}$, we deduce from estimate (19) that the second summand is of order $h^{1/2}$. To estimate the first one, using Lemma 2.10, the optimality of $\bar u_h$, the fact that $\mathcal{I}_h \bar u \in U_{h,\mathrm{ad}}$, (32) and (33) we have that

$$\begin{aligned}
\frac{1}{2}\|y_{\bar u_h} - \bar y\|^2_{L^2(\Omega)} &\leq J(\bar u_h) - J(\bar u) \\
&= \Big(J(\bar u_h) - J_h(\bar u_h)\Big) + \Big(J_h(\bar u_h) - J_h(\mathcal{I}_h \bar u)\Big) \\
&\quad + \Big(J_h(\mathcal{I}_h \bar u) - F(\mathcal{I}_h \bar u) + F(\mathcal{I}_h \bar u) - J(\bar u)\Big) \\
&\leq \Big(F(\bar u_h) + \mu j(\bar u_h) - F_h^{\mathcal{I}_h}(\bar u_h) - \mu j_h(\bar u_h)\Big) \\
&\quad + \Big(F_h^{\mathcal{I}_h}(\mathcal{I}_h \bar u) + \mu j_h(\mathcal{I}_h \bar u) - F(\mathcal{I}_h \bar u) + F(\mathcal{I}_h \bar u) - F(\bar u) - \mu j(\bar u))\Big) \\
&\leq \Big(F(\bar u_h) - F_h^{\mathcal{I}_h}(\bar u_h)\Big) + \Big(F_h^{\mathcal{I}_h}(\mathcal{I}_h \bar u) - F(\mathcal{I}_h \bar u)\Big) + \Big(F(\mathcal{I}_h \bar u) - F(\bar u))\Big)
\end{aligned}$$

Estimates for the first and second terms of order $h^{1/2}$ follow as in (29). To estimate the third term, we use Theorem 2.3 and estimate (17)

$$\begin{aligned} F(\mathcal{I}_h\bar u)-F(\bar u) &= \int_\Omega \Big((y_{\mathcal{I}_h\bar u}-y_d)^2-(\bar y-y_d)^2\Big)dx \\ &= \int_\Omega \Big((y_{\mathcal{I}_h\bar u}-\bar y)(y_{\mathcal{I}_h\bar u}+\bar y-2y_d)Big)dx \\ &\le \|y_{\mathcal{I}_h\bar u}-\bar y\|_{L^2(\Omega)}\|y_{\mathcal{I}_h\bar u}+\bar y-2y_d\|_{L^2(\Omega)} \\ &\le C\|\mathcal{I}_h\bar u-\bar u\|_{H^{-1/2}(\Gamma)} \le Ch^{1/2}. \end{aligned}$$

□

Theorem 5.6. *Let $\bar u \in U_{\rm ad}$ be the solution of* (P) *and let $\bar u_h \in U_{h,\rm ad}$ be the solution of* (P_h). *Suppose that the structure assumption* (H) *is satisfied. Then, there exists $C > 0$ such that*

$$\|\bar u-\bar u_h\|_{L^1(\Gamma)} \le Ch^{1/4}.$$

Proof. Testing the enhanced first order optimality condition (11) of (P) for $u = \bar u_h$ and the first order optimality condition (37) of the discrete problem (P_h) for $u_h = \mathcal{I}_h\bar u \in U_{h,\rm ad}$, we have

$$\begin{aligned} (-\partial_{\nu_{A^*}}\bar\varphi, \bar u_h-\bar u)_\Gamma + \quad \mu j(\bar u_h) - \quad \mu j(\bar u) &\ge \frac{1}{4(\beta-\alpha)^2}\|\bar u_h-\bar u\|^2_{L^1(\Gamma)}, \\ (-\partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \mathcal{I}_h\bar u-\bar u_h)_\Gamma + \mu j_h(\mathcal{I}_h\bar u) - \mu j_h(\bar u_h) &\ge 0. \end{aligned}$$

Adding up both inequalities and taking into account (32) and (33) we have

$$\begin{aligned} \frac{1}{4(\beta-\alpha)}\|\bar u_h-\bar u\|^2_{L^1(\Gamma)} &\le (-\partial_{\nu_{A^*}}\bar\varphi, \bar u_h-\bar u)_\Gamma + (-\partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \mathcal{I}_h\bar u-\bar u_h)_\Gamma \\ &= (-\partial_{\nu_{A^*}}\bar\varphi + \partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \bar u_h-\mathcal{I}_h\bar u)_\Gamma + (-\partial_{\nu_{A^*}}\bar\varphi, \mathcal{I}_h\bar u-\bar u)_\Gamma. \end{aligned} \tag{38}$$

For the second term, we have from Lemma 2.1 and estimate (17) that

$$(-\partial_{\nu_{A^*}}\bar\varphi, \mathcal{I}_h\bar u-\bar u)_\Gamma \le \|\partial_{\nu_{A^*}}\bar\varphi\|_{H^{1/2}(\Gamma)}\|\mathcal{I}_h\bar u-\bar u\|_{H^{-1/2}(\Gamma)} \le Ch^{1/2}.$$

Next we insert $\partial_{\nu_{A^*}}\varphi_{\bar u_h}$ and $\bar u$ into the first term to obtain

$$\begin{aligned} (-\partial_{\nu_{A^*}}\bar\varphi + \partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \bar u_h-\mathcal{I}_h\bar u)_\Gamma &= (-\partial_{\nu_{A^*}}\bar\varphi + \partial_{\nu_{A^*}}\varphi_{\bar u_h}, \bar u_h-\mathcal{I}_h\bar u)_\Gamma \\ &\quad + (-\partial_{\nu_{A^*}}\varphi_{\bar u_h} + \partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \bar u_h-\mathcal{I}_h\bar u)_\Gamma \\ &= (-\partial_{\nu_{A^*}}\bar\varphi + \partial_{\nu_{A^*}}\varphi_{\bar u_h}, \bar u_h-\bar u)_\Gamma \\ &\quad + (-\partial_{\nu_{A^*}}\bar\varphi + \partial_{\nu_{A^*}}\varphi_{\bar u_h}, \bar u-\mathcal{I}_h\bar u)_\Gamma \\ &\quad + (-\partial_{\nu_{A^*}}\varphi_{\bar u_h} + \partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \bar u_h-\bar u)_\Gamma \\ &\quad + (-\partial_{\nu_{A^*}}\varphi_{\bar u_h} + \partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h, \bar u-\mathcal{I}_h\bar u)_\Gamma = I+II+III+IV. \end{aligned}$$

As in (31), using (8) the first term is negative:

$$I = -\|y_{\bar u_h}-\bar y\|^2_{L^2(\Omega)} \le 0.$$

An estimate for the second term follows from Lemma 2.1 and (17):

$$II \le \| -\partial_{\nu_{A^*}}\bar\varphi + \partial_{\nu_{A^*}}\varphi_{\bar u_h}\|_{H^{1/2}(\Gamma)} \|\mathcal{I}_h\bar u - \bar u\|_{H^{-1/2}(\Gamma)} \le Ch^{1/2}.$$

To estimate the third term, we first use (25) and that $\|\bar u_h - \bar u\|_{L^\infty(\Gamma)} \le \beta - \alpha$ and in a second step Young's inequality for $p = 4/3$, $q = 4$, to deduce the existence of $\tilde C > 0$, that may depend on α and β but is independent of h, such that

$$\begin{aligned} III \le & \| -\partial_{\nu_{A^*}}\varphi_{\bar u_h} + \partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h\|_{L^2(\Gamma)} \|\bar u_h - \bar u\|_{L^2(\Gamma)} \\ \le & Ch^{1/2}(\beta-\alpha)^{1/2}\|\bar u_h - \bar u\|^{1/2}_{L^1(\Gamma)} \le \tilde C h^{2/3} + \frac{1}{8(\beta-\alpha)}\|\bar u_h - \bar u\|^2_{L^1(\Gamma)}. \end{aligned}$$

For the fourth term we use Lemma 2.1, (26) and (19)

$$IV \le \left(\| -\partial_{\nu_{A^*}}\varphi_{\bar u_h}\|_{H^{1/2}(\Gamma)} + \|\partial^{\mathcal{I}_h}_{\nu_{A^*}}\bar\varphi_h\|_{H^{1/2}(\Gamma)}\right) \|\mathcal{I}_h\bar u - \bar u\|_{H^{-1/2}(\Gamma)} \le Ch^{1/2}.$$

The result follows gathering all the estimates and taking the term $\frac{1}{8(\beta-\alpha)}\|\bar u_h - \bar u\|^2_{L^1(\Gamma)}$ to the left hand side of (38). □

6 Optimization algorithms

Let us consider a Tikhonov regularization of (P). For $\varepsilon > 0$, consider

$$\min_{u\in U_{\rm ad}} J^\varepsilon(u) := J(u) + \frac{\varepsilon}{2}(u,u)_\Gamma, \tag{P$^\varepsilon$}$$

For every $\varepsilon > 0$, problem (P$^\varepsilon$) has a unique solution $u^\varepsilon \in U_{\rm ad}$. We compute (an approximation of) u^{ε_n}, for a sequence $\{\varepsilon_n\} \searrow 0$. Finally, we approach $\bar u \approx u^{\varepsilon_N}$ for some ε_N small; see [19]. At each step, we use an adaptation of the active set algorithm described in [23, Algorithm 2] taking the solution of the previous step as initial guess. For $n = 1$ we take 0 as starting point. Let us briefly describe the algorithm to solve (P$^\varepsilon$).

For every $\varepsilon > 0$, there exists $\lambda^\varepsilon \in \partial j(u^\varepsilon)$ such that

$$(-\partial_{\nu_{A^*}}\varphi_{u^\varepsilon} + \mu\lambda^\varepsilon + \varepsilon u^\varepsilon, u - u^\varepsilon)_\Gamma \ge 0 \ \forall u \in U_{\rm ad}.$$

For $u \in L^2(\Gamma)$, denote $\phi = \partial_{\nu_{A^*}}\varphi_u \in W^{1-1/p,p}(\Gamma) \hookrightarrow C(\Gamma)$ and define the active sets as

$$\begin{aligned} \mathbb{A}_\alpha =& \{x\in\Gamma:\ \phi(x)+\mu \le \varepsilon\alpha\}, \\ \mathbb{J}_- =& \{x\in\Gamma:\ \varepsilon\alpha < \phi(x)+\mu < 0\}, \\ \mathbb{A}_0 =& \{x\in\Gamma:\ |\phi(x)| \le \mu\}, \\ \mathbb{J}_+ =& \{x\in\Gamma:\ 0 < \phi(x)-\mu < \varepsilon\beta\}, \\ \mathbb{A}_\beta =& \{x\in\Gamma:\ \varepsilon\beta \le \phi(x)-\mu\}. \end{aligned} \tag{39}$$

Noticing that on $\mathbb{J}_\pm$ we have that $\lambda^\varepsilon = \pm 1$, that the active sets cover all Γ, and that they are pairwise disjoint, we can write the active set algorithm described in Algorithm 1. Notice that (40) can be written as an unconstrained differentiable linear-quadratic optimal control problem.

Algorithm 1: Active set algorithm for problem (P^ε).

Initialize $k=0$, choose initial point u^k, and compute $y^k = y_{u^k}$ and $\varphi^k = \varphi_{u^k}$
repeat
  Compute $\phi^k = \partial_{\nu_{A^*}} \varphi^k$
  Compute the active sets using (39)
  Solve for $u^{k+1}, y^{k+1}, \varphi^{k+1}$,

$$\begin{aligned} y^{k+1} &= y_{u^{k+1}} & \varepsilon u^{k+1} &= \partial_{\nu_{A^*}} \varphi^{k+1} - \mu \text{ on } \mathbb{J}_+ \\ \varphi^{k+1} &= \varphi_{u^{k+1}} & \varepsilon u^{k+1} &= \partial_{\nu_{A^*}} \varphi^{k+1} + \mu \text{ on } \mathbb{J}_- \\ & & u^{k+1} &= \alpha \text{ on } \mathbb{A}_\alpha \\ & & u^{k+1} &= 0 \text{ on } \mathbb{A}_0 \\ & & u^{k+1} &= \beta \text{ on } \mathbb{A}_\beta \end{aligned} \tag{40}$$

  Set $k = k+1$
until *convergence*

Algorithm 2: Fixed point iteration to solve (40). Variational discretization.

Data: $\pi_h \in \{\mathcal{P}_h, \mathcal{I}_h\}$
Define $u_{\mathbb{A}} = \alpha$ in $\mathbb{A}_\alpha$, $u_{\mathbb{A}} = \beta$ in $\mathbb{A}_\beta$, $u_{\mathbb{A}} = 0$ elsewhere
Initialize $i = 0$, $u^i = u_{\mathbb{A}}$
repeat
  Compute $\phi^i = \partial^h_{\nu_{A^*}} \varphi_h(u^i)$
  $u^{i+1} = \frac{1}{\varepsilon}(\phi^i + \mu)$ on $\mathbb{J}_-$, $u^{i+1} = \frac{1}{\varepsilon}(\phi^i - \mu)$ on $\mathbb{J}_+$, $u^{i+1} = u_{\mathbb{A}}$ on $\mathbb{A}$
  $i = i+1$
until *convergence*

To this, denote $\mathbb{A} = \mathbb{A}_\alpha \cup \mathbb{A}_0 \cup \mathbb{A}_\beta$, define $u_{\mathbb{A}} = \alpha$ in $\mathbb{A}_\alpha$, $u_{\mathbb{A}} = \beta$ in $\mathbb{A}_\beta$, $u_{\mathbb{A}} = 0$ elsewhere and consider the linear space $V = \{u \in L^2(\Gamma) : \ u = 0 \text{ on } \mathbb{A}\}$. Then (40) is equivalent to

$$\begin{aligned} &\text{Find } u_{\mathbb{J}} = \arg\min_{u \in V} \frac{1}{2}\|y_u - (y_d - y_{u_{\mathbb{A}}})\|^2_{L^2(\Omega)} + \mu \int_{\mathbb{J}_+} u dx - \mu \int_{\mathbb{J}_-} u dx + \frac{\varepsilon}{2}(u,u)^2_\Gamma; \\ &u^{k+1} = u_{\mathbb{A}} + u_{\mathbb{J}}. \end{aligned}$$

The reader is referred to [16] for optimization algorithms for differentiable linear-quadratic Dirichlet control problems.

Adaptation of Algorithm 1 to the variational discretization described in Section 4 is straightforward. One just has to select the discretization method $\pi_h \in \{\mathcal{P}_h, \mathcal{I}_h\}$ and replace $\partial_{\nu_{A^*}}$ by the corresponding discrete normal derivative $\partial^h_{\nu_{A^*}}$. To solve the system in (40), a fixed point iteration can be used. See Algorithm 2. It is worth noticing that if we know that the solution is bang-bang or bang-off-bang it is possible to adapt the algorithm described in [14] to solve directly ($\mathrm{P_v}$); see Algorithm 3.

Algorithm 3: Variational discretization. Problem with bang-bang or bang-off-bang solution.

Data: $\pi_h \in \{\mathcal{P}_h, \mathcal{I}_h\}$

Initialize $k = 0$, choose initial point u^k

repeat

  Compute $u_h = \pi_h u^k$

  Use u_h to compute $y^{k+1} = y_h(u^k)$

  Use y^{k+1} to compute $\varphi^{k+1} = \varphi_h(u^k)$

  Compute $\phi^{k+1} = \partial^h_{\nu_{A^*}} \varphi^{k+1}$

  Set $u^{k+1}(x) = \begin{cases} \alpha & \text{if } \phi^{k+1}(x) < -\mu \\ 0 & \text{if } |\phi^{k+1}(x)| < \mu \\ \beta & \text{if } \phi^{k+1}(x) > \mu \end{cases}$

  Set $k = k + 1$

until *convergence*

To solve the full discrete problem, the proper Tikhonov regularization makes use of the inner product introduced in (13). For $\varepsilon > 0$, we consider

$$\min_{u_h \in U_{h,\mathrm{ad}}} J_{h,\varepsilon}(u_h) = J_h(u_h) + \frac{\varepsilon}{2}(u_h, u_h)_h. \qquad (\mathrm{P}_{h,\varepsilon})$$

This approach is known as the *mass lumping*. It has been used, in the framework of distributed controls, in [9] or [18, eq. (4.32)] for problems with sparsity promoting terms. A thorough study for problems without sparsity promoting terms can be found in [22].

For $u_h = \sum_{j=1}^{N_\Gamma} u_j e_j$, denote $\phi_h = \mathcal{I}_h \partial^{\mathcal{I}_h}_{\nu_{A^*}} \varphi^{\mathcal{I}_h}_h(u_h) = \sum_{j=1}^{N_\Gamma} \phi_j e_j$. We define the active sets related to u_h as

$$\begin{aligned} \mathbb{A}_\alpha &= \{j : \ \phi_j + \mu \le \varepsilon\alpha\}, \\ \mathbb{J}_- &= \{j : \ \varepsilon\alpha < \phi_j + \mu < 0\}, \\ \mathbb{A}_0 &= \{j : \ |\phi_j| \le \mu\}, \\ \mathbb{J}_+ &= \{j : \ 0 < \phi_j - \mu < \varepsilon\beta\}, \\ \mathbb{A}_\beta &= \{j : \ \varepsilon\beta \le \phi_j - \mu\}. \end{aligned}$$

Adaptation of Algorithm 1 is now straightforward. In this case, it is possible to solve the discrete version of system (40) directly. Denote $\mathbb{I}$ and $\mathbb{B}$ the sets of interior and boundary nodes, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{B}$ the stiffness (related to the operator A), mass and boundary mass matrices for the finite element method and let $\mathcal{D}$ be the diagonal matrix of size N_Γ such that $\mathcal{D}_{jj} = (e_j, 1)_\Gamma$ for $j \in \mathbb{B}$. We also use $\mathcal{I}$ for the identity matrix. For different sets of indexes $\mathbb{L}$, $\mathbb{K}$, we denote $\mathcal{X}_{\mathbb{L},\mathbb{K}}$ the submatrix of $\mathcal{X}$ formed by the rows indexed by $\mathbb{L}$ and the columns indexed by $\mathbb{K}$. As in MATLAB, we use the colon notation ":" for all the indexes. Vectors representing the discrete functions are denoted in boldface, $\boldsymbol{u}$, $\boldsymbol{y}$, $\boldsymbol{\varphi}$, $\boldsymbol{y}_d$, $\mathbf{1}$, and $\mathbf{0}$. We denote $\mathcal{C} = \mathcal{D}^{-1}\mathcal{B}$ so that the Cartesen interpolant of u_h can be computed by means of $\mathcal{C}\boldsymbol{u}$. Using the vector $\boldsymbol{\xi}$ to represent the discrete normal derivative introduced in Theorem 3.1, the discrete version of system in (40) can be written

$$\begin{aligned}
\mathcal{K}_{\mathbb{I},\mathbb{I}}\boldsymbol{y}_{\mathbb{I}} &= -\mathcal{K}_{\mathbb{I},\mathbb{B}}\mathcal{C}\boldsymbol{u}\\
\mathcal{K}_{\mathbb{I},\mathbb{I}}\boldsymbol{\varphi}_{\mathbb{I}} &= \mathcal{M}_{\mathbb{I},\mathbb{I}}\boldsymbol{y}_{\mathbb{I}} + \mathcal{M}_{\mathbb{I},\mathbb{B}}\mathcal{C}\boldsymbol{u} - \mathcal{M}_{\mathbb{I},:}\boldsymbol{y}_d\\
\mathcal{D}\boldsymbol{\xi} &= \mathcal{K}_{\mathbb{B},\mathbb{I}}\boldsymbol{\varphi}_{\mathbb{I}} - \mathcal{M}_{\mathbb{B},\mathbb{I}}\boldsymbol{y}_{\mathbb{I}} - \mathcal{M}_{\mathbb{B},\mathbb{B}}\mathcal{C}\boldsymbol{u} + \mathcal{M}_{\mathbb{B},:}\boldsymbol{y}_d\\
\varepsilon\mathcal{I}_{\mathbb{J}_-,\mathbb{B}}\boldsymbol{u} &= \mathcal{I}_{\mathbb{J}_-,\mathbb{B}}\mathcal{C}\boldsymbol{\xi} + \mathbf{1}\mu\\
\varepsilon\mathcal{I}_{\mathbb{J}_+,\mathbb{B}}\boldsymbol{u} &= \mathcal{I}_{\mathbb{J}_+,\mathbb{B}}\mathcal{C}\boldsymbol{\xi} - \mathbf{1}\mu\\
\mathcal{I}_{\mathbb{A}_\alpha,\mathbb{B}}\boldsymbol{u} &= \mathbf{1}\alpha \qquad \mathcal{I}_{\mathbb{A}_0,\mathbb{B}}\boldsymbol{u} = \mathbf{0} \qquad \mathcal{I}_{\mathbb{A}_\beta,\mathbb{B}}\boldsymbol{u} = \mathbf{1}\beta
\end{aligned}$$

Finally, we can do $\boldsymbol{\phi} = \mathcal{C}\boldsymbol{\xi}$ to compute the active sets of the next step.

7 Examples

Let Ω be $(0,1)^2$, $A = -\Delta$, $f = 0$. All the examples are solved using a uniform mesh with 256 edges per side.

Example 7.1. Bang-off-bang optimal control

Consider $y_d(x) = 2\text{sign}(x_1 - x_2)$, $-\alpha = \beta = 0.5$ and $\mu = 0.218$. All three approaches are tested: variational with L^2-projection, variational with Carstensen interpolant and full discretization approach ($\varepsilon = 1.7 \times 10^{-15}$). We obtain the following optimal values for the different approximations.

$$J_{\text{v}}^{\mathcal{P}_h}(\bar{u}_{\text{v}}^{\mathcal{P}_h}) \approx 1.817457 \quad J_{\text{v}}^{\mathcal{I}_h}(\bar{u}_{\text{v}}^{\mathcal{I}_h}) \approx 1.817462 \quad J_h(\bar{u}_h) \approx 1.817463$$

Graphs with the optimal control and states, as well as the discrete normal derivatives used, can be found in Figure 1. We clearly observe a bang-off-bang optimal control. You may notice how the jump points appear naturally for the variational approach, (figures 1d and 1e), and also how the use of Carstensen interpolant (Figure 1b) prevents the optimal state from overshooting the control constraints on the boundary, a phenomenon which occurs for the $L^2(\Gamma)$ projection (Figure 1a).

Example 7.2. Optimal control with singular arcs

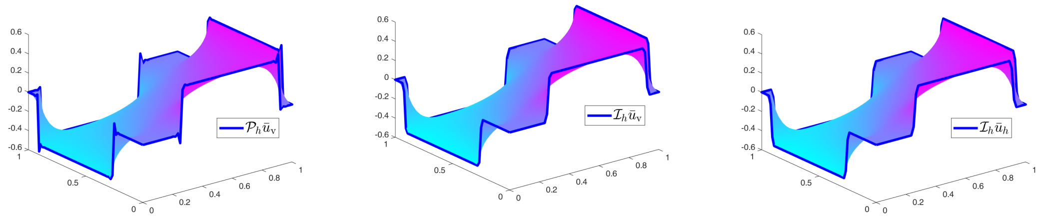
We consider the example presented for the first time in [13, Section 8]. Consider $y_d(x) = |x|^{-2/3}$, $\alpha = -1$ and $\beta = 2$. In that reference, (P^ε) is solved for $\varepsilon = 1$ (and of course $\mu = 0$). The bounds are not attained by the solution and the value of the optimum for that problem is $J^{\varepsilon=1}(u^{\varepsilon=1}) \approx 0.97383$.

We solve problem (P) for $\mu = 0.218$ using the full discrete approach ($\varepsilon = 4.1 \times 10^{-13}$). We obtain $J_h(\bar{u}_h) = 0.87889$. Graphical representation of the solution, where the singular arcs can be clearly seen, is presented in Figure 2.

Example 7.3. A doubtful case

We take the same data as in Example 7.2 but with the bounds $-\alpha = \beta = 0.5$. For the full discretization approach ($\varepsilon = 6.5 \times 10^{-15}$) we obtain the solution shown in Figure 3c with

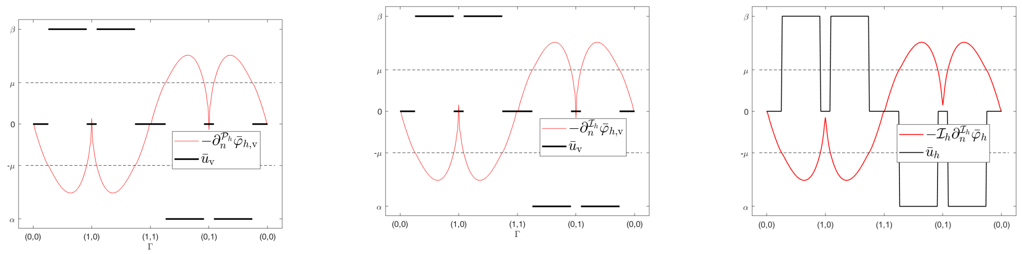
$$J_h(\bar{u}_h) \approx 0.99842.$$



(a) Variational approach with $L^2(\Gamma)$-projection. Optimal state.

(b) Variational approach with Carstensen interpolation. Optimal state.

(c) Full discretization. Optimal state.



(d) Variational approach with $L^2(\Gamma)$-projection. Optimal control and negative discrete normal derivative of the adjoint state.

(e) Variational approach with Carstensen quasi-interpolation. Optimal control and negative new discrete normal derivative of the adjoint state.

(f) Full discretization. Optimal control. Optimal control and Carstensen interpolant of the negative new discrete normal derivative of the adjoint state.

Figure 1: Results for Example 7.1. Solutions of bang-off-bang type.

We seemingly observe a bang-off-bang optimal control, but $\bar u_h$ has a few components $\bar u_j \notin \{\alpha, 0, \beta\}$, marked with black dots in Figure 3c.

Then we try to use Algorithm 3 to look for a bang-off-bang solution of the variational discretization, but it loops forever between two admissible controls $u_{\rm v}^1$ and $u_{\rm v}^2$, changing the switching points back and forth; see figures 3a and 3b. This does not seem an effect of the mesh size, because the switching points of both functions are not near one another. To fix ideas, we set $\pi_h = \mathcal{I}_h$. We obtain the following values.

$$J_{\rm v}(u_{\rm v}^1) \approx 1.00852 \quad J_{\rm v}(u_{\rm v}^2) \approx 1.00323$$

One may notice that the value of $J_h(\bar u_h)$ is smaller than $J_{\rm v}(u_{\rm v}^i)$, for $i = 1, 2$. But, for any $\bar u_{\rm v}$ solution of $(\mathrm{P}_{\rm v})$, using (32), we have

$$J_{\rm v}(\bar u_{\rm v}) \le J_{\rm v}(\bar u_h) = F_h^{\mathcal{I}_h}(\bar u_h) + \mu j(\bar u_h) \le F_h^{\mathcal{I}_h}(\bar u_h) + \mu j_h(\bar u_h) = J_h(\bar u_h).$$

This fact, together with the non-convergence of Algorithm 3, may be an indication that the optimal control is not of the bang-off-bang type.

Experimental orders of convergence. As we said in the introduction, the obtained orders of convergence are far from being optimal. Just to give a sense of this, we have measured some

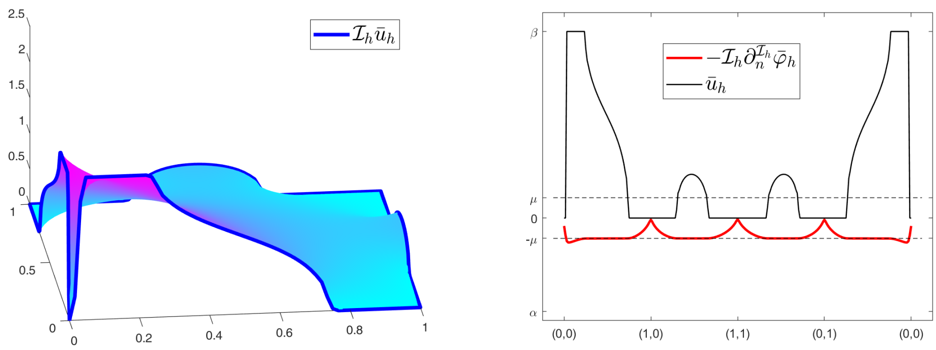
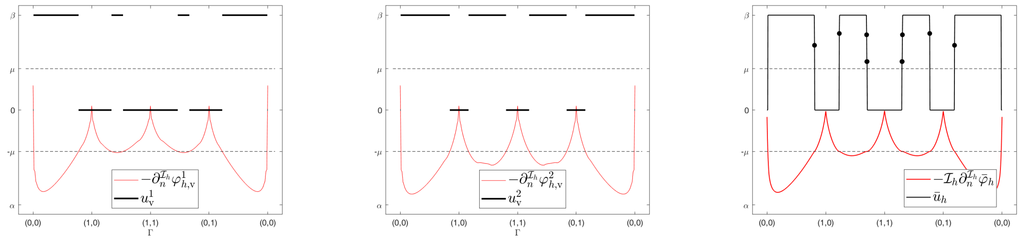


Figure 2: Results for Example 7.2. Solution with singular arcs. Full discretization.



(a) Variational discretization. Control u^1_{v}.

(b) Variational discretization. Control u^2_{v}.

(c) Full discretization. Optimal control.

Figure 3: Results for Example 7.3. A doubtful case.

errors and orders of convergence in the previous examples. We define

$$e_k^y = \|\bar{y}_{h_k} - \bar{y}_{h_{k+1}}\|_{L^2(\Omega)},\ e_k^u = \|\bar{u}_{h_k} - \bar{u}_{h_{k+1}}\|_{L^1(\Gamma)},$$

for the full discretization and analogously for the variational discretization and we report on the experimental order of convergence (EOC) obtained as the slope of the regression line of $\log h_k$ *vs.* $\log e_k$.

For Example 7.1, the state EOC for the variational approach with Carstensen interpolation is 1.04, while the control EOC is 1.01; for the full discretization, we obtain 0.91 and 0.99. In Example 7.2, the control has singular arcs, so we only report on the state EOC, which is 0.78. It is clear that these numbers are far from the expected 1/3 or 1/4 given by the theoretical results.

A On the regulairty of the conormal derivative

Let $\phi_g \in H^2(\Omega) \cap H^1_0(\Omega)$ be the solution of (2). Let us prove in detail that $\partial_{\nu_{A^*}}\phi_g \in H^{1/2}(\Gamma)$.

Let us denote $(S_j)_{0\le j\le N_S}$ the vertices of Ω, numbered counterclockwise and with the convention that $S_0 = S_{N_S}$. Each side of Γ is denoted $\Gamma_j = [S_{j-1}, S_j]$, where $1 \le j \le N_S$. The trace of $\nabla\phi_g$ on Γ belongs to $(H^{1/2}(\Gamma))^2$. Thus the trace of $\nabla\phi_g$ on Γ_j belongs to $(H^{1/2}(\Gamma_j))^2$ and the

Lipschitz regularity of the coefficients a_{ij} implies that $\partial_{\nu_{A^*}}\phi_g \in H^{1/2}(\Gamma_j)$. Moreover we have the estimate

$$\|\partial_{\nu_{A^*}}\phi_g\|_{H^{1/2}(\Gamma_j)} \le C\|g\|_{L^2(\Omega)}.$$

To show that $\partial_{\nu_{A^*}}\phi_g$ belongs to $H^{1/2}(\Gamma)$ we have to analyze the behaviour at the corners $S_j = \Gamma_j \cap \Gamma_{j+1}$. Following [15], we parametrize Γ_{j+1} by setting $x_j(\sigma) = S_j + \frac{\sigma}{|\Gamma_{j+1}|}(S_{j+1} - S_j)$ with $0 \le \sigma \le |\Gamma_{j+1}|$, and Γ_j by $x_j(-\sigma) = S_j - \frac{\sigma}{|\Gamma_j|}(S_j - S_{j-1})$ with $0 \le \sigma \le |\Gamma_j|$. For $0 \le \sigma \le \delta_j = \min\{|\Gamma_j|, |\Gamma_{j+1}|\}$, $x_j(\sigma) \in \Gamma_{j+1}$, $x_j(-\sigma) \in \Gamma_j$ and $|x_j(\sigma) - S_j| = |x_j(-\sigma) - S_j| = \sigma$. According to Theorem 1.5.2.3.c in [15], to prove that $\partial_{\nu_{A^*}}\phi_g \in H^{1/2}(\Gamma_j \cup \Gamma_{j+1})$, we have to show that

$$\int_0^{\delta_j} \frac{|\partial_{\nu_{A^*}}\phi_g(x_j(\sigma)) - \partial_{\nu_{A^*}}\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma < +\infty. \tag{41}$$

First notice that $\|\phi_g\|_{H^2(\Omega)} \le C\|g\|_{L^2(\Omega)}$. Since $\phi_g \in H^2(\Omega)$, then $\nabla\phi_g \in (H^1(\Omega))^2$, and the usual trace theorem says that $\partial_i\phi_g \in H^{1/2}(\Gamma)$ for $i = 1, 2$. As is shown in the first part of the proof of Theorem 1.5.2.3.c) in [15], this implies that

$$\int_0^{\delta_j} \frac{|\partial_i\phi_g(x_j(\sigma)) - \partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma < +\infty, \tag{42}$$

for $i = 1, 2$. We are going to transform the integral in (41) into a combinations of integrals involving the partial derivatives. To do that, without loss of generality, we can suppose that Γ_j is on the negative part of the x axis, S_j is at the origin and $\Gamma_{j+1} \subset \{(-\sigma\, n_2, \sigma\, n_1) \mid\ 0 \le \sigma\}$, so that the normal and tangent vectors to Γ_j and Γ_{j+1} respectively are $\nu_j = (0, -1)^T$, $\tau_j = (1, 0)^T$ and $\nu_{j+1} = (n_1, n_2)^T$, $\tau_{j+1} = (-n_2, n_1)^T$ where $n_1 > 0$ and $n_1^2 + n_2^2 = 1$. The functions $r, s, \gamma_1, \gamma_2 \in C^{0,1}(\Gamma_j \cup \Gamma_{j+1})$ defined as

$$\begin{aligned} \gamma_2 &= -a_{22}, & s &= a_{12} + \frac{n_2+1}{n_1}a_{22}, \\ \gamma_1 &= n_1 a_{11} + n_2 a_{21} + n_2 s, & r &= -\gamma_1 - a_{21}, \end{aligned}$$

satisfy that

$$\begin{aligned} \mathcal{A}^T\nu_j &= r\tau_j \quad + \gamma_1 e_1 + \gamma_2 e_2 \text{ on } \Gamma_j \\ \mathcal{A}^T\nu_{j+1} &= s\tau_{j+1} + \gamma_1 e_1 + \gamma_2 e_2 \text{ on } \Gamma_{j+1}, \end{aligned}$$

where $\mathcal{A}^T$ is the transpose matrix of

$$\mathcal{A}(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},\ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} \partial_{\nu_{A^*}}\phi_g(x_j(\sigma)) - \partial_{\nu_{A^*}}\phi_g(x_j(-\sigma)) = {} & s(x_j(\sigma))\partial_\tau\phi_g(x_j(\sigma)) - r(x_j(-\sigma))\partial_\tau\phi_g(x_j(-\sigma)) \\ & + \gamma_1(x_j(\sigma))\partial_1\phi_g(x_j(\sigma)) - \gamma_1(x_j(-\sigma))\partial_1\phi_g(x_j(-\sigma)) \\ & + \gamma_2(x_j(\sigma))\partial_2\phi_g(x_j(\sigma)) - \gamma_2(x_j(-\sigma))\partial_2\phi_g(x_j(-\sigma))). \end{aligned}$$

Since $\phi_g = 0$ on Γ, the tangential derivatives in that expression are zero. Therefore, we only have to prove for $i = 1, 2$ that

$$\int_0^{\delta_j} \frac{|\gamma_i(x_j(\sigma))\partial_i\phi_g(x_j(\sigma)) - \gamma_i(x_j(-\sigma))\partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma < +\infty.$$

To prove this, we first insert the term $\gamma_i(x_j(\sigma))\partial_i\phi_g(x_j(-\sigma))$ and apply Young's inequality. Next we apply the fundamental theorem of Calculus and take advantage of the Lipschitz regularity of γ_i to obtain

$$\begin{aligned}
&\int_0^{\delta_j} \frac{|\gamma_i(x_j(\sigma))\partial_i\phi_g(x_j(\sigma)) - \gamma_i(x_j(-\sigma))\partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma \\
&\le 2\int_0^{\delta_j} \frac{|\gamma_i(x_j(\sigma))\partial_i\phi_g(x_j(\sigma)) - \gamma_i(x_j(\sigma))\partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma \\
&\quad + 2\int_0^{\delta_j} \frac{|\gamma_i(x_j(\sigma))\partial_i\phi_g(x_j(-\sigma)) - \gamma_i(x_j(-\sigma))\partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma \\
&\le 2\int_0^{\delta_j} \gamma_i(x_j(\sigma))^2 \frac{|\partial_i\phi_g(x_j(\sigma)) - \partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma \\
&\quad + 2\int_0^{\delta_j} \frac{|\gamma_i(x_j(\sigma)) - \gamma_i(x_j(-\sigma))|^2}{\sigma} |\partial_i\phi_g(x_j(-\sigma))|^2 d\sigma \\
&\le 2\|\gamma_i\|^2_{L^\infty(\Gamma_{j+1})} \int_0^{\delta_j} \frac{|\partial_i\phi_g(x_j(\sigma)) - \partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma \\
&\quad + 2\int_0^{\delta_j} \frac{|\int_{-\sigma}^{\sigma} \gamma'(x_j(s))ds|^2}{\sigma} |\partial_i\phi_g(x_j(-\sigma))|^2 d\sigma \\
&\le 2\|\gamma_i\|^2_{L^\infty(\Gamma_{j+1})} \int_0^{\delta_j} \frac{|\partial_i\phi_g(x_j(\sigma)) - \partial_i\phi_g(x_j(-\sigma))|^2}{\sigma} d\sigma \\
&\quad + 8\|\gamma_i\|^2_{C^{0,1}(\Gamma_j\cup\Gamma_{j+1})} \delta_j \|\partial_i\phi_g\|^2_{L^2(\Gamma_j)}
\end{aligned}$$

which is finite thanks to (42) and the fact that the trace of $\partial_i\phi_g$ is in $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$.

Making the same analysis for each corner, we have proved the claimed regularity.

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