Received: 9 October 2020 | Accepted: 30 April 2021
DOI: 10.1002/mma.7525

SPECIAL ISSUE PAPER

WILEY

On a class of nonlocal parabolic equations of Kirchhoff type: Nonexistence of global solutions and blow-up

Uğur Sert[1] | **Sergey Shmarev**[2]

[1]Department of Mathematics, Faculty of Science, Hacettepe University, Beytepe, Ankara, Turkey

[2]Mathematics Department, University of Oviedo, c/ Federico García Lorca 18, Oviedo, 33007, Spain

Correspondence
Sergey Shmarev, Mathematics Department, University of Oviedo, c/ Federico García Lorca 18, 33007, Oviedo, Spain.
Email: shmarev@uniovi.es

Communicated by: P. Hastö

Funding information
Ministerio de Ciencia e Innovación, España, Grant/Award Number: MTM2017-87162-P; Tübitak, the Scientific and Technological Research Council of Turkey, Grant/Award Number: 2219 scholarship programme

We study the homogeneous Dirichlet problem for the degenerate parabolic equation of the Kirchhoff type

$$u_t - a(\|\nabla u\|_2^2)\Delta u = b(\|u\|_2^2)|u|^{q(x,t)-2}u \text{ in } Q_T = \Omega \times (0,T),$$

where $T > 0$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a smooth bounded domain. The exponent $q(x,t)$ is a measurable function in Q_T with values in an interval $[q^-, q^+] \subset (1,\infty)$. The coefficients $a(\cdot)$, $b(\cdot)$ are continuous functions defined on $\mathbb{R}_+$. It is assumed that $a(s) \to 0$ or $a(s) \to \infty$ as $s \to 0^+$; therefore, the equation degenerates or becomes singular as $\| \nabla u(t)\|_2 \to 0$. We prove the local in time solvability of the problem and derive sufficient conditions of the finite time blow-up of the nonnegative solutions. The upper and lower estimates on the blow-up moment are found.

KEYWORDS
blow-up, Kirchhoff-type problem, local existence, singular parabolic equation, variable nonlinearity

MSC CLASSIFICATION
35K55; 35B44; 35K67; 35K99

1 | INTRODUCTION

This paper addresses the questions of local in time existence and blow-up of solutions of a nonlinear parabolic equation with nonlocal terms. We consider the Dirichlet problem

$$\begin{cases} u_t - a(\|\nabla u\|_2^2)\Delta u = b(\|u\|_2^2)|u|^{q(x,t)-2}u \text{ in } Q_T = \Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$, $T > 0$ is a given finite number, a and b are real-valued functions defined on $\mathbb{R}_+$, and $q(x,t)$ is a measurable function of the argument $z = (x,t) \in Q_T$ with values in an interval $[q^-, q^+] \subset (1,\infty)$.

Equation (1) falls into the class of nonlocal evolution equations, in which the arguments of some terms are functionals of the unknown function. Such equations are often termed the Kirchhoff type equations. This is because an equation (of hyperbolic type) with one of the coefficients given by the Dirichlet energy integral of the unknown function was first proposed by Kirchhoff in 1883 as a model of the transversal oscillation of a string. In this model, the change of the string length caused by oscillation was taken into account.

Nonlocal equations of Kirchhoff-type appear in various context and have been studied by many researchers; see, e.g. previous studies[1–9] and references therein. Among other applications, nonlocal PDEs arise in the mathematical modeling of migration of a bacteria population in a container,[1,10,11] in the combustion theory,[12,13] and in medicine.[14]

The questions of existence, uniqueness, and asymptotic behavior of solutions of the initial and boundary-value problems for the equations

$$u_t - a(l(u))\Delta u = f(x,t), \qquad u_t - a(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = f(x,t),$$

were studied in the series of works.[5–7,15,16] The function a is a continuous function whose argument $l(u)$ is a linear continuous functional on $L^2(\Omega)$, or a continuously differentiable function of the argument $\|\nabla u\|^2_{L^2(\Omega)}$. In these works, the equation is nondegenerate: there exist positive constants $0 < m \le M < \infty$ such that

$$m \le a(s) \le M \ \forall s \in \mathbb{R}. \tag{2}$$

Local or global in time existence of solutions of the diffusion equation with a nonlocal diffusion coefficient, and with a lower order term depending on the solution, was studied[4,17,18] under the nondegeneracy assumption (2). The degenerate case $m = 0$ was considered in Ackleh and Ke.[19]

Global in time solvability of problem (1) for the degenerate equation, i.e., without assumption (2), was proven in Sert and Shmarev[20] under suitable conditions on the growth and regularity of the functions $a(\cdot)$ and $b(\cdot)$ and for a specific range of the variable exponent $q(x,t)$.

Local in time existence of solutions to parabolic equations of Kirchhoff type and the blow-up were studied by a number of authors. In Han and Li[21] and Han et al,[22] the potential well method was applied to study the blow-up of solutions of the equation

$$u_t - \operatorname{div}\left[\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\nabla u\right] = |u|^{q-2}u,$$

with positive constants a, b and a constant exponent $q > 3$. Mingqi et al[23] deal with the class of nonlocal fractional parabolic equations of Kirchhoff type

$$\begin{cases} u_t + M([u]_s^2)(-\Delta)^s u = |u|^{p-2}u, \ \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 \ \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+ \\ u(x,0) = u_0(x) \ \text{in } \Omega \end{cases},$$

where $0 < s < 1 < p < \infty$. The diffusion coefficient $M(\cdot)$ is a continuous function depending on the Gagliardo seminorm $[u]_s$ and can be zero at the origin. It is assumed that $M(s) \ge ms^{\theta-1}$ for all $s \ge 0$ and some constants $m > 0$, $\theta > 1$. Moreover, there exists a constant $\mu \ge 1$ such that

$$\mu \int_0^\sigma M(\tau)\,d\tau \ge M(\sigma)\sigma \ \text{ for all } \ \sigma \ge 0.$$

Mingqi et al[23] prove local in time solvability of the Dirichlet problem and find conditions of the finite-time blow-up of nonnegative solutions.

The Neumann problem for the nonlocal semilinear parabolic equation

$$u_t - \Delta u = |u|^p - \frac{1}{|\Omega|}\int_\Omega u^p\,dx, \ p = const > 1$$

was studied in Jazar and Kiwan.[24] The authors used the convexity arguments to show that the solution blows-up if the initial datum satisfies the conditions

$$\int_\Omega u_0\,dx = 0 \text{ and } E(u_0) = \int_\Omega \left(\frac{1}{2}|\nabla u_0|^2 - \frac{1}{p+1}|u_0|^{p+1}\right) dx \le 0.$$

In Niculescu and Rovenŷţa,[25] this result was generalized to the equations

$$u_t - \Delta u = f(|u|) - \frac{1}{|\Omega|}\int_\Omega f(|u|)\,dx, \ \ (x,t) \in \Omega \times (0, t^*)$$

with the null Neumann boundary condition.

The systems of parabolic equations

$$\partial_t u_i - a_i(p_i(u_1), q_i(u_2))\Delta u_i = f_i(u_1, u_2), \qquad i = 1, 2,$$

where p_i, q_i are linear functionals over $L^{\gamma_{p_i}}(\Omega_{p_i})$, $L^{\gamma_{q_i}}(\Omega_{q_i})$, $\Omega_{p_i}, \Omega_{q_i} \subseteq \Omega$, and a_i are continuous functions, are studied in Ferreira and de Oliveira.[8] It is shown that the system has a global in time solution if a_i satisfy condition (2) and f_i are Lipschitz-continuous functions. If in condition (2) $m = 0$, the solution exists locally in time. The finite time blow-up is proved in the case of convex f_i.

There is a series of papers devoted to study the questions of nonexistence of global in time solutions and the blow-up in solutions of nonlocal equations with variable nonlinearity. The first result is due to Pinasco.[26] He considered the homogeneous Dirichlet problem for the semilinear parabolic equation

$$u_t = \Delta u + f(x, u) \ \text{ in } \ Q_T,$$

with local or nonlocal source terms of the form

$$f(x, u) = a(x)u^{p(x)} \text{ or } f(x, u) = a(x)\int_\Omega u^{p(y)}(y, t)\,dy.$$

An adaptation of the eigenfunction argument of Kaplan[27] shows that for $p^- = \inf_\Omega p(x) > 1$, there exist positive initial functions such that the corresponding solutions blow-up in finite time.

A more general class of nonlocal equations,

$$u_t = \Delta u + b(x, t)u^{p(x,t)}\int_\Omega u^{q(x,t)}\,dx, \ \ (x, t) \in \Omega \times (0, T),$$

was considered in Liu and Yang.[28] The authors employ various methods to show the existence of blow-up solutions with negative or positive initial energy in the equation with $q(x, t) \le 1$. Moreover, asymptotics of the blow-up solutions are obtained. Part of the results in Liu and Yang[28] requires a very special choice of the initial function.

In the present paper, we consider equation (1) with the nonlocal coefficient $a(\cdot)$ not satisfying condition (2) and with the variable nonlocal source. For the sake of convenience of presentation, let us describe the results for the prototype of equation (1):

$$u_t = C_a\|\nabla u\|_{2,\Omega}^{\tau-2}\Delta u + C_b\|u\|_{2,\Omega}^{\beta}|u|^{q(x,t)-2}u, \tag{3}$$

with constant parameters $\tau > 1$, $\beta \ge 0$, $C_a > 0$, $C_b > 0$, and the exponent $q \in C^0(\bar{Q}_T)$, $q \in [q^-, q^+] \subset (1, \infty)$, $q^\pm = const$.

1. We prove first that for every nonnegative $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, there exists $T > 0$ such that problem (1) has a nonnegative bounded solution in the space

$$\mathbf{V}(Q_T) \equiv \{u(x, t) : \ u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^0([0, T]; H_0^1(\Omega)),\ u_t \in L^2(Q_T)\}$$

(a local in time solution) which can be continued to the maximal interval of existence wherein $\|u(\cdot,t)\|_{\infty,\Omega}$ remains bounded. The proof of existence follows the proof of the global in time existence given in Sert and Shmarev.[20] We regularize equation (1) and obtain a solution of the degenerate problem as the limit of solutions of the regularized problems. In turn, the solutions of the regularized problems are obtained as the limits of the sequences of finite-dimensional Galerkin's approximations (Theorem 1).

2. Let us define the blow-up moment t^* as follows:

$$t^* = \sup\{t > 0 : \|u(\cdot,s)\|_{2,\Omega} < \infty \quad \forall s < t\}.$$

In Section 3, we derive the conditions of finite time blow-up for the nonnegative solutions of the *singular* equation (1), i.e., under the assumption that $a(s) \to \infty$ as $s \to 0^+$. This is done by means of adaptation of the eigenfunction method of Kaplan. Every nonnegative solution of equation (3) blows up in a finite time, provided that the initial function is sufficiently "large," and the exponents of nonlinearity satisfy the conditions

$$\tau \in (1,2), \quad \beta \geq 0, \quad \beta + q^- > 2.$$

Moreover, since the exponent $q(x,t)$ is a function, the solution blows up in a finite time if these conditions are fulfilled only on a part of the cylinder Q_T (Theorems 2 and 3).

3. In Section 4, we consider the *degenerate* equation (1) with $q \equiv q(x)$. The eigenfunction method ceases to be applicable when the equation is degenerate, that is, $a(s) \to 0$ as $s \to 0^+$. It is shown that every nonstationary solution of the degenerate equation (1) blows up in a finite time, provided that the initial energy is nonpositive. For the protopype equation (3), this condition reads

$$E(u_0) \equiv \frac{C_a}{\tau}\|\nabla u_0\|_2^\tau - C_b\|u_0\|_2^\beta \int_\Omega \frac{u_0^{q(x)}}{q(x)}\,dx \leq 0, \quad \beta \geq 0, \quad q^- \geq \tau$$

(Theorem 4). In the proof, we use monotonicity of the functional $E(u(t))$ and a differential inequality for the function $\|u(t)\|_{2,\Omega}$.

4. In Section 5, we consider equation (3) in the borderline case $q(x) \equiv 2$ and in the general case (Theorems 5 and 6). In the case $q \equiv 2$, equation (3) admits an explicit solution, which illustrates sharpness of the results obtained.
5. Finally, in Section 6, we derive a lower estimate on the life span of the solution of problem (1). We prove that in the case $2 < q^+ < 2\min\left\{\frac{n}{n-2}, 1+\frac{\tau}{n}\right\}$, the norm $\|u(t)\|_{2,\Omega}$ remains bounded on a finite interval $(0,t^*)$, while in the case $q^+ + \beta + n\left(\frac{1}{\tau} - \frac{1}{2}\right)(q^+ - 2) \leq 2$, it is bounded for all $t < \infty$ (Theorem 7).

Throughout the text, we use the standard notation for the Lebesgue and Sobolev spaces and the corresponding norms. For the functions $v(x,t)$ defined on the cylinder $\Omega \times (0,T)$, we write $\|v(t)\|_2 = \|v(\cdot,t)\|_{L^2(\Omega)}$. The symbol C is used to denote the constants which may be calculated or estimated through the data but whose exact values are unimportant. The value of C may change from line to line even inside the same formula.

2 | EXISTENCE OF LOCAL IN TIME SOLUTIONS

In this section, we derive the conditions of existence of a local in time solution of problem (1).

(H.1) (i) $a : (0,\infty) \mapsto (0,\infty)$, $b : [0,\infty) \mapsto \mathbb{R}$,

$$a(s) \in C^0(0,\infty), \; b(s) \in C^0[0,\infty),$$
$$a(s^2)s \in C^0[0,\infty) \cap C^1(0,\infty), \; \left(a(s^2)s\right)' \geq 0 \text{ for } s > 0;$$

(ii) there exist constants $\tau > 1$, $\sigma > 1$, $\beta \geq 0$, $C_a > 0$, $C_\sigma > 0$, $C_b \geq 0$, $\nu \geq 0$ such that for all $s \geq 0$

$$C_a s^{\frac{\tau}{2}} \leq a(s)s \leq C_\sigma\left(s^{\frac{\sigma}{2}} + s\right), \quad 0 \leq b(s) \leq C_b s^{\frac{\beta}{2}} + \nu.$$

Throughout the rest of the section, we assume that $q(x,t)$ is a measurable function such that

$$1 < q^- \le q(z) \le q^+ < \infty \ \forall \text{a.e.} z \in Q_T$$

with some constants $q^\pm$.

Definition 1. A function $u(z)$ is called a local solution of problem (1) if there exists $\theta > 0$ such that

(i) $u \in \mathbf{V}(Q_\theta)$;
(ii) for every test-function $\phi \in \mathbf{V}(Q_\theta)$

$$\int_{Q_\theta} \left(u_t\phi + a(\|\nabla u\|^2_{2,\Omega})\nabla u \cdot \nabla\phi - b(\|u\|^2_{2,\Omega})|u|^{q(z)-2}u\phi\right) dz = 0; \tag{4}$$

(iii) for every $\psi \in L^2(\Omega)$

$$\int_\Omega (u(x,t) - u_0(x))\,\psi(x)\,dx \to 0 \ \text{as} \ t \to 0^+.$$

2.1 | Regularized nondegenerate problem

Let us fix numbers $\epsilon \in (0,1)$, $M \ge 1$ and consider the Dirichlet problem for the nondegenerate nonlocal equation with the bounded reaction term: find $t^* > 0$ and $u \in \mathbf{V}(Q_{t^*})$ such that

$$\begin{cases} u_t - a_\epsilon(\|\nabla u\|_2^2)\Delta u = b(\|u\|_2^2)f_M(u) \ \text{in} \ Q_{t^*}, \\ u = 0 \ \text{on} \ \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \ \text{in} \ \Omega \end{cases} \tag{5}$$

with the diffusion coefficient

$$a_\epsilon(s) = a(\epsilon + s).$$

The reaction term $f_M(s)$ is defined by the equality

$$f_M(u) = \min\left\{|u^+|^{q(z)-2}u^+, M^{q(z)-1}\right\}, \ u^+ = \max\{0,u\}.$$

The solution of problem (5) is understood in the sense of Definition 1: we look for a number $t^* > 0$ and a function $u \in \mathbf{V}(Q_{t^*})$ such that for every test-function $\phi \in \mathbf{V}(Q_{t^*})$

$$\int_{Q_{t^*}} \left(\phi u_t + a_\epsilon(\|\nabla u\|_2^2)\nabla u \cdot \nabla\phi\right) dz = \int_{Q_{t^*}} b(\|u\|_2^2)f_M(u)\phi\,dz$$

and for every $\zeta \in L^2(\Omega)$

$$\int_\Omega (u(x,t) - u_0(x))\zeta(x)\,dx \to 0 \ \text{as} \ t \to 0.$$

2.2 | Galerkin's approximations

Let us denote by $\{\varphi_k\}_{k=1}^\infty \subset H_0^1(\Omega)$, $\{\lambda_k\}$ the eigenfunctions and eigenvalues of the Dirichlet problem for the Laplace equation: $\varphi_k \in H_0^1(\Omega)$

$$(\nabla\varphi_k, \nabla\varphi)_{2,\Omega} = \lambda_k(\varphi_k, \varphi)_{2,\Omega} \ \forall\varphi \in H_0^1(\Omega).$$

The system $\{\varphi_k\} \subset H_0^1(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$, and $\left\{\frac{1}{\sqrt{\lambda_i}}\varphi_i\right\}$ is an orthonormal basis of $H_0^1(\Omega)$. The solution of problem (5) is sought as the limit of the sequence of functions

$$u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)\varphi_i(x),$$

where the coefficients $g_{im}(t)$ are to be defined. Since the set $\{\varphi_k\}$ is dense in $H_0^1(\Omega)$, for every $u_0 \in H_0^1(\Omega)$,

$$u_0^m = \sum_{k=1}^{m} (u_0, \varphi_k)_{2,\Omega}\varphi_k \overset{H_0^1(\Omega)}{\to} u_0 \text{ as } m \to \infty.$$

The functions g_{km} satisfy the initial conditions

$$g_{km}(0) = (u_0, \varphi_k)_{2,\Omega}, \; k = 1, \dots, m.$$

The functions $g_{km}(t)$ are defined from the system of nonlinear ordinary differential equations

$$(\partial_t u_m, \varphi)_{2,\Omega} + a_\epsilon\left(\|\nabla u_m\|_2^2\right)(\nabla u_m, \nabla\varphi)_{2,\Omega} = \int_\Omega b\left(\|u_m\|_2^2\right) f_M(u_m)\varphi \, dx. \tag{6}$$

Taking $\varphi = \varphi_j$ in (6), we find that (6) is equivalent to the Cauchy problem for the system of m ordinary differential equations for the functions $g_{jm}(t)$:

$$\begin{aligned} &g'_{jm}(t) = -\lambda_j a_\epsilon\left(\|\nabla u_m\|_2^2\right) g_{jm}(t) + b(\|u_m\|_2^2)\int_\Omega f_M(u_m)\varphi_j \, dx, \\ &g_{jm}(0) = (u_0, \varphi_j)_{2,\Omega}, \; j = 1, \dots, m. \end{aligned} \tag{7}$$

Since the functions a, b, f_M are continuous with respect to $g_{im}(t)$, it follows from the Carathéodory existence theorem that for every finite m the Cauchy problem (7) has a solution on an interval $(0, T_m)$.

2.3 | A priori estimates

We will need the following auxiliary assertion.

Lemma 1. *If $y(t)$ satisfies the inequality*

$$y' \le Ay^\alpha + B, \; \alpha > 1, \; y(0) = y_0 > 0, \; A, B = const > 0, \tag{8}$$

then

$$y(t) \le \frac{1}{\left(z_0^{1-\alpha} - 2A(\alpha-1)t\right)^{\frac{1}{\alpha-1}}} \text{ for } t < t^*$$

with

$$t^* = \frac{z_0^{1-\alpha}}{2A(\alpha-1)}, \; z_0 = \max\left\{y_0, (B/A)^{\frac{1}{\alpha}}\right\}.$$

Proof. Let us consider the function

$$z(t) = \frac{1}{\left(z_0^{1-\alpha} - 2A(\alpha-1)t\right)^{\frac{1}{\alpha-1}}},$$

which satisfies the conditions

$$z' = 2Az^{\alpha} \ \text{for} \ t < t^*, \ \ z(0) = z_0 > y_0.$$

Let us choose $z_0 = \max\{y_0, (B/A)^{1/\alpha}\}$ and notice that by virtue of the equation $z'(t) > 0$, that is, $z(t) > z_0$ for all $t \in (0, t^*)$. Subtracting the equation for $z(t)$ from inequality (8), for the function $Y = y - z$ we obtain the differential inequality

$$\begin{aligned} Y'(t) &\leq \alpha A \int_0^1 (\theta y + (1-\theta)z)^{\alpha-1}\, d\theta\, Y(t) + (B - Az^{\alpha}) \\ &\leq \alpha A \left(\int_0^1 (\theta y + (1-\theta)z)^{\alpha-1}\, d\theta \right) Y(t) \ \text{for} \ t < t^*. \end{aligned}$$

Since $Y(0) \leq 0$, the assertion follows from the Gronwall lemma. □

Corollary 1. *If a function y(t) satisfies inequality (8) with* $\alpha = 1$, *it follows from the Gronwall lemma that*

$$y(t) \leq y_0 e^{At} + \frac{B}{A}\left(e^{At} - 1\right).$$

The case $\alpha \in (0, 1)$ *reduces to the case* $\alpha = 1$ *by means of the Young inequality:*

$$y'(t) \leq Ay(t) + (A + B),$$

whence

$$y(t) \leq y_0 e^{At} + \left(1 + \frac{B}{A}\right)\left(e^{At} - 1\right).$$

For the sake of simplicity of notation, throughout the rest of this subsection, we denote $u \equiv u_m$ with a fixed index m.

Lemma 2. *The functions u satisfy the estimates*

$$\sup_{(0,t^*)} \|u(t)\|_2 \leq g(M),$$

where

$$\begin{aligned} &\beta < 1: \ \ g(M) = \left(\|u_0\|_2 + \tfrac{\nu}{C_b} + 1 \right) \exp\left(C_b |\Omega|^{\frac{1}{2}} M^{q^+-1} t^* \right) - \left(1 + \tfrac{\nu}{C_b} \right), \ t^* < \infty \text{ is arbitrary,} \\ &\beta = 1: \ \ g(M) = \left(\|u_0\|_2 + \tfrac{\nu}{C_b} \right) \exp\left(C_b |\Omega|^{\frac{1}{2}} M^{q^+-1} t^* \right) - \tfrac{\nu}{C_b}, \ t^* < \infty \text{ is arbitrary,} \\ &\beta > 1: \ \ g(M) = \frac{\max\left\{ \|u_0\|_2, \frac{\nu}{C_b} \right\}}{\left(1 - 2\max^{\beta-1}\left\{ \|u_0\|_2, \frac{\nu}{C_b} \right\} (\beta-1) C_b |\Omega|^{\frac{1}{2}} M^{q^+-1} t^* \right)^{\frac{1}{\beta-1}}} \end{aligned}$$

with any t^* *satisfying the inequality*

$$\frac{1}{t^*} > 2\left(\max\left\{ \|u_0\|_2, \frac{\nu}{C_b} \right\} \right)^{\beta-1} (\beta - 1) C_b |\Omega|^{\frac{1}{2}} M^{q^+-1}.$$

These estimates do not depend on m and ϵ.

Proof. Multiplying each of Equations (7) by g_{jm} and summing the results for $j = 1, \ldots, m$, we obtain the estimate

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + a_{\epsilon}(\|\nabla u\|_2^2)\|\nabla u\|_2^2 = b(\|u\|_2^2) \int_{\Omega} f_M(u) u\, dx \leq \left(C_b \|u\|_2^{\beta} + \nu \right) |\Omega|^{\frac{1}{2}} M^{q^+-1} \|u\|_2.$$

Omitting the second term on the left-hand side, we obtain the differential inequality for the function $y(t) = \|u(t)\|_2$:

$$y'(t) \le \left(C_b y^\beta(t) + \nu\right) |\Omega|^{\frac{1}{2}} M^{q^+-1} \equiv A y^\beta(t) + B. \tag{9}$$

The assertion follows from Lemma 1 and Corollary 1 because $y(t)$ satisfies inequality (9) with $A = C_b|\Omega|^{\frac{1}{2}} M^{q^+-1}$, $B = \nu|\Omega|^{\frac{1}{2}} M^{q^+-1}$, and $y_0 = \sum_{i=1}^m u_{0i}^2$. □

Lemma 3. *Let t^* be the number from Lemma 2. The functions $u = u_m$ satisfy the uniform in m and ϵ estimates*

$$\|\partial_t u\|_{2,Q_{t^*}} + \sup_{(0,t^*)} \int_0^{\|\nabla u\|_2^2} a_\epsilon(s)\,ds \le Ct^* \tag{10}$$

with the constant C depending on $q^\pm$, β, τ, C_b, $\|u_0\|_{H_0^1(\Omega)}$, M, $g(M)$ and $\int_0^{\|\nabla u_0\|_2^2} a(s)\,ds$.

Proof. Multiplication of jth equation of (7) by g'_{jm} and summation in $j = 1, 2, \dots, m$ leads to the inequality

$$\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\int_0^{\|\nabla u\|_2^2(t)} a_\epsilon(s) ds\right) = b(\|u\|_2^2) \int_\Omega f_M(u) u_t\,dx$$
$$\le b(\|u\|_2^2)\|f_M(u)\|_2 \|u_t\|_2 \le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2} b^2(\|u\|_2^2)\|f_M(u)\|_2^2.$$

The first term on the right-hand side can be absorbed in the left-hand side. To estimate the second term, we notice that since $f_M^2(u) \le M^{2(q^+-1)}$, then

$$b^2(\|u\|_2^2)\|f_M(u)\|_2^2 \le \left(C_b\|u\|_2^\beta + \nu\right)^2 M^{2(q^+-1)}|\Omega|.$$

Estimate (10) follows from Lemma 2 after integration in t. □

Corollary 2. *If $\|u_0\|_2^2 = \mu > 0$, the number t^* can be chosen so small that Lemmas 2 and 3 are fulfilled and, moreover, for all sufficiently large m, the functions $u = u_m$ satisfy the inequalities*

$$\|u(t)\|_2^2 \ge \frac{\mu}{2} \ \textit{for} \ t \in (0, t^*).$$

Proof. By the choice of the initial functions, $\|u_m(\cdot, 0)\|_2^2 = \sum_{j=1}^m g_{jm}^2(0) \to \mu$ as $m \to \infty$. By Lemmas 2 and 3, for every $t < t^*$

$$\|u(t)\|_2^2 = \|u_m(\cdot, 0)\|_2^2 + 2\int_0^t \int_\Omega u(s) u_t(s)\,ds$$
$$\ge \|u_m(\cdot, 0)\|_2^2 - 2\|u_t\|_{2,Q_t}\|u\|_{2,Q_t} \ge \|u_m(\cdot, 0)\|_2^2 - C(M)g(M)\sqrt{t} \ge \frac{\mu}{2}$$

provided that t^* is sufficiently small. □

Lemma 4. *One may choose t^* so small that*

$$\sup_{(0,t^*)} \|\nabla u(t)\|_2^2 + \int_0^{t^*} a_\epsilon(\|\nabla u\|_2^2)\|\Delta u(t)\|_2^2 dt \le C. \tag{11}$$

with a constant $C = C(C_a, C_b, C_*, q^+, M, \beta, \|\nabla u_0\|_2)$ *which does not depend on* m *and* ϵ.

Proof. By virtue of assumptions **(H.1)** on $a(\cdot)$ and Corollary 2,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + a_\epsilon(\|\nabla u\|)\|\Delta u\|_2^2 &= -b(\|u\|_2^2)\int_\Omega f_M(u)\Delta u\,dx \\
&\leq \left(C_b\|u\|_2^\beta + \nu\right)\|\Delta u\|_2\|f_M(u)\|_2 \\
&\leq \frac{1}{2}a_\epsilon(\|\nabla u\|)\|\Delta u\|_2^2 + \frac{1}{2}\frac{\left(C_b\|u\|_2^\beta + \nu\right)^2(\epsilon + \|\nabla u\|_2^2)\|f_M(u)\|_2^2}{a_\epsilon(\|\nabla u\|_2^2)\left(\epsilon + \|\nabla u\|_2^2\right)} \\
&\leq \frac{1}{2}a_\epsilon(\|\nabla u\|)\|\Delta u\|_2^2 + \frac{1}{2C_a}\frac{\left(C_b\|u\|_2^\beta + \nu\right)^2(\epsilon + \|\nabla u\|_2^2)\|f_M(u)\|_2^2}{\left(\epsilon + C_*\|u\|_2^2\right)^{\frac{\tau}{2}}} \\
&\leq \frac{1}{2}a_\epsilon(\|\nabla u\|)\|\Delta u\|_2^2 + \frac{1}{2C_a}\frac{\left(C_b\|u\|_2^\beta + \nu\right)^2(\epsilon + \|\nabla u\|_2^2)\|f_M(u)\|_2^2}{\left(\epsilon + C_*\frac{\mu}{2}\right)^{\frac{\tau}{2}}} \\
&\leq \frac{1}{2}a_\epsilon(\|\nabla u\|)\|\Delta u\|_2^2 + \frac{2^{\frac{\tau}{2}-1}}{(\mu C_*)^{\frac{\tau}{2}}}C_a\left(C_b\|u\|_2^\beta + \nu\right)^2(\epsilon + \|\nabla u\|_2^2)\|f_M(u)\|_2^2
\end{aligned}$$

with the constant C_* from the Poincaré inequality. It follows that

$$\frac{d}{dt}\|\nabla u\|_2^2 + a_\epsilon(\|\nabla u\|_2^2)\|\Delta u\|_2^2 \leq C\left(g^\beta(M) + \nu\right)^2 M^{2(q^+-1)}\left(1 + \|\nabla u\|_2^2\right),$$

$C = C(\mu, \tau, C_a, C_b, C_*, |\Omega|)$, and (11) follows from Corollary 1 applied to the function $\|\nabla u\|_2^2$. □

Corollary 3. *In the conditions of Lemma 4,* $\|\Delta u\|_{2,Q_t}^2 \leq C$ *for the sufficiently small* t.

Proof. It suffices to plug into (11) the inequality

$$a_\epsilon(\|\nabla u\|_2^2) \geq C_a(\epsilon + \|\nabla u\|_2^2)^{\frac{\tau}{2}-1} \geq C_a \begin{cases} (C_*\mu)^{\frac{\tau}{2}-1} & \text{if } \tau \geq 2, \\ (1 + C^2)^{\frac{\tau}{2}-1} & \text{if } \tau < 2 \end{cases}$$

with the constant $C \geq \sup_{(0,t^*)}\|\nabla u(t)\|_2^2$. □

2.4 | Passing to the limit as $m \to \infty$

The a priori estimates of the previous section can be summarized in the following way: there exists a number t^* and a constant C, independent of m and ϵ, such that

$$\begin{aligned}
&\sup_{(0,t^*)}\|u_m(t)\|_2^2 + \sup_{(0,t^*)}\|\nabla u_m(t)\|_2^2 + \|\partial_t u_m\|_{2,Q_{t^*}}^2 + \|\Delta u_m\|_{2,Q_{t^*}}^2 \leq C, \\
&\inf_{(0,t^*)}\|u_m(t)\|_2^2 \geq \frac{1}{2}\|u_0\|_2^2.
\end{aligned} \tag{12}$$

The uniform estimates (12) are sufficient to prove the convergence of the sequence $\{u_m\}$ to a solution of problem (5) with $\epsilon > 0$ and $M \geq 1$. We omit the detailed proof which can be found in Sert and Shmarev.[20, Proof of Lemma 3.1] The only difference is that instead of the term $|u_m|^{q(z)-2}u_m$, one has to deal with $f_M(u_m)$, which is nonnegative and bounded by definition.

Lemma 5. *Let the functions a, b satisfy conditions* ***(H.1)****. For every $\epsilon>0$, $M\geq 1$, $u_0\in H^1_0(\Omega)$, there exists $t^*>0$ such that problem (5) has at least one solution $u(z)\in \mathbf{V}(Q_{t^*})$. The solution satisfies estimates (12) with an independent of ϵ constant C.*

Let us show now that if $0\leq u_0\leq L$ a.e. in Ω, then the constructed solution is a.e. nonnegative and remains a.e. bounded for small times. Let $u=u_\epsilon$ be the solution of problem (5) obtained as the limit of the sequence of Galerkin's approximations.

Lemma 6. *Let $u\equiv u_\epsilon$ be the solution of problem (5) constructed in Lemma 5. If $0\leq u_0\leq L$ a.e. in Ω, then there exists t^* so small that $0\leq u(z)\leq 2L$ a.e. in $\Omega\times[0,t^*]$.*

Proof. Assume that u is a solution of problem (5) in a cylinder $\Omega\times(0,t^*)$. Let us take $u^-=\min\{0,u\}\leq 0$ for the test function. Since $2u^-u_t=((u^-)^2)_t$ and $(\nabla u^-,\nabla u)_{2,\Omega}=\|\nabla u^-\|^2_{2,\Omega}$, then

$$\int_{Q_{t^*}}\left(\frac{1}{2}((u^-)^2)_t+a_\epsilon(\|\nabla u\|_2^2)|\nabla u^-|^2\right)dz=\int_{Q_{t^*}} b(\|u\|_2^2)f_M(u)u^-\,dz\leq 0.$$

Dropping the nonnegative term on the left-hand side, we find that for every $t\in[0,t^*]$,

$$\|u^-(t)\|_2^2\leq \|u^-(0)\|_2^2=0 \text{ if } u_0\geq 0 \text{a.e. in } \Omega.$$

The solution of problem (5) satisfies the estimates of Lemma 2,

$$\|u(t)\|_2\leq g(M)<\infty \text{ in } Q_{t^*}=\Omega\times(0,t^*),$$

whence

$$b(\|u\|_2^2)\leq C_b g^\beta(M)+\nu\equiv B(M) \text{ in } Q_{t^*}.$$

Let us fix positive constants K, L and consider the function

$$V=\frac{L}{1-Kt}, \text{ in } Q_T=\Omega\times(0,T) \text{ with } T=\min\{t^*,1/K\}.$$

A straightforward computation shows that

$$V_t-a_\epsilon(\|\nabla u\|_2^2)\Delta V=\frac{KL}{(1-Kt)^2}\geq KL\geq BM^{q^+-1} \text{ in } Q_T,$$

provided that $KL\geq BM^{q^+-1}$. Since $V>0$ and $u=0$ on the lateral boundary of Q_T, then $\max\{u-V,0\}=0$ on $\partial\Omega\times(0,T)$, while at the initial moment,

$$u(x,0)-V(0)\leq 0 \text{ if } L\geq \|u_0\|_{\infty,\Omega}.$$

Let L be chosen according to this condition. Increasing, if needed, the constant K, we find that for every $\phi\in L^2(0,T;H^1_0(\Omega))$, $\phi\geq 0$ a.e. in Q_T,

$$\begin{aligned}\int_{Q_T}\left(\phi(u-V)_t+a_\epsilon(\|\nabla u\|_2^2)\nabla(u-V)\cdot\nabla\phi\right)dz&=\int_{Q_T}\left(b(\|u\|_2^2)f_M(u)-\frac{KL}{(1-Kt)^2}\right)\phi\,dz\\ &\leq\int_{Q_T}\left(BM^{q^+-1}-\frac{KL}{(1-Kt)^2}\right)\phi\,dz\leq 0.\end{aligned}$$

The function $\phi = \max\{0, u - V\} \equiv (u - V)^+ \geq 0$ is an admissible test-function. Moreover, since $\phi_t \in L^2(Q_T)$, the previous inequality reads

$$\int_{Q_T} \left(\frac{1}{2}(((u-V)^+)^2)_t + a_\epsilon(\|\nabla u\|_2^2)|\nabla(u-V)^+|^2 \right) dz \leq 0,$$

whence

$$\|(u-V)^+\|_2^2 \leq \|(u_0 - L)^+\|_2^2 = 0.$$

It follows that $\|(u-V)^+\|_2^2(t) = 0$ for every $t \in (0,T)$, that is, $u \leq V$ a.e in Q_T.

Let us choose $L = 1 + \|u_0\|_{\infty,\Omega}$, $T = \min\{1/(2K), t^*\}$, and claim that for all $t \in (0,T)$

$$\frac{L}{1-Kt} \leq M, \quad \frac{KL}{(1-Kt)^2} > BM^{q^+-1}.$$

These conditions are fulfilled if we take $M = 2L$ and

$$KL \geq BM^{q^+-1} \iff K \geq 2^{q^+-1}L^{q^+-2}\left(C_b g^\beta(2L) + \nu\right).$$

□

2.5 | Passing to the limit as $\epsilon \to 0$.

An immediate consequence of Lemma 6 is that if $0 \leq u_0 \leq L$ a.e. in Ω, then the constructed solution of problem (5) satisfies the inequalities $0 \leq u(z) \leq 2L = M$ a.e. in $\Omega \times (0,T)$. It follows that

$$u^+ = u, \quad f_M(u) = \min\{|u^+|^{q(z)-2}u^+, M^{q(z)-1}\} = u^{q(z)-1} \text{ in } Q_T;$$

that is, the function u is a nonnegative solution of the problem

$$\begin{cases} u_t - a_\epsilon(\|\nabla u\|_2^2)\Delta u = b(\|u\|_2^2)u^{q(z)-1} \text{ in } Q_T, \\ u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 \text{ in } \Omega. \end{cases} \tag{13}$$

The proof of the existence of a local in time nonnegative solution of the degenerate problem (1) is an imitation of the proof of Sert and Shmarev.[20, Theorem 2.1] It consists in choosing a convergent subsequence of the sequence $\{u_{\epsilon_k}\}$ of solutions of problem (13) and relies on the monotonicity of $a(s)s$, continuity of $b(s)$, and the uniform in ϵ_k a priori estimates (12) for the solutions of problems (13).

Theorem 1. *Let the functions $a(\cdot)$, $b(\cdot)$ satisfy conditions **(H.1)**. Assume that $q(z) \in [q^-, q^+] \subset (1, \infty)$ and that $q(z)$ is continuous on the closure of every cylinder $Q_T = \Omega \times (0,T)$ of finite height T. Then, for every $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq 0$ a.e. in Ω and $u_0 \not\equiv 0$, there exists $T > 0$ such that problem (1) has a nonnegative solution $u \in \mathbf{V}(Q_T)$. This solution satisfies estimates (12) and*

$$\sup_{(0,T)} \|u(\cdot,t)\|_{\infty,\Omega} \leq 2\|u_0\|_{\infty,\Omega}.$$

The properties of the constructed solution allow one to take $u(x,T)$ for the initial datum and continue $u(x,t)$ to an interval (T,T'). Iterating, we continue the solution to the maximal interval of existence $(0,T_0)$, $T_0 \equiv T(u_0)$, which we define as follows:

(a) for every $T < T(u_0)$ problem (1) has a solution $u \in \mathbf{V}(Q_T)$;
(b) $\|u(t)\|_{2,\Omega} \to \infty$ as $t \to T(u_0)^-$.

3 | BLOW-UP IN SOLUTIONS OF SINGULAR EQUATIONS

By Theorem 1, problem (1) with the coefficients $a(\cdot)$, $b(\cdot)$ satisfying conditions **(H.1)** and a nontrivial initial datum $0 \leq u_0(x) \in L^\infty(\Omega) \cap H_0^1(\Omega)$ has a bounded nonnegative local in time solution $u \in \mathbf{V}(Q_T)$ with some $T > 0$ depending on the data, which can be extended to the maximum existence interval $(0, T(u_0))$. We want to find conditions on the structure of equation (1) and the initial function which guarantee nonexistence of global in time solutions.

Definition 2. The nonnegative solution $u(x,t)$ blows up in a finite time if there exists a moment $T_0 \equiv T(u_0) < \infty$ such that

$$\|u(\cdot,t)\|_{2,\Omega} \to \infty \text{ as } t \to T_0^-.$$

We begin with the study of equation (1) which can be regarded as a singular diffusion equation with the diffusion coefficient $a(s) \to \infty$ as $s \to 0^+$. In this case, to find sufficient conditions of blow-up, we will adjust the classical eigenvalue method of Kaplan.[27] Let $\lambda > 0$, $\phi \in H_0^1(\Omega)$ be the first (positive) eigenvalue and the corresponding positive eigenfunction of the Dirichlet problem for the Laplace operator in Ω: $\phi \in H_0^1(\Omega)$

$$(\nabla\phi, \nabla\psi)_{2,\Omega} = \lambda(\phi, \psi)_{2,\Omega} \ \ \forall \psi \in H_0^1(\Omega). \tag{14}$$

The function ϕ is normalized by the condition $\int_\Omega \phi\, dx = 1$. Let us introduce the following functions and constants:

$$\mu(t) := \int_\Omega u(x,t)\phi(x)\, dx, \quad q^- = \operatorname*{ess\,inf}_{z \in Q_T} q(z).$$

It is easy to see that the solution must blow-up in a finite time if there exists a moment t_0 such that $\mu(t) \to \infty$ as $t \to t_0^-$:

$$\|u(\cdot,t)\|_{2,\Omega}\|\phi\|_{2,\Omega} \geq \int_\Omega u\phi\, dx = \mu(t) \to \infty \text{ as } t \to t_0.$$

We assume that the following conditions are fulfilled:

$$\begin{cases} \text{there exist constants } \sigma > 1,\ \beta \geq 0,\ C_\sigma > 0,\ C_b' > 0 \text{ such that} \\ \quad a(s)s \leq C_\sigma \left(s^{\frac{\sigma}{2}} + s\right),\ b(s) \geq C_b' s^{\frac{\beta}{2}} \text{ for all } s \geq 0, \\ u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega),\ u_0 \geq 0 \text{ a.e. in } \Omega \end{cases} \tag{15}$$

(notice the exponents β and σ in conditions (**H.1**) and (15) need not coincide),

$$\begin{aligned} &\text{(a)}\ \ q(x,t) \in L^\infty(Q_T), \qquad 2 \leq q^- \leq q(x,t) \leq q^+ \text{ a.e. in } Q_T \text{ with some constants } q^\pm, \\ &\text{(b)}\ \ f(\mu(0)) = A\mu^{\beta+q^--1}(0) - B > 0, \qquad \mu(0) = \int_\Omega u_0\phi\, dx, \end{aligned} \tag{16}$$

with the constants

$$\begin{aligned} &A = \frac{C'}{2}, \qquad C' \equiv \frac{C_b'}{\|\phi\|_2^\beta}, \qquad C \equiv \left(\frac{\sqrt{\lambda}}{\|\phi\|_2}\right)^{\sigma-2}, \\ &B = (\lambda C C_\sigma)^{\frac{\beta+q^--1}{\beta+q^--\sigma}} \left(\frac{6}{C'}\right)^{\frac{\sigma-1}{\beta+q^--\sigma}} + (\lambda C_\sigma)^{\frac{\beta+q^--1}{\beta+q^--2}} \left(\frac{6}{C'}\right)^{\frac{1}{\beta+q^--2}} + 6^{\frac{\beta}{q^--1}} C'. \end{aligned} \tag{17}$$

Theorem 2. *Assume that the data of problem (1) satisfy conditions (15) and (16). If*

$$\beta \geq 0,\ q^- \geq 2,\ \sigma \in (1,2],\ \beta + q^- > 2,$$

then every nonnegative solution of (1) blows up at a finite moment T_0. The blow-up moment T_0 can be estimated through the data:

$$T_0 \le \frac{2}{A}\frac{\mu^{2-\beta-q^-}(0)}{\beta+q^- - 2} + \frac{1}{f(\mu(0))}\left(\frac{2B}{A}\right)^{\frac{1}{\beta+q^- -1}}.$$

Remark 1. Under the assumption $\beta + q^- > 2$ inequality in (16) (b) is fulfilled for every positive constants A, B and the sufficiently large $\mu(0)$: $\mu(0) > (B/A)^{\frac{1}{\beta+q^- -1}}$.

3.1 | The differential inequality for $\mu(t)$

Let u be a nonnegative strong solution of problem (1) in a cylinder Q_T: for every test function $\phi \in H_0^1(\Omega)$ and $t, t+h < T$

$$\int_t^{t+h}\int_\Omega (u_t\phi + a(\|\nabla u\|_2^2)\nabla u\cdot\nabla\phi - b(\|u\|_2^2)u^{q(x,t)-1}\phi)\,dxdt = 0.$$

Let us take the first eigenfunction ϕ of problem (14) for the test function, divide the resulting equality by h, and then send $h \to 0$. By the Lebesgue differentiation theorem, for a.e. $t < T$

$$\begin{aligned}\mu'(t) &= -a(\|\nabla u\|_2^2)\int_\Omega \nabla u\cdot\nabla\phi\,dx + b(\|u\|_2^2)\int_\Omega u^{q(x,t)-1}\phi\,dx \\ &= -a(\|\nabla u\|_2^2)\lambda\mu + b(\|u\|_2^2)\int_\Omega u^{q(x,t)-1}\phi\,dx.\end{aligned} \tag{18}$$

Let us fix $t \in (0, T)$. Splitting the domain Ω into the parts where $u \ge 1$ or $u < 1$ for a.e. $x \in \Omega$, we find that

$$\begin{aligned}\int_\Omega u^{q(x,t)-1}\phi\,dx &= \int_{\Omega\cap\{u\ge1\}} u^{q(x,t)-1}\phi\,dx + \int_{\Omega\cap\{u<1\}} u^{q(x,t)-1}\phi\,dx \ge \int_{\Omega\cap\{u\ge1\}} u^{q^- -1}\phi\,dx \\ &= \int_{\Omega\cap\{u\ge1\}} u^{q^- -1}\phi\,dx + \int_{\Omega\cap\{u<1\}} u^{q^- -1}\phi\,dx - \int_{\Omega\cap\{u<1\}} u^{q^- -1}\phi\,dx \\ &\ge \int_\Omega u^{q^- -1}\phi\,dx - \int_{\Omega\cap\{u<1\}} u^{q^- -1}\phi\,dx \\ &\ge \int_\Omega u^{q^- -1}\phi\,dx - \int_\Omega \phi\,dx.\end{aligned} \tag{19}$$

Let $q^- > 2$. By the reverse Hölder's inequality with the exponent $p = \frac{1}{q^- -1} \in (0, 1)$

$$\int_\Omega u^{q^- -1}\phi\,dx = \int_\Omega u^{q^- -1}\phi^{q^- -1}\phi^{2-q^-}\,dx \ge \left(\int_\Omega u\phi\,dx\right)^{q^- -1}\left(\int_\Omega \phi\,dx\right)^{2-q^-} = \mu^{q^- -1}(t).$$

If $q^- = 2$, then

$$\int_\Omega u^{q^- -1}\phi\,dx = \mu(t).$$

Gathering these relations with (19), we arrive at the inequality

$$\int_\Omega u^{q(x,t)-1}\phi\,dx \ge \mu^{q^- -1}(t) - 1, \qquad q^- \ge 2. \tag{20}$$

On the other hand,

$$0 \le \mu^2(t) = \left(\int_\Omega u\phi\, dx \right)^2 \le \|\phi\|_2^2 \|u\|_2^2. \tag{21}$$

According to (15), (21) yields the inequality

$$b(\|u\|_2^2) \ge C_b' \|u\|_2^\beta \ge C' \mu^\beta(t), \tag{22}$$

whence, by virtue of (20) and (22),

$$b(\|u\|_2^2) \int_\Omega u^{q(x,t)-1} \phi\, dx \ge C' \mu^{\beta+q^- -1}(t) - C' \mu^\beta(t). \tag{23}$$

By (15),

$$a(\|\nabla u\|_2^2) \le C_\sigma(\|\nabla u\|_2^{\sigma-2} + 1), \tag{24}$$

where $1 < \sigma \le 2$ by assumption. Since $\lambda = \lambda_1$ is the minimal eigenvalue of problem (14), it follows from the Parseval identity that

$$\|u\|_2^2 = \sum_{i=1}^\infty u_i^2 = \sum_{i=1}^\infty \frac{1}{\lambda_i}(\lambda_i u_i^2) \le \frac{1}{\lambda_1} \sum_{i=1}^\infty \lambda_i u_i^2 = \frac{1}{\lambda} \|\nabla u\|_2^2.$$

Plugging this inequality into (21), we find that

$$\|\nabla u\|_2^{\sigma-2} \le \lambda^{\frac{\sigma-2}{2}} \|u\|_2^{\sigma-2} \le C\mu^{\sigma-2}(t). \tag{25}$$

Gathering (24) and (25), we find that

$$a(\|\nabla u\|_2^2) \le CC_\sigma \mu^{\sigma-2}(t) + C_\sigma,$$

whence

$$-a(\|\nabla u\|_2^2)\lambda\mu \ge -\lambda CC_\sigma \mu^{\sigma-1}(t) - \lambda C_\sigma \mu(t). \tag{26}$$

Using (23) and (26) in (18), we arrive at the differential inequality for $\mu(t)$:

$$\mu'(t) \ge C' \mu^{\beta+q^- -1}(t) - \lambda CC_\sigma \mu^{\sigma-1}(t) - \lambda C_\sigma \mu(t) - C' \mu^\beta(t). \tag{27}$$

Let us use the Young inequality in the form: for $\gamma < \beta + q^- - 1$ and $\delta > 0$

$$\delta\mu^\gamma = \left(\frac{C'}{6} \mu^{\beta+q^- -1} \right)^{\frac{\gamma}{\beta+q^- -1}} \left(\frac{6\delta^{\frac{\beta+q^- -1}{\gamma}}}{C'} \right)^{\frac{\gamma}{\beta+q^- -1}} \le \frac{C'}{6} \mu^{\beta+q^- -1} + \left(\frac{6\delta^{\frac{\beta+q^- -1}{\gamma}}}{C'} \right)^{\frac{\gamma}{\beta+q^- -(\gamma+1)}}.$$

Applying this inequality with $\gamma = \sigma - 1 < \beta + q^- - 1$, $\gamma = 1 < \beta + q^- - 1$ and $\gamma = \beta < \beta + q^- - 1$, we estimate the nonpositive terms on the right-hand side of (27) in the following way:

$$\begin{aligned}
\lambda CC_\sigma \mu^{\sigma-1}(t) &\le \frac{C'}{6} \mu^{\beta+q^- -1}(t) + (\lambda CC_\sigma)^{\frac{\beta+q^- -1}{\beta+q^- -\sigma}} \left(\frac{6}{C'} \right)^{\frac{\sigma-1}{\beta+q^- -\sigma}}, \\
\lambda C_\sigma \mu(t) &\le \frac{C'}{6} \mu^{\beta+q^- -1}(t) + (\lambda C_\sigma)^{\frac{\beta+q^- -1}{\beta+q^- -2}} \left(\frac{6}{C'} \right)^{\frac{1}{\beta+q^- -2}}, \\
C' \mu^\beta(t) &\le \frac{C'}{6} \mu^{\beta+q^- -1}(t) + 6^{\frac{\beta}{q^- -1}} C' \text{ if } \beta > 0.
\end{aligned}$$

Substituting these inequalities into (27), we obtain the inequality

$$\mu'(t) \geq f(\mu(t)) \equiv A\mu^{\beta+q^- -1}(t) - B \tag{28}$$

with the constants $A = \frac{C'}{2}$ and B from (17).

3.2 | Proof of Theorem 2

The function $f(\mu)$ is continuous with respect to μ, strictly monotone increasing and convex because

$$f'(\mu) = A(\beta + q^- - 1)\mu^{\beta+q^- -2} > 0, \;\; f''(\mu) = A(\beta + q^- - 1)(\beta + q^- - 2)\mu^{\beta+q^- -3} > 0$$

for $\mu > 0$. By Taylor's formula and assumption (16)

$$f(\mu(t)) = f(\mu(0)) + f'(\mu(0))(\mu(t) - \mu(0)) + \frac{1}{2}f''(s)(\mu(t) - \mu(0))^2 > f'(\mu(0))(\mu(t) - \mu(0)).$$

Using this inequality in (28) and plugging the assumption $\mu(0) > 0$, from (28) we obtain

$$\mu'(t) \geq f'(\mu(0))(\mu(t) - \mu(0)).$$

A straightforward integration leads to the inequality $\mu(t) \geq \mu(0) > 0$ for all $t \in [0, T)$. It follows that $\mu'(t) \geq f(\mu(t)) \geq f(\mu(0))$, whence

$$\mu(t) \geq \mu(0) + f(\mu(0))t. \tag{29}$$

Let us denote

$$t_0 = \sup\{s > 0 : \;\; \mu(t) < \infty \;\forall t \in (0, s)\},$$

assume $t_0 = \infty$, and show that this assumption leads to a contradiction. Inequality (29) allows one to find $\theta > 0$ such that

$$f(\mu(t)) \equiv A\mu^{\beta+q^- -1}(t) - B \geq \frac{A}{2}\mu^{\beta+q^- -1}(t) \text{ for all } \; t \geq \theta. \tag{30}$$

Indeed, since $\mu(t) \geq \mu(0)$ and $f(\mu)$ is monotone increasing, it is sufficient to take

$$\theta = \frac{1}{f(\mu(0))}\left(\frac{2B}{A}\right)^{\frac{1}{\beta+q^- -1}}.$$

The function $\mu(t)$ satisfies the inequality

$$\mu'(t) \geq \frac{A}{2}\mu^{\beta+q^- -1}(t), \;\; \text{for } \; t \in (\theta, t_0),$$

which follows from (28) and (30). Dividing the both parts by $\mu^{\beta+q^- -1}(t)$ and integrating, we have

$$J(\mu(t)) := \int_{\mu(0)}^{\mu(t)} \frac{ds}{s^{\beta+q^- -1}} \geq \int_{\mu(\theta)}^{\mu(t)} \frac{ds}{s^{\beta+q^- -1}} \geq \frac{A}{2}(t - \theta) \;\; \text{for all} \;\; t \in (\theta, t_0).$$

Since we assumed $t_0 = \infty$, we arrive at the contradiction: the integral $J(\infty)$ is convergent, while $\mu(t) \to \infty$ as $t \to \infty$ due to (29), whence

$$\infty > \frac{1}{\beta + q^- - 2} \frac{1}{\mu^{\beta+q^- -2}(0)} = J(\infty)$$
$$\geq J(\mu(t)) = \int_{\mu(0)}^{\mu(t)} \frac{ds}{s^{\beta+q^- -1}} \geq \int_{\mu(\theta)}^{\mu(t)} \frac{ds}{s^{\beta+q^- -1}} \geq \frac{A}{2}(t-\theta) \to \infty \text{ as } t \to \infty.$$

The same inequality written in the form

$$J(\infty) \geq \frac{A}{2}(t-\theta) \text{ for all } t \in (\theta, t_0)$$

provides the upper estimate on t_0:

$$t_0 \leq \theta + \frac{2}{A} \frac{\mu^{2-\beta-q^-}(0)}{\beta + q^- - 2}.$$

3.3 | Regional blow-up

A nonnegative solution of problem (1) may blow up in a finite time if the conditions of Theorem 2 are fulfilled only on a part of the problem domain. Let $u \geq 0$ be a solution of problem (1) in a cylinder $\Omega \times (0, T)$. Let $D \subset \Omega$, $\partial D \in C^1$, be a subdomain of Ω and λ, ϕ be the first eigenvalue and eigenfunction of the Dirichlet problem for the Laplace operator in D:

$$-\Delta\phi = \lambda\phi \text{ in } D, \ \phi = 0 \text{ on } \partial D. \tag{31}$$

We normalize ϕ by the condition $\int_D \phi \, dx = 1$. Introduce the functions

$$\Phi(x) = \begin{cases} \phi & \text{in } D \\ 0 & \text{in } \bar{\Omega} \setminus D. \end{cases} \quad \text{and} \quad \mu(t) = \int_D u\phi \, dx \equiv \int_\Omega u\Phi \, dx.$$

For every $\psi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla\psi \cdot \nabla\Phi \, dx = \int_D \nabla\psi \cdot \nabla\phi \, dx = \int_{\partial D} \psi(\nabla\phi \cdot \mathbf{n}) \, dx - \int_D \psi\Delta\phi \, dx$$
$$= \int_{\partial D} \psi(\nabla\phi \cdot \mathbf{n}) \, dx - \lambda \int_D \psi\phi \, dx,$$

where $\mathbf{n}$ denotes the exterior normal vector to ∂D. Since

$$\|u(t)\|_{2,\Omega} \|\phi\|_{2,D} \geq \|u(t)\|_{2,D} \|\phi\|_{2,D} \geq \int_D u\phi \, dx = \mu(t),$$

the solution of problem (1) must blow up if the function $\mu(t)$ becomes infinite in a finite time.

The function Φ can be taken for the test-function in (4). Arguing as in the derivation of (18), we find that for a.e. $t \in (0, T)$

$$\begin{aligned} \mu'(t) &= \int_\Omega u_t \Phi \, dx = \int_D u_t \phi \, dx = -a(\|\nabla u\|_{2,\Omega}^2) \int_\Omega \nabla u \cdot \nabla\Phi \, dx + b(\|u\|_{2,\Omega}^2) \int_\Omega u^{q(x,t)-1} \Phi \, dx \\ &= -\lambda a(\|\nabla u\|_{2,\Omega}^2)\mu(t) - a(\|\nabla u\|_{2,\Omega}^2) \int_{\partial D} u(\nabla\phi, \mathbf{n}) \, dS + b(\|u\|_{2,\Omega}^2) \int_D u^{q(x,t)-1} \phi \, dx, \end{aligned} \tag{32}$$

where $\mathbf{n}$ denotes the outward normal to ∂D. Since $\phi > 0$ in D, then $\nabla\phi \cdot \mathbf{n} \le 0$ on ∂D and for the nonnegative solution u

$$-\int_{\partial D} u(\nabla\phi, \mathbf{n})\, dS \ge 0.$$

By virtue of (24), (25), and the inequality $\|u\|_{2,D} \le \|u\|_{2,\Omega}$, for $\sigma \le 2$ we obtain

$$\begin{aligned} a(\|\nabla u\|_{2,\Omega}^2) &\le C_\sigma(\|\nabla u\|_{2,\Omega}^{\sigma-2} + 1) \le C_\sigma\left(\lambda^{\frac{\sigma-2}{2}}\|u\|_{2,\Omega}^{\sigma-2} + 1\right) \\ &\le C_\sigma\left(\lambda^{\frac{\sigma-2}{2}}\|u\|_{2,D}^{\sigma-2} + 1\right). \end{aligned}$$

Since $\mu(t) \le \|\phi\|_{2,D}\|u\|_{2,D}$, then

$$-\lambda a(\|\nabla u\|_{2,\Omega}^2)\mu(t) \ge -\lambda C_\sigma\left(\lambda^{\frac{\sigma-2}{2}}\|\phi\|_{2,D}^{2-\sigma}\mu^{\sigma-2}(t) + 1\right)\mu(t).$$

By the same token,

$$b(\|u\|_{2,\Omega}^2) \ge C_b'\|u\|_{2,\Omega}^\beta \ge C_b'\|u\|_{2,D}^\beta \ge C_b'\|\phi\|_{2,D}^{-\beta}\mu^\beta(t).$$

The last term on the right-hand side of (32) is estimated from below by analogy with (18) and (19):

$$\int_D u^{q(x,t)-1}\phi\, dx \ge \mu^{q^- -1}(t) - 1.$$

Gathering these estimates, we arrive at the ordinary differential inequality for $\mu(t)$ in the following form (cf. with (18)):

$$\begin{aligned} \mu'(t) &\ge -a(\|\nabla u\|_{2,\Omega}^2)\lambda\mu + b(\|u\|_{2,\Omega}^2)\int_D u^{q(x,t)-1}\phi\, dx \\ &\ge \frac{C_b'}{\|\phi\|_{2,D}^\beta}\mu^{\beta+q^- -1}(t) - \frac{C_b'}{\|\phi\|_{2,D}^\beta}\mu^\beta(t) - \left(\lambda^{\frac{\sigma}{2}}C_\sigma\|\phi\|_{2,D}^{2-\sigma}\right)\mu^{\sigma-1}(t) - \lambda C_\sigma\mu(t) \\ &\ge A_D\mu^{\beta+q^- -1}(t) - B_D. \end{aligned} \tag{33}$$

A_D and B_D are the constants defined by formulas (17) in which ϕ, λ are the first eigenfunction and eigenvalue of problem (31) in the domain D. The analysis of inequality (33) leads to the following assertion.

Theorem 3. *Let the data of problem (1) satisfy conditions (15) with $\beta \ge 0, C_b' > 0, \sigma \in (1,2]$, and $q(x,t) \in L^\infty(Q_T)$, $1 < \kappa \le q(x,t) \le q^+$ a.e. in Q_T with some constants κ, q^+. Assume that there exists a subdomain $D \subset \Omega$ such that $\partial D \in C^2$,*

$$ess \inf_{D\times(0,T)} q(x,t) = q^- \ge 2,$$

and

$$f(\mu(0)) = A_D\mu^{\beta+q^- -1}(0) - B_D > 0, \quad \mu(0) = \int_D u_0\phi\, dx.$$

If $\sigma \in (1,2]$ and $\beta + q^- > 2$, then every nonnegative solution of (1) blows up at a finite moment t_0. The blow-up moment t_0 can be estimated through the data:

$$t_0 \le \frac{2}{A_D}\frac{\mu^{2-\beta-q^-}(0)}{\beta+q^- -2} + \frac{1}{f(\mu(0))}\left(\frac{2B_D}{A_D}\right)^{\frac{1}{\beta+q^- -1}}.$$

4 | BLOW-UP IN SOLUTIONS OF DEGENERATE EQUATIONS

Let $u \in \mathbf{V}(Q_T)$ be a nonnegative solution of the problem

$$\begin{cases} u_t - a(\|\nabla u\|_2^2)\Delta u = b(\|u\|_2^2)u^{q(x)-1} \text{ in } Q_T, \\ u = 0 \text{ on } \partial\Omega \times (0,T), \; u(x,0) = u_0 \text{ in } \Omega, \end{cases} \tag{34}$$

where $0 < T < T_0$ and $(0, T_0)$, $T_0 = T(u_0)$, is the maximal existence interval of this solution. We admit that equation (34) may be degenerate, that is, $a(s) \to 0$ as $s \to 0^+$. In this situation, the eigenfunction method ceases to be applicable; for this reason, we adapt the techniques based on the study of a suitably chosen energy functional.

Let us take the solution $u \in \mathbf{V}(Q_T)$ for the test function in identify (4): for all $t, t+h \in [0,T]$, $h > 0$,

$$\frac{1}{2h} \|u(t)\|_2^2\Big|_t^{t+h} + \frac{1}{h}\int_t^{t+h} a(\|\nabla u(s)\|_2^2)\|\nabla u(s)\|_2^2 \, ds = \frac{1}{h}\int_t^{t+h} b(\|u(s)\|_2^2)\int_\Omega u^{q(x)}(x,s)\,dxds.$$

Letting $h \to 0^+$ and using the Lebesgue differentiation theorem, we conclude that for a.e. $t \in (0,T)$,

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u^2\,dx\right) = -a(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 + b(\|u(t)\|_2^2)\int_\Omega u^{q(x)}(x,t)\,dx. \tag{35}$$

Let us make use of the following auxiliary assertion.

Lemma 7 (Lemmas 2.5 and 2.7 of Antontsev & Shmarev[29]). *Let $\partial\Omega \in C^2$. If $u \in \mathbf{V}(Q_T)$, then $\nabla u \in C([0,T]; L^2(\Omega))$ after possible redefining on a set of zero measure in $(0,T)$ and*

$$\int_t^{t+h}\int_\Omega u_t \Delta u\,dz + \frac{1}{2}\int_\Omega |\nabla u(s)|^2\,dx\Big|_{s=t}^{s=t+h} = 0 \;\forall t, t+h \in (0,T).$$

It follows that for a.e. $t \in (0,T)$

$$\frac{d}{dt}\left(\|\nabla u(t)\|_2^2\right) = 2\int_\Omega u_t \Delta u\,dx.$$

For the nonnegative solution $u \in \mathbf{V}(Q_T)$ and every $\eta \in L^2(Q_T)$,

$$\int_t^{t+h}\int_\Omega \eta\left(u_t - a(\|\nabla u\|_2^2)\Delta u - b(\|u\|_2^2)u^{q(x)-1}u\right)\,dz = 0.$$

Choosing $\eta = u_t \in L^2(Q_T)$ and using Lemma 7, we conclude that for a.e. $t \in (0,T)$,

$$\|u_t(t)\|_{2,\Omega}^2 = -\frac{1}{2}\frac{d}{dt}\left(\int_0^{\|\nabla u(t)\|_2^2} a(s)\,ds\right) + b(\|u\|_2^2)\int_\Omega u^{q(x)-1}u_t\,dx. \tag{36}$$

Let us introduce the following functions:

$$f(t)=\frac{1}{2}\int_0^t\int_\Omega u^2(x,\tau)\,dz,$$

$$E(t)=\frac{1}{2}\int_0^{\|\nabla u\|_2^2} a(s)\,ds-b(\|u(t)\|_2^2)\int_\Omega \frac{u^{q(x)}}{q(x)}\,dx.$$

Assume that the following conditions hold:

(B.1) there exists a constant $\theta>0$ such that

$$(sa(s))'\le \theta a(s)\ \text{ for all }\ s\in\mathbb{R}_0^+, \tag{37}$$

(B.2)

$$q^-\ge 2\theta>1,\ \ b(s)\in C^1(0,\infty)\ \text{ with }\ b'(s)\ge 0,$$

(B.3)

$$E(0)=\frac{1}{2}\int_0^{\|\nabla u_0\|_2^2} a(s)\,ds-b(\|u_0\|_2^2)\int_\Omega \frac{u_0^{q(x)}}{q(x)}\,dx\le 0.$$

By Theorem 1, there is $T_0>0$ such that problem (1) has a local in time solution on every interval $[0,T]$ with $T<T_0$. We show now that under conditions **(B.1)–(B.3)**, the number T_0 must be finite.

Theorem 4. *If conditions* ***(B.1)–(B.3)*** *are fulfilled, then every nonstationary solution of problem (34)* $u\in \mathbf{V}(Q_T)$ *blows up in a finite time.*

The proof of Theorem 4 is based on the following technical lemmas.

Lemma 8. *If conditions* ***(B.1)*** *and* ***(B.2)*** *are fulfilled, then every nonnegative solution* $u\in\mathbf{V}(Q_T)$ *of problem (34) satisfies the inequality*

$$E(t)+\int_0^t\int_\Omega u_t^2\,dxd\tau\le E(0)v(t)\ \ \textit{for a.e.}\ t\in(0,T) \tag{38}$$

with a bounded function $v(t)\ge 1$.

Proof. Differentiation of $E(t)$ yields the equality

$$E'(t)=\frac{1}{2}\frac{d}{dt}\left(\int_0^{\|\nabla u\|_2^2} a(s)\,ds\right)-b(\|u\|_2^2)\int_\Omega u^{q(x)-1}u_t\,dx-2b'\left(\|u\|_2^2\right)\left(\int_\Omega \frac{u^{q(x)}}{q(x)}\,dx\right)\left(\int_\Omega uu_t\,dx\right).$$

By virtue of (36), this equality can be written in the form

$$E'(t)=-\int_\Omega u_t^2\,dx-C(t)\int_\Omega uu_t\,dx \tag{39}$$

with the coefficient

$$C(t)=2b'(\|u\|_2^2)\int_\Omega \frac{u^{q(x)}}{q(x)}\,dx\ge 0\ \text{a.e. in}\,(0,T).$$

Due to (35),

$$-\frac{1}{2\theta}\int_{\Omega} uu_t\,dx = \frac{1}{2\theta}a(\|\nabla u\|_2^2)\|\nabla u\|_2^2 - \frac{b(\|u\|_2^2)}{2\theta}\int_{\Omega} u^{q(x)}\,dx, \tag{40}$$

with the constant θ from assumption (37). On the other hand, integrating by parts and using (37), we obtain the inequality

$$\int_0^t a(s)\,ds \; = a(t)t - \int_0^t sa'(s)\,ds \geq a(t)t - (\theta-1)\int_0^t a(s)\,ds,$$

which can be written in the form

$$\theta\left(\int_0^t a(s)\,ds\right) \geq a(t)t, \text{ for all } t \geq 0. \tag{41}$$

By (41),

$$\left(\frac{1}{2\theta}\right) a(\|\nabla u\|_2^2)\|\nabla u\|_2^2 \leq \frac{1}{2}\int_0^{\|\nabla u\|_2^2} a(s)\,ds.$$

Gathering this inequality with (40), we obtain

$$\begin{aligned} -\frac{1}{2\theta}\int_{\Omega} uu_t\,dx &\leq \frac{1}{2}\int_0^{\|\nabla u\|_2^2} a(s)\,ds - \frac{q^-}{2\theta}b(\|u\|_2^2)\int_{\Omega}\frac{u^{q(x)}}{q(x)}\,dx \\ &\leq \frac{1}{2}\int_0^{\|\nabla u\|_2^2} a(s)\,ds - b(\|u\|_2^2)\int_{\Omega}\frac{u^{q(x)}}{q(x)}\,dx = E(t). \end{aligned} \tag{42}$$

It follows from (42) and (39) that

$$E'(t) + \int_{\Omega} u_t^2\,dx \leq C(t)E(t).$$

Since $C(t) \in L^1(0,T)$, it follows from the Gronwall inequality that

$$E(t) \leq E(0)\exp\left(\int_0^t C(s)\,ds\right),$$

whence

$$E'(t) + \int_{\Omega} u_t^2\,dx \leq E(0)C(t)\exp\left(\int_0^t C(s)\,ds\right) = E(0)\frac{d}{dt}\left(\exp\left(\int_0^t C(s)\,ds\right)\right).$$

Integrating this inequality and taking into account that $C(t) \geq 0$ for a.e. $t \in (0,T)$, we obtain (38) with the coefficient

$$\nu(t) = \exp\left(\int_0^t C(s)\,ds\right) \geq 1.$$

□

Lemma 9. *Under the conditions of Theorem 4, the inequality*

$$\theta\int_0^t\int_\Omega u_t^2\,dxd\tau \le \frac{f''(t)}{2} \tag{43}$$

is fulfilled for a.e. $t\in(0,T)$.

Proof. Let $u\in\mathbf{V}(Q_T)$ be a solution of problem (34). For a.e. $t\in(0,T)$

$$f'(t)=\frac{1}{2}\int_\Omega u^2(x,t)\,dx,$$
$$f''(t)=\int_\Omega uu_t\,dx=-a(\|\nabla u\|_2^2)\|\nabla u\|_2^2+b(\|u\|_2^2)\int_\Omega u^{q(x)}\,dx.$$

From (42) we find

$$\frac{1}{2}f''(t)\ge-\theta E(t). \tag{44}$$

Adding (44) to (38) with $E(0)\le 0$ due to assumption **(B.3)**, we obtain the needed inequality:

$$\theta\int_0^t\int_\Omega u_t^2\,dz\le\theta E(0)\nu(t)+\frac{f''(t)}{2}\le\frac{f''(t)}{2}.$$

□

4.1 | Proof of Theorem 4

Given a solution u of problem (34), let us denote

$$t_0=\sup\left\{t>0\,:\,\|u(t)\|_{2,\Omega}<\infty\ \text{ for all }\ t<t_0\right\}.$$

Let us assume, for contradiction, that $t_0=\infty$. It follows from (43) and the definition of $f(t)$ that

$$\begin{aligned}f'(t)-f'(0)&=\int_0^t f''(s)\,ds=\int_0^t\int_\Omega uu_t\,dxds\\&\le\left(\int_0^t\int_\Omega u^2\,dxds\right)^{\frac{1}{2}}\left(\int_0^t\int_\Omega u_t^2\,dxds\right)^{\frac{1}{2}}\\&\le\left(\frac{1}{2\theta}\right)^{\frac{1}{2}}(f(t))^{\frac{1}{2}}\left(f''(t)\right)^{\frac{1}{2}},\end{aligned}$$

whence

$$\left[f'(t)-f'(0)\right]^2\le\frac{1}{2\theta}f(t)f''(t). \tag{45}$$

Since u is a nonstationary solution of problem (34), and $f(t)$ is a monotone increasing function of t, it is necessary that there exist $\epsilon>0$ and a moment t_ϵ such that $f''(t)\ge\epsilon$ for all $t\ge t_\epsilon$. Assuming the contrary, we deduce from (43) that u must be independent of t, which is impossible. By (43), $f'(t)$ is a nondecreasing function and, by the intermediate value theorem,

$$f'(t)-f'(0)\ge f'(t)-f'(t_\epsilon)=f''(\xi)(t-t_\epsilon)\ge\epsilon\frac{t}{2}\ \text{ for }\ t\ge 2t_\epsilon,$$

which means that

$$f'(t) \to \infty \text{ and } f(t) \to \infty \text{ as } t \to \infty. \tag{46}$$

According to (46), for every $0 < \mu < 1$ there exists a moment t_μ such that for all $t \geq t_\mu$

$$\mu\left(f'(t)\right)^2 \leq \left[f'(t) - f'(0)\right]^2,$$

whence, by (45),

$$\mu\left(f'(t)\right)^2 \leq \frac{1}{2\theta} f(t)f''(t) \text{ for all } t \geq t_\mu. \tag{47}$$

Let us consider the function $F(t) \equiv f^{-r}(t)$ with some $r > 0$ to be chosen. By (47),

$$\begin{aligned} F''(t) &= r(f(t))^{-(r+2)}\left[(r+1)\left(f'(t)\right)^2 - f(t)f''(t)\right] \\ &\leq r(f(t))^{-(r+2)}\left(\frac{r+1}{2\mu\theta} - 1\right) f(t)f''(t). \end{aligned}$$

By assumption **(B.2)** $2\theta > 1$, hence, one may choose $\mu \in (0,1)$ and $r \in (0, 2\mu\theta - 1)$ such that $F''(t) < 0$ for all $t > t_\mu$. The function $F(t)$ is concave, nonincreasing, and positive for $t > t_\mu$, which is impossible because

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} f^{-r}(t) = 0$$

due to (46).

5 | A MODEL EQUATION

Let us assume that

$$a(s) = C_a s^{\frac{\tau-2}{2}}, \quad b(s) = C_b s^{\frac{\beta}{2}} \text{ with positive constants } C_a,\ C_b,\ \tau > 1,\ \beta \geq 0$$

and consider the following model version of problem (1):

$$\begin{cases} u_t - C_a\|\nabla u\|_2^{\tau-2}\Delta u = C_b\|u\|_2^{\beta}|u|^{q(x)-2}u \text{ in } Q_T, \\ u = 0 \text{ on } \partial\Omega \times (0,T),\ u(x,0) = u_0 \text{ in } \Omega. \end{cases} \tag{48}$$

If we assume that $q(x) \equiv 2$ and $\tau > 2$, then the existence of a local in time solution of problem (48) follows from Theorem 1 but Theorems 2 and 4 become inapplicable. Theorem 2 requires the inequality $\sigma = \tau \in (1,2]$, and Theorem 4 relies on condition (37) which now has the form $2\theta = \tau > 2$ and is incompatible with **(B.2)**:

$$2 = q^- \geq 2\theta = \tau > 2.$$

5.1 | Nonexistence of global solutions in the case $q(x) \equiv 2$

Sufficient conditions of nonexistence of a global solution of the model problem (48) can be obtained by imitation of the proof of Theorem 4 but with the energy functional defined in a different way. Let us consider the functional and the function

$$\widetilde{E}(t) = \frac{C_a}{\tau}\|\nabla u(t)\|_2^{\tau} - \frac{C_b}{\beta+2}\|u(t)\|_2^{\beta+2},$$

$$f(t) = \frac{1}{2}\int_0^t \int_\Omega u^2\,dz.$$

By the straightforward computation,

$$\begin{aligned}
\|u_t\|_{2,\Omega}^2 &= -\frac{C_a}{2}\|\nabla u\|_2^{\tau-2}\frac{d}{dt}\left(\|\nabla u\|_{2,\Omega}^2\right) + \frac{C_b}{2}\|u\|_{2,\Omega}^{\beta}\frac{d}{dt}\left(\|u\|_{2,\Omega}^2\right) \\
&= \frac{d}{dt}\left(-\frac{C_a}{\tau}\|\nabla u\|_{2,\Omega}^{\tau} + \frac{C_b}{\beta+2}\|u\|_{2,\Omega}^{\beta+2}\right) = -\widetilde{E}'(t), \\
f''(t) &= \int_\Omega u u_t\,dx = -C_a\|\nabla u\|_{2,\Omega}^{\tau} + C_b\|u\|_2^{\beta+2} \\
&= -\tau\widetilde{E}(t) + C_b\left(1-\frac{\tau}{\beta+2}\right)\|u\|_2^{\beta+2} \\
&\geq -\tau\widetilde{E}(t) \qquad \text{if } \beta+2 \geq \tau.
\end{aligned}$$

It follows that (cf. with (38) and (44))

$$\widetilde{E}(t) + \int_0^t\int_\Omega u_t^2\,dz \leq \widetilde{E}(0), \quad f''(t) \geq -\tau\widetilde{E}(t).$$

Following the proof of Theorem 4, we find that the function $f(t) = \frac{1}{2}\|u\|_{2,\Omega\times(0,t)}^2$ satisfies inequalities (43) and (47) with θ substituted by $\frac{\tau}{2}$. The analysis of the latter inequality does not require any changes.

Theorem 5. *If $\widetilde{E}(0) \leq 0$, $\beta > 0$, $q(x) \equiv 2$ and $\beta+2 \geq \tau$, then every nonnegative nonstationary solution $u \in \mathbf{V}(Q_T)$ of problem (48) blows up in a finite time.*

5.2 | Explicit solutions

In the case $q(x) \equiv 2$, $\beta > 0$ and $\tau = \beta+2$, problem (48) admits the explicit solution in separate variables which furnishes an illustration to the assertion of Theorem 5. Let $\phi(x)$ and $\lambda > 0$ be the first (positive) eigenfunction and the corresponding eigenvalue of the Dirichet problem for the Laplace operator in Ω, $\|\phi\|_{L^2(\Omega)} = 1$, $\|\nabla\phi\|_{L^2(\Omega)}^2 = \lambda$. The function $U = \phi(x)\psi(t)$ is a solution of problem (48) with $q(x) \equiv 2$ and the initial function $U_0(x) = C_0\phi(x)$, $C_0 = const > 0$, if $\psi(t)$ is a solution of the ordinary equation

$$\psi'(t) + \left(C_a\lambda^{\frac{\beta+2}{2}} - C_b\right)\psi^{\beta+1}(t) = 0, \quad \psi(0) = C_0. \tag{49}$$

Equation (49) can be explicitly integrated:

$$\psi^{\beta}(t) = \frac{C_0^{\beta}}{1-\beta C_0^{\beta}\left(C_b - C_a\lambda^{\frac{\beta+2}{2}}\right)t} \quad \text{for } 0 \leq t < t_0$$

with

$$t_0 = \begin{cases} \left(\beta C_0^{\beta}\left(C_b - C_a\lambda^{\frac{\beta+2}{2}}\right)\right)^{-1} & \text{if } C_b > C_a\lambda^{\frac{\beta+2}{2}}, \\ \infty & \text{otherwise.}\end{cases}$$

The energy functional of the solution U has the form

$$\widetilde{E}(0) = \frac{C_0^{\beta+2}}{\beta+2}\left(C_a\lambda^{\frac{\beta+2}{2}} - C_b\right).$$

By virtue of (49), for $C_a\lambda^{\frac{\beta+2}{2}} = C_b$, we have $\phi'(t) = 0$, and $U(x,t)$ renders a stationary solution. If $C_a\lambda^{\frac{\beta+2}{2}} < C_b$, then $\psi(t) \nearrow \infty$ as $t \nearrow t_0$, and the corresponding solution U blows up as $t \to t_0^-$.

5.3 | Nonexistence of global solutions in the case $q(x) \not\equiv 2$

Let us assume that $\beta \geq 0$ and $q(x) \not\equiv 2$. Introduce the energy functional

$$\widehat{E}(t) = \frac{C_a}{\tau}\|\nabla u\|_2^\tau - C_b\|u\|_2^\beta \int_\Omega \frac{u^{q(x)}}{q(x)}\,dx.$$

Theorem 6. *If $\widehat{E}(0) \leq 0$, $\beta \geq 0$ and $q^- \geq \tau$, then every nonstationary nonnegative solution of problem (48) $u \in \mathbf{V}(Q_T)$ blows up in a finite time.*

The proof imitates the proof of Theorem 4 with the obvious changes due to the choice of the energy functional $\widehat{E}(t)$. By virtue of (35) and (36),

$$\begin{aligned}
\int_\Omega u_t^2\,dx &= -\frac{C_a}{\tau}\frac{d}{dt}\left(\|\nabla u\|_2^\tau\right) + C_b\|u\|_2^\beta \int_\Omega u^{q(x)-1}u_t\,dx,\\
-\frac{1}{\tau}\int_\Omega uu_t\,dx &= \frac{C_a}{\tau}\|\nabla u\|_2^\tau - \frac{C_b}{\tau}\|u\|_2^\beta \int_\Omega u^{q(x)}\,dx\\
&\leq \frac{C_a}{\tau}\|\nabla u\|_2^\tau - \frac{q^-}{\tau}C_b|u\|_2^\beta \int_\Omega \frac{u^{q(x)}}{q(x)}\,dx\\
&\leq \frac{C_a}{\tau}\|\nabla u\|_2^\tau - C_b\|u\|_2^\beta \int_\Omega \frac{u^{q(x)}}{q(x)}\,dx = \widehat{E}(t) \text{ if } q^- \geq \tau.
\end{aligned} \tag{50}$$

Taking the derivative of $\widehat{E}(t)$ and plugging (50), we find that

$$\begin{aligned}
\widehat{E}'(t) &= \frac{C_a}{\tau}\frac{d}{dt}\left(\|\nabla u\|_2^\tau\right) - C_b\|u\|_2^\beta \int_\Omega u^{q(x)-1}u_t\,dx - \beta C_b\|u\|_2^{\beta-2}\left(\int_\Omega \frac{u^{q(x)}}{q(x)}\,dx\right)\left(\int_\Omega uu_t\,dx\right)\\
&\leq -\int_\Omega u_t^2\,dx + C(t)\widehat{E}(t), \qquad C(t) = \tau\beta C_b\|u\|_2^{\beta-2}\left(\int_\Omega \frac{u^{q(x)}}{q(x)}\,dx\right).
\end{aligned}$$

By Lemma 8,

$$\widehat{E}(t) + \int_0^t\int_\Omega u_t^2\,dxds \leq \widehat{E}(0)\nu(t) \text{ with } \nu(t) = \exp\left(\int_0^t C(s)\,ds\right) \geq 1.$$

The rest of the proof is the literal repetition of the proof of Theorem 4.

5.4 | Reaction terms of general form

The assertions of Theorems 1 and 6 remain true if in the reaction term $C_b\|u\|_2^\beta$ is substituted by an arbitrary function $b(\|u\|_2^2)$ that satisfies conditions **(H.1)** (ii) and **(B.2)**. For example, one may take

$$b(s) = \ln\left(1 + s^{\frac{\gamma}{2}}\right),\ b(s) = 1 - e^{-s^{\frac{\gamma}{2}}} \text{ with } \gamma > 0.$$

The corresponding energy functional has the form

$$\widehat{E}(t) = \frac{C_a}{\tau}\|\nabla u\|_2^\tau - b(\|u\|_2^2)\int_\Omega \frac{u^{q(x)}}{q(x)}\,dx.$$

6 | A LOWER ESTIMATE FOR THE TIME OF EXISTENCE OF THE SOLUTION

Let us assume that the functions $a(\cdot)$, $b(\cdot)$, the exponent $q(z)$, and the initial function u_0 satisfy the conditions of Theorem 1. Then, problem (1) has a local solution $u \in \mathbf{V}(Q_T)$, and the function $y(t) = \|u(t)\|_2^2$ satisfies identity (36), which yields the inequality

$$\frac{1}{2}y'(t) + C_a\|\nabla u(t)\|_2^{\tau} \leq C_b\left(y^{\frac{\beta}{2}}(t) + \nu\right)\int_{\Omega} u^{q(x)}\,dx. \tag{51}$$

Let $2 < q^+ < \frac{2n}{n-2}$. Since $u \in \mathbf{V}(Q_T)$, then by the Sobolev embedding theorem $u(\cdot,t) \in L^{q^+}(\Omega)$ for a.e. $t \in (0,T)$, and by Young's inequality,

$$\int_{\Omega} u^{q(x)}\,dx \leq \|u\|_{q^+}^{q^+} + 1.$$

Let us apply the Gagliardo-Nirenberg inequality: for every $v \in L^{q^+}(\Omega) \cap H_0^1(\Omega)$

$$\|v\|_{q^+}^{q^+} \leq C_*^{q^+}\|\nabla v\|_2^{q^+\alpha}\|v\|_2^{q^+(1-\alpha)} \quad \text{with } \alpha = n\left(\frac{1}{2} - \frac{1}{q^+}\right) \in (0,1)$$

and independent of v constant C_*. We assume that

$$\alpha q^+ < \tau \iff q^+ < 2\left(1 + \frac{\tau}{n}\right),$$

and apply the Young inequality:

$$\begin{aligned}
C_b\|u\|_2^{\beta}\int_{\Omega} u^{q^+}\,dx &\leq C_bC_*^{q^+}\|\nabla u\|_2^{\alpha q^+}\|u\|_2^{\beta+q^+(1-\alpha)} \\
&= \left(\left(\frac{C_a}{2}\|\nabla u\|_2^{\tau}\right)^{\frac{1}{\tau}}\right)^{n\frac{q^+-2}{2}}(C_bC_*^{q^+})\left(\frac{2}{C_a}\right)^{n\frac{q^+-2}{2\tau}}\|u\|_2^{q^++\beta-n\frac{q^+-2}{2}} \\
&\leq \frac{C_a}{2}\|\nabla u\|_2^{\tau} + \mu\|u\|_2^{2\gamma}
\end{aligned}$$

with

$$\gamma = \left(q^+ + \beta - n\frac{q^+-2}{2}\right)\frac{\tau}{2\tau - n(q^+-2)}, \qquad \mu = \left[(C_bC_*^{q^+})\left(\frac{2}{C_a}\right)^{n\frac{q^+-2}{2\tau}}\right]^{\frac{1}{1-n\frac{q^+-2}{2\tau}}}.$$

From (51), we obtain the following differential inequality for $y(t)$:

$$\frac{1}{2}y'(t) + \frac{C_a}{2}\|\nabla u(t)\|_2^{\tau} \leq \mu y^{\gamma}(t) + C_b\|u\|_2^{\beta} + \nu C_b\int_{\Omega} u^{q(x)}\,dx.$$

The last term here can be estimated in a similar way:

$$\begin{aligned}
\nu C_b\int_{\Omega} u^{q(x)} &\leq \nu C_b\left(1 + \|u\|_{q^+}^{q^+}\right) \\
&\leq \nu C_b + \nu C_bC_*^{q^+}\|\nabla u\|_2^{n\frac{q^+-2}{2}}\|u\|_2^{q^+-n\frac{q^+-2}{2}} \\
&\leq \nu C_b + \frac{C_a}{2}\|\nabla u\|_2^{\tau} + \theta\|u\|_2^{2\delta}
\end{aligned}$$

with the constants

$$\delta = \left(q^+ - n\frac{q^+-2}{2}\right)\frac{\tau}{2\tau - n(q^+-2)} \in (0,\gamma), \qquad \theta = \mu\nu^{\frac{2\tau}{2\tau-n(q^+-2)}}.$$

Since $q^+ < \frac{2n}{n-2}$, then $\beta < 2\gamma$ and by Young's inequality

$$C_b\|u\|_2^{\beta} \le C_b y^{\gamma}(t) + C_b.$$

The resulting inequality has the form

$$y'(t) \le Ay^{\gamma}(t) + B \text{ for } t>0, \quad y(0) = \|u_0\|_2^2, \tag{52}$$

$$A = C_b + \mu\left(1+\nu^{\frac{2\tau}{2\tau-n(q^+-2)}}\right), \quad B = (1+\nu)C_b + \theta.$$

Theorem 7. *Let the data of problem (1) satisfy the conditions of Theorem 1 and*

$$2 < q^+ < 2\min\left\{\frac{n}{n-2}, 1+\frac{\tau}{n}\right\}.$$

(i) *If* $q^+ + \beta + n\left(\frac{1}{\tau}-\frac{1}{2}\right)(q^+-2) > 2$ *(which is equivalent to the inequality* $\gamma > 1$*), then* $\|u(t)\|_2^2$ *is bounded for every* $t < t^*$ *where*

$$t^* = \frac{z_0^{1-\gamma}}{2A(\gamma-1)}, \qquad z_0 = \max\left\{y_0, (B/A)^{\frac{1}{\gamma}}\right\}.$$

(ii) *If* $q^+ + \beta + n\left(\frac{1}{\tau}-\frac{1}{2}\right)(q^+-2) \le 2$*, then* $\|u(t)\|_2^2$ *remains bounded for all* $t < \infty$.

The proof is an immediate consequence of Lemma 1: the functions satisfying inequality (52) remain bounded on an interval $(0, t^*)$ with t^* defined by the initial datum $y(0)$ and the parameters γ, A, B. The condition $\gamma \le 1$ guarantees global boundedness of $\|u(t)\|_2$. It is easy to check that the condition $\gamma \le 1$ cannot be fulfilled under the conditions of Theorem 2 or 4.

ACKNOWLEDGEMENTS

The first author of the work was supported by the 2219 scholarship programme of Tübitak, the Scientific and Technological Research Council of Turkey. The second author acknowledges the support of the Research Grant MTM2017-87162-P, Spain.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Uğur Sert https://orcid.org/0000-0003-4783-6983
Sergey Shmarev https://orcid.org/0000-0002-1783-5165

REFERENCES

1. Anderson JR, Deng K. Global existence for degenerate parabolic equations with a non-local forcing. *Math Methods Appl Sci*. 1997;20(13):1069-1087.
2. Bebernes J, Eberly D. *Mathematical Problems From Combustion Theory*, *Applied Mathematical Sciences*. Vol. 83: Springer-Verlag, New York; 1989.
3. Bernstein S. Sur une classe d'équations fonctionnelles aux dérivées partielles. *Bull Acad Sci URSS Sér Math [Izvestia Akad Nauk SSSR]*. 1940;4:17-26.
4. Caraballo T, Herrera-Cobos M, Marín-Rubio P. Asymptotic behaviour of nonlocal *p*-Laplacian reaction-diffusion problems. *J Math Anal Appl*. 2018;459(2):997-1015.
5. Chipot M, Lovat B. On the asymptotic behaviour of some nonlocal problems. *Positivity*. 1999;3(1):65-81.

6. Chipot M, Lovat B. Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems. *Dyn Contin Discrete Impuls Syst Ser A Math Anal*. 2001;8(1):35-51. Advances in quenching.
7. Chipot M, Molinet L. Asymptotic behaviour of some nonlocal diffusion problems. *Appl Anal*. 2001;80(3-4):279-315.
8. Ferreira J, de Oliveira HB. Parabolic reaction-diffusion systems with nonlocal coupled diffusivity terms. *Discrete Contin Dyn Syst*. 2017;37(5):2431-2453.
9. Souplet P. Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source. *J Differential Equations*. 1999;153(2):374-406.
10. Chipot M, Rodrigues J-F. On a class of nonlocal nonlinear elliptic problems. *RAIRO Modél Math Anal Numér*. 1992;26(3):447-467.
11. Furter J, Grinfeld M. Local vs. nonlocal interactions in population dynamics. *J Math Biol*. 1989;27(1):65-80.
12. Hilhorst D, Rodrigues J-F. On a nonlocal diffusion equation with discontinuous reaction. *Adv Differential Equations*. 2000;5(4-6):657-680.
13. Pao CV. Blowing-up of solution for a nonlocal reaction-diffusion problem in combustion theory. *J Math Anal Appl*. 1992;166(2):591-600.
14. Chaplain MAJ, Lachowicz M, Szymańska Z, Wrzosek D. Mathematical modelling of cancer invasion: the importance of cell-cell adhesion and cell-matrix adhesion. *Math Models Methods Appl Sci*. 2011;21(4):719-743.
15. Chipot M, Valente V, Vergara Caffarelli G. Remarks on a nonlocal problem involving the Dirichlet energy. *Rend Sem Mat Univ Padova*. 2003;110:199-220.
16. Zheng S, Chipot M. Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms. *Asymptot Anal*. 2005;45(3-4):301-312.
17. Caraballo T, Herrera-Cobos M, Marín-Rubio P. Global attractor for a nonlocal p-Laplacian equation without uniqueness of solution. *Discrete Contin Dyn Syst Ser B*. 2017;22(5):1801-1816.
18. Dawidowski L. The quasilinear parabolic Kirchhoff equation. *Open Math*. 2017;15(1):382-392.
19. Ackleh AS, Ke L. Existence-uniqueness and long time behavior for a class of nonlocal nonlinear parabolic evolution equations. *Proc Amer Math Soc*. 2000;128(12):3483-3492.
20. Sert U, Shmarev S. On a degenerate nonlocal parabolic equation with variable source. *J Math Anal Appl*. 2020;484(1):123695.
21. Han Y, Li Q. Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy. *Comput Math Appl*. 2018;75(9):3283-3297.
22. Han Y, Gao W, Sun Z, Li H. Upper and lower bounds of blow-up time to a parabolic type Kirchhoff equation with arbitrary initial energy. *Comput Math Appl*. 2018;76(10):2477-2483.
23. Mingqi X, Rădulescu VD, Zhang B. Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions. *Nonlinearity*. 2018;31(7):3228-3250.
24. Jazar M, Kiwan R. Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions. *Ann Inst H Poincaré Anal Non Linéaire*. 2008;25(2):215-218.
25. Niculescu CP, Rovenţa I. Large solutions for semilinear parabolic equations involving some special classes of nonlinearities. *Discrete Dyn Nat Soc*. 2010;11:491023.
26. Pinasco JP. Blow-up for parabolic and hyperbolic problems with variable exponents. *Nonlinear Anal*. 2009;71(3-4):1094-1099.
27. Kaplan S. On the growth of solutions of quasi-linear parabolic equations. *Comm Pure Appl Math*. 1963;16:305-330.
28. Liu B, Yang J. Blow-up properties in the parabolic problems with anisotropic nonstandard growth conditions. *Z Angew Math Phys*. 2016;67(1):Art. 13, 26.
29. Antontsev S, Shmarev S. On a class of fully nonlinear parabolic equations. *Adv Nonlinear Anal*. 2019;8(1):79-100.

How to cite this article: Sert U, Shmarev S. On a class of nonlocal parabolic equations of Kirchhoff type: Nonexistence of global solutions and blow-up. *Math Meth Appl Sci*. 2021;1–27. https://doi.org/10.1002/mma.7525