Sistemas diferenciales quasi-homogéneos en el plano y en el espacio

Universidad de Oviedo
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TESIS DOCTORAL

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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

1.- Título de la Tesis	
Español: Sistemas diferenciales quasi-homogéneos en el plano y en el espacio	Inglés: Two and three dimensional quasi-homogeneous differential systems

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RESUMEN (en español)

Esta memoria se concibió con el propósito general de profundizar en los conocimientos existentes respecto de los sistemas diferenciales quasi-homogéneos. Esta clase de sistemas polinomiales tiene interesantes propiedades, y en el caso particular de dimensión 2 han sido estudiados por numerosos investigadores. No así en dimensión 3 y superiores. La tesis se presenta bajo el formato de compendio de publicaciones, y consta de tres artículos interrelacionados entre sí.
En el plano, se proporciona una clasificación de los quasi-homogéneos sobre la base del concepto de vector peso, especialmente en términos del vector peso mínimo. Además, se obtiene el número exacto de diferentes formas de sistemas quasi-homogéneos de un grado arbitrario, poniéndose de manifiesto una relación entre este número y la función indicatriz de Euler. Finalmente, se aportan implementaciones en software para algunos de los resultados anteriores.
En el espacio, se desarrolla un algoritmo que proporciona todas las formas de sistemas quasi-homogéneos de un grado dado. En la práctica, con este algoritmo se ponen a disposición de los investigadores todos los sistemas quasi-homogéneos tridimensionales. Haciendo uso del mismo, se enumeran todos los sistemas de este tipo de grado 2, y posteriormente se hace un estudio de la integrabilidad analítica de dichos sistemas.
Siguiendo con la integrabilidad, se ha realizado un análisis completo del estado del arte del llamado método de Yoshida, un sistema que en muchas ocasiones puede ser de utilidad para determinar las integrales primeras analíticas de sistemas quasi-homogéneos en cualquier dimensión.
Los resultados obtenidos a lo largo de la investigación y presentados en esta memoria han sido publicados en varias revistas de alto nivel y se han presentado como ponencias en diversos congresos internacionales para dotarlos de la mayor difusión posible. Por otra parte, los desarrollos sofware implementados han sido puestos a disposición de la comunidad científica internacional a través de la página web https://matemat51.epv.uniovi.es/gsd.

RESUMEN (en Inglés)

This report was conceived with the general purpose of deepening the existing knowledge regarding quasi-homogeneous differential systems. This class of polynomial systems has interesting properties, and in the particular case of dimension 2 they have been studied by numerous researchers. Not so in dimension 3 and higher. The thesis is presented under the format of a compendium of publications, and consists of three interrelated articles.
On the plane, a classification of quasi-homogeneous is provided on the basis of the weight vector concept, especially in terms of the minimum weight vector. Furthermore, the exact number of different forms of quasi-homogeneous systems of an arbitrary degree is obtained, showing a relationship between this number and the Euler totient function. Finally, software implementations are provided for some of the above results.
In space, an algorithm is developed that provides all forms of quasi-homogeneous systems of a given degree. In practice, this algorithm makes all quasi-homogeneous three-dimensional systems available to researchers. Making use of it, all the systems of this type of grade 2 are listed, and later a study of the analytical integrability of these systems is made.
Continuing with integrability, a complete analysis of the state of the art of the so-called Yoshida

method has been carried out, a system that on many occasions can be useful to determine the first analytical integrals of quasi-homogeneous systems in any dimension.
The results obtained throughout the investigation and presented in this report have been published in several high-level journals and have been presented as papers at various international conferences to give them the widest possible dissemination. On the other hand, the implemented software developments have been made available to the international scientific community through the website https://matemat51.epv.uniovi.es/gsd.

SR. PRESIDENTE DE LA COMISIÓN ACADÉMICA DEL PROGRAMA DE DOCTORADO EN MATEMÁTICAS Y ESTADÍSTICA

Sistemas diferenciales quasi-homogéneos en el plano y en el espacio

Memoria que presenta para optar al título de Doctor en Matemáticas

Dirigida por los Doctores

Jesús Suárez Pérez del Río

María Belén García Fernández

Programa de Doctorado en Matemáticas y Estadística

Febrero de 2021

A María, Antón y Mario

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El extenso recorrido que culmina con la presentación de esta tesis arranca muchos años antes de formalizar mi matrícula en el programa de doctorado de la Universidad de Oviedo. Mis padres me transmitieron el valor de la cultura, y de la importancia de hacer las cosas que nos gustan, más allá de que aporten réditos materiales o no. Sin su estímulo, dudo que me hubiese aventurado en esta tesis poco convencional, hecha "por amor al arte" y con mi vida profesional ya asentada.

Creo que también le debo algo a la Universidad de Santiago de Compostela, y en particular a su Facultad de Matemáticas. Al contrario que otras personas con una clara vocación, ingresé en la USC sin grandes pretensiones; pero salí de allí cautivado para siempre por la profundidad de las matemáticas. De esta institución, quiero recordar a Juan José Nieto Roig, con quien me inicié en trabajos de investigación, y a Gerardo Rodríguez, el profesor que me dio a conocer los Sistemas Dinámicos.

Con referencias muy escasas sobre mi validez investigadora, José Ángel Rodríguez, *Chachi*, me abrió la puerta al Grupo de Sistemas Dinámicos de la Universidad de Oviedo, algo que siempre le agradeceré.

A lo largo de casi toda mi trayectoria como doctorando, he tenido el privilegio de colaborar con Jaume Llibre, de la UAB. De Jaume es innecesario mencionar su autoridad académica, así que solo quiero hacer constancia aquí de su amabilidad y sencillez de trato, sea cual sea la condición de la persona con la que tenga relación profesional.

No se a ciencia cierta como son otros directores de tesis, pero mi sensación es que he tenido mucha suerte con los míos.

Belén es una calculadora humana, una garantía para la fiabilidad de cualquier cómputo o razonamiento matemático que pase por sus manos. Pero asimismo, su positividad y optimismo ante cualquier contratiempo han supuesto un apoyo muy importante para mí. Siempre con mensajes de ánimo, y a pesar de su forzosa y prematura jubilación, ha seguido implicada hasta el final en el desarrollo de esta tesis.

He recorrido un largo camino al lado de Jesús. Durante todos estos años de investigación, congresos, estancias, despachos y aeropuertos, él ha sido un generoso maestro del que he aprendido mucho en el plano académico, pero aún más en el personal. Desde la jubilación de Belén, Jesús ha cargado con el grueso de la dirección de la tesis, y con su impulso y apoyo incondicional ha demostrado una encomiable confianza en mí[1].

Finalizo con los de casa, los que han asistido al aspecto más árido de este proyecto. Cuando todo esto empezó, Mario todavía se recreaba tranquilo dentro de la barriga de María, y Antón apenas comenzaba a vislumbrar que era eso de las letras y los números. Ahora, y no me sorprende con esa madre, se comen el mundo. Esta etapa se acaba... gracias a los tres por acompañarme en el camino.

[1] Jesús ha implicado en la tesis incluso a su familia, y quiero dar las gracias a su hijo Carlos.

Resumen

Esta memoria tiene como objetivo fundamental profundizar en los conocimientos existentes acerca de los sistemas diferenciales quasi-homogéneos. Esta clase de sistemas polinomiales posee propiedades importantes y, en el caso particular de dimensión 2, han sido estudiados desde diversos puntos de vista por parte de varios autores. Sin embargo, es escaso el número de publicaciones relativas a dimensiones superiores a 2.

En la memoria se proporciona una clasificación de los sistemas quasi-homogéneos en el plano sobre la base del concepto de vector peso y, más concretamente, en términos del vector peso mínimo. A continuación, se obtiene el número exacto de formas diferentes de sistemas quasi-homogéneos de un grado arbitrario, y se pone de manifiesto una relación entre este número y la función Indicatriz de Euler. Además, se aportan implementaciones en software para algunos de los resultados anteriores.

Por otra parte, en el caso del espacio se desarrolla un algoritmo que proporciona todas las formas normales de sistemas quasi-homogéneos maximales de un grado dado, lo que en la práctica equivale a determinarlos todos. Utilizando el algoritmo, se enumeran todos los sistemas de este tipo de grado 2, y posteriormente se hace un estudio de la integrabilidad analítica de dichos sistemas.

El estudio de la integrabilidad ha llevado a realizar un análisis completo del estado del arte del llamado método de Yoshida, un procedimiento que en determinadas ocasiones es de utilidad para obtener las integrales primeras analíticas de sistemas quasi-homogéneos de cualquier dimensión.

La tesis se presenta bajo el formato de compendio de publicaciones, y consta de tres artículos interrelacionados entre sí.

Los resultados obtenidos a lo largo de la investigación y expuestos en esta memoria han sido publicados en revistas incluidas en el *Science Citation Index* y se han presentado como ponencias en diversos congresos internacionales para dotarlos de la mayor difusión posible. Por otra parte, los desarrollos de software implementados han sido puestos a disposición de la comunidad científica a través de la página web https://matemat51.epv.uniovi.es/gsd.

Abstract

The main objective of this report is to deepen the existing knowledge about quasi-homogeneous differential systems. This class of polynomial systems has important properties and, in the particular case of dimension 2, they have been studied from various points of view by several authors. However, the number of publications relating to dimensions greater than 2 is scarce.

The report provides a classification of quasi-homogeneous systems in the plane based on the concept of the weight vector and, more specifically, in terms of the minimum weight vector. Furthermore, the exact number of different forms of quasi-homogeneous systems of an arbitrary degree is obtained, and a relationship between this number and the Euler *totient* function is revealed. In addition, software implementations are provided for some of the above results.

On the other hand, in the case of space, an algorithm is developed that provides all normal forms of maximal quasi-homogeneous systems of a given degree, which in practice amounts to determining all of them. Using the algorithm, all systems of this type with grade 2 are listed, and subsequently a study is made of the analytical integrability of these systems.

The study of integrability has led to a complete analysis of the state of the art of the so-called Yoshida method, a procedure that on certain occasions is useful to obtain the analytical first integrals of quasi-homogeneous systems of any dimension.

The thesis is presented under the format of a compendium of publications, and consists of three interrelated articles.

The results obtained throughout the investigation and presented in this report have been published in journals included in the *Science Citation Index* and have been presented as reports at various international congresses to provide them with the widest possible dissemination. On the other hand, the implemented software developments have been made available to the scientific community through the website https://matemat51.epv.uniovi.es/gsd.

Índice

Capítulo 1

Introducción

1.1. Sistemas dinámicos

Movimiento y ecuaciones diferenciales

El estudio del movimiento de los cuerpos ha atraído a la humanidad desde los orígenes del pensamiento científico. En el caso particular de la Mecánica Celeste, civilizaciones de la Edad Antigua como la egipcia o la mesopotámica ya establecieron conjeturas acerca del misterioso desplazamiento de los objetos del Sistema Solar, fundamentadas en la observación de la repetición de ciclos. El salto cualitativo más importante en la comprensión de estos fenómenos se produce con el desarrollo del Cálculo Infinitesimal en el siglo XVII. Esta nueva rama de la Matemática, en conjunción con las leyes de la Física, se convertiría en la herramienta más adecuada para el estudio y la comprensión del movimiento.

En aquella época, el enfoque del Cálculo era el de modelizar la realidad a través de ecuaciones diferenciales. A pesar de tratarse, en muchas ocasiones, de ecuaciones de gran simplicidad y elegancia, su resolución analítica resultaba ser una tarea ardua o incluso inabarcable. La ingente labor de los más grandes matemáticos de los siglos XVIII y XIX desembocó en una teoría completa para las ecuaciones diferenciales ordinarias lineales, pero los sistemas no lineales permanecieron fundamentalmente inaccesibles, más allá de aproximaciones a soluciones de problemas simples muy cercanos (métodos de perturbación).

Henri Poincaré

Dicho planteamiento cuantitativo es parcialmente reemplazado a finales del siglo XIX gracias a las aportaciones de Henri Poincaré [18]. El matemático francés desvincula el estudio de las ecuaciones diferenciales de la mera resolución analítica y, combinando el Análisis con la Geometría, se centra en la topología de las soluciones. El cálculo efectivo de las mismas pasa a un segundo plano, y así surge lo que se conoce como teoría cualitativa, que da origen a la actual disciplina de los Sistemas Dinámicos.

Poincaré concibe un sistema dinámico como un campo de vectores en un espacio de fases, de forma que las soluciones serán curvas regulares tangentes en cada uno de sus puntos al vector con origen en dicho punto. En esencia un sistema dinámico está definido por un conjunto de elementos (los puntos del espacio de fases), cada uno de ellos con un estado (su posición), junto con una regla que especifica cómo evolucionan esos estados en el tiempo (el campo de vectores). La descripción geométrica global de todas las soluciones, el llamado *retrato de fases* del sistema, aporta valiosa información sobre la existencia de situaciones estacionarias, elementos atractores o repulsores, o comportamientos periódicos.

Aunque la perspectiva clásica se centra en el análisis de ecuaciones diferenciales ordinarias, las cuales derivan en los denominados sistemas dinámicos continuos, aparecen otro tipo de sistemas dinámicos, llamados discretos, que consisten en el estudio cualitativo de ecuaciones en diferencias o de iteraciones de un difeomorfismo definido sobre una variedad diferenciable. Estos últimos modelizan situaciones en las que el cambio ocurre en intervalos discretos de tiempo. Ambos tipos, continuos y discretos, están estrechamente relacionados, como ya se pone de manifiesto en los trabajos de Poincaré. De manera más formal, dado un espacio topológico X y un grupo aditivo G, un *sistema dinámico* sobre X es una terna (X, G, Φ) donde

$$\Phi : X \times G \to X$$

es una aplicación continua tal que:

a) $\Phi(x, 0) = x, \qquad \forall x \in X$

b) $\Phi(x, s+t) = \Phi(\Phi(x,t), s), \qquad \forall s, t \in G,\ \forall x \in X$

El espacio X es el *espacio de fase*s, y del sistema dinámico se dice que es *continuo* o *discreto* según que G sea $\mathbb{R}$ o $\mathbb{Z}$. En el caso continuo, se denomina *flujo* a la aplicación $\Phi(x, t)$.

Teoría del Caos

A lo largo de todo el pasado siglo y de lo que va del presente, la teoría de Sistemas Dinámicos se ha desarrollado notablemente. Un gran número de matemáticos y grupos de investigación han continuado con el trabajo emprendido por Poincaré, destacando las contribuciones de Birkhoff [2], Andronov y Pontriagin [1], Lorenz [16], Smale [20], Moser [17] y Hénon [13]. En la actualidad se tienen en cuenta tanto los aspectos cualitativos como los cuantitativos, y la asociación de ambos enfoques contribuye a una comprensión plena de los Sistemas Dinámicos. Tanto los estudios puramente teóricos como aquellos apoyados en la experimentación con software de cálculo numérico concluyen que, independientemente de la sencillez de las ecuaciones, el comportamiento asintótico de las soluciones puede revelar trayectorias de gran complejidad. Este tipo de dinámica *caótica* se pone de manifiesto, por ejemplo, con la presencia de los denominados *atractores extraños*[1]. En 1963, E. L. Lorenz [16] encuentra de forma casual un sencillo sistema polinomial de grado 2 con un comportamiento de esta naturaleza, que ha pasado a la posteridad como el experimento que da origen a la Teoría del Caos.

En relación con lo anterior, una de las aportaciones más significativas del estudio de los Sistemas Dinámicos ha sido un cambio de concepción en cuanto a la naturaleza del Universo. La existencia de atractores extraños da lugar a una ruptura definitiva con el modelo determinista que durante siglos dominó la Física. A pesar de los importantes obstáculos que para el determinismo habían supuesto la Mecánica Estadística y la Mecánica Cuántica, aún se podían encontrar justificaciones que permitían suponer que para modelos simples el comportamiento sería siempre predecible y regular. Los sistemas caóticos muestran que este punto de vista no es realista.

Los Sistemas Dinámicos ofrecen a partir de aquí un contexto abstracto, un marco teórico y una metodología que permiten sean aplicados a la modelización de problemas en todas las áreas científicas y tecnológicas. En esta memoria se realizan aportaciones en el marco de la clasificación y de las técnicas de integrabilidad de la clase de los sistemas diferenciales quasi-homogéneos, que dan lugar a sistemas dinámicos continuos.

1.2. Sistemas diferenciales polinomiales

Sistemas y campos vectoriales

Un *sistema polinomial de ecuaciones diferenciales en* $\mathbb{R}^n$ se define por

$$\frac{dx_i}{dt} = \dot{x}_i = P_i(\mathbf{x}), \qquad \mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n, \qquad i = 1, ..., n, \tag{1.1}$$

donde las funciones P_i pertenecen al anillo de polinomios sobre $\mathbb{R}$ en las variables reales $x_1, ..., x_n$, y las funciones $x_i(t)$ son reales de variable real, todo ello para $i = 1, ..., n$. Se denomina *grado* del sistema al valor $h = \max\{h_1, ..., h_n\}$, siendo h_i el grado del polinomio P_i para todo i entre 1 y n. Asociado al sistema (1.1) se tiene el *campo vectorial polinomial*

[1] Por atractor se entiende un conjunto compacto invariante del espacio fase que contiene una órbita densa y que posee un recinto de atracción de interior no vacío. Si además la órbita es exponencialmente expansiva, i.e., los errores en las condiciones iniciales se amplifican exponencialmente con el tiempo, se dice que es un atractor extraño.

$X = (P_1, ..., P_n)$. Este campo vectorial define un sistema dinámico de tipo continuo. De aquí en adelante, se denotará por X indistintamente al sistema o al correspondiente campo vectorial. Finalmente, por $X^k = \left(P_1^k, ..., P_n^k\right)$ se denotará la parte homogénea de grado k del sistema, con k variando de 0 a h.

Obsérvese que los sistemas que se están englobando en esta definición están constituidos por ecuaciones diferenciales de primer orden, autónomas y con coeficientes constantes. Además, el número de funciones desconocidas $x_1, ..., x_n$ coincide con el número de ecuaciones del sistema.

Soluciones y órbitas

Una *solución* del sistema (1.1) es una función $F(t) = (\phi_1(t), ..., \phi_n(t))$, definida en un intervalo $I \in \mathbb{R}$ y con valores en $\mathbb{R}^n$, tal que para todo $t \in I$ se tiene que

$$\frac{d}{dt}\phi_i(t) = P_i(\phi_1(t), ..., \phi_n(t)), \qquad i = 1, ..., n.$$

Dado que las funciones P_i, por ser polinómicas, son lo suficientemente regulares, los teoremas de existencia y unicidad para ecuaciones diferenciales aseguran que para cada punto $\mathbf{x_0} \in \mathbb{R}^n$ existe una única solución $F_{\mathbf{x_0}}(t)$ de (1.1) definida para $t \in I$, y tal que $F_{\mathbf{x_0}}(0) = \mathbf{x_0}$.

Las soluciones $F_{\mathbf{x}}(t)$ del sistema diferencial (1.1) definen el flujo del correspondiente sistema dinámico continuo asociado, flujo que viene dado por $\Phi(x, t) = F_{\mathbf{x}}(t)$, con $\mathbf{x} \in \mathbb{R}^n$ y $t \in I$.

Se denomina *órbita* del punto $\mathbf{x_0} \in \mathbb{R}^n$ al conjunto $\{F_{\mathbf{x_0}}(t) : t \in I\}$. La órbita de $\mathbf{x_0}$ define una curva en $\mathbb{R}^n$, llamada *curva integral*, que representa la trayectoria que describe el punto $\mathbf{x_0}$ al evolucionar con el tiempo. En el caso $n = 2$, los mencionados retratos de fases son representaciones geométricas en el plano del comportamiento asintótico de las curvas integrales más significativas del sistema, indicando con una flecha el sentido de recorrido de la órbita a través del tiempo.

Las órbitas más sencillas de un sistema se corresponden con sus puntos singulares. Se dice que $\mathbf{x_0} \in \mathbb{R}^n$ es un *punto singular o crítico* de (1.1) si se verifica que $P_i(\mathbf{x_0}) = 0$ para todo i entre 1 y n. En estos casos las órbitas son triviales, puesto que $F_{\mathbf{x_0}}(t) = \mathbf{x_0}$ para todo $t \in I$, lo que significa que el punto se mantiene fijo en la dinámica inducida por el sistema diferencial.

Otras órbitas de gran importancia en el estudio de los sistemas diferenciales son las soluciones periódicas. Se dice que $\{F_{\mathbf{x}}(t) : t \in I\}$ es una *órbita periódica* de (1.1) si existe un número real $T > 0$, denominado *periodo* de la órbita, tal que $F_{\mathbf{x}}(t + T) = F_{\mathbf{x}}(t)$ para todo $t \in I$. En aquellos casos en los que una órbita periódica está aislada, es denominada *ciclo límite*[2].

[2]La segunda parte del conocido Problema 16 de Hilbert se plantea la obtención del

Por otra parte, se dice que una curva $f(x_1, ..., x_n) = 0$ en $\mathbb{R}^n$ es una *curva invariante* del sistema (1.1) si contiene las órbitas de todos sus puntos, es decir, si para cada solución $F_{\mathbf{x}}(t)$ del sistema verificando $f(F_{\mathbf{x}}(0)) = 0$, se tiene que $f(F_{\mathbf{x}}(t)) = 0$ para todo $t \in I$.

1.3. Integrabilidad

Integrales primeras

El cálculo explícito de las soluciones de un sistema diferencial polinomial, tratado desde el punto de vista puramente analítico, es por lo general una tarea de gran dificultad, o incluso inabordable. Una alternativa a la búsqueda de las soluciones es el estudio de las integrales primeras. Se define *integral primera* del sistema (1.1) en un abierto $\Omega \subseteq \mathbb{R}^n$ como aquella función no constante $H : \Omega \to \mathbb{R}$ que permanece constante sobre toda solución $F_{\mathbf{x}}(t)$ de (1.1); es decir, $H(F_{\mathbf{x}}(t)) = c \in \mathbb{R}$ para todos los valores de t tales que $F_{\mathbf{x}}(t) \in \Omega$. Nótese que, de acuerdo con esta definición, las curvas $H(\mathbf{x}) = c$ son invariantes para el sistema, para todo $c \in \mathbb{R}$.

Una integral primera es *global* cuando $\Omega = \mathbb{R}^n$, y se designa como *polinomial* (resp. *analítica*) en caso de que H sea una función polinomial (resp. analítica).

Proposición 1. *Dadas H_1 y H_2, ambas integrales primeras en Ω de (1.1), se verifica que las funciones $\lambda H_1 + \mu H_2$, $H_1 H_2$ y $\frac{H_1}{H_2}$, con $\lambda, \mu \in \mathbb{R}$, son también integrales primeras del sistema en sus respectivos dominios de definición.*

Demostración. Se deduce fácilmente de la definición de integral primera. □

Proposición 2. *Dada una función analítica $H : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, H es una integral primera analítica de (1.1) en Ω si y solo si el gradiente de H es perpendicular al campo vectorial de (1.1), es decir, si*

$$\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} P_i \equiv 0$$

en Ω.

Demostración. Si H es una integral primera de (1.1), entonces para cualquier solución $F(t) = (\phi_1(t), ..., \phi_n(t))$ definida sobre Ω se tiene que $H(F(t))$ es constante. Derivando respecto de t se obtiene

$$\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} (\phi_1(t), ..., \phi_n(t)) \frac{\partial \phi_i}{\partial t}(t) = 0,$$

número máximo y la posición relativa de los ciclos límite de los sistemas diferenciales polinomiales de grado n.

y utilizando que $F(t)$ es una solución de (1.1), se llega a

$$\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} (\phi_1(t), ..., \phi_n(t)) P_i (\phi_1(t), ..., \phi_n(t)) = 0,$$

lo que, teniendo en cuenta que todo punto $\mathbf{x} \in \Omega$ está en la imagen de alguna solución del sistema, demuestra la primera implicación. La implicación recíproca se prueba invirtiendo el proceso. □

En dimensión 2, los sistemas hamiltonianos $X = (P_1, P_2)$ son un ejemplo clásico de sistemas con integral primera. En este caso existe una función $H : \mathbb{R}^2 \to \mathbb{R}$, llamada *Hamiltoniano*, tal que $P_1 = \frac{\partial H}{\partial x_2}$ y $P_2 = -\frac{\partial H}{\partial x_1}$, y por tanto la propia función H constituye una integral primera de X.

Integrabilidad completa

Un conjunto de funciones reales sobre $\mathbb{R}^n$ $H_1, ..., H_r$ se dice *funcionalmente independiente* si el rango de la matriz $r \times n$

$$\begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \cdots & \frac{\partial H_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial H_r}{\partial x_1} & \cdots & \frac{\partial H_r}{\partial x_n} \end{pmatrix} (\mathbf{x})$$

toma el valor r en todos los puntos $\mathbf{x} \in \mathbb{R}^n$ donde la matriz está definida, con la excepción a lo sumo de un subconjunto de medida de Lebesgue cero. Geométricamente, esto se traduce en que los respectivos gradientes $\nabla H_i(\mathbf{x})$ son linealmente independientes en todos los puntos $\mathbf{x} \in \mathbb{R}^n$. En caso contrario, las funciones se denominan *funcionalmente dependientes*.

Un sistema diferencial polinomial en $\mathbb{R}^n$ (1.1) se dice *completamente integrable* cuando posee $n-1$ integrales primeras analíticas funcionalmente independientes. En caso de tener alguna integral primera, pero en un número inferior a $n-1$, el sistema se denomina *parcialmente integrable*.

El interés en la propiedad de integrabilidad completa radica en el hecho de que si para un sistema determinado se puede demostrar la existencia de $H_1, ..., H_{n-1}$ integrales primeras funcionalmente independientes, entonces las órbitas quedan determinadas por la intersección de los conjuntos invariantes $\mathcal{F}_i = \left\{ H_i^{-1}(t) \mid t \in \mathbb{R} \right\}$, $i = 1, ..., n-1$. Incluso el caso de la integrabilidad parcial es provechoso, ya que conocer una integral primera implica conocer superficies de dimensión $n-1$ invariantes para las órbitas, y el comportamiento de estas últimas sobre las superficies puede así ser estudiado con la ventaja de haber reducido en una dimensión el problema original.

Factores integrantes en el plano

En el estudio de la integrabilidad de un sistema diferencial $X = (P_1, P_2)$ en $\mathbb{R}^2$ son de especial relevancia los llamados factores integrantes. Una función analítica $R : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ no idénticamente nula en Ω se dice *factor integrante* del sistema X si se verifica que para todo $(x_1, x_2) \in \Omega$,

$$\frac{\partial (P_1 \cdot R)}{\partial x_1} = -\frac{\partial (P_2 \cdot R)}{\partial x_2}. \tag{1.2}$$

En tal caso, el sistema es completamente integrable en Ω, ya que se tiene la integral primera

$$H(x_1, x_2) = \int (R \cdot P_1)(x_1, x_2)\, dx_2 + f(x_1),$$

donde $f(x_1)$ se determina considerando que $\frac{\partial H}{\partial x_1}(x_1, x_2) = -(P_2 \cdot R)(x_1, x_2)$. Obsérvese que el caso trivial en el que R es una constante da lugar a los sistemas hamiltonianos.

Por otra parte, si $V : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ es una función verificando

$$\frac{\partial V}{\partial x_1} P_1 + \frac{\partial V}{\partial x_2} P_2 = V \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} \right) \tag{1.3}$$

para todo $(x_1, x_2) \in \Omega$, a V se le denomina *inverso de factor integrante* del sistema X. Es fácilmente comprobable que V verificará (1.3) si y solo si $R = \frac{1}{V}$ satisface (1.2) en $\{(x_1, x_2) \in \Omega : V(x_1, x_2) \neq 0\}$, por lo que la existencia de un inverso de factor integrante también implica integrabilidad completa en $\mathbb{R}^2$, salvo para aquellos puntos donde dicha función se anule.

1.4. Sistemas quasi-homogéneos

Conceptos básicos

Sea $\mathbb{Z}^+$ el conjunto de los enteros positivos y $\mathbb{R}^+$ el conjunto de los reales positivos. Un sistema diferencial polinomial (1.1) en $\mathbb{R}^n$ se dice *quasi-homogéneo* si existen $s_1, ..., s_n, d \in \mathbb{Z}^+$ tales que para cualquier $\alpha \in \mathbb{R}^+$, la condición

$$P_i(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n) = \alpha^{s_i - 1 + d} P_i(x_1, ..., x_n). \tag{1.4}$$

se verifica para $i = 1, ..., n$.

En este caso, $\mathbf{v} = (s_1, ..., s_n, d)$ se denomina *vector peso* del sistema (1.1), los valores $s_1, ..., s_n$ son llamados *exponentes peso* del sistema, y d es el *grado peso* del sistema respecto de los citados exponentes peso. El número de vectores peso de un sistema quasi-homogéneo siempre es infinito, como muestra el siguiente resultado:

Proposición 3. *Si* $\mathbf{v} = (s_1, ..., s_n, d)$ *es un vector peso de un sistema quasi-homogéneo (1.1), entonces para cualquier valor* $k \in \mathbb{Z}^+$*, también* $\mathbf{w} = (ks_1, ..., ks_n, k(d-1)+1)$ *es un vector peso de (1.1).*

Demostración. Sean $k \in \mathbb{Z}^+$ y $\alpha \in \mathbb{R}^+$. Por ser $\mathbf{v}$ un vector peso de (1.1), se tiene que tomando $\beta = \alpha^k \in \mathbb{R}^+$, se verifica

$$P_i\left(\alpha^{ks_1}x_1, ..., \alpha^{ks_n}x_n\right) = P_i\left(\beta^{s_1}x_1, ..., \beta^{s_n}x_n\right) =$$

$$= \beta^{s_i-1+d}P_i\left(x_1, ..., x_n\right) = \alpha^{ks_i+k(d-1)}P_i\left(x_1, ..., x_n\right)$$

para todo $i \in \{1, ..., n\}$. □

La proposición anterior facilita un modo de agrupar los vectores peso de un sistema quasi-homogéneo: se llama *familia de proporciones* $(\lambda_2, ..., \lambda_n) \in (\mathbb{Q}^+)^{n-1}$ al conjunto del vectores peso de (1.1) que verifican $s_1 = \lambda_i s_i$ para $i = 2, ..., n$. Nótese que en un mismo sistema puede existir más de una familia de vectores peso. Un ejemplo de ello es el sistema plano

$$P_1\left(x_1, x_2\right) = x_1^2 x_2, \qquad P_2\left(x_1, x_2\right) = x_1 x_2^2, \tag{1.5}$$

para el cual cualquier vector de la forma $(s_1, s_2, s_1 + s_2 + 1)$, con s_1 y s_2 enteros positivos, constituye un vector peso.

Además, dentro del conjunto de vectores peso de un sistema quasi-homogéneo es posible definir una relación de orden parcial: dados dos vectores peso $\mathbf{v} = (s_1, ..., s_n, d)$ y $\mathbf{w} = (t_1, ..., t_n, e)$, se afirma que $\mathbf{v} \leq \mathbf{w}$ cuando $s_i \leq t_i$ para $i = 1, ..., n$ y $d \leq e$. Como muestra el ejemplo (1.5), no se trata necesariamente de una relación de orden total.

Cuando existe un vector peso $\mathbf{v_m}$ verificando que $\mathbf{v_m} \leq \mathbf{v}$ para todo vector peso $\mathbf{v}$ del sistema, se denomina *vector peso mínimo* a $\mathbf{v_m}$.

Los sistemas quasi-homogéneos son una generalización de los homogéneos. Un sistema (1.1) se dice *homogéneo de grado* h si en él todas las funciones polinomiales P_i son de grado h, o equivalentemente, si dado cualquier $\alpha \in \mathbb{R}$, $P_i(\alpha\mathbf{x}) = \alpha^h P_i(\mathbf{x})$ para todo $\mathbf{x} \in \mathbb{R}^n$ y para todo $i = 1, ..., n$. Por tanto, un sistema homogéneo de grado h es, en particular, un sistema quasi-homogéneo que tiene por vector peso a $\mathbf{v} = (k, ..., k, k(h-1)+1)$, para todo $k \in \mathbb{Z}^+$. No obstante, históricamente se introdujeron primero los sistemas homogéneos; partiendo de la idea de homogeneidad e introduciendo diferentes pesos para las variables, el concepto de quasi-homogeneidad surgió después de manera natural.

En los trabajos [7] y [8], que forman parte de esta memoria, se establecen algoritmos para la determinación de todos los sistemas quasi-homogéneos de un grado dado. Para ello se centran en la búsqueda de aquellos sistemas que sean maximales. Un sistema quasi-homogéneo (1.1) se dice *maximal* si cualquier nuevo monomio que se añada a su estructura manteniendo el grado h del sistema hace que este deje de ser quasi-homogéneo. Conociendo los sistemas maximales, se pueden determinar todos los sistemas quasi-homogéneos, ya que el resto son casos particulares de maximales en los que alguno de los monomios ha sido anulado.

Invariantes por similitud

En algunos de los primeros trabajos sobre el tema, Yoshida y otros [11, 23] no trataron estrictamente con sistemas quasi-homogéneos, sino con otro tipo de sistemas de ecuaciones diferenciales denominados *invariantes por similitud*. Esta clase de sistemas se caracteriza por la existencia de ciertos número racionales g_1, ..., g_n tales que se verifica

$$P_i\left(\varepsilon^{g_1}x_1,...,\varepsilon^{g_n}x_n\right) = \varepsilon^{g_i+1}P_i\left(x_1,...,x_n\right) \tag{1.6}$$

es verificado para cualquier $\varepsilon \in \mathbb{R}^+ = \mathbb{R} \setminus \{0\}$ y para $i = 1,...,n$. Los invariantes por similitud poseen dos propiedades fundamentales: en primer lugar, son invariantes bajo la transformación

$$t \to \varepsilon^{-1}t,\ x_1 \to \varepsilon^{g_1}x_1, ...,\ x_n \to \varepsilon^{g_n}x_n \tag{1.7}$$

para cualquier constante $\varepsilon \in \mathbb{R}^+$. Por otra parte, si el sistema de ecuaciones

$$P_i\left(c_1,...,c_n\right) = -g_i c_i, \qquad i = 1,...,n,$$

tiene alguna solución no nula $(c_1,...,c_n)$, entonces, asociada a ella, se conoce un tipo especial de solución del sistema diferencial:

$$\varphi\left(t\right) = \left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right).$$

Hay que señalar que las condiciones (1.4) y (1.6) son equivalentes si se hace $g_i = s_i/\left(d-1\right)$ y se toma $\varepsilon = \alpha^{d-1}$. Como consecuencia, los sistemas quasi-homogéneos pertenecen al conjunto de los invariantes por similitud siempre que cuenten con algún vector peso $(s_1,...,s_n,d)$ con $d \neq 1$. Se verifica entonces que todos los vectores peso de la familia a la que pertenece $(s_1,...,s_n,d)$ dan lugar a los mismos valores g_1, ..., g_n.

Descomposición polinómica

Los sistemas quasi-homogéneos no deben ser confundidos con los polinomios quasi-homogéneos. Un polinomio sobre $\mathbb{R}$, $P\left(x_1,...,x_n\right)$ es *quasi-homogéneo* con *exponente peso* $\mathbf{s} = (s_1,...,s_n) \in \left(\mathbb{Z}^+\right)^n$ y *grado peso* $k \in \mathbb{Z}^+$ cuando para cualquier $\alpha \in \mathbb{R}^+$, se tiene que

$$P\left(\alpha^{s_1}x_1,...,\alpha^{s_n}x_n\right) = \alpha^k P\left(x_1,...,x_n\right). \tag{1.8}$$

Obsérvese que, en virtud de lo anterior, los sistemas quasi-homogéneos con vector peso $\mathbf{v} = (s_1,...,s_n,d)$ serán aquellos en los que su componente

i-ésima verifique ser un polinomio quasi-homogéneo con exponente peso $\mathbf{s} = (s_1, ..., s_n)$ y grado peso $s_i - 1 + d$, para todo $i = 1, ..., n$.

El conjunto de los polinomios quasi-homogéneos con exponente peso $\mathbf{s} = (s_1, ..., s_n)$ y grado peso k está constituido por las funciones de la forma

$$\sum_{(e_1,...,e_n)\in D} A x_1^{e_1} ... x_n^{e_n},$$

donde A representa coeficientes reales arbitrarios, nulos o no, y D representa la colección de soluciones no negativas de la ecuación diofántica $s_1 e_1 + \cdots + s_n e_n = k$.

Es obvio que un polinomio quasi-homogéneo con exponente peso $\mathbf{s} = (s_1, ..., s_n)$ y grado peso k es también un polinomio quasi-homogéneo de exponente peso $p\mathbf{s}$ y grado peso pk, para cualquier valor $p \in \mathbb{Z}^+$. Por tanto, puede asumirse que $mcd\,(s_1, ..., s_n) = 1$.

Proposición 4. *Dados los polinomios quasi-homogéneos* $P\,(x_1, ..., x_n)$, $Q\,(x_1, ..., x_n)$, *ambos con exponente peso* $\mathbf{s} = (s_1, ..., s_n)$ *y grado peso* k_1, *y* $R\,(x_1, ..., x_n)$, *con exponente peso* $\mathbf{s} = (s_1, ..., s_n)$ *y grado peso* k_2, *se verifican las siguientes propiedades:*

a) Todos los monomios que constituyen P *son, en particular, polinomios de exponente peso* $\mathbf{s} = (s_1, ..., s_n)$ *y grado peso* k_1.

b) El polinomio $P + Q$ *tiene exponente peso* $\mathbf{s} = (s_1, ..., s_n)$ *y grado peso* k_1.

c) El polinomio $P \cdot R$ *tiene exponente peso* $\mathbf{s} = (s_1, ..., s_n)$ *y grado peso* $k_1 + k_2$.

d) El polinomio $\frac{\partial P}{\partial x_i}$ *tiene exponente peso* $\mathbf{s} = (s_1, ..., s_n)$ *y grado peso* $k_1 - s_i$, *para todo* $i = 1, ..., n$.

Demostración. Las tres primeras afirmaciones se demuestran de manera sencilla en base a la definición de polinomio quasi-homogéneo. Se probará la última afirmación. Se parte de que para cualquier $\alpha \in \mathbb{R}^+$,

$$P\,(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n) = \alpha^{k_1} P\,(x_1, ..., x_n)\,.$$

Reordenando términos y derivando respecto de x_i se deduce que

$$\frac{\partial}{\partial x_i} P\,(x_1, ..., x_n) = \alpha^{-k_1} \frac{\partial}{\partial x_i} P\,(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n)\,\alpha^{s_i},$$

lo que lleva a que

$$\frac{\partial P}{\partial x_i}\,(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n) = \alpha^{k_1 - s_i} \frac{\partial P}{\partial x_i}\,(x_1, ..., x_n)\,,$$

y esto demuestra el resultado d). □

Proposición 5. *Dada una función analítica $H : \Omega \in \mathbb{R}^n \to \mathbb{R}$ y un vector arbitrario $\mathbf{s} = (s_1, ..., s_n) \in (\mathbb{Z}^+)^n$, es posible expresar H de forma única como $H = \sum_{k \in I} H_k$, donde cada H_k es un polinomio quasi-homogéneo con exponente peso $\mathbf{s}$ y grado peso $k \in \mathbb{Z}^+$, e I un subconjunto de enteros positivos.*

Demostración. Cualquier monomio $Ax_1^{\beta_1}...x_n^{\beta_n}$, con $A \in \mathbb{R}$ y $\beta_1, ..., \beta_n \in \mathbb{N}$, es en particular un polinomio quasi-homogéneo de exponente peso $\mathbf{s}$ y grado peso $k = \sum_{i=1}^{n} s_i\beta_i$. Como función analítica, H es una suma (finita o infinita) de monomios, lo que, uniendo en un polinomio P_k todos los monomios de grado peso k, demuestra la existencia y unicidad de la descomposición $H = \sum_{k \in I} H_k$. □

Integrabilidad y sistemas quasi-homogéneos

La siguiente propiedad fundamental respecto de la integrabilidad de los sistemas quasi-homogéneos ha sido probada por Llibre y Zhang [15] para el caso de las integrales primeras polinomiales. Se demuestra aquí de manera similar para las analíticas.

Proposición 6. *Sea (1.1) un sistema quasi-homogéneo en $\mathbb{R}^n$ con vector peso $\mathbf{v} = (s_1, ..., s_n, d)$, y sea $H : \Omega \in \mathbb{R}^n \to \mathbb{R}$ una función analítica cuya descomposición única en polinomios quasi-homogéneos de exponente peso $\mathbf{s} = (s_1, ..., s_n)$ es $H = \sum_{k \in I} H_k$. Entonces H es una integral primera de (1.1) si y solo si cada polinomio H_k es una integral primera de (1.1).*

Demostración. Como $H = \sum_{k \in I} H_k$ es una integral primera analítica, se tiene que

$$\sum_{i=1}^{n} \left(\sum_{k \in I} \frac{\partial H_k}{\partial x_i} \right) P_i \equiv 0,$$

y en consecuencia

$$\sum_{k \in I} \left(\sum_{i=1}^{n} \frac{\partial H_k}{\partial x_i} P_i \right) \equiv 0. \tag{1.9}$$

Como H_k y P_i son polinomios quasi-homogéneos de exponente peso $\mathbf{s} = (s_1, ..., s_n)$ y grados pesos respectivos k y $s_i - 1 + d$, la Proposición 4 determina que $\sum_{i=1}^{n} \frac{\partial H_k}{\partial x_i} P_i$ es un polinomio quasi-homogéneo de exponente peso $\mathbf{s} = (s_1, ..., s_n)$ y grado peso $k - 1 + d$, para todo $k \in I$. Por tanto, y también debido a la Proposición 4, todos los monomios que constituyen $\sum_{i=1}^{n} \frac{\partial H_k}{\partial x_i} P_i$ tienen grado peso $k - 1 + d$, por lo que no pueden coincidir con ningún monomio de $\sum_{i=1}^{n} \frac{\partial H_r}{\partial x_i} P_i$ con $r \in I$, $r \neq k$. Esto, junto con (1.9), implica que

$$\sum_{i=1}^{n} \frac{\partial H_k}{\partial x_i} P_i \equiv 0 \qquad \forall k \in I,$$

lo que prueba la condición necesaria. La condición suficiente es obvia en virtud de la Proposición 1. □

De la proposición anterior se deduce uno de los resultados más relevantes en relación con la integrabilidad de los sistemas quasi-homogéneos: el estudio de las integrales primeras analíticas se reduce a la búsqueda de aquellas integrales primeras polinomiales quasi-homogéneas que tengan los mismos exponentes peso que el sistema.

Corolario 7. *Un sistema quasi-homogéneo (1.1) en* $\mathbb{R}^n$ *con vector peso* $\mathbf{v} = (s_1, ..., s_n, d)$ *tiene una integral primera analítica si y solo si tiene una integral primera polinomial quasi-homogénea con exponente peso* $\mathbf{s} = (s_1, ..., s_n)$.

Algunos de los escasos procedimientos para la integración de sistemas quasi-homogéneos se basan en esta propiedad. En particular el método de Yoshida, que hemos analizado en profundidad en el artículo [9] incorporado en esta memoria, se centra en la búsqueda de integrales primeras polinomiales de tipo quasi-homogéneo. Esta técnica se basa en la correspondencia entre ciertos valores característicos de dichas integrales primeras y otros valores inherentes al sistema diferencial, los llamados exponentes de Kowalevskaya.

Mientras que para dimensiones $n \geq 3$ las referencias en cuanto a la integrabilidad de sistemas quasi-homogéneos son escasas, en el caso particular del plano existe más bibliografía disponible. En García *et al.* [10] se prueba la existencia de un inverso de factor integrante para cualquier sistema quasi-homogéneo plano, lo que se fundamenta en los dos resultados que a continuación se reproducen:

Lema 8 (Fórmula de Euler generalizada)**.** *Sea* $X = (P_1, P_2)$ *un sistema quasi-homogéneo en* $\mathbb{R}^2$ *con vector peso* $\mathbf{v} = (s_1, s_2, d)$. *Entonces se verifica que*

$$s_1 x_1 \frac{\partial P_i}{\partial x_1} + s_2 x_2 \frac{\partial P_i}{\partial x_2} = (s_i - 1 + d)\, P_i, \tag{1.10}$$

para $i = 1, 2$.

Demostración. Como X es quasi-homogéneo con vector peso $\mathbf{v} = (s_1, s_2, d)$, entonces para $i = 1, 2$ se cumple

$$P_i\left(\alpha^{s_1} x_1, \alpha^{s_2} x_2\right) = \alpha^{s_i - 1 + d} P_i\left(x_1, x_2\right).$$

Derivando esta expresión respecto de α, se obtiene

$$s_1 \alpha^{s_1 - 1} x_1 \frac{\partial P_i}{\partial x_1}\left(\alpha^{s_1} x_1, \alpha^{s_2} x_2\right) + s_2 \alpha^{s_2 - 1} x_2 \frac{\partial P_i}{\partial x_2}\left(\alpha^{s_1} x_1, \alpha^{s_2} x_2\right) =$$

$$= (s_i - 1 + d)\, \alpha^{s_i - 2 + d} P_i\,(x_1, x_2)\,,$$

con lo que tomando $\alpha = 1$ se concluye la demostración. □

Proposición 9. *Sea $X = (P_1, P_2)$ un sistema quasi-homogéneo en $\mathbb{R}^2$ con vector peso $\mathbf{v} = (s_1, s_2, d)$. Entonces se verifica que la función*

$$V\,(x_1, x_2) = s_1 x_1 P_2 + s_2 x_2 P_1 \tag{1.11}$$

es un inverso de factor integrante de X.

Demostración. De acuerdo con la definición (1.3), una función $V : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ será un inverso de factor integrante si se cumple

$$M = \frac{\partial V}{\partial x_1} P_1 + \frac{\partial V}{\partial x_2} P_2 - V \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} \right) \equiv 0.$$

Utilizando a continuación la expresión (1.11), se obtiene que

$$M = \left(s_1 P_2 + s_1 x_1 \frac{\partial P_2}{\partial x_1} - s_2 x_2 \frac{\partial P_1}{\partial x_1} \right) P_1 + \left(s_1 x_1 \frac{\partial P_2}{\partial x_2} - s_2 P_1 - s_2 x_2 \frac{\partial P_1}{\partial x_2} \right) P_2 -$$

$$- \left(s_1 x_1 P_2 - s_2 x_2 P_1 \right) \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} \right).$$

Por último, teniendo en cuenta (1.10), se llega a que

$$M = (s_2 - 1 + d)\, P_2 P_1 - (s_1 - 1 + d)\, P_1 P_2 + s_1 P_1 P_2 - s_2 P_1 P_2 \equiv 0.$$

□

En el artículo [9] hemos proporcionado contraejemplos que muestran que, en dimensión 3, los sistemas quasi-homogéneos no mantienen la anterior propiedad de integrabilidad del plano.

1.5. Función φ de Euler

Definición y propiedades

Por su trascendencia en la elaboración de esta memoria, se analizan aquí las propiedades más importantes de la *función indicatriz φ de Euler*. Dicha función se define, para cada $n \in \mathbb{Z}^+$, como el número de enteros positivos menores o iguales que n y coprimos con n:

$$\varphi\,(n) = |\left\{ r \in \mathbb{Z}^+ \;:\; 1 \leq r \leq n\;,\; mcd\,(n, r) = 1 \right\}| \tag{1.12}$$

Leonhard Euler es quien la define en primera instancia en 1763, denotándola con la letra π y con la particularidad de que $\pi\,(1) = 0$.

Posteriormente, en su tratado *Disquisitiones Arithmeticae* de 1801, Gauss establece la notación con la letra φ y fija la definición actual (1.12), en la que $\varphi(1) = 1$. El término *totient*, denominación alternativa para referirse a esta función, es acuñado por Sylvester [3] en 1879.

Es trivial que cuando $p \in \mathbb{Z}^+$ es primo, $\varphi(p) = p-1$, ya que todo número primo es coprimo con todos sus anteriores. En base a ello y por inducción en k, también se prueba fácilmente que

$$\varphi\left(p^k\right) = p^k\left(1 - \frac{1}{p}\right) \tag{1.13}$$

para el caso de que p sea primo y k un número entero positivo.

En el ámbito de la Teoría de Números, una función $f : A \subseteq \mathbb{Z} \to \mathbb{Z}$ se dice *multiplicativa* si $f(1) = 1$ y además $f(a \cdot b) = f(a) \cdot f(b)$ en el caso de que a y b sean primos entre sí. Obsérvese que una función multiplicativa queda completamente definida con tan solo conocer su comportamiento sobre las potencias de los números primos. La siguiente propiedad fue establecida por Hardy y Wright [12], y se aporta aquí una demostración alternativa:

Proposición 10. *La función indicatriz φ de Euler es multiplicativa.*

Demostración. En esta prueba se denota por (u, v) al máximo común divisor de u y v, de forma que $(u, v) = 1$ equivale a que u y v son coprimos. En varios puntos de esta demostración se utilizarán las siguientes propiedades conocidas de divisibilidad:

- $(u, vw) = 1 \iff (u, v) = 1$ y $(u, w) = 1$
- $(u, v) = 1 \iff (u, v\,(mod\,u)) = 1$

Sean a, b enteros positivos coprimos, y sea $c = ab$. Denotaremos por A (respectivamente B, C) el conjunto de enteros positivos menores o iguales que a (respectivamente, menores o iguales que b, c) y coprimos con a (respectivamente, coprimos con b, c). Se trata de demostrar que el cardinal de C coincide con el de $A \times B$, para lo cual se define una aplicación $f : C \to A \times B$ tal que

$$f(z) = (z\,(mod\,a)\,, z\,(mod\,b)) \tag{1.14}$$

En primer lugar, f está bien definida: si $z \in C$, entonces

$$(z, c) = 1 \Rightarrow (z, a) = 1 \Rightarrow (z\,(mod\,a)\,, a) = 1 \Rightarrow z\,(mod\,a) \in A,$$

y de forma idéntica se prueba que $z\,(mod\,b) \in B$.

En segundo lugar, se verá que f es una biyección. Dado $(x, y) \in A \times B$, puesto que $(a, b) = 1$, el Teorema Chino del Resto garantiza que existe un valor único $z \in \{1, 2, ..., c\}$ tal que es solución del sistema de congruencias

$$\begin{aligned} z &\equiv x\,(mod\,a)\,, \\ z &\equiv y\,(mod\,b)\,. \end{aligned} \tag{1.15}$$

Además, $z \in C$, ya que en caso contrario $(z, c) \neq 1$ y entonces o bien $(z, a) \neq 1$, o bien $(z, b) \neq 1$ (o ambos). En el primer supuesto,

$$(z, a) \neq 1 \Rightarrow (x \,(mod\, a)\,, a) \neq 1 \Rightarrow (x, a) \neq 1 \Rightarrow x \notin A.$$

El segundo supuesto también lleva a contradicción. El carácter biyectivo de f se deriva de que, en base a (1.14) y (1.15), se tiene que $f(z) = (x, y)$. □

A través del resultado anterior se puede establecer una fórmula para $\varphi(n)$, sea cual sea el valor de n:

Proposición 11. *Dado $n \in \mathbb{Z}^+$,*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \tag{1.16}$$

donde el producto recorre los distintos números primos p que dividen a n.

Demostración. Sea $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ la descomposición única en primos de n. Utilizando repetidamente la multiplicidad de φ (Proposición 10) y la propiedad (1.13),

$$\varphi(n) = \varphi\left(p_1^{k_1}\right) \varphi\left(p_2^{k_2}\right) \cdots \varphi\left(p_s^{k_s}\right) =$$

$$= p_1^{k_1}\left(1 - \tfrac{1}{p_1}\right) p_2^{k_2}\left(1 - \tfrac{1}{p_2}\right) p_s^{k_s}\left(1 - \tfrac{1}{p_s}\right) = n \prod_{p|n}\left(1 - \tfrac{1}{p}\right).$$

□

La siguiente propiedad es debida a Gauss:

Proposición 12. *Dado $n \in \mathbb{Z}^+$,*

$$n = \sum_{q|n} \varphi(q)\,,$$

donde la suma recorre los distintos enteros positivos q que dividen a n.

Demostración. Partiendo del conjunto $F = \left\{\frac{k}{n}\right\}_{k=1}^{n}$, de cardinal n, se reducen las fracciones que lo componen y se obtiene $R = \left\{\frac{p_i}{q_i}\right\}_{i=1}^{n}$, donde los diferentes q_i coinciden exactamente con los divisores positivos de n y p_i es coprimo con q_i para todo $i = 1, ..., n$. Así, se puede reescribir R de la forma $R = \bigcup_{q|n} D^q$, siendo

$$D^q = \left\{\frac{p_j^q}{q}\right\}_{j=1}^{m^q},$$

donde q representa un divisor de n, p_j^q los numeradores de las fracciones de R con denominador q, y m^q $(1 \leq m^q \leq n)$ el número de fracciones de R con denominador q.

Sea $q = \frac{n}{d}$ un divisor positivo de n cualquiera, con $1 \leq d \leq n$ entero. Se tiene:

- p_j^q es coprimo con q para todo $\frac{p_j^q}{q} \in D^q$.
- Si p es un entero positivo menor o igual que q y coprimo con q, como $pd \leq qd = n$, se tiene que $\frac{pd}{n} \in F$, y por tanto, reduciendo con factor común d, $\frac{p}{q} \in D^q$.
- Dados $i, j \in \{1, ..., m^q\}$, si $i \neq j$ entonces $p_i^q \neq p_j^q$, ya que si no darían lugar a la misma fracción de R.

En consecuencia, $m^q = \varphi(q)$, y como el cardinal de R es n, el resultado queda demostrado. □

Aplicaciones

La función indicatriz de Euler aparece con mucha frecuencia dentro de la Teoría de Números elemental, como por ejemplo en el *Teorema de Euler*[3]:

Teorema 13 (de Euler)**.** *Dados a, $b \in \mathbb{Z}^+$ coprimos,*

$$a^{\varphi(b)} \equiv 1 \, (mod \, b) \, .$$

La función φ está también relacionada con el problema matemático conocido como *conjetura de Goldbach,* sobre la expresión de todo número entero mayor o igual a 4 como la suma de dos números primos. Hoy en día, los cálculos numéricos han llevado a demostrar que la conjetura es cierta hasta $4 \cdot 10^{18}$ [19], pero aún no hay una prueba general. Para obtener esta demostración, es importante señalar que la conjetura también puede ser formulada como la existencia, para todo entero positivo m, de primos p_1 y p_2 tales que $\varphi(p_1) + \varphi(p_2) = 2m$. Esta formulación equivalente nos permite estudiar los aspectos más generales problema, incluyendo el caso en el que p_1 y p_2 no son números primos.

Asimismo, en la última sección de sus *Disquisitiones*, Gauss demuestra que un polígono regular de n lados se puede construir con regla y compás si $\varphi(n)$ es una potencia de 2.

En un ámbito más práctico, el sistema de encriptación RSA, desarrollado en 1979 por Rivest, Shamir y Adleman, es el algoritmo más utilizado en el mundo tanto para cifrar como para firmar digitalmente. El sistema RSA

[3] Este resultado es una generalización del *Pequeño Teorema de Fermat*: Si p es primo y $a \in \mathbb{N}$ no es divisible entre p, entonces $a^{p-1} \equiv 1 \, (mod \, p)$.

implica elegir dos enormes números primos p_1 y p_2, calcular $n = p_1p_2$ y $k = \varphi(n)$, y encontrar dos números e y d tales que $ed \equiv 1(mod\, k)$. Los números n y e (la *clave pública*) se divulgan al público y d (la *clave privada*) se mantiene oculta. Un mensaje, representado por un entero m tal que $0 < m < n$, se codifica calculando $S = me(mod\, n)$ y se descifra calculando $t = Sd(mod\, n)$. Romper el sistema de cifrado equivale por tanto a conocer $k = \varphi(n)$, o lo que es lo mismo, a factorizar n, ya que por (1.16), $\varphi(n) = (p_1 - 1)(p_2 - 1)$. Si los primos p_1 y p_2 son grandes, la factorización requiere tal esfuerzo computacional, que constituye una tarea insalvable en la práctica.

Obsérvese que, inversamente, conocer $\varphi(n)$ también nos proporciona los valores de p_1 y p_2: $\varphi(n) = (p_1 - 1)(p_2 - 1)$ equivale a que $p_1 + p_2 = n - \varphi(n) + 1$, y conocido este valor junto con la realción $p_1p_2 = n$, se obtienen p_1 y p_2 mediante la resolución de la sencilla ecuación cuadrática $x^2 - (p_1 + p_2)\, x + p_1p_2 = 0$.

1.6. Algoritmos

Impacto y desarrollo

Se entiende por *algoritmo* (un término cuyo origen está en el matemático persa de los siglos VIII-IX Al-Khowârizmî) un conjunto ordenado y finito de operaciones que permite hallar la solución de un problema. Además, este proceso suele ser iterativo. La Algoritmia, la ciencia que se dedica al diseño y análisis de los algoritmos, se interesa principalmente por aquellos cálculos que por su complejidad necesitan del apoyo de una máquina de cómputo. Pero también son algoritmos los métodos que se enseñan en las escuelas para realizar a mano sumas, restas, multiplicaciones o divisiones. No en vano, algunos de los algoritmos más famosos de la Historia, como el de Euclides para calcular el máximo común divisor de dos enteros, o la Criba de Eratóstenes para hallar números primos, tienen más de 2000 años de antigüedad.

En el proceso de ejecución de un algoritmo no cabe ninguna decisión subjetiva, ni tampoco deben de entrar en juego factores como la intuición o la creatividad. Debe constituir, por el contrario, una serie de instrucciones con el nivel de detalle suficiente como para que el usuario tenga claro en todo momento cual es el siguiente paso que tiene que llevar a cabo.

Como se ha dicho, la búsqueda de algoritmos eficientes es muy anterior a la época de las computadoras. Matemáticos de todas las épocas, como Gauss, Leibnitz, Babbage, Lovelace o Turing, han puesto sus esfuerzos al servicio del tema, hasta que el desarrollo exponencial de las capacidades de los ordenadores, a partir de los años 50 del pasado siglo, trae consigo una revolución en la teoría de los algoritmos. En las últimas tres décadas, aspectos como la computación cuántica, que a diferencia de la clásica funciona sobre los principios de la mecánica cuántica, han dado lugar al desafío de desarrollar algoritmos específicamente para esta clase de computadoras. En nuestros días, los principales organismos internacionales del ámbito de la Ciencia y la Ingeniería coinciden en que el estudio, diseño y construcción de algoritmos es una necesidad ineludible para numerosos aspectos de la vida. Herramientas como *Google*, *Facebook*, *Amazon* o *Netflix*, que en cierto modo controlan las vidas de muchas personas, no se podrían entender sin sus algoritmos subyacentes de selección automatizada de la información.

Pseudocódigo

Es necesario fijar un modo para la descripción de los algoritmos. Los lenguajes naturales, como el español o el inglés, no están adaptados para ello, pues necesitan de largos y ambiguos textos para detallar las estructuras de control más sencillas. Por otra parte, optar por un lenguaje de programación concreto implica una elección *a priori* que condicionará al programador a la hora de llevar a cabo el proceso de implementación.

El algoritmo publicado en el artículo [8], correspondiente a esta tesis, es presentado en *pseudocódigo*, también denominado *lenguaje de descripción algorítmica*. Dicho lenguaje no está diseñado para ser ejecutado por una computadora, sino para la lectura humana. Se trata de un primer borrador con el que representar de manera sencilla las estructuras de control propias del algoritmo, pero prescindiendo de detalles que no son esenciales y con independencia de cualquier lenguaje de programación. Para su posterior ejecución, ha de ser traducido a un lenguaje en particular.

Esta neutralidad respecto de los lenguajes de programación favorece la claridad del algoritmo, ya que el programador puede concentrarse en la lógica y en las estructuras de control y no preocuparse de las reglas de un lenguaje especifico. Por otra parte, un algoritmo descrito en pseudocódigo necesita mucho menos espacio y es más fácil de modificar si se descubren errores que una vez ya está codificado en un lenguaje de programación.

Las ventajas citadas convierten al pseudocódigo en el medio más comúnmente utilizado para documentar algoritmos, ya sea en artículos científicos, en libros de texto o en las fases iniciales de desarrollo del software.

No se cuenta con un modelo estándar para el lenguaje pseudocódigo, tal que haya sido aceptado como patrón por la mayoría de los estamentos internacionales tecnológicos, científicos y educativos. No obstante, las diferentes variantes (véanse por ejemplo [4], [5] o [6]) presentan unas sintaxis muy similares. En esta tesis se ha optado por un pseudocódigo basado en vocablos de la lengua inglesa y diseñado para *LaTeX* [6], cuyas principales estructuras se describen a continuación:

i) Datos de entrada

Mediante una instrucción *Input* al principio de cada algoritmo, se indican los datos de entrada del mismo. Por ejemplo,

Input: grado del sistema (n)

señala que la semilla que el algoritmo necesita para funcionar es el grado del sistema.

ii) Datos de salida

La sentencia anterior suele ir acompañada de una instrucción *Output*, para clarificar los datos de salida. Por ejemplo,

Output: vector peso mínimo del sistema ($\mathbf{w_m}$)

indica que, como resultado de su ejecución, el algoritmo devolverá el vector peso mínimo del sistema.

iii) Funciones

Las funciones permiten independizar del resto del algoritmo series de secuencias que por sí mismas formen una unidad, posibilitando así una programación modular. Una función puede ser invocada en múltiples ocasiones a lo largo de la ejecución del algoritmo, por lo que son una herramienta adecuada para codificar procesos repetitivos. Reciben uno o varios datos de entrada, que se declaran entre paréntesis al principio, y generan una salida, que se indica al final del cuerpo de la función, tras la palabra *return*. Por ejemplo,

```
Function calcularWm(B_i, B_j, B_p)
    instrucciones;
    return w_m
end
```

representa una función de nombre calcularWm que tiene por entrada los valores B_i, B_j, B_p y devuelve el valor $\mathbf{w_m}$.

iv) Asignación de datos

La asignación de un valor a una variable se indica en pseudocódigo mediante un símbolo de flecha ($\leftarrow$). La flecha apunta desde el valor asignado hacia la variable que se está actualizando. Por ejemplo,

```
d ← 2πr
```

establece el valor $2\pi r$ para la variable d.

v) Control condicional

Las estructuras de control condicional verifican que se ejecuten o no instrucciones en función de si una determinada afirmación es verdadera o falsa. Permiten así la toma de decisiones en el desarrollo del algoritmo. Por ejemplo,

```
if condición then
| instrucciones
```

hará que se lleven a cabo *instrucciones* si y solo si *condición* es verdadera. También se pueden presentar estructuras condicionales múltiples, en las que se evalúan varias condiciones sucesivamente. Por ejemplo,

```
if condición_A then
| instrucciones_A
else if condición_B then
| instrucciones_B
else
| instrucciones_C
end
```

ejecutará *instrucciones*$_{\mathrm{A}}$ si se cumple *condición*$_{\mathrm{A}}$. En caso contrario, se ejecutará *instrucciones*$_B$ si se cumple *condición*$_{\mathrm{B}}$. Si tampoco se verifica *condición*$_{\mathrm{B}}$, se ejecutará *instrucciones*$_{\mathrm{C}}$.

vi) Control iterativo

Las estructuras de control iterativo representan la ejecución de un bloque de instrucciones un número prefijado de veces. Por ejemplo,

```
for i ← 1 to N do
| instrucciones
end
```

ejecuta *instrucciones* N veces, empleando como índice la variable i.

vii) Anidamiento

Toda estructura de control puede constituir una instrucción más dentro de otra estructura superior, de forma que se forman configuraciones anidadas con diferentes niveles. Por ejemplo,

```
for i ← 1 to N do
│ if condición then
│ │ instrucciones_A
│ else
│ │ instrucciones_B
│ end
end
```

ejecuta N veces la estructura de control condicional en la que se evalúa *condición,* empleando como índice la variable i.

Algoritmia y sistemas quasi-homogéneos

En 2013 Belén García, Jaume Llibre y Jesús Suárez Pérez del Río publican el artículo *Planar quasi-homogeneous polynomial differential systems and their integrability* [10], el cual constituye uno de los principales puntos de partida para el desarrollo de esta tesis. En dicho trabajo se estudian los sistemas diferenciales quasi-homogéneos en dimensión 2, y se proporciona un algoritmo para obtener todos los sistemas de esta clase de un grado dado. Haciendo uso del propio algoritmo se obtienen todos los campos vectoriales quasi-homogéneos de grados 2 y 3. Además de esto, en [10] se tratan cuestiones referentes a la integrabilidad de los quasi-homogéneos planos, como la demostración de que todos ellos tienen una integral primera de tipo Liouville, o la caracterización de todos aquellos de grado 2 o 3 con integral primera polinomial, racional o analítica.

El mencionado trabajo ha sido citado hasta el momento en varias docenas de artículos, algunos de los cuales (ver por ejemplo [14], [21]) hacen una aplicación explícita de este algoritmo para generar listados completos de sistemas quasi-homogéneos.

Por su parte, Xiong y Han [22] han publicado un algoritmo de características similares, también para determinar la lista de sistemas quasi-homogéneos en el plano, pero en el cual el *input* no es el grado, sino el grado peso. Adicionalmente, estos autores proporcionan algunas condiciones necesarias para la existencia de centros en los sistemas quasi-homogéneos.

En la linea de estas publicaciones, en los trabajos [7] y [8] aportamos algoritmos para el conteo y la determinación de sistemas quasi-homogéneos, tanto en el plano como en el espacio.

MATLAB

Debido a su complejidad computacional, tanto el algoritmo publicado por García *et al.* [10] como los algoritmos desarrollados en el artículo [7] son especialmente indicados para su implementación mediante el uso de lenguajes de programación para computadoras. Se han desarrollado los dos resultados mencionados, utilizando para ello el lenguaje de programación M, incluido en la herramienta de software matemático MATLAB.

MATLAB es un producto propietario de la compañía The Mathworks, creado por el matemático y programador de computadoras Cleve Moler en 1984. Se ha optado por MATLAB, de entre las múltiples opciones en cuanto a lenguajes de programación, por tratarse de un software multiplataforma especialmente apropiado para los cálculos matemáticos, y muy extendido en universidades y centros de investigación y desarrollo. Se estima que MATLAB es empleado por más de un millón de personas en ámbitos académicos y empresariales.

Los citados algoritmos codificados en MATLAB se encuentran disponibles para su libre descarga en la dirección web

https://matemat51.epv.uniovi.es/gsd

1.7. Estructura de capítulos

Tras esta Introducción, en la que se ha llevado a cabo un repaso del estado del arte de los aspectos matemáticos de relevancia en esta memoria, en el Capítulo 2 se detallan los objetivos que se establecieron como punto de partida de la misma. El Capítulo 3 constituye la parte más importante de la memoria, ya que en él se reproducen los tres artículos que componen esta tesis, bajo el formato de compendio de publicaciones. Cada artículo se organiza en una sección, y en otra sección adicional al final se proporcionan los indicadores relativos a la calidad de las publicaciones. El Capítulo 4 agrupa la discusión de resultados, las conclusiones y las lineas futuras de trabajo. A continuación se detalla la bibliografía utilizada, aunque hay que subrayar que cada uno de los tres artículos cuenta a su vez con su propia bibliografía. La memoria se cierra con el capítulo de Anexos, en los que se recopilan los diferentes documentos, impresos e informes de inclusión preceptiva.

Capítulo 2

Objetivos

El objetivo general de esta tesis consistía en ampliar los conocimientos existentes acerca de los sistemas diferenciales quasi-homogéneos, principalmente en dimensiones 2 y 3. Para ello se establecieron los siguientes objetivos específicos:

- Establecer algoritmos para la determinación de todos los sistemas diferenciales quasi-homogéneos.
- Implementar los algoritmos anteriores en algún lenguaje de programación concreto.
- Contabilizar el número de sistemas diferenciales quasi-homogéneos en función de su grado.
- Desarrollar en profundidad conceptos relacionados con la organización de los sistemas quasi-homogéneos (vector peso mínimo, familias de vectores peso, sistemas maximales...).
- Estudiar en qué situaciones se produce la unicidad del vector peso mínimo.
- Estudiar la integrabilidad de sistemas quasi-homogéneos en dimensiones superiores a 2.
- Analizar el grado de utilidad del método de Yoshida para la integración de sistemas quasi-homogéneos.

Capítulo 3

Artículos publicados

Esta tesis doctoral se presenta bajo la modalidad de compendio de publicaciones. Siguiendo este formato, la tesis está constituida por tres trabajos relacionados con la misma línea de investigación (los sistemas diferenciales quasi-homogéneos) y publicados por el doctorando con posterioridad al inicio de los estudios de doctorado, en revistas incluidas en el *Science Citation Index*. En este capítulo se recoge una copia de los tres artículos tal y como han sido publicados en sus respectivas revistas, respetándose así la maquetación original de cada una de ellas. Las referencias son las siguientes:

García, B., Lombardero, A. y Pérez del Río, J.S., *Classification and counting of planar quasi-homogeneous differential systems through their weight vectors*, Qual. Theory Dyn. Syst. **17**, 541–561 (2018).

García, B., Llibre, J., Lombardero, A. y Pérez del Río, J.S., *An algorithm for providing the normal forms of spatial quasi-homogeneous polynomial differential systems*, Journal of Symbolic Computation **95**, 1–25 (2019).

García, B., Llibre, J., Lombardero, A. y Pérez del Río, J.S., *Analytic integrability of quasi-homogeneous systems via the Yoshida method*, Journal of Symbolic Computation **104**, 960–980 (2021).

3.1. Classification and counting of planar quasi-homogeneous differential systems through their weight vectors

Los sistemas quasi-homogéneos tienen interesantes propiedades y han sido estudiados desde diferentes enfoques: integrabilidad, centros, ciclicidad, estabilidad estructural, retratos de fase, ciclos límite, etc. En particular, García, Llibre y Pérez del Río [10] estudiaron estos sistemas desde el punto de vista de su clasificación, elaborando un algoritmo para generar, para cada grado fijado, todos los sistemas de este tipo en el plano.

El objetivo del artículo que se reproduce a continuación era, por una parte, proporcionar un método de clasificación de los sistemas quasi-homogéneos planos en función de sus vectores peso mínimos, a la vez que desarrollar en profundidad este último concepto. Por otra parte, se pretendía abordar la tarea de contabilización de esta clase de sistemas, así como proporcionar una implementación en un lenguaje de programación del citado algoritmo de [10] .

Además de alcanzar los objetivos anteriores, en este trabajo se probó la unicidad del vector peso mínimo para los sistemas quasi-homogéneos planos y se halló una relación entre el número de estos sistemas y la función indicatriz φ de Euler. El algoritmo fue codificado en MATLAB, proporcionándose como un anexo *online* al artículo.

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Qualitative Theory of Dynamical Systems

Classification and Counting of Planar Quasi-Homogeneous Differential Systems Through Their Weight Vectors

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Abstract The quasi-homogeneous systems have important properties and they have been studied from various points of view. In this work, we provide the classification of quasi-homogeneous systems on the basis of the weight vector concept, especially in terms of the minimum weight vector, which is proved to be unique for any quasi-homogeneous system. Later we obtain the exact number of different forms of non-homogeneous quasi-homogeneous systems of arbitrary degree, proving a nice relation between this number and Euler's totient function. Finally, we provide software implementations for some of the above results, and also for the algorithm, recently published by García et al., that generates all the quasi-homogeneous systems.

Keywords Quasi-homogeneous polynomial differential system · Weight vector · Euler's totient function

1 Introduction and Main Results

We consider polynomial differential systems in the plane, of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

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where $P,\ Q \in \mathbb{C}[x, y]$ and $(P, Q) \neq (0, 0)$. The dots denote derivatives with respect to the independent variable t, which can be real or complex, and with $\mathbb{C}[x, y]$ we mean the ring of polynomials with coefficients in $\mathbb{C}$ and complex variables x and y. We say that the system (1) has *degree* n if $n = \max\{l, m\}$, where l and m are the degrees of the component polynomials P and Q, respectively. Finally, with $X_k = (P_k(x, y), Q_k(x, y))$ we denote the homogeneous part of degree k of the system, k varying between 0 and n.

A polynomial differential system (1), denoted by $\mathsf{S}(P, Q)$ or by S when it does not lead to confusion, is *quasi-homogeneous* if there exist three positive integers s_1, s_2, d verifying that for any $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$,

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1+d-1}P(x, y), \quad Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2+d-1}Q(x, y).$$

In this case, $\mathbf{v} = (s_1, s_2, d)$ is denominated a *weight vector* of the system S, s_1 and s_2 are called *weight exponents* of S, and d is the *weight degree* of the system relative to the weight exponents s_1 and s_2.

The quasi-homogeneous systems are a generalization of the homogeneous systems, a system class that has been profusely studied. A system (1) is called *homogeneous of degree* n if P and Q are homogeneous polynomials of degree n in the variables x and y. We consider the zero polynomial as homogeneous of any degree, so that in this definition it is possible for the system components P or Q to be zero, provided that $(P, Q) \neq (0, 0)$.

By the definition of homogeneous polynomial, a polynomial differential system is homogeneous of degree n if and only if for any $\alpha \in \mathbb{R}$,

$$P(\alpha x, \alpha y) = \alpha^n P(x, y), \quad Q(\alpha x, \alpha y) = \alpha^n Q(x, y).$$

Therefore, homogeneous systems are particularly quasi-homogeneous, and the set of the homogeneous systems of degree n coincides exactly with the set of the quasi-homogeneous systems that have the vector $(1, 1, n)$ as one of their weight vectors.

The quasi-homogeneous polynomial differential systems specifically homogeneous have been investigated by many authors, the quadratic homogeneous have been studied in [11,27–29,32,35], the cubic homogeneous in [7], and the homogeneous systems of arbitrary degree in [4,7,9,24]. In the aforementioned works one can find an algorithm for the study of phase portraits of homogeneous polynomial vector fields of any degree, the classification of all phase portraits of homogeneous polynomial vector fields of degree 2 and 3, the algebraic classification of homogeneous polynomial vector fields and the characterization of the structural stability of the homogeneous polynomial vector fields.

The quasi-homogeneous polynomial differential systems (particularly those non-homogeneous) have also been studied by several authors from different points of view, such as integrability [2,6,13,16,20,26,34,36], structural stability [30], cyclicity [14], centers [1,25], normal forms [3] or limit cycles [2,8,21,23].

Recently, García et al. [12] have published an algorithm to obtain all non-homogeneous quasi-homogeneous polynomial differential systems of a given degree. Using this algorithm they have obtained all quadratic and cubic systems in this class.

Subsequently, the authors of [22] and [33], also using the algorithm described in [12], have classified all systems of degree 4 and 5 respectively. Other applications of [12], related to centers and integrability, can be seen in [5,15].

With this work we try to deepen the study and classification of quasi-homogeneous polynomial differential systems from the standpoint of weight vectors, while we consolidate a new line of research regarding this type of differential systems.

In the set of weight vectors of a quasi-homogeneous system S, a binary relation $\leq$ can be defined as follows: given two weight vectors $\mathbf{w} = (s_1, s_2, d)$ and $\bar{\mathbf{w}} = \left(\bar{s_1}, \bar{s_2}, \bar{d}\right)$, we say that

$$\mathbf{w} \leq \bar{\mathbf{w}} \tag{2}$$

when $s_1 \leq \bar{s_1}$, $s_2 \leq \bar{s_2}$ and $d \leq \bar{d}$ are verified. It is clearly a partial order (reflexive, symmetric and transitive) on the set of weight vectors of S.

We say that a weight vector $\mathbf{w}_m = \left(s_1^*, s_2^*, d^*\right)$ of a quasi-homogeneous system S is the *minimum weight vector* S if for any weight vector $\mathbf{w} = (s_1, s_2, d)$ of S is verified that $\mathbf{w}_m \leq \mathbf{w}$, or equivalently, if $s_1^* \leq s_1$, $s_2^* \leq s_2$ and $d^* \leq d$. Note that in [12] this concept was named *minimal* weight vector of the system.

The binary relation $\leq$ is not a total order, because for example (1, 3, 4) and (2, 2, 3) are weight vectors in the system

$$\dot{x} = xy, \;\; \dot{y} = y^2, \tag{3}$$

but not comparable. However, we will show that the set of weight vectors of any quasi-homogeneous polynomial differential system without null components has minimum for the order relation $\leq$, which constitutes the first of the main results of this work:

Theorem A *A quasi-homogeneous polynomial differential system* $\mathsf{S}\,(P, Q)$ *verifying that* $P \cdot Q \neq 0$ *has minimum weight vector.*

The minimum weight vector of the system can be used as an element for algebraic classification of the quasi-homogeneous systems of certain degree n. In this paper we develop such classification, and we conclude that $(1, 1, n)$ is the minimum weight vector of the system for every homogeneous system of degree n without null components. This is not the case for non-homogeneous quasi-homogeneous systems. As one can see in [12], in this last set there exists a large variety of both algebraic forms and minimum weight vectors.

To obtain the exact number of non-homogeneous quasi-homogeneous systems we will use a classical tool of number theory, *Euler's totient function* that we denote by φ. This function is defined, for each $n \in \mathbb{Z}^+$, as the number of positive integers less than or equal to n that are coprime with n:

$$\varphi\,(n) = |\,\{r \in \mathbb{N} \;:\; 1 \leq r \leq n \;,\; \gcd\,(n, r) = 1\}\,|, \tag{4}$$

and its main properties are shown, for example, in [18]. The function φ is strongly related to important mathematical problems such as, for example, the well-known Goldbach's conjecture about the expression of each integer greater than or equal to 4 as the sum of two primes. Nowadays, the numerical computations have led to prove that the conjecture is true up to $4 \cdot 10^{18}$ (see [31]) but there is no a general proof yet. In order to obtain this proof, it is important to remark that the conjecture can be also formulated as the existence, for every positive integer m, of primes p and q such that $\varphi(p) + \varphi(q) = 2m$. This equivalent formulation allows us to study the more general problem in which p and q are not necessarily primes (see, for example, [17]).

For other applications of the function φ in different fields of mathematics, one can see, for example, [19].

Using the function φ we can set the second of the main results of this paper.

Theorem B *The number $c(n)$ of non-homogeneous quasi-homogeneous polynomial differential systems of degree n is given by:*

$$\text{(i)} \quad c(1) = 0, \quad c(2) = 6, \quad c(3) = 16 \tag{5}$$

$$\text{(ii)} \quad c(n) = 2c(n-1) - c(n-2) + 2\varphi(n+1) \tag{6}$$

or, equivalently,

$$c(n) = 10n - 14 + 2\sum_{k=4}^{n}\sum_{j=5}^{k+1}\varphi(j) \tag{7}$$

for $n \geq 4$, where φ is Euler's totient function.

The paper is organized as follows: in Sect. 2.1 we develop the concept of family of weight vectors of a quasi-homogeneous system, and we derive some properties thereof. This is utilized in Sect. 2.2 for a complete classification of quasi-homogeneous polynomial differential systems based on their weight vectors. The section concludes with the proof of Theorem A.

Section 3 is devoted to the proof of Theorem B.

Section 4 focuses on the software implementation of some of the contents of this work and of García et al. paper [12]. To do it, we have used the programming language included in the software package MATLAB. Specifically, we have implemented the algorithm published in [12] in a program that returns in .tex format all non-homogeneous quasi-homogeneous systems of a n degree provided by the user. We also developed a program based on Theorem B, dedicated exclusively to count systems. In both cases we provide the corresponding web links in order to download this software.

2 Weight Vectors of Quasi-Homogeneous Polynomial Differential Systems

2.1 Weight Vector Families

We recall from [12] a fundamental relationship between the coefficients and the weight vectors of a quasi-homogeneous polynomial differential system. We write the P and Q system polynomials separated into their homogeneous parts:

$$P(x, y) = \sum_{j=0}^{l} P_j(x, y), \quad \text{where} \quad P_j(x, y) = \sum_{i=0}^{j} a_{i,j-i} x^i y^{j-i}, \tag{8}$$

and

$$Q(x, y) = \sum_{j=0}^{m} Q_j(x, y), \quad \text{where} \quad Q_j(x, y) = \sum_{i=0}^{j} b_{i,j-i} x^i y^{j-i}. \tag{9}$$

Proposition 1 *Given a quasi-homogeneous system* S*:*

1. *If the coefficient* $a_{i,j-i}$ *of* P *is not null, then for every weight vector* (s_1, s_2, d) *of* S *is verified:*
$$(i-1)\, s_1 + (j-i)\, s_2 = d - 1 \tag{10}$$
2. *If the coefficient* $b_{i,j-i}$ *of* Q *is not null, then for every weight vector* (s_1, s_2, d) *of* S *is verified:*
$$i s_1 + (j-i-1)\, s_2 = d - 1 \tag{11}$$

Proof For any polynomial system (1) we have, according to (8), that given positive integers s_1, s_2 and $\alpha > 0$:

$$\begin{aligned} P\left(\alpha^{s_1} x, \alpha^{s_2} y\right) &= \sum_{j=0}^{l} P_j\left(\alpha^{s_1} x, \alpha^{s_2} y\right) = \sum_{j=0}^{l} \sum_{i=0}^{j} a_{i,j-i} \alpha^{i s_1} x^i \alpha^{(j-i) s_2} y^{j-i} \\ &= \sum_{j=0}^{l} \sum_{i=0}^{j} a_{i,j-i} \alpha^{i s_1 + (j-i) s_2} x^i y^{j-i}. \end{aligned} \tag{12}$$

If in addition the system is quasi-homogeneous, it shall fulfill that given $\alpha > 0$ and for any of its weight vectors (s_1, s_2, d):

$$P\left(\alpha^{s_1} x, \alpha^{s_2} y\right) = \alpha^{s_1 + d - 1} P(x, y) = \alpha^{s_1 + d - 1} \sum_{j=0}^{l} \sum_{i=0}^{j} a_{i,j-i} x^i y^{j-i}. \tag{13}$$

Then, by (12) and (13):

$$a_{i,j-i} \alpha^{i s_1 + (j-i) s_2} = a_{i,j-i} \alpha^{s_1 + d - 1}$$

for any weight vector (s_1, s_2, d) of the system, and therefore, if $a_{i,j-i} \neq 0$, (10) is satisfied. The equality (11) can be proved similarly using a permutation between (x, P) and (y, Q). □

Definition 2 Given a quasi-homogeneous system S and $\lambda \in \mathbb{Q}^+$, the weight vector family of the system with ratio λ is defined as the set of weight vectors of S where the proportion between the exponents s_1 and s_2 is λ,

$$F_{\mathsf{S}}(\lambda) := \left\{ (s_1, s_2, d) \ \textit{weight vector of} \ \mathsf{S} \ : \ \frac{s_1}{s_2} = \lambda \right\}.$$

Note that a system may have more than one family of weight vectors. In fact, we will prove that the only two possible assumptions for a quasi-homogeneous system are to have a unique family of weight vectors or, in some homogeneous cases, to have infinite families. An example of the second situation is the system

$$\dot{x} = x^2 y, \ \dot{y} = xy^2, \tag{14}$$

for which any vector $(a, b, a + b + 1)$, a and b being positive integers, forms a weight vector.

Proposition 3 *Given a weight vector* $(s_1, s_2, d) \in F_{\mathsf{S}}(\lambda)$ *and* $r = \frac{p}{q} \in \mathbb{Q}^+$ *with* p *and* q *coprime, a vector* (rs_1, rs_2, d^*) *is also a* S *weight vector if and only if* q *divides* $\gcd(s_1, s_2)$ *and* $d^* = r(d-1) + 1$. *Furthermore, this new vector also belongs to* $F_{\mathsf{S}}(\lambda)$.

Proof Note first that $(rs_1, rs_2, r(d-1) + 1)$ is a vector of positive integers if and only if q divides s_1, s_2 and $d - 1$, or equivalently when q divides $\gcd(s_1, s_2, d-1)$. Ever since, by (10), $d - 1$ is a linear combination of s_1 and s_2, it is enough that q divides $\gcd(s_1, s_2)$.

As (s_1, s_2, d) is a weight vector, for any $\alpha > 0$, one has that

$$P(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{s_1+d-1} P(x, y).$$

On the other hand $r \in \mathbb{Q}^+$, whereby $\alpha^r > 0$, and consequently

$$\begin{aligned} P\left(\alpha^{rs_1} x, \alpha^{rs_2} y\right) &= P\left(\left(\alpha^r\right)^{s_1} x, \left(\alpha^r\right)^{s_2} y\right) \\ &= \left(\alpha^r\right)^{s_1+d-1} P(x, y) = \alpha^{rs_1+r(d-1)} P(x, y). \end{aligned}$$

Therefore, (rs_1, rs_2, d^*) is a weight vector if and only if $d^* = r(d-1) + 1$. The system component Q provides an equivalent conclusion. Finally, $(rs_1, rs_2, r(d-1) + 1)$ also belongs to the family of (s_1, s_2, d) because the weight exponents keep the same proportion $\frac{s_1}{s_2}$. □

The above proposition provides a method to, given a weight vector, get infinite different weight vectors of the same family, simply by varying r through $\mathbb{Z}^+$. Therefore we conclude that any weight vector family has infinite elements.

The proposition also supplies a procedure for obtaining new vectors by reducing the weight exponents just dividing them by a common factor $q \in \mathbb{Z}^+$ which also divides $d - 1$. Obviously in this way we will not obtain an infinite number of vectors, but a finite quantity never greater than the number of common divisors of s_1 and s_2.

Note that if (s_1, s_2, d) and (s_1, s_2, d^*) are weight vectors in the same system S, then it is concluded that $d = d^*$ simply by taking $r = 1$. That is to say, given a weight vector in a system with fixed exponents, the weight degree is uniquely determined.

Proposition 4 *If* $\mathbf{w} = (s_1, s_2, d)$ *and* $\bar{\mathbf{w}} = \left(\bar{s_1}, \bar{s_2}, \bar{d}\right)$ *are weight vectors belonging the same family* $F_{\mathsf{S}}(\lambda)$ *of a quasi-homogeneous system* S, *the following statements are equivalent:*

(i) $s_1 \leq \bar{s_1}$.
(ii) $s_2 \leq \bar{s_2}$.
(iii) $d \leq \bar{d}$.

Proof The equivalence between (i) and (ii), when $s_1 = \lambda s_2$ and $\bar{s_1} = \lambda \bar{s_2}$ with $\lambda > 0$, is obvious. Let us see the equivalence of (i) and (iii).

Suppose $P \neq 0$. As $\mathbf{w}$ and $\bar{\mathbf{w}}$ are weight vectors of $F_{\mathsf{S}}(\lambda)$, for any $\alpha, \ \beta > 0$:

$$P\left(\alpha^{s_1} x, \alpha^{k s_1} y\right) = \alpha^{s_1} \alpha^{d-1} P(x, y) \tag{15}$$

and

$$P\left(\beta^{\bar{s_1}} x, \beta^{k \bar{s_1}} y\right) = \beta^{\bar{s_1}} \beta^{\bar{d}-1} P(x, y), \tag{16}$$

where $k = \frac{1}{\lambda}$.

Setting $\beta = \alpha^{\frac{s_1}{\bar{s_1}}} > 0$, (16) becomes:

$$P\left(\alpha^{s_1} x, \alpha^{k s_1} y\right) = \alpha^{s_1} \alpha^{\frac{s_1}{\bar{s_1}}(\bar{d}-1)} P(x, y). \tag{17}$$

Then, by (15) and (17), and because P is not zero in the whole plane,

$$\alpha^{d-1} = \alpha^{\frac{s_1}{\bar{s_1}}(\bar{d}-1)}.$$

Exponential functions are injective, whereby $d - 1 = \frac{s_1}{\bar{s_1}}\left(\bar{d} - 1\right)$. In conclusion, $\frac{s_1}{\bar{s_1}} \leq 1$ if and only if $d \leq \bar{d}$.

If $P = 0$, then $Q \neq 0$, and the result is demonstrated in a similar way. □

This result is not necessarily true when the two weight vectors belong to different families, as we already noted in the Introduction when we mentioned that both $(1, 3, 4)$ and $(2, 2, 3)$ are weight vectors of the quasi-homogeneous system (3).

However, the above proposition allows to apply the order relation (2) to any pair of weight vectors of a family $F_{\mathsf{S}}(\lambda)$: given $\mathbf{w} = (s_1, s_2, d)$ and $\bar{\mathbf{w}} = \left(\bar{s_1}, \bar{s_2}, \bar{d}\right)$ belonging to $F_{\mathsf{S}}(\lambda)$, we can say that $\mathbf{w} \leq \bar{\mathbf{w}}$ if any condition (i), (ii) or (iii) of Proposition 4 is true. If we impose such restriction, and contrary to what happens when we applied it to the entire set of weight vectors of a system, $\leq$ is a total order relation, and consequently any weight vector family will be a totally ordered set. Now we will prove the existence of its minimum.

Corollary 5 *Any weight vector family $F_{\mathsf{S}}(\lambda)$ of a quasi-homogeneous system S has one minimum with respect to the total order relation $\leq$, which we denote by $\mathbf{w}_{m(\lambda)}$.*

Proof To find the minimum of $F_{\mathsf{S}}(\lambda)$ in regard to $\leq$ is equivalent to finding the minimum of all weight exponents s_1 existing in the family, since there is only one vector in $F_{\mathsf{S}}(\lambda)$ for each s_1. This minimum can always be found because the weight exponents are positive integers. □

Proposition 6 *If* $\mathbf{v} = (s_1, s_2, d) \in F_{\mathsf{S}}(\lambda)$*, then* $\mathbf{v}$ *is the minimum vector of its family if and only if* s_1 *and* s_2 *are coprime.*

Proof Let $\mathbf{v}_{m(\lambda)} = (s_1, s_2, d)$ be the minimum weight vector of the family, and let us suppose that s_1 and s_2 share a common divisor $q > 1$. If $P \neq 0$, then there is a not null coefficient $a_{i,j-i}$, and by Proposition 1, $(i-1)s_1 + (j-i)s_2 = d-1$, then q would also divide $d-1$, and using $r = \frac{1}{q}$ in Proposition 3, $\mathbf{w}_{m(\lambda)}$ would not be the family minimum. If $P = 0$, then $Q \neq 0$ and it is proved identically.

On the other hand, the fact that the weight exponents are coprime implies, also by Proposition 3, that they can not be reduced more. Neither can the weight degree, due to Proposition 4. □

It is clear that any weight vector of a family $F_{\mathsf{S}}(\lambda)$ can be obtained by multiplying the weight exponents s_1, s_2 of any other vector by a given positive rational number, since all share the same ratio λ. Now we will see that starting from the minimum vector of the family $\mathbf{w}_{m(\lambda)}$, all $F_{\mathsf{S}}(\lambda)$ can be generated exclusively by multiplications by positive integers.

Proposition 7 *If* $\mathbf{w}_{m(\lambda)} = \left(s_1^*, s_2^*, d^*\right)$ *is the minimum weight vector in a family* $F_{\mathsf{S}}(\lambda)$*, then*

$$F_{\mathsf{S}}(\lambda) = \left\{\left(rs_1^*, rs_2^*, r\left(d^*-1\right)+1\right) : r \in \mathbb{Z}^+\right\}.$$

Proof Let $\mathbf{v} = (s_1, s_2, d) \in F_{\mathsf{S}}(\lambda)$. As s_1 and s_2 keep the same proportion λ as s_1^* and s_2^*, a positive rational number $r = \frac{p}{q}$, with p and q coprime, necessarily has to exist, such that $s_1 = rs_1^*$ and $s_2 = rs_2^*$. On the other hand, by Proposition 3, d must be $r(d^*-1)+1$.

If $r = \frac{p}{q}$ is a non-integer rational, then $q > 1$ and, also by Proposition 3, q must be a common divisor of s_1^* and s_2^*. Then, by Proposition 6, $\mathbf{w}_{m(\lambda)}$ would not be the $F_{\mathsf{S}}(\lambda)$ minimum vector.

This proves the inclusion "$\subset$". The inclusion "$\supset$" follows directly from Proposition 3. □

As a consequence, we get that the weight degree d^* of the minimum vector $\mathbf{w}_{m(\lambda)}$ determines the character of all the family weight degrees:

Corollary 8 *If* $F_{\mathsf{S}}(\lambda)$ *is a vector family of a quasi-homogeneous system* S*, either all vectors that compose it have weight degree* $d = 1$*, or all have weight degree* $d > 1$.

A minimum weight vector of the system $\mathbf{w}_m = \left(s_1^*, s_2^*, d^*\right)$ verifies $s_1^* \leq s_1, s_2^* \leq s_2$ and $d^* \leq d$ for any other weight vector $\mathbf{w} = (s_1, s_2, d)$ of S, so it must be a vector that minimizes all existing family minimum vectors $\mathbf{w}_{m(\lambda)}$ in S. The uniqueness of $\mathbf{w}_m$ would be obvious, but to demonstrate their existence in almost all quasi-homogeneous systems, it is required to study in detail this class of systems, which we will develop in the next section.

2.2 Homogeneous and Non-homogeneous Quasi-Homogeneous Polynomial Differential Systems

Proposition 9 *Given a polynomial differential system* (1)*, the following statements are equivalent:*

(i) *The system is homogeneous.*
(ii) *The system has a weight vector* $\mathbf{v} = (s_1, s_2, d)$ *where* $s_1 = s_2$.
(iii) *The system has a weight vector family* $F_{\mathsf{S}}(\lambda)$ *with ratio* $\lambda = 1$.

Proof The assertions (ii) and (iii) are clearly equivalent. It is also clear that (i) implies (ii), just by taking the vector $(1, 1, n)$. We will demonstrate the implication (ii) $\Rightarrow$ (i).

We consider that the system has a weight vector $\mathbf{v} = (s, s, d)$. We will assume that $P \neq 0$, so there is a coefficient $a_{i,j-i} \neq 0$ (in case of $Q \neq 0$ the proof is similar). This, by Proposition 1, implies

$$(i-1)\,s + (j-i)\,s = d - 1 \Rightarrow (j-1)\,s = d - 1$$

and therefore s divides $d-1$. Then, by Proposition 3 it follows that $\left(1, 1, \frac{d-1}{s} + 1\right)$ is a weight vector of the system, so it will be homogeneous. □

Corollary 10 *A quasi-homogeneous polynomial differential system is non-homogeneous if and only if all its weight vectors* $\mathbf{v} = (s_1, s_2, d)$ *verify* $s_1 \neq s_2$.

However, as we have seen in the example (14), not all weight vectors of a homogeneous system must necessarily have its two weight exponents equal.

Now we state two results for the classification of homogeneous systems based on their vector families. In this classification we consider all existing homogeneous systems, not only the maximal (maximal in the sense given in [12]). In fact, there is only one maximal homogeneous system of degree n, the one that owns all possible monomials both in P and Q (a total of $n+1$ monomials in each component). All other homogeneous systems can be considered particular cases of it simply by making some of its coefficients zero.

Proposition 11 *A homogeneous polynomial differential system of degree n with more than one nonzero monomial in any of its components P or Q verifies that* $F_{\mathsf{S}}(1) = \left\{(s, s, (n-1)\,s + 1) \ : \ s \in \mathbb{Z}^+\right\}$ *is the only weight vector family of the system.*

Proof First, it has to be noted by Proposition 9 that $F_{\mathsf{S}}(1)$ must be a weight vector family of the system. We will prove that in fact it is unique.

Suppose that component P has two different monomials. Then there exist $i, j \in \{0, 1, \ldots, n\}$, $i \neq j$, such that $a_{i,n-i} \neq 0$ and $a_{j,n-j} \neq 0$. Then, based on Proposition 1, for any weight vector (s_1, s_2, d) of the system it is true that:

$$(i-1)\,s_1 + (n-i)\,s_2 = d - 1, \tag{18}$$

$$(j-1)\,s_1 + (n-j)\,s_2 = d - 1, \tag{19}$$

and combining both equations the result is $(i - j)(s_1 - s_2) = 0$. As $i \neq j$, we obtain $s_1 = s_2 = s$, and from (18) and (19) it follows that the weight degree d must be $(n - 1)s + 1$.

If there are two nonzero monomials in the Q component, the proof is similar. □

Proposition 12 *A homogeneous polynomial differential system of degree n that has at most one nonzero monomial in both components P and Q fits within one and only one of the following situations:*

(i) *If P or Q are null, then the system has an infinite number of weight vector families, and given any positive rational λ, the $F_S(\lambda)$ family exists.*

(ii) *If $PQ \neq 0$, for the system*

$$P(x, y) = a_{i,n-i} x^i y^{n-i},$$
$$Q(x, y) = b_{j,n-j} x^j y^{n-j},$$

for certain $i, j \in \{0, 1, \ldots, n\}$, we distinguish between:

(ii-a) *If $j = i - 1$, when $i > 0$, then the system has an infinite number of weight vector families, and given any positive rational λ, the $F_S(\lambda)$ family exists.*

(ii-b) *If $j \neq i - 1$, then $F_S(1) = \left\{(s, s, (n-1)s + 1) : s \in \mathbb{Z}^+\right\}$ is the only weight vector family of the system.*

Proof First, it is noted that the three mentioned cases cover the full range of possibilities in terms of algebraic structure of a system of the mentioned characteristics.

(i) Let us suppose $Q = 0$, so that there is $i \in \{0, 1, \ldots, n\}$ such that the system is

$$P(x, y) = a_{i,n-i} x^i y^{n-i},$$
$$Q(x, y) = 0,$$

with $a_{i,n-i} \neq 0$ as the only condition. By Proposition 1, for any weight vector (s_1, s_2, d), $d = (i - 1)s_1 + (n - i)s_2 + 1$, with this being the only condition for all the system weight vectors. Let $\lambda = \frac{p}{q} \in \mathbb{Q}^+$, p and q being coprime positive integers. Then the family whose minimum vector is $\mathbf{w}_{m(\lambda)} = (p, q, (i - 1)p + (n - i)q + 1)$ has ratio λ.

The case $P = 0$ is demonstrated similarly.

(ii) When $a_{i,n-i}$ and $b_{j,n-j}$ are nonzero, Proposition 1 implies that for any weight vector $\mathbf{v} = (s_1, s_2, d)$ of the system it is verified that

$$(i - 1)s_1 + (n - i)s_2 = d - 1 \tag{20}$$

and

$$js_1 + (n - j - 1)s_2 = d - 1. \tag{21}$$

The equalities (20) and (21) translate into $(i - j - 1)(s_1 - s_2) = 0$, and so we distinguish:

(ii-a) If $j = i - 1$, being $i > 0$, then (20) and (21) mean the same, and $d = (i-1)\,s_1 + (n-i)\,s_2 + 1$ is the only condition for weight vectors of the system (s_1, s_2, d), so we are in the same situation as in (i).
(ii-b) If $j \neq i - 1$, then $s_1 = s_2 = s$ for any weight vector (s_1, s_2, d), and therefore $F_{\mathrm{S}}\,(1)$ is the only family. Also, for a weight exponents s, by (20) and (21) it follows that the weight degree d has to be $(n-1)\,s + 1$.

□

From Propositions 11 and 12, which classify all the homogeneous polynomial systems, it is deduced an important property for such systems:

Proposition 13 *All homogeneous polynomial differential systems of degree n verifying $P \cdot Q \neq 0$ have $\mathbf{w}_m = (1, 1, n)$ as the minimum vector of the system.*

Proof In systems with a single family $F_{\mathrm{S}}\,(1)$, the assertion is obvious.

Systems with an infinite number of families have at most one nonzero monomial in both components, and $\mathbf{v} = (1, 1, n)$ is a weight vector of the system that clearly minimizes the weight exponents $s_1,\ s_2$. Then, let $\mathbf{w} = (p, q, d)$ be any other weight vector, whereby $p \geq 1,\ q \geq 1$. As $P \neq 0$, there is exactly one coefficient $a_{i,n-i} \neq 0$, that must verify $i > 0$, since the case $a_{0,n} \neq 0$ implies $Q = 0$ when there is at most one nonzero monomial in Q. So, by (10):

$$d = (i-1)\,p + (n-i)\,q + 1 \geq (i-1) + (n-i) + 1 = n$$

and therefore $d \geq n$, so in conclusion $(1, 1, n)$ is the minimum vector of the system.

□

Remark 14 The previous Proposition does not cover the trivial case of homogeneous systems with a null component. However, most of these systems also have $(1, 1, n)$ as minimum vector. As it follows from the above demonstration, the unique exceptions are

$$\dot{x} = a_{0,n}\,y^n,\ \ \dot{y} = 0, \tag{22}$$

and

$$\dot{x} = 0,\ \ \dot{y} = b_{n,0}\,x^n, \tag{23}$$

being the degree $n > 1$. These two are the only n-degree quasi-homogeneous polynomial differential systems in the plane with no minimum vector of the system. It is easy to prove the nonexistence of minimun in this two cases, because $(1, 1, n)$ and $(n, 1, 1)$ are weight vectors of (22) and $(1, 1, n)$ and $(1, n, 1)$ are weight vectors of (23).

The set of quasi-homogeneous systems is not limited to those homogeneous, as can be seen by noting that $\dot{x} = xy,\ \dot{y} = y^2 + x$ is a quasi-homogeneous system with weight vector $(2, 1, 2)$, but non-homogeneous. This leads to a division of the quasi-homogeneous systems into two mutually exclusive groups: one of those who are particularly homogeneous and other of those that not are. The second group is studied in later sections of this paper and in [12], where, fixed a n degree, is provided an algorithm to obtain all. Also from [12] it can draw conclusions about their weight vector families. There, these systems are classified according to their weight degree d is equal to or greater than 1 in all their vectors, and the following results are obtained:

Proposition 15 *A non-homogeneous quasi-homogeneous polynomial differential system of degree n has a unique weight vector family, and fits into one and only one of the following situations:*

(i) *If any weight vector* $\mathbf{v} = (s_1, s_2, d)$ *of the system verifies that* $d = 1$ *and* $s_1 > s_2$, *then the minimum weight vector of the system is* $\mathbf{w}_m = (n, 1, 1)$.

(ii) *If any weight vector* $\mathbf{v} = (s_1, s_2, d)$ *of the system verifies that* $d = 1$ *and* $s_1 < s_2$, *then the minimum weight vector of the system is* $\mathbf{w}_m = (1, n, 1)$.

(iii) *If any weight vector* $\mathbf{v} = (s_1, s_2, d)$ *of the system verifies that* $d > 1$ *and* $s_1 > s_2$, *then the minimum weight vector of the system is*

$$\mathbf{w}_m = \left(\frac{k+t}{\gcd(t,k)}, \frac{k}{\gcd(t,k)}, 1 + \frac{(n-1)\,k + (p-1)\,t}{\gcd(t,k)} \right)$$

where

$$X_n = (P_n(x,y), Q_n(x,y)) = \left(a_{p,n-p} x^p y^{n-p},\ b_{p-1,n-p+1} x^{p-1} y^{n-p+1} \right),$$

being $a_{p,n-p}^2 + b_{p-1,n-p+1}^2 \neq 0$, *is the highest homogeneous part (with n degree) of the system,*

$$\begin{aligned} X_{n-t} &= (P_{n-t}(x,y), Q_{n-t}(x,y)) \\ &= \left(a_{q,n-t-q} x^q y^{n-t-q},\ b_{q-1,n-t-q+1} x^{q-1} y^{n-t-q+1} \right), \end{aligned}$$

being $a_{q,n-t-q}^2 + b_{q-1,n-t-q+1}^2 \neq 0$, *is any other non-zero homogeneous part (with* $n - t$ *degree) of the system, and* $k = q - p$.

(iv) *If any weight vector* $\mathbf{v} = (s_1, s_2, d)$ *of the system verifies that* $d > 1$ *and* $s_1 < s_2$, *then the minimum weight vector of the system is*

$$\mathbf{w}_m = \left(\frac{k-t}{\gcd(t,k)}, \frac{k}{\gcd(t,k)}, 1 + \frac{(n-1)\,k - (p-1)\,t}{\gcd(t,k)} \right)$$

where

$$X_n = (P_n(x,y), Q_n(x,y)) = \left(a_{p,n-p} x^p y^{n-p},\ b_{p-1,n-p+1} x^{p-1} y^{n-p+1} \right),$$

being $a_{p,n-p}^2 + b_{p-1,n-p+1}^2 \neq 0$, *is the highest homogeneous part (with n degree) of the system,*

$$\begin{aligned} X_{n-t} &= (P_{n-t}(x,y), Q_{n-t}(x,y)) \\ &= \left(a_{q,n-t-q} x^q y^{n-t-q},\ b_{q-1,n-t-q+1} x^{q-1} y^{n-t-q+1} \right), \end{aligned}$$

being $a_{q,n-t-q}^2 + b_{q-1,n-t-q+1}^2 \neq 0$*, is any other non-zero homogeneous part (with* $n - t$ *degree) of the system, and* $k = p - q$*.*

Proof The existence of a single weight vector family in this class of systems follows from [12, Propositions 9–10]. Note that any quasi-homogeneous system (and particularly non-homogeneous) with a unique family is necessarily located in one of the four listed options: according to Corollary 8, it must be $d = 1$ or $d > 1$ for all its vectors; also, because it is not homogeneous, $s_1 \neq s_2$ (Corollary 10), and inequalities $s_1 > s_2$ and $s_1 < s_2$ are also mutually exclusive, and only one of them is fulfilled for all vectors of the family.

Paragraph (i) is demonstrated in [12, Proposition 9]. These systems are those of the form

$$\dot{x} = a_{0n}y^n + a_{10}x, \quad \dot{y} = b_{01}y, \tag{24}$$

for any degree $n \geq 2$.

Paragraph (ii) is also deduced from [12, Proposition 9]. These are the symmetric systems of those of type (24):

$$\dot{x} = a_{10}x, \quad \dot{y} = b_{n0}x^n + b_{01}y, \tag{25}$$

for any degree $n \geq 2$.

Meanwhile, paragraphs (iii) and (iv) have been proved in [12, Proposition 10]. □

Thus, we have now everything we needed to prove the main result of the section:

Proof (Proof of Theorem A) A quasi-homogeneous polynomial differential system verifying $P \cdot Q \neq 0$, either it is homogeneous or it is not. If so, Proposition 13 ensures that $\mathbf{w}_m = (1, 1, n)$. If it is not, Proposition 15 shows that the system must also have a minimum vector of the system. □

Remark 16 We already noted that given a quasi-homogeneous system the only possibilities are either to have a single weight vector family or to have an infinite number of families. As we have seen, systems with an infinite number of families, always homogeneous, have a weight vector family for any ratio $\lambda \in \mathbb{Q}^+$. Regarding unique-family systems, which can be both homogeneous and non-homogeneous, it is possible find one whose family be $F_\mathrm{S}(\lambda)$, whatever $\lambda \in \mathbb{Q}^+$:

- With $F_\mathrm{S}(1)$ as unique family, we have the homogeneous system of those seen in Propositions 11 and 12 (ii-b), as for example:

$$\dot{x} = x^2 + y^2, \quad \dot{y} = xy,$$

  or

$$\dot{x} = y^2, \quad \dot{y} = x^2.$$

- With $F_\mathrm{S}(n)$ as unique family, being n a positive integer greater than 1, we have the non-homogeneous systems of type (24).

- With $F_{\mathsf{S}}\left(\frac{1}{n}\right)$ as unique family, being n a positive integer greater than 1, we have the non-homogeneous systems of type (25).
- With $F_{\mathsf{S}}\left(\frac{a}{b}\right)$ as unique family, being a and b coprime integers greater than 1, we have, for example, the following non-homogeneous system:

$$\dot{x} = y^{a-1}, \quad \dot{y} = x^{b-1}.$$

This is a quasi-homogeneous system with minimum vector of the system $(a, b, ab - a - b + 1)$ and a unique weight vector family of ratio $\frac{a}{b}$.

3 Number of Non-homogeneous Quasi-Homogeneous Polynomial Differential Systems

In [12] the authors provide a general algorithm for get all non-homogeneous quasi-homogeneous polynomial differential systems of certain degree n, but without addressing their count. The aim of this section is to contribute with a formula that provide the number of systems of this class according to the degree n.

Lemma 17 *If $n > 1$, Euler's totient function (4) matches the number of elements of the set*

$$\left\{(t, k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \ : \ t + k = n \ , \ \gcd(t, k) = 1\right\}.$$

Proof Note first that if $n > 1$, $\varphi(n)$ matches the number of elements of the set

$$A_n = \{r \in \mathbb{N} \ : \ 1 \leq r < n \ , \ \gcd(n, r) = 1\}.$$

Then, we establish the bijection between A_n and

$$B_n = \left\{(t, k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \ : \ t + k = n \ , \ \gcd(t, k) = 1\right\}$$

defined by

$$\omega : r \in A_n \to (r, n - r) \in B_n.$$

The ω application is well defined: as it holds that $1 \leq r < n$, then $(r, n - r) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, and because n and r are coprime, the Euclidean algorithm ensures that $1 = \gcd(n, r) = \gcd(r, n - r)$. Besides, ω is trivially injective, and also surjective: given $(t, k) \in B_n$, then $t \in \{1, \ldots, n - 1\}$ and $k = n - t$. If $\gcd(n, t) = p > 1$, then p divides t and n, and therefore k. As a consequence, $\gcd(t, k) \neq 1$, against our hypothesis. Thereupon $t \in A_n$, and $(t, k) = \omega(t)$. □

Proposition 18 *The number $c_1(n)$ of non-homogeneous quasi-homogeneous polynomial differential systems of degree n such that for every weight vector $\mathbf{v} = (s_1, s_2, d)$ it is verified that $s_1 > s_2$ and $d > 1$ is given by:*

(i) $c_1(1) = 0, \quad c_1(2) = 2, \quad c_1(3) = 7,$

(ii) $c_1(n) = 2c_1(n-1) - c_1(n-2) + \varphi(n+1) \quad for\ n \geq 4,$

where φ is Euler's totient function.

Proof The specific cases $c_1(1) = 0$, $c_1(2) = 2$ and $c_1(3) = 7$ are demonstrated in [12], therefore (i) is proved. Let's prove (ii).

As proved in [12, Proposition 6], any quasi-homogeneous system on which $s_1 \neq s_2$ for one of its vectors meets that all homogeneous parts of P and Q components have at most one nonzero monomial. Particularly, let p the only value in the set $\{0, 1, \ldots, n-1\}$ such that the highest-degree homogeneous part of the system, X_n, is defined by $\left(a_{p,n-p}x^p y^{n-p}, b_{p-1,n-p+1}x^{p-1}y^{n-p+1}\right)$ if $p \neq 0$ and $\left(a_{0,n}y^n, 0\right)$ if $p = 0$. Let's enumerate the number of systems of degree n by counting independently how many systems there are for each possible value of p. For reasons of calculation, we define $M_n(a)$ $(1 \leq a \leq n)$ as the set of all existing systems of degree n that verify $p = n - a$. Also, let $m_n(a)$ be the number of elements of $M_n(a)$, so that

$$c_1(n) = \sum_{a=1}^{n} m_n(a). \tag{26}$$

As is proved in [12], there is a bijection between $M_n(a)$, being $n \geq 2$, and the set of maximal compatible linear equation systems composed of the equation

$$e_{n-a}^0[0] \equiv (n-a-1)s_1 + as_2 + 1 = d$$

and of one or more equations of the type

$$e_{n-a}^t[k] \equiv (n-a+k-1)s_1 + (a-t-k)s_2 + 1 = d$$

being s_1, s_2 the unknowns, d a parameter that can be considered as a constant, and t, k positive integers verifying $t \in \{1, \ldots, a\}$ and $k \in \{1, \ldots, a-t+1\}$, or equivalently, satisfying the constraints

$$2 \leq t + k \leq a + 1. \tag{27}$$

Also in [12, Proposition 12] it is observed that the necessary and sufficient condition for the compatibility of the system formed by equations $e_{n-a}^0[0]$, $e_{n-a}^{t_1}[k_1]$ and $e_{n-a}^{t_2}[k_2]$ is

$$k_1 t_2 = k_2 t_1. \tag{28}$$

Therefore, the count of the quasi-homogeneous systems in $M_n(a)$ set is equivalent to counting the aforementioned maximal compatible linear systems of equations, and under (27) and (28) this is equivalent to counting the equivalence classes of the set defined by

$$E(a) = \left\{(t,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \ : \ 2 \leq t + k \leq a + 1\right\} \tag{29}$$

subject to the equivalence relation

$$(t,k) \approx (r,s) \Longleftrightarrow ts = kr.$$

So, $m_n(a) = |E(a)/\approx|$. We note that the value of $m_n(a)$ only depends on a and is independent of the degree n, so we will denote it by $m(a)$ and (26) is rewritten as

$$c_1(n) = \sum_{a=1}^{n} m(a), \tag{30}$$

and, hence,

$$m(n) = c_1(n) - c_1(n-1). \tag{31}$$

In order to obtain the value of $c_1(n)$, first we note that according to (29) it follows that

$$E(a) = E(a-1) \bigcup G(a),$$

being $G(a) = \{(t,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : t + k = a + 1\}$, and therefore $m(a)$ is obtained by adding to $m(a-1)$ the number of new equivalence classes arising from $G(a)$.

We claim that $(t,k) \in G(a)$ will form a new equivalence class in $E(a)$ if and only if t and k are coprime:

1. If $\gcd(t,k) = 1$, then we suppose that already exists $(t_1, k_1) \in E(a)$ being $tk_1 = t_1 k$. Then, by Euclid's theorem, t must divide t_1 and k must divide k_1, so $t + k \le t_1 + k_1$. Being $t + k = a + 1$ and $(t_1, k_1) \in E(a)$, it can only be true the equality $t + k = t_1 + k_1$, which along with that t divides t_1 and k divides k_1, implies $t = t_1$ and $k = k_1$.

2. If $\gcd(t,k) = p > 1$, then there are t_1, k_1 positive integers satisfying $t = pt_1$ and $k = pk_1$, whereby $tk_1 = t_1 k$, and (t,k) belongs to the same equivalence class that (t_1, k_1).

In conclusion, and making use of Lemma 17:

$$m(a) = m(a-1) + |\{(t,k) \in G(a) : \gcd(t,k) = 1\}| = m(a-1) + \varphi(a+1), \tag{32}$$

and therefore, by (30), (31) and (32):

$$\begin{aligned} c_1(n) = \sum_{a=1}^{n-1} m(a) + m(n) &= c_1(n-1) + m(n) \\ &= c_1(n-1) + m(n-1) + \varphi(n+1) \\ &= c_1(n-1) + (c_1(n-1) - c_1(n-2)) + \varphi(n+1) \\ &= 2c_1(n-1) - c_1(n-2) + \varphi(n+1). \end{aligned}$$

One of our hypothesis was that the degree n should be greater than or equal to 2, so that the above formula works for values $n \ge 4$. □

Proof (Proof of Theorem B) Specific cases $c(1) = 0$, $c(2) = 6$ and $c(3) = 16$ are demonstrated in [12], considering that there only partial counts are made. This proves (i). Let's prove (ii).

Following the classification given to the quasi-homogeneous systems in Corollary 8, we distinguish between systems with weight degree $d > 1$ in all vector, and systems with weight degree $d = 1$ in all vector. On the other hand, the fact that we are dealing with non-homogeneous systems ensures (see Corollary 10) that weight exponents of any vector do not match ($s_1 \neq s_2$), so we also will differentiate between $s_1 > s_2$ and $s_1 < s_2$:

(a) Case $d > 1$:

Let $c_1(n)$ be the number of n degree systems of this type which verify $s_1 > s_2$ for every weight vector, being $n \geq 4$. By Proposition 18 we know that

$$c_1(n) = 2c_1(n-1) - c_1(n-2) + \varphi(n+1).$$

The situation $s_1 < s_2$ is symmetrical and counts the same number of quasi-homogeneous systems of degree n. Thereby, $c(k) = 2c_1(k)$ for any $k \geq 4$, and we conclude:

$$\begin{aligned} c(n) &= 2(2c_1(n-1) - c_1(n-2) + \varphi(n+1)) \\ &= 2c(n-1) - c(n-2) + 2\varphi(n+1). \end{aligned} \tag{33}$$

(b) Case $d = 1$:

As shown in [12, Proposition 9], this situation is reduced, for counting purposes, to two particular cases, whatever the degree n. With $s_1 > s_2$ we have the system

$$\begin{aligned} P(x, y) &= a_{0n}y^n + a_{10}x, \\ Q(x, y) &= b_{01}y, \end{aligned}$$

and with $s_1 < s_2$, the system

$$\begin{aligned} P(x, y) &= a_{10}x, \\ Q(x, y) &= b_{n0}x^n + b_{01}y. \end{aligned}$$

Adding these two systems increase by 2 units all values $c(k)$, but does not vary the recursive formula (33).

We are going to demonstrate the analytical formula (7) by induction: we check that it is valid for $n = 2, 3$, keeping in mind that the summations where $n < 4$ are zero. Let's suppose (7) true for $n-2$ and $n-1$, i.e.:

$$c(n-2) = 10(n-2) - 14 + 2\sum_{k=4}^{n-2}\sum_{j=5}^{k+1}\varphi(j), \tag{34}$$

$$c(n-1) = 10(n-1) - 14 + 2\sum_{k=4}^{n-1}\sum_{j=5}^{k+1}\varphi(j). \tag{35}$$

Then, using (33), (34) and (35) we have:

$$
\begin{aligned}
c(n) &= 2(10(n-1)-14) - (10(n-2)-14) \\
&\quad + 4\sum_{k=4}^{n-1}\sum_{j=5}^{k+1}\varphi(j) - 2\sum_{k=4}^{n-2}\sum_{j=5}^{k+1}\varphi(j) + 2\varphi(n+1) \\
&= 10n - 14 + 4\sum_{k=4}^{n-2}\sum_{j=5}^{k+1}\varphi(j) - 2\sum_{k=4}^{n-2}\sum_{j=5}^{k+1}\varphi(j) + 4\sum_{j=5}^{n}\varphi(j) + 2\varphi(n+1) \\
&= 10n - 14 + 2\sum_{k=4}^{n-2}\sum_{j=5}^{k+1}\varphi(j) + 4\sum_{j=5}^{n}\varphi(j) + 2\varphi(n+1) \\
&= 10n - 14 + 2\sum_{k=4}^{n-2}\sum_{j=5}^{k+1}\varphi(j) + 2\sum_{j=5}^{n}\varphi(j) + 2\sum_{j=5}^{n+1}\varphi(j) \\
&= 10n - 14 + 2\sum_{k=4}^{n}\sum_{j=5}^{k+1}\varphi(j)\,.
\end{aligned}
$$

□

4 Implementations in MATLAB

The algorithm published by García et al. [12], which provides all non-homogeneous quasi-homogeneous systems of a given degree, and the result stated in Theorem B of this article, which counts them, are both particularly suitable for implementation using computer programming languages. We have developed the two mentioned results by using the *M* programming language, included in the mathematical software tool MATLAB.

MATLAB is a proprietary product of the company Mathworks, created in 1984 by the mathematician and computer programmer Cleve Moler. We have chosen MATLAB, from among the many options for programming languages, because it is a software platform especially suitable for mathematical calculations, and remarkably widespread in universities and research centers. It is estimated that MATLAB is utilized by more than one million people in academia and business.

4.1 The Program qh.m

The *qh.m* program prompts the user for a degree n, and returns a numbered list containing all quasi-homogeneous polynomial differential systems of degree n which particularly are not homogeneous, together with the minimum weight vector of each system. The output is embodied in a LaTeX format document (.tex), that is created in the same folder where the program is located, under the name *quasihomog_degn.tex* and being n the degree of the systems. Information is also provided to the user via the execution console, associated with the systems construction process, such as the full list of maximal systems of equations used. In order not to slow execution times, and as they are unnecessary, symbolic variables are never used. The program is based on the algorithm of [12], which only provides, among the systems mentioned, those that verify $s_1 > s_2$ in all its weight vectors (s_1, s_2, d). This computerized version of the algorithm also returns the symmetric systems, with $s_1 < s_2$ for every weight vector, in addition to the $s_1 = s_2$ systems, so the list is complete.

Using the R2014a version of MATLAB, the program is limited to system degrees below 385. Starting from this value there will be a memory capacity error, because its execution requires the creation of huge storage matrices. We have implemented on other program version that allows for degrees up to 5500, but at the cost of dramatically increasing run times. Even in the version we provide, the large amount of computation required causes run times above the minute from grade 30 onwards.

The program consists of three modules: the aforementioned *qh.m* script performs the main functions, and calls the *addMonomial.m* function, which is responsible for building the LaTeX code for each monomial of a system, and also calls *printLatexSyst.m*, which writes each of the systems in the .tex output document.

4.2 The Program counterQH.m

Although *qh.m* already performs the systems counting, it has the problem of execution times for higher degrees. The *counterQH.m* program receives a degree n from the user and is limited to counting the number of non-homogeneous quasi-homogeneous polynomial differential systems of that degree, without stopping at their algebraic construction. For this it employs the expression (5) in Theorem B of this article. Being a recursive formula, it needs almost no internal memory allocation to work, and so the program has no limits on the degree n introduced, except those set by the user based on the execution times. In any case, these are very low: for degrees below 10,000 the response is almost instantaneous, the degree 100,000 executes in less than 30 s, and the degree 1,000,000 in less than 5 min.

As program output, no new document is created, and it is provided through the execution console. In addition to the total number of n-degree systems, the same information is provided for systems of lower degree.

This program is also divided into three modules: *counterQH.m* is the main script, and that is where the recursive formula (5) is implemented. Meanwhile, *counterQH.m* calls the *eulersTotient.m* function, which is responsible for calculating the values of Euler's totient function $\varphi(n)$ by the formula, published in [18],

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

in which $p \mid n$ represents each different prime number p that divides n. As noted in [10], all prime numbers greater than 3 may be expressed on the form $6k \pm 1$, with $k \in \mathbb{N}$. Checking these values is one of the most efficient iterative solutions when factoring a number through successive divisions is required. This task is delegated to *primeFactors.m* function, which returns the list of all prime divisors of n without repetitions.

Both programs can be freely downloaded at this web address: http://xixon.epv.uniovi.es/gsd.

As an example of use of *counterQH.m*, we show the number of non-homogeneous quasi-homogeneous systems of a few different degrees:

n	n° systems	n	n° systems	n	n° systems
1	0	11	420	25	3940
2	6	12	534	50	28,414
3	16	13	660	100	214,802
4	34	14	802	500	25,634,774
5	56	15	960	1000	203,858,766
6	90	16	1150	5000	25,360,684,978
7	132	17	1352	1e+4	2.0276e+11
8	186	18	1590	1e+5	2.0265e+14
9	248	19	1844	1e+6	2.0264e+17
10	330	20	2122	1e+7	2.0264e+20

We remark that the number of systems grows asymptotically as n^3.

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3.2. An algorithm for providing the normal forms of spatial quasi-homogeneous polynomial differential systems

Varios autores han contribuido al desarrollo de algoritmos para la determinación y enumeración de los sistemas quasi-homogéneos existentes: los anteriormente citados trabajos de García *et al.* [10] y Xiong [22], además del artículo [7] de esta tesis, profundizan en este tema, pero todos ellos restringiéndose a sistemas bidimensionales. Se planteó por tanto, como objetivo de este artículo, la elaboración de un algoritmo tal que, fijado un entero positivo n, devolviera todos los sistemas quasi-homogéneos tridimensionales de grado n. Al contrario que en el plano, para los sistemas quasi-homogéneos en $\mathbb{R}^3$ se disponía de poca literatura científica y, antes de la aparición de este trabajo no había sido publicado ningún método que permitiese determinarlos todos.

Además del mencionado objetivo, y en un plano más teórico, en este artículo se identificó una familia de sistemas quasi-homogéneos, denominados *maximales*, a partir de los cuales se podían deducir todos los demás. Estos sistemas maximales en $\mathbb{R}^3$ verifican poseer siempre un vector peso mínimo, que además es único dentro de los sistemas quasi-homogéneos de su grado, actuando así como «código de identificación» del sistema. Finalmente, utilizando la implementación del algoritmo, se obtuvo la lista de todos los sistemas quasi-homogéneos maximales de grado 2 en $\mathbb{R}^3$.

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An algorithm for providing the normal forms of spatial quasi-homogeneous polynomial differential systems

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ABSTRACT

Quasi-homogeneous systems, and in particular those 3-dimensional, are currently a thriving line of research. But a method for obtaining all fields of this class is not yet available. The weight vectors of a quasi-homogeneous system are grouped into families. We found the maximal spatial quasi-homogeneous systems with the property of having only one family with minimum weight vector. This minimum vector is unique to the system, thus acting as identification code. We develop an algorithm that provides all normal forms of maximal 3-dimensional quasi-homogeneous systems for a given degree. All other 3-dimensional quasi-homogeneous systems can be trivially deduced from these maximal systems. We also list all the systems of this type of degree 2 using the algorithm. With this algorithm we make available to the researchers all 3-dimensional quasi-homogeneous systems.

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1. Introduction

We deal with spatial polynomial differential systems of the form

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z) \tag{1}$$

where $P, Q, R \in \mathbb{C}[x, y, z]$, with degrees n_1, n_2, n_3 respectively. As usual $\mathbb{C}[x, y, z]$ denotes the ring of polynomials in x, y and z with coefficients in $\mathbb{C}$. The dot denotes derivative with respect to an independent variable t, which can be real or complex. We say that the *degree* of the system is $n = \max\{n_1, n_2, n_3\}$. From now on a polynomial differential system (1) will be denoted by $\mathsf{S}(P, Q, R)$, or by S when it does not lead to any confusion.

Let $\mathbb{Z}^+$ denote the set of positive integers, and $\mathbb{R}^+$ the set of positive reals. A polynomial differential system $\mathsf{S}(P, Q, R)$ is *quasi-homogeneous* (from here on, simply QH) if there exist $s_1, s_2, s_3, d \in \mathbb{Z}^+$ such that for an arbitrary $\alpha \in \mathbb{R}^+$,

$$\begin{aligned} P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \alpha^{s_1-1+d}P(x, y, z), \\ Q(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \alpha^{s_2-1+d}Q(x, y, z), \\ R(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \alpha^{s_3-1+d}R(x, y, z). \end{aligned} \tag{2}$$

We call s_1, s_2 and s_3 *weight exponents* of S, and d the *weight degree* with respect to the weight exponents s_1, s_2 and s_3.

Suppose that a system S is QH, with weight exponents s_1, s_2 and s_3 and with weight degree d. In this case we state that $\mathbf{w} = (s_1, s_2, s_3, d)$ is a *weight vector* of the system S.

In the set of weight vectors of a QH system S it is possible to define a *partial order* relation as follows: given two weight vectors of S, $\mathbf{w} = (s_1, s_2, s_3, d)$ and $\mathbf{v} = (s_1^*, s_2^*, s_3^*, d^*)$, we write that $\mathbf{w} \le \mathbf{v}$ when

$$s_1 \le s_1^*, \; s_2 \le s_2^*, \; s_3 \le s_3^*, \; d \le d^*. \tag{3}$$

We say that a weight vector $\mathbf{w}_m$ is the *minimum weight vector* of the QH system S if for any other weight vector $\mathbf{w}$ of the system S it is verified that $\mathbf{w}_m \le \mathbf{w}$.

A QH system is called *maximal* if any new monomial added to its structure maintaining the degree of the system prevents it to be QH.

As an example consider the polynomial differential system

$$\begin{aligned} \dot{x} &= xyz + x^2, \\ \dot{y} &= y^2z + xy, \\ \dot{z} &= yz^2 + xz. \end{aligned} \tag{4}$$

It is a QH system with weight vector $(2, 1, 1, 3)$, as can be seen from (2). But this system is not maximal, because it can be completed to

$$\begin{aligned} \dot{x} &= xyz + y^3 + x^2, \\ \dot{y} &= y^2z + xz^2 + xy, \\ \dot{z} &= yz^2 + y^2 + xz, \end{aligned} \tag{5}$$

which still is QH with the weight vector $(3, 2, 1, 4)$.

We will focus our study on the maximal QH systems, considering the rest as particular cases of these in which some monomials are zero. A non-maximal system will possess all the weight vectors of those maximal systems to which it can be completed, and perhaps other new weight vectors. All the weight vectors of (5), i.e. the set $\{(3a, 2a, a, 3a + 1) : a \in \mathbb{Z}^+\}$, are also weight vectors of (4), besides others as the mentioned $(2, 1, 1, 3)$.

The QH systems are a generalization of the homogeneous systems, to which they contain as a particular case. Several reasons motivate their study. For example, if a system S is QH with weight vector $\mathbf{v} = (s_1, s_2, s_3, d)$, being $d > 1$, then S is invariant under the changes of variable $x_i \to \alpha^{w_i}x_i$, $t \to \alpha^{-1}t$, for any $\alpha \in \mathbb{R}^+$, where $w_i = s_i/(d - 1)$ for $i = 1, 2, 3$. In addition, the structural properties of QH systems allow to find their possible analytic first integrals through the Kowalevski exponents, see Yoshida (1983).

In the literature many authors have made contributions to this field, and in recent times it generates an increasing interest. The integrability has been studied extensively, highlighting the contributions of García et al. (2013), Kozlov (2015) and Llibre and Zhang (2002). In Liang et al. (2014) the phase portraits are studied. The centers and limit cycles are discussed in Geng and Lian (2015), Li and Wu (2016), Tang et al. (2015) and Xiong et al. (2015). Chiba (2015) and Yoshida (1983) have explored the Kowalevski exponents. Other topics such as the period function of the sum of two quasi-homogeneous (Álvarez et al., 2017), or the isochronicity and normal forms (Han and Romanovski, 2012) have also been treated recently. On the other hand, García et al. (2018) have studied the classification and counting of this class of systems in dimension 2. Although the mentioned previous papers deal with systems in the plane, the area of QH systems in the space has recently begun to be explored, as shown by the works of Huang and Zhao (2012), devoted to the limit set of trajectories, and Liang and Torregrosa (2016), which studies the centers of a certain class of 3-dimensional QH systems.

The objective of this work is to develop an algorithm that provides, given a degree n supplied by the user, all normal forms of existing spatial QH polynomial differential systems of degree n. As we have said, we will restrict ourselves to maximal QH systems. A similar objective, but for systems in the plane, have been carried out in García et al. (2013). However, to the best of our knowledge, there is no work focused on supplying the complete set of 3-dimensional QH. We provide such an algorithm in the present paper, which will be of valuable assistance for the development of future works in the field of study of polynomial differential systems

This work is organized as follows. In Section 2 we present some properties about weight vectors of QH systems, besides providing some concepts like the weight vector family. Section 3 deals with the particular case of homogeneous QH systems. In Section 4 we introduce the concept of brick, that is the unitary element with which we will later build the QH systems. Section 5 contains some of the most important theoretical results, such as Theorem 15, which states that the maximal QH systems have a single family of weight vectors, or the fact that the minimum weight vector can be constituted as a unique identifier in this type of systems. Section 6 contains the main practical results used directly by the algorithm, which is described in pseudocode in Section 7. The work is closed with the list of all QH systems of degree $n=2$, obtained by applying the algorithm, see Section 8.

2. Some results on weight vectors

Given a QH system $\mathrm{S}(P, Q, R)$, where

$$P=\sum_{k=0}^{n_1} P_k, \quad Q=\sum_{k=0}^{n_2} Q_k, \quad R=\sum_{k=0}^{n_3} R_k, \tag{6}$$

we define its *homogeneous parts of degree k*, P_k, Q_k and R_k as:

$$P_k(x,y,z)=\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1} a_{p_1 p_2 k-p_1-p_2} x^{p_1} y^{p_2} z^{k-p_1-p_2} \quad (k=1,2,\dots,n_1), \tag{7}$$

$$Q_k(x,y,z)=\sum_{q_1=0}^{k}\sum_{q_2=0}^{k-q_1} b_{q_1 q_2 k-q_1-q_2} x^{q_1} y^{q_2} z^{k-q_1-q_2} \quad (k=1,2,\dots,n_2),$$

$$R_k(x,y,z)=\sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1} c_{t_1 t_2 k-t_1-t_2} x^{t_1} y^{t_2} z^{k-t_1-t_2} \quad (k=1,2,\dots,n_3),$$

being the coefficients $a_{p_1 p_2 k-p_1-p_2}$, $b_{q_1 q_2 k-q_1-q_2}$ and $c_{t_1 t_2 k-t_1-t_2}$ complex numbers.

Now we are going to obtain some properties of the coefficients of the QH systems. The equations included in the following result provide information of great relevance about the structure of the QH systems, because they are the key for determining whether a monomial is present in the system or not.

Proposition 1. *Given a QH system* $\mathsf{S}(P, Q, R)$*, being* P_k*,* Q_k *and* R_k *its homogeneous parts of degree* k*, and* $p_1, p_2, q_1, q_2, t_1, t_2, p_1+p_2, q_1+q_2, t_1+t_2 \in \{0, 1, \ldots, k\}$*, then*

$$a_{p_1 p_2 k-p_1-p_2} \neq 0 \Rightarrow (p_1-1)s_1 + p_2 s_2 + (k-p_1-p_2)s_3 = d-1, \tag{8}$$

$$b_{q_1 q_2 k-q_1-q_2} \neq 0 \Rightarrow q_1 s_1 + (q_2-1)s_2 + (k-q_1-q_2)s_3 = d-1, \tag{9}$$

$$c_{t_1 t_2 k-t_1-t_2} \neq 0 \Rightarrow t_1 s_1 + t_2 s_2 + (k-t_1-t_2-1)s_3 = d-1, \tag{10}$$

for any weight vector $\mathbf{w} = (s_1, s_2, s_3, d)$ *of* S*. If* S *is a maximal system, then the three reciprocal implications are also true.*

Proof. We will do the proof for the coefficients of P_k, because the proofs for the coefficients of Q_k and R_k are identical. Let $\{\mathbf{w}_i\}_{i\in I}$ be the weight vector set of the QH system S.

Due to (6) and (7) we have that

$$P(x, y, z) = \sum_{k=0}^{n_1} \sum_{p_1=0}^{k} \sum_{p_2=0}^{k-p_1} a_{p_1 p_2 k-p_1-p_2} x^{p_1} y^{p_2} z^{k-p_1-p_2}$$

satisfies

$$P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) =$$
$$= \sum_{k=0}^{n_1} \sum_{p_1=0}^{k} \sum_{p_2=0}^{k-p_1} a_{p_1 p_2 k-p_1-p_2} \alpha^{p_1 s_1 + p_2 s_2 + (k-p_1-p_2)s_3} x^{p_1} y^{p_2} z^{k-p_1-p_2}.$$

Then it follows from the fact that S is QH (see (2)) that

$$a_{p_1 p_2 k-p_1-p_2} \alpha^{p_1 s_1 + p_2 s_2 + (k-p_1-p_2)s_3} = a_{p_1 p_2 k-p_1-p_2} \alpha^{s_1 - 1 + d}, \tag{11}$$

for all coefficients, for all $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$ and for any $\alpha \in \mathbb{R}^+$. Then we fix a coefficient $a_{p_1 p_2 k-p_1-p_2}$, corresponding to the k-degree monomial

$$a_{p_1 p_2 k-p_1-p_2} x^{p_1} y^{p_2} z^{k-p_1-p_2}. \tag{12}$$

Due to (11), if $a_{p_1 p_2 k-p_1-p_2} \neq 0$,

$$(p_1-1)s_1 + p_2 s_2 + (k-p_1-p_2)s_3 = d-1 \tag{13}$$

is necessarily fulfilled for all $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$, and thus the necessary condition ($\Rightarrow$) is proved.

We now study the sufficient condition ($\Leftarrow$) supposing that S is maximal. For the fixed values p_1, p_2, k, we have that (13) meets for any $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$. Suppose, by reductio ad absurdum, that $a_{p_1 p_2 k-p_1-p_2} = 0$, that is the monomial (12) is not present in P_k. Let us see that a new monomial can be added to P_k by maintaining the QH character of the system. The new system $\mathsf{S}'\left(P', Q', R'\right)$ will be

$$P'(x, y, z) = P(x, y, z) + x^{p_1} y^{p_2} z^{k-p_1-p_2},$$
$$Q' = Q, \qquad R' = R.$$

Then if we take any weight vector of S, $\mathbf{w} \in \{\mathbf{w}_i\}_{i \in I}$, we have due to (13) and since S is QH, that

$$\begin{aligned} & P'\left(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z\right) = \\ & \quad = P\left(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z\right) + \alpha^{p_1 s_1 + p_2 s_2 + (k-p_1-p_2)s_3} x^{p_1} y^{p_2} z^{k-p_1-p_2} \\ & \quad = P\left(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z\right) + \alpha^{s_1-1+d} x^{p_1} y^{p_2} z^{k-p_1-p_2} \\ & \quad = \alpha^{s_1-1+d}\left[P(x, y, z) + x^{p_1} y^{p_2} z^{k-p_1-p_2}\right] \\ & \quad = \alpha^{s_1-1+d} P'(x, y, z), \end{aligned}$$

for an arbitrary $\alpha \in \mathbb{R}^+$. As this same property is also fulfilled for Q' and R', we get that S$'$ is also a QH system. As a consequence S was not maximal, a contradiction. □

Corollary 2. *If a QH system* S *has a monomial of degree k, then there exist $x_1, x_2 \in \{-1, 0, ..., k\}$, $-1 \leq x_1 + x_2 \leq k$, verifying*

$$x_1 s_1 + x_2 s_2 + (k - x_1 - x_2 - 1)s_3 = d - 1$$

for any weight vector (s_1, s_2, s_3, d) of S.

Proof. It is deduced from Proposition 1 by taking into account that $p_i, q_i, t_i \in \{0, 1, ..., k\}$ for $i = 1, 2$. □

Corollary 3. *If a system* S (P, Q, R) *is QH, then $P_0 = Q_0 = R_0 = 0$.*

Proof. The equivalence (8) shows that if $P_0 = a_{0,0,0}$ is not null, then $d \leq 0$, a contradiction. Also, if Q_0 or R_0 are different from zero, we obtain the same conclusion from (9) and (10). □

The following results are a direct consequence of Corollary 2, taking into account that a system must have some coefficient different from zero.

Remark 4. (i) A necessary condition for a vector $\mathbf{w} = (s_1, s_2, s_3, d) \in \left(\mathbb{Z}^+\right)^4$ to be a weight vector of some QH system is that $\gcd(s_1, s_2, s_3)$ be a divisor of $d - 1$.

(ii) Given a QH system S, the weight degree d of any weight vector is uniquely determined by the weight exponents s_1, s_2 and s_3.

The following result proves that the set of weight vectors of a QH system is infinite, and also provides a method for constructing new weight vectors from a given one.

Proposition 5. *Given a weight vector (s_1, s_2, s_3, d) of a QH system* S *and $r = \frac{p}{q} \in \mathbb{Q}^+$ with p and q co-prime, the vector (rs_1, rs_2, rs_3, d^*) is also a weight vector of* S *if and only if q divides $\gcd(s_1, s_2, s_3)$ and $d^* = r(d-1) + 1$.*

Proof. Note first that $(rs_1, rs_2, rs_3, r(d-1)+1)$ is a vector of positive integers if and only if q divides $\gcd(s_1, s_2, s_3, d-1)$. Taking into account that $\gcd(s_1, s_2, s_3)$ is a divisor of $d-1$, it is enough that q divides $\gcd(s_1, s_2, s_3)$.

As (s_1, s_2, s_3, d) is a weight vector, for any $\alpha > 0$, we have that

$$P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_1+d-1} P(x, y, z).$$

On the other hand $r \in \mathbb{Q}^+$, whereby $\alpha^r > 0$, and consequently

$$\begin{aligned} P(\alpha^{rs_1}x, \alpha^{rs_2}y, \alpha^{rs_3}z) &= P((\alpha^r)^{s_1} x, (\alpha^r)^{s_2} y, (\alpha^r)^{s_3} z) \\ &= (\alpha^r)^{s_1+d-1} P(x, y, z) \\ &= \alpha^{rs_1+r(d-1)} P(x, y, z). \end{aligned}$$

Therefore (rs_1, rs_2, rs_3, d^*) will be a weight vector if and only if $d^* = r(d-1)+1$. Similar conclusions hold for Q and R. □

Given a QH system S and $\lambda,\ \mu \in \mathbb{Q}^+$, the *weight vector family* $F_{\mathsf{S}}(\lambda,\ \mu)$ *of* S *with ratio* $(\lambda,\ \mu)$ is defined as the set of weight vectors of S where the proportion between the exponents s_1 and s_2 is λ and the proportion between the exponents s_1 and s_3 is μ:

$$F_{\mathsf{S}}(\lambda,\ \mu) = \left\{ (s_1, s_2, s_3, d) \text{ weight vector of } \mathsf{S}\ :\ \frac{s_1}{s_2} = \lambda \text{ and } \frac{s_1}{s_3} = \mu \right\}.$$

Note that in this definition it is not relevant the value that can take the weight degree d, which is uniquely determined by s_1, s_2 and s_3. Moreover, if we fix the system S and the family $F_{\mathsf{S}}(\lambda,\ \mu)$, the first weight exponent s_1 of a weight vector uniquely determines the rest of the vector, because $s_2 = s_1/\lambda$, $s_3 = s_1/\mu$ and, as we said before, the weight degree d depends functionally on s_1, s_2 and s_3.

However, fixed two values λ and μ, and given two systems S and T, it can happen that families $F_{\mathsf{S}}(\lambda,\ \mu)$ and $F_{\mathsf{T}}(\lambda,\ \mu)$ are different. In this case the weight exponents of the vectors match but not necessarily the weight degrees.

Given a weight vector family $F_{\mathsf{S}}(\lambda,\ \mu)$, a weight vector that minimizes the rest of vectors of the family in the sense of the order relation (3) is called the *family generator* and we denote it by $\mathbf{g}_{(\lambda,\mu)}$. Now we will prove that $\mathbf{g}_{(\lambda,\mu)}$ exists for every family. As it happens in two dimensions (García et al., 2018), we have:

Proposition 6. *Given a weight vector family* $F_{\mathsf{S}}(\lambda,\ \mu)$ *of a QH system* S*, it is verified that:*

(i) The family generator $\mathbf{g}_{(\lambda,\mu)}$ *exists.*
(ii) Given $\mathbf{w} = (s_1^*, s_2^*, s_3^*, d^*) \in F_{\mathsf{S}}(\lambda,\ \mu)$*, then* $\mathbf{g}_{(\lambda,\mu)} = \mathbf{w}$ *if and only if* $\gcd(s_1^*, s_2^*, s_3^*) = 1$.

Proof. Let $\mathbf{w} = (s_1^*, s_2^*, s_3^*, d^*)$ be the only weight vector of the family $F_{\mathsf{S}}(\lambda,\ \mu)$ that verifies $s_1^* \le s_1$ for every $(s_1, s_2, s_3, d) \in F_{\mathsf{S}}(\lambda,\ \mu)$. This weight vector always exists, because the weight exponents are positive integers; and it is unique, as two weight vectors of $F_{\mathsf{S}}(\lambda,\ \mu)$ with the same weight exponent s_1^* are identical. We will see that $\mathbf{w}$ is the generator of $F_{\mathsf{S}}(\lambda,\ \mu)$. Let $\mathbf{v} = (s_1, s_2, s_3, d)$ be any weight vector of $F_{\mathsf{S}}(\lambda,\ \mu)$. From $s_1^* \le s_1$ we easily follow that $s_2^* = s_1^*/\lambda \le s_1/\lambda = s_2$ and $s_3^* = s_1^*/\mu \le s_1/\mu = s_3$. To prove $d^* \le d$, we suppose $P \ne 0$. As $\mathbf{w}$ and $\mathbf{v}$ are weight vectors of $F_{\mathsf{S}}(\lambda,\ \mu)$, for any $\alpha,\ \beta > 0$ we have:

$$P\left(\alpha^{s_1^*}x, \alpha^{\frac{s_1^*}{\lambda}}y, \alpha^{\frac{s_1^*}{\mu}}z\right) = \alpha^{s_1^*-1+d^*}P(x,y,z) \tag{14}$$

and

$$P\left(\beta^{s_1}x, \beta^{\frac{s_1}{\lambda}}y, \beta^{\frac{s_1}{\mu}}z\right) = \beta^{s_1-1+d}P(x,y,z). \tag{15}$$

Then, setting $\beta = \alpha^{\frac{s_1^*}{s_1}} > 0$, (15) becomes:

$$P\left(\alpha^{s_1^*}x, \alpha^{\frac{s_1^*}{\lambda}}y, \alpha^{\frac{s_1^*}{\mu}}z\right) = \alpha^{s_1^*+\frac{s_1^*}{s_1}(d-1)}P(x,y,z). \tag{16}$$

So, by (14) and (16), and since P is not zero in the whole plane, we have

$$\alpha^{d^*-1} = \alpha^{\frac{s_1^*}{s_1}(d-1)}.$$

The exponential function is injective, hence $d^* - 1 = \frac{s_1^*}{s_1}(d-1)$. As a conclusion,

$$s_1^* \le s_1 \Leftrightarrow d^* \le d. \tag{17}$$

If $P=0$, then $Q\neq 0$ or $R\neq 0$, and the result is proved in a similar way. Thus $\mathbf{w}\leq\mathbf{v}$ and accordingly $\mathbf{w}=\mathbf{g}_{(\lambda,\mu)}$.

Now let $\mathbf{g}_{(\lambda,\mu)}=(s_1^*,s_2^*,s_3^*,d^*)$ be the family generator, and we suppose that s_1, s_2 and s_3 share a common divisor $q>1$. Making use of $r=1/q$ in Proposition 5, $\mathbf{g}_{(\lambda,\mu)}$ would not be the family generator.

On the other hand, the fact that the weight exponents are coprime implies, also by Proposition 5, that they cannot be reduced more. Neither the weight degree, due to (17). □

The reason for calling such vector of $F_{\mathrm{S}}(\lambda,\ \mu)$ *family generator* is clear when we observe that the whole family can be constructed based on $\mathbf{g}_{(\lambda,\mu)}$ multiples. It is clear from the above results that

$$F_{\mathrm{S}}(\lambda,\ \mu)=\left\{\left(as_1^*,as_2^*,as_3^*,a\left(d^*-1\right)+1\right)\,:\,a\in\mathbb{Z}^+\right\},$$

being $\mathbf{g}_{(\lambda,\mu)}=(s_1^*,s_2^*,s_3^*,d^*)$ the family generator of $F_{\mathrm{S}}(\lambda,\ \mu)$.

As a consequence a weight vector family is always contained in an unidimensional linear variety of $\mathbb{R}^4$ passing through the point $(0,0,0,1)$. The linear variety that contains the family generated by (s_1^*,s_2^*,s_3^*,d^*) is x_0+L, where $x_0=(0,0,0,1)$ and $L=\mathrm{Span}\left(s_1^*,s_2^*,s_3^*,d^*-1\right)$ is the vector subspace with basis $\left(s_1^*,s_2^*,s_3^*,d^*-1\right)$. For the same reason, a dimension 1 variety of $\mathbb{R}^4$ cannot contain two different families, even if the families belong to distinct systems. Or it contains a unique whole family, or it does not contain any.

The next question is how many weight vector families can have a QH system. There are many examples of systems with more than one family of weight vectors. As an example, system (4) has $\left\{(a,b,a-b,a+1):a,b\in\mathbb{Z}^+ \textit{ and } a>b\right\}$ as a set of weight vectors, which means that $F\left(a/b,\ a/\left(a-b\right)\right)$ is a weight vector family of (4) for every pair $a,b\in\mathbb{Z}^+$ verifying $a>b$. Therefore this is a case of infinite number of families.

The fact that a system S has more than one weight vector family implies that the existence of the minimum weight vector $\mathbf{w}_m$ of S is not guaranteed. If it exists, it should be the minimum of all family generators, and this minimum may not be reached. However in systems with a single family $F_{\mathrm{S}}(\lambda,\ \mu)$ we have that $\mathbf{g}_{(\lambda,\mu)}=\mathbf{w}_m$. We will show in Theorem 15 that the maximal QH systems, the main object of our study, always fulfill the property of having a unique family, and therefore they have minimum vector of the system.

3. The homogeneous maximal case

We say that a polynomial differential system $\mathrm{S}(P,Q,R)$ is *homogeneous of degree n* if

$$\begin{aligned}
P(\alpha x,\alpha y,\alpha z)&=\alpha^n P(x,y,z),\\
Q(\alpha x,\alpha y,\alpha z)&=\alpha^n Q(x,y,z),\\
R(\alpha x,\alpha y,\alpha z)&=\alpha^n R(x,y,z)
\end{aligned}$$

for every $\alpha\in\mathbb{R}$. This is equivalent to verifying that all the monomials that constitute S are of degree n. All homogeneous polynomial differential systems belong to the set of QH systems. In order to verify this property, it is enough to take $(1,1,1,n)$, or any of its multiples, as weight vector. We will now see that the reciprocal is also true, whereby the homogeneous systems of degree n are totally determined as those having $(1,1,1,n)$ as a weight vector.

Proposition 7. *If the system* $\mathrm{S}(P,Q,R)$ *is QH and* $\mathbf{w}=(s,s,s,d)$ *is a weight vector of* S, *then the system is homogeneous.*

Proof. If n is the degree of S, we can apply Corollary 2 to the weight vector $\mathbf{w}$ and $k=n$ to obtain $(n-1)s=d-1$, so $s-1+d=ns$. Therefore, from (2) we get

$$\begin{aligned}
P(\alpha^s x,\alpha^s y,\alpha^s z)&=\alpha^{ns} P(x,y,z),\\
Q(\alpha^s x,\alpha^s y,\alpha^s z)&=\alpha^{ns} Q(x,y,z),\\
R(\alpha^s x,\alpha^s y,\alpha^s z)&=\alpha^{ns} R(x,y,z).
\end{aligned}\tag{18}$$

Now we observe that any positive parameter β can be written in the form α^s just taking $\alpha = \beta^{\frac{1}{s}} > 0$ and therefore, by using (18), the system verifies $P(\beta x, \beta y, \beta z) = \beta^n P(x, y, z)$, $Q(\beta x, \beta y, \beta z) = \beta^n Q(x, y, z)$ and $R(\beta x, \beta y, \beta z) = \beta^n R(x, y, z)$, that is, the system is homogeneous. □

The system constructed with all possible monomials of degree n is the only maximal homogeneous system of degree n. We will denote this system by $\mathsf{H_n}$. The only weight vector family of $\mathsf{H_n}$ is

$$F_{\mathsf{H_n}}(1,1) = \left\{(a, a, a, a(n-1)+1) : a \in \mathbb{Z}^+\right\},$$

being $\mathbf{w}_m = (1, 1, 1, n)$ the minimum weight vector. Again non-maximal homogeneous systems are very varied and generally have more weight vectors than the maximal one.

4. Bricks and compatibility

We will use the weight vectors to list the set of maximal inhomogeneous QH systems. With the aim of reduce the total number of cases, in what follows we will consider only those weight vectors (s_1, s_2, s_3, d) that verify

$$s_1 \geq s_2 \geq s_3. \tag{19}$$

Also, taking into account Proposition 7, and with the aim of determining just the inhomogeneous systems, we impose that the condition

$$s_1 > s_3 \tag{20}$$

must be also verified. That is, we are going to construct the maximal systems that have among their weight vectors someone satisfying (19) and (20). In this way we simplify our study, significantly reducing the number of systems to study. The rest of systems are symmetrical to those obtained with these restrictions, without doing more than permutations on the variables x, y and z. As an example, if we find system (5), whose weight vectors, the set $\left\{(3a, 2a, a, 3a+1) : a \in \mathbb{Z}^+\right\}$, verify the restrictions (19) and (20), we are automatically finding five more systems, as there are six possible permutations of the variables x, y and z. One of these systems is obtained by permuting the variables x and y, and has as weight vectors the set $\left\{(2a, 3a, a, 3a+1) : a \in \mathbb{Z}^+\right\}$:

$$\begin{aligned} \dot{x} &= x^2 z + y z^2 + xy, \\ \dot{y} &= xyz + x^3 + y^2, \\ \dot{z} &= x z^2 + x^2 + yz. \end{aligned}$$

In order to simplify the equations involved in the process, and taking into account the restrictions (19), we define the new variables $\bar{s_1}$, $\bar{s_2}$, $\bar{s_3}$ and $\bar{d}$ as follows:

$$\bar{s_1} = s_1 - s_3, \quad \bar{s_2} = s_2 - s_3, \quad \bar{s_3} = s_3, \quad \bar{d} = d - 1 + s_3. \tag{21}$$

It follows from (19), (20), (21) and from the fact that the weight vectors are made of positive integers, that these new variables verify the constraints

$$\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d} \in \mathbb{Z}, \tag{22}$$

$$\bar{s_1} \geq \bar{s_2} \geq 0, \tag{23}$$

$$\bar{s_1} > 0, \tag{24}$$

$$\bar{d} \geq \bar{s_3} > 0. \tag{25}$$

Given a weight vector $\mathbf{w} = (s_1, s_2, s_3, d)$ that satisfies (19) and (20), the new vector $\bar{\mathbf{w}} = (\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ obtained by the change of variables (21) is called the *transformed vector of* $\mathbf{w}$, and we denote the implicit bijection by $\bar{\mathbf{w}} = t(\mathbf{w})$. A transformed vector always verifies the conditions

(22), (23), (24) and (25). Reciprocally, given a transformed vector $\bar{\mathbf{w}} = (\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$, its corresponding weight vector $\mathbf{w} = t^{-1}(\bar{\mathbf{w}}) = (s_1, s_2, s_3, d)$ can be obtained from the transformation:

$$s_1 = \bar{s_1} + \bar{s_3}, \quad s_2 = \bar{s_2} + \bar{s_3}, \quad s_3 = \bar{s_3}, \quad d = \bar{d} - \bar{s_3} + 1. \tag{26}$$

The transformed vectors are also grouped into families. That is the weight vector family $F_{\mathsf{S}}(\lambda, \mu) = \left\{(as_1, as_2, as_3, a(d-1)+1) : a \in \mathbb{Z}^+\right\}$ is transformed by t into the set

$$\bar{F_{\mathsf{S}}}(\lambda, \mu) = \left\{\left(a\bar{s_1}, a\bar{s_2}, a\bar{s_3}, a\bar{d}\right) : a \in \mathbb{Z}^+\right\},$$

which is called the *transformed weight vector family of* $F_{\mathsf{S}}(\lambda, \mu)$. While a weight vector family was contained in a straight line of $\mathbb{R}^4$ passing through the point $(0, 0, 0, 1)$, a transformed weight vector family exists within a more simple unidimensional subspace of $\mathbb{R}^4$. As previously, a subspace of $\mathbb{R}^4$ of dimension 1 cannot contain two different transformed weight vector families.

The following result, an improved and simplified version of Corollary 2, will be an important tool in this work, and shows the close relationship between the monomials of a QH system and a certain type of homogeneous linear equations.

Proposition 8. *A maximal QH system* S *has a monomial of degree* k *if and only if there exist* $x_1, x_2 \in \{-1, 0, \dots, k\}$, $-1 \le x_1 + x_2 \le k$, *verifying*

$$x_1\bar{s_1} + x_2\bar{s_2} + k\bar{s_3} = \bar{d} \tag{27}$$

for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ *of* S.

Proof. Using of the change of variables (21), the equivalences of Proposition 1 can be rewritten for transformed vectors, as

$$\begin{aligned}
&a_{p_1p_2k-p_1-p_2} \neq 0 \Leftrightarrow (p_1-1)\bar{s_1} + p_2\bar{s_2} + k\bar{s_3} = \bar{d},\\
&b_{q_1q_2k-q_1-q_2} \neq 0 \Leftrightarrow q_1\bar{s_1} + (q_2-1)\bar{s_2} + k\bar{s_3} = \bar{d},\\
&c_{t_1t_2k-t_1-t_2} \neq 0 \Leftrightarrow t_1\bar{s_1} + t_2\bar{s_2} + k\bar{s_3} = \bar{d}.
\end{aligned}$$

Therefore, the proof is completed by simply taking into account that S is a maximal system and $p_i, q_i, t_i \in \{0, 1, \dots, k\}$ for $i = 1, 2$. □

As a consequence of Proposition 8, we have an interesting property of the maximal QH systems: certain monomials of the same degree k, belonging to each of the three homogeneous parts P_k, Q_k and R_k, are related. That is, if one of them appears in a given maximal system, the other two also appear.

Corollary 9. *Given a maximal QH system* $\mathsf{S}(P, Q, R)$ *of degree* n, $1 \le k \le n$, $x_1, x_2 \in \{0, 1, \dots, k-1\}$, $0 \le x_1 + x_2 \le k-1$, *then*

$$a_{x_1+1, x_2, k-x_1-x_2-1} \neq 0 \Leftrightarrow b_{x_1, x_2+1, k-x_1-x_2-1} \neq 0 \Leftrightarrow c_{x_1, x_2, k-x_1-x_2} \neq 0$$

Proof. By Proposition 8, the three inequalities of the statement are equivalents to the verification of equation (27) for every transformed vector of S. □

Fixed a degree k, some monomials of P_k, Q_k and R_k are not related with other monomials. They appear freely within the QH maximal systems. This happens because they have zeros as exponents of the variables x, y or z. These are the following:

1. Monomials of the homogeneous part P_k with exponent 0 in the variable x, as

$$a_{0p_2k-p_2}y^{p_2}z^{k-p_2} \quad (0 \le p_2 \le k),$$

which are present in the system when the equation $-\bar{s_1} + p_2\bar{s_2} + k\bar{s_3} = \bar{d}$ is verified for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$.

2. Monomials of the homogeneous part Q_k with exponent 0 in the variable y, as

$$a_{p_1 0 k-p_1} x^{p_1} z^{k-p_1} \quad (0 \le p_1 \le k),$$

which are present in the system when the equation $p_1\bar{s_1} - \bar{s_2} + k\bar{s_3} = \bar{d}$ is verified for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$.

3. Monomials of the homogeneous part R_k with exponent 0 in the variable z, as

$$a_{p_1 k-p_1 0} x^{p_1} y^{k-p_1} \quad (0 \le p_1 \le k),$$

which are present in the system when the equation $p_1\bar{s_1} + p_2\bar{s_2} + k\bar{s_3} = \bar{d}$, with $p_1 + p_2 = k$, is verified for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$.

From now on, when we speak of a *brick* of a system, we will refer to one of these sets of linked monomials, that are the simplest constituent elements of the maximal QH systems. We denote by $[x_1, x_2; k]$ the brick associated with the equation (27). Bricks can contribute to the system with three monomials, one in each of the components P, Q and R, or with only one if they are in any of the special situations studied before. The brick $[x_1, x_2; k]$ contributes to the component P with the monomial

$$a_{x_1+1, x_2, k-x_1-x_2-1} x^{x_1+1} y^{x_2} z^{k-x_1-x_2-1},$$

to the component Q with the monomial

$$b_{x_1, x_2+1, k-x_1-x_2-1} x^{x_1} y^{x_2+1} z^{k-x_1-x_2-1},$$

and to the component R with the monomial

$$c_{x_1, x_2, k-x_1-x_2} x^{x_1} y^{x_2} z^{k-x_1-x_2}.$$

Although as we said two of these monomials may be null. When necessary, we will summarize the contributions of $[x_1, x_2; k]$ with

$$(P, Q, R) = \left(x^{x_1+1} y^{x_2} z^{k-x_1-x_2-1}, x^{x_1} y^{x_2+1} z^{k-x_1-x_2-1}, x^{x_1} y^{x_2} z^{k-x_1-x_2} \right).$$

We denote by B_k the set of bricks of degree k, meaning *brick of degree k* those $[x_1, x_2; k]$ that contribute with monomials of such degree. From the constraints for x_1 and x_2 stated in Proposition 8 we deduce that given $k \in \mathbb{Z}^+$ we have

$$\mathsf{B}_\mathsf{k} = \left\{ [x_1, x_2; k] \,:\, x_1, x_2 \in \mathbb{Z} \quad \text{and} \quad -1 \le x_1, x_2, x_1 + x_2 \le k \right\}. \tag{28}$$

Table 1 shows the bricks of B_1 together with their associated equations and the contributions to the maximal systems in which they are present.

For a fixed degree k the set of bricks B_k can be represented graphically in the plane by simply taking into account the restrictions on the integers x_1, x_2, which will act as abscissa and ordinate in this graphic, respectively. In the region of the plane corresponding to B_3, shown in Fig. 1, it is observed that the bricks that contribute with a single monomial correspond to the border points of the region.

Proposition 10. *An inhomogeneous QH system of degree n can be constructed with*

$$\frac{n^3}{6} + 2n^2 + \frac{29}{6}n$$

different bricks.

Table 1
The set of bricks of degree 1 (B_1).

$[x_1, x_2; 1]$	$x_1\bar{s_1} + x_2\bar{s_2} + \bar{s_3} = \bar{d}$	(P, Q, R)
$[-1, 0; 1]$	$-\bar{s_1} + \bar{s_3} = \bar{d}$	$(z, 0, 0)$
$[0, -1; 1]$	$-\bar{s_2} + \bar{s_3} = \bar{d}$	$(0, z, 0)$
$[-1, 1; 1]$	$-\bar{s_1} + \bar{s_2} + \bar{s_3} = \bar{d}$	$(y, 0, 0)$
$[0, 0; 1]$	$\bar{s_3} = \bar{d}$	(x, y, z)
$[1, -1; 1]$	$\bar{s_1} - \bar{s_2} + \bar{s_3} = \bar{d}$	$(0, x, 0)$
$[0, 1; 1]$	$\bar{s_2} + \bar{s_3} = \bar{d}$	$(0, 0, y)$
$[1, 0; 1]$	$\bar{s_1} + \bar{s_3} = \bar{d}$	$(0, 0, x)$

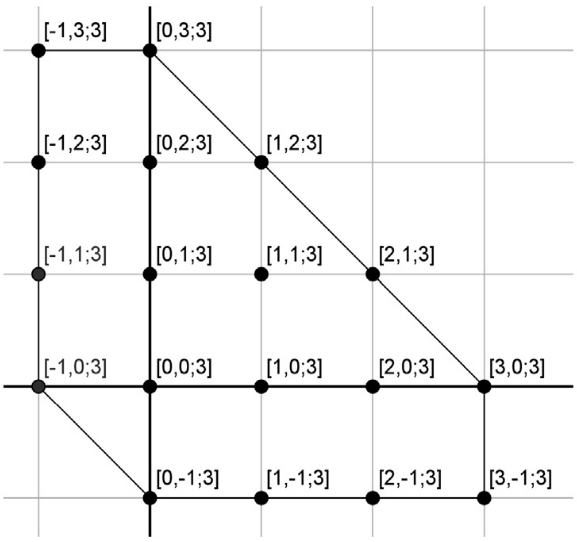

Fig. 1. B_3 region in the plane.

Proof. An inhomogeneous QH system of degree n can be constructed with the bricks of the sets B_k, $k \in \{1, ..., n\}$. The cardinal of B_k, which matches the number of bricks of H_k, is

$$2 + 3 + ... + (k + 2) + (k + 1) = \frac{(k+1)(k+6)}{2}.$$

Then, the total number of available bricks is

$$\sum_{k=1}^{n} \frac{(k+1)(k+6)}{2} = \frac{n^3}{6} + 2n^2 + \frac{29}{6}n. \quad \square$$

As an example, the maximal system (5) is split into the bricks that compose it. This QH system is built of five bricks with degrees running from 2 to 3.

$\dot{x}$	$=$	$+xyz$		$+y^3$	$+x^2$	
$\dot{y}$	$=$	$+y^2z$	$+xz^2$		$+xy$	
$\dot{z}$	$=$	$+yz^2$			$+xz$	$+y^2$
Brick:		$[0, 1; 3]$	$[1, -1; 3]$	$[-1, 3; 3]$	$[1, 0; 2]$	$[0, 2; 2]$

In the next two results we will study when two given bricks can coexist in the same QH system, that is, their compatibility. It is obvious that in this sense there must be restrictions. Otherwise, there would only be one possible maximal QH system of degree n, constituted by the $\frac{n^3}{6} + 2n^2 + \frac{29}{6}n$ bricks mentioned in Proposition 10. We will start by analyzing the compatibility of a pair of bricks, understanding for *compatible bricks* those that can coexist in a QH system, and *incompatible* those that cannot do so under any circumstances. Note that brick compatibility is not equivalent to the compatibility of their respective associated equations (27), because the equations can have common solutions $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$, but without satisfying the conditions (22), (23), (24) and (25).

Proposition 11. *Let $x_1, x_2, y_1, y_2, k, p \in \mathbb{Z}$ be such that $-1 \le x_1, x_2, x_1 + x_2 \le k$, $-1 \le y_1, y_2, y_1 + y_2 \le p$ and $0 < p < k$. The bricks $[x_1, x_2; k]$ and $[y_1, y_2; p]$ are compatible in an inhomogeneous QH system if and only if $Y_1 > 0$, or $Y_1 + Y_2 > 0$, being $Y_i = y_i - x_i$, $i = 1, 2$.*

Proof. First we note that the compatibility of $[x_1, x_2; k]$ and $[y_1, y_2; p]$ within the same inhomogeneous QH system means that there is some transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ that satisfy the system

$$x_1\bar{s_1} + x_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0,$$
$$y_1\bar{s_1} + y_2\bar{s_2} + p\bar{s_3} - \bar{d} = 0.$$

This system has the infinite set of solutions

$$\{(\alpha,\ \beta,\ (Y_1\alpha + Y_2\beta)/(k-p),\ (X_1\alpha + X_2\beta)/(k-p))\ :\ \alpha, \beta \in \mathbb{R}\},$$

where $Y_i = y_i - x_i$ and $X_i = ky_i - px_i$ for $i = 1, 2$. But the solutions that count for compatibility are those being a transformed vector, i.e., those verifying (22), (23), (24) and (25).

We start by proving that the compatibility implies $Y_1 > 0$ or $Y_1 + Y_2 > 0$, that is, if $Y_1 \le 0$ and $Y_1 + Y_2 \le 0$ then there are no values of α and β in the conditions of statements (22), (23), (24) and (25). Since $k - p$ is positive, it is sufficient to prove that $Y_1\alpha + Y_2\beta$ is negative or zero, so consequently statement (25) does not hold. If $Y_1 \le 0$ and also $Y_2 \le 0$, then $Y_1\alpha + Y_2\beta \le 0$ for all $\alpha, \beta \in \mathbb{N}$. Otherwise, if $Y_1 \le 0$ and $Y_2 > 0$, from condition $Y_1 + Y_2 \le 0$ it is deduced that $Y_1 \le -Y_2$ and since $\beta - \alpha \le 0$, it follows that $Y_1\alpha + Y_2\beta \le (-Y_2)\alpha + Y_2\beta = Y_2(\beta - \alpha) \le 0$ for every $\alpha, \beta \in \mathbb{N}$.

To study the reciprocal implication we distinguish two cases:

Case $Y_1 > 0$. Let $\alpha = k - p$ and $\beta = 0$, so we obtain the solution $(k - p, 0, Y_1, X_1)$. It is clear that conditions (22), (23), (24) are verified with these values, and also that $\bar{s_3} = Y_1 > 0$. To prove $\bar{d} \ge \bar{s_3}$, note that $\bar{d} - \bar{s_3} = X_1 - Y_1 = (k-1)\,y_1 - (p-1)\,x_1$ and that $Y_1 > 0$ implies $y_1 > x_1 \ge -1$, so $y_1 \ge 0$. All this, together with $0 \le p - 1 < k - 1$, means that $(k-1)\,y_1 \ge (p-1)\,x_1$, so $\bar{d} \ge \bar{s_3}$.

Case $Y_1 + Y_2 > 0$. Now we set $\alpha = \beta = k - p$, obtaining the solution $(k - p, k - p, Y_1 + Y_2, X_1 + X_2)$, and the proof is identical as in the previous case, although now $\bar{d} - \bar{s_3} = (X_1 + X_2) - (Y_1 + Y_2) = (k-1)\,(y_1 + y_2) - (p-1)\,(x_1 + x_2)$, and $Y_1 + Y_2 > 0$ implies $y_1 + y_2 > x_1 + x_2 \ge -1$, so $\bar{d} \ge \bar{s_3}$. □

Note that, in addition to the transformed vector obtained in the proof, there is not a single family of transformed vectors, but there are infinitely many families, contained in a two-dimensional vector subspace of $\mathbb{R}^4$.

We have studied the compatibility for two bricks of distinct degrees. On the other hand, two bricks of the same degree k are always compatible; even all bricks of degree k are mutually compatible, giving rise to the maximal homogeneous system H_k. But this is a compatibility that, in the search for exclusively inhomogeneous systems, we are not interested in. The following result establishes the conditions of compatibility between bricks of the same degree in the case of inhomogeneous systems. Note that previous papers (García et al., 2013, 2018) have shown that the mentioned type of compatibility does not exist in two-dimensional QH systems.

Proposition 12. *Let $x_1, x_2, y_1, y_2, k \in \mathbb{Z}$ be with $-1 \le x_1, x_2, y_1, y_2, x_1 + x_2, y_1 + y_2 \le k$, and $k > 1$. Two different bricks $[x_1, x_2; k]$ and $[y_1, y_2; k]$ are compatible in an inhomogeneous QH system if and only if $Y_1 = 0$, or $-Y_2/Y_1 \ge 1$, being $Y_i = y_i - x_i$ for $i = 1, 2$.*

Proof. First we note that the compatibility of $[x_1, x_2; k]$ and $[y_1, y_2; k]$ within the same inhomogeneous QH system means that there is some transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ that satisfy the system

$$x_1\bar{s_1} + x_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0,$$
$$y_1\bar{s_1} + y_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0.$$

The matrix expression of the system, after Gaussian transformations, adopts the form

$$\begin{pmatrix} x_1 & x_2 & k & -1 \\ Y_1 & Y_2 & 0 & 0 \end{pmatrix}.$$

The bricks are not the same, so it must be verified that $Y_1^2 + Y_1^2 \neq 0$. As a conclusion the rank of the system is 2, and the set of solutions is

$$\{(Y_2\alpha,\ -Y_1\alpha,\ \beta,\ (Y_2x_1 - Y_1x_2)\alpha + k\beta)\ :\ \alpha, \beta \in \mathbb{R}\}, \tag{29}$$

where $Y_i = y_i - x_i$ for $i = 1, 2$.

We start by proving that the compatibility of the two bricks implies $Y_1 = 0$, or $-Y_2/Y_1 \geq 1$, or equivalently, that when $-Y_2/Y_1 < 1$ is verified, being $Y_1 \neq 0$, then the bricks are incompatible. The solutions $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ have to be of the form (29), so when $-Y_2/Y_1 < 1$ is verified we have $|\bar{s_2}| > |\bar{s_1}|$. In that case $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ cannot be a transformed vector, and therefore there is no compatibility between the bricks.

To prove the reciprocal implication we distinguish two cases:

Case $Y_1 = 0$. The solutions are in this case

$$\{(Y_2\alpha,\ 0,\ \beta,\ Y_2x_1\alpha + k\beta)\ :\ \alpha, \beta \in \mathbb{R}\},$$

where $Y_2 \neq 0$. Setting $\alpha = 1/Y_2$ and $\beta = 1$ we obtain the solution $(1, 0, 1, x_1 + k)$. Taking into account that $x_1 \geq -1$ and that $k > 1$, the solution is a vector that satisfies the requirements (22), (23), (24) and (25), so it is a transformed vector.

Case $-Y_2/Y_1 \geq 1$. Note that in this case $Y_2 \neq 0$ and $0 < -Y_1/Y_2 \leq 1$. To find a transformed vector, we will take from the solutions (29) the case $\alpha = 1/Y_2$, $\beta = 1$:

$$(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d}) = \left(1, \frac{-Y_1}{Y_2}, 1, x_1 - \frac{Y_1}{Y_2}x_2 + k\right).$$

This vector trivially satisfies the conditions (23) and (24) and fulfills $\bar{s_3} > 0$ of condition (25). Therefore we only need to prove $\bar{d} \geq \bar{s_3}$, that is,

$$x_1 - \frac{Y_1}{Y_2}x_2 + k \geq 1.$$

If $x_2 \geq 0$, as $x_1 \geq 1$ and $k \geq 2$, then $x_1 - Y_1x_2/Y_2 + k \geq 1 + (-Y_1/Y_2)x_2 \geq 1$.

In the case $x_2 = -1$, then $x_1 \geq 0$ because $x_1 + x_2 \geq -1$, and as well $k \geq 2$. Then $x_1 - Y_1x_2/Y_2 + k = x_1 - (-Y_1/Y_2) + k \geq 2 - (-Y_1/Y_2) \geq 1$.

Finally if the vector obtained is not formed by integers, it is sufficient to multiply it by $|Y_2|$ to meet (22) and accordingly obtain a transformed vector. □

Remark 13. Proposition 12 does not study the compatibility between pairs of bricks of degree 1, that is, the bricks of B_1. If we study case by case all the possible compatibilities, we observe that here are some special situations that slightly modify the previous result. In general, Proposition 12 is true, but with the following exceptions:

1. The brick $[-1, 0; 1]$ has as associated equation $-\bar{s_1} + \bar{s_3} = \bar{d}$, which is incompatible with the requirements (24) and (25). Therefore, this brick can never exist in a QH system.
2. The compatibility between the bricks $[0, -1; 1]$ and $[-1, 1; 1]$ is not verified, contrary to what would be deduced from the application of Proposition 12.

5. w_m as a unique identifier of maximal QH systems

Lemma 14. *Given a brick* $[x_1, x_2; k]$ *belonging to* B_k*, it is satisfied that:*

(i) $[x_1, x_2+1; k]$, $[x_1, x_2-1; k]$*, or both belong to* B_k*.*
(ii) $[x_1+1, x_2-1; k]$, $[x_1-1, x_2+1; k]$*, or both belong to* B_k*.*

Proof. Let $k \in \mathbb{Z}^+$ be and $[x_1, x_2; k] \in \mathsf{B}_\mathsf{k}$.

From the definition (28) we know that the bricks of B_k are exactly those $[p, q; k]$ that verify the conditions

$$(a)\ p, q \in \mathbb{Z}, \quad (b)\ -1 \le p \le k, \quad (c)\ -1 \le q \le k, \quad (d)\ -1 \le p+q \le k.$$

(i) Let $[x_1, x_2; k] = [x_1, -1; k] \in \mathsf{B}_\mathsf{k}$. Then (a) and (d) imply $0 \le x_1 \le k$, so $[x_1, x_2+1; k] = [x_1, 0; k] \in \mathsf{B}_\mathsf{k}$.

On the other hand, if $[x_1, x_2; k] \in \mathsf{B}_\mathsf{k}$, being $0 \le x_2 \le k$, then $[x_1, x_2-1; k]$ belongs to B_k, except if $x_1 = -1$ and $x_2 = 0$. But in such a case $[x_1, x_2+1; k] = [-1, 1; k] \in \mathsf{B}_\mathsf{k}$.

(ii) Let $[x_1, x_2; k] \in \mathsf{B}_\mathsf{k}$ where $x_1 > x_2$. Then, by (b) and (c), $0 \le x_1 \le k$ and $-1 \le x_2 \le k-1$. As a consequence $[x_1-1, x_2+1; k]$ verifies (a), (b), (c) and (d), so $[x_1-1, x_2+1; k] \in \mathsf{B}_\mathsf{k}$.

In a similar way we can prove that $[x_1+1, x_2-1; k] \in \mathsf{B}_\mathsf{k}$ when $x_1 < x_2$.

Now let $[x_1, x_2; k] \in \mathsf{B}_\mathsf{k}$ being $x_1 = x_2$. By (d), we have that $-1 \le 2x_1,\ 2x_2 \le k$, so $0 \le x_1,\ x_2 \le \frac{k}{2}$, and being $k \ge 1$, we conclude $0 \le x_1,\ x_2 \le k-1$. Thus $[x_1+1, x_2-1; k]$, $[x_1-1, x_2+1; k] \in \mathsf{B}_\mathsf{k}$. □

Theorem 15. *A maximal QH system has a unique weight vector family.*

Proof. The case of maximal homogeneous systems has already been studied in section 3. The maximal homogeneous system of degree n, H_n, only has the weight vector family $F_{\mathsf{H}_\mathsf{n}}(1, 1)$.

We will therefore analyze the case of inhomogeneous maximal systems. An inhomogeneous QH system of degree n must have at least two compatible bricks: one is of the degree of the system n, $[x_1, x_2; n]$, and another of degree $m < n$, $[y_1, y_2; m]$. Being compatible, we know from Proposition 11 that either $Y_1 > 0$ or $Y_1 + Y_2 > 0$ must occur, with $Y_i = y_i - x_i$ for $i = 1, 2$. Distinguishing these two cases we will show that there is always a third brick compatible with the two previous ones, so that it can be added to the system. In addition, we will see that any maximal system formed from these three bricks has a unique family of weight vectors.

Case $Y_1 > 0$: By Lemma 14 we know that $[x_1, x_2+1; n]$, $[x_1, x_2-1; n]$, or both, belong to B_n. We suppose that $[x_1, x_2+1; n] \in \mathsf{B}_\mathsf{n}$, because the other case is identical, and we will see that $[x_1, x_2+1; n]$ is compatible with $[x_1, x_2; n]$ and $[y_1, y_2; m]$. The system of equations associated with these three bricks is

$$\begin{aligned} x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} &= 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} &= 0, \\ x_1\bar{s_1} + (x_2+1)\,\bar{s_2} + n\bar{s_3} - \bar{d} &= 0. \end{aligned} \tag{30}$$

Solving it we observe that it is a system of rank 3 with the following infinite set of solutions of dimension 1:

$$\left\{\left(\alpha,\ 0,\ \frac{Y_1}{n-m}\alpha,\ \left(\frac{n}{n-m}Y_1 + x_1\right)\alpha\right)\ :\ \alpha \in \mathbb{R}\right\}.$$

Using the inverse transformation (26), we obtain the solution set

$$\left\{\left(\left(\frac{Y_1}{n-m}+1\right)\alpha,\ \frac{Y_1}{n-m}\alpha,\ \frac{Y_1}{n-m}\alpha,\ \left(\frac{n-1}{n-m}Y_1 + x_1\right)\alpha + 1\right)\ :\ \alpha \in \mathbb{R}\right\},$$

of the corresponding system (30) in the variables s_1, s_2, s_3 and d.

This space of solutions has dimension 1, and therefore can contain at most one weight vector family. We will see that it contains that family: if we set $\alpha = n - m$, the solution of the obtained system is a weight vector. This solution is

$$(s_1, s_2, s_3, d) = (Y_1 + n - m,\ Y_1,\ Y_1,\ (n-1)\,Y_1 + (n-m)\,x_1 + 1)\,,$$

whose components are integers. In order to (s_1, s_2, s_3, d) be a weight vector, we have to check that $s_1 \geq s_2 \geq s_3 > 0$ and $d > 0$. Since $Y_1 > 0$ and $n > m$, the first is evident. To prove $d > 0$, we observe that being $n > m \geq 1$, $Y_1 > 0$ and $x_1 \geq -1$, the unique problematic case could appear when $x_1 = -1$. When it happens, we have that $d = (n-1)\,(y_1+1) - (n-m) + 1 = (n-1)\,y_1 + m > 0$, because $y_1 > x_1 \geq -1$.

It is possible to ask, since we deal with maximal systems, if adding a fourth brick, and therefore another equation to system (30), can reduce the number of solutions, thus losing the obtained family.

If we add the brick $[z_1, z_2; p]$ we have the system of equations in the variables $\bar{s_1}$, $\bar{s_2}$, $\bar{s_3}$ and $\bar{d}$

$$\begin{aligned}
x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} &= 0,\\
y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} &= 0,\\
x_1\bar{s_1} + (x_2+1)\,\bar{s_2} + n\bar{s_3} - \bar{d} &= 0,\\
z_1\bar{s_1} + z_2\bar{s_2} + p\bar{s_3} - \bar{d} &= 0.
\end{aligned}$$

If this new system of equations maintains the rank 3, it has the same solutions as (30) and therefore the corresponding QH system has a unique family of vectors.

If the system reaches rank 4, it would be a determined compatible system with single solution $(0, 0, 0, 0)$, which cannot be a transformed vector because it does not satisfy (24) nor (25). Thus any system containing these four bricks can be QH. Therefore as we add bricks to the system while maintaining the quality of being QH, we maintain the rank 3 in the corresponding systems of equations, and also maintain the existence of a unique family of weight vectors.

Case $Y_1 + Y_2 > 0$: This case is proved almost identically to the case $Y_1 > 0$, so we will only point out the differences. Now the third brick will be $[y_1+1, y_2-1; m]$ or $[y_1-1, y_2+1; m]$, depending on which of the two exists (see Lemma 14). If we assume that the first one exists, the system of equations is

$$\begin{aligned}
x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} &= 0,\\
y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} &= 0,\\
(y_1+1)\,\bar{s_1} + (y_2-1)\,\bar{s_2} + m\bar{s_3} - \bar{d} &= 0,
\end{aligned}$$

whose solutions are

$$\left\{\left(\alpha,\ \alpha,\ \frac{Y_1+Y_2}{n-m}\alpha,\ \left(\frac{n}{n-m}\,(Y_1+Y_2) + x_1 + x_2\right)\alpha\right)\ :\ \alpha \in \mathbb{R}\right\}.$$

Using the inverse transformation (26), we obtain the solutions in the variables s_1, s_2, s_3 and d:

$$\left\{\left((A+1)\,\alpha,\ (A+1)\,\alpha,\ A\alpha,\ \left(\frac{n-1}{n-m}\,(Y_1+Y_2) + x_1 + x_2\right)\alpha + 1\right)\ :\ \alpha \in \mathbb{R}\right\},$$

where $A = (Y_1+Y_2)\,/\,(n-m)$.

As in the previous case if we take $\alpha = n - m$ we obtain the particular solution

$$\begin{aligned}
&s_1 = s_2 = Y_1 + Y_2 + n - m,\\
&s_3 = Y_1 + Y_2,\\
&d = (n-1)\,(Y_1+Y_2) + (n-m)\,(x_1+x_2) + 1.
\end{aligned}$$

This solution consists of integers, verifies $s_1 \geq s_2 \geq s_3 > 0$ and verifies $d > 0$ providing that $Y_1 + Y_2 > 0$ implies $y_1 + y_2 > x_1 + x_2 \geq -1$. Therefore, it is a weight vector. □

Corollary 16. *A maximal QH system is made up of at least three bricks.*

Proof. As we have seen in Proposition 10, the homogeneous maximal system H_n has exactly $(k+1)(k+6)/2$ bricks, which exceeds 2 for every $n > 0$.

An inhomogeneous maximal QH system must have at least two bricks of different degrees. In the proof of Theorem 15 it is showed that there is always another compatible brick that can be added to the system. □

Corollary 17. *A maximal QH system always has a minimum weight vector of the system.*

Proof. Let S be the system, and $F_\mathsf{S}(\lambda, \mu)$ its unique family of weight vectors. Then, $\mathbf{w}_m$=$\mathbf{g}_{(\lambda,\mu)}$. □

Corollary 18. *Given a maximal QH system* S*, a weight vector* (s_1, s_2, s_3, d) *is the minimum weight vector of* S *if and only if* $\gcd(s_1, s_2, s_3) = 1$.

Proof. The proof is an easy consequence of Theorem 15 and Proposition 6. □

An important consequence of Theorem 15 is that a weight vector of a maximal QH system S verifies conditions (19) and (20) if and only if all other vectors of S verify them. Because of this we avoid the possibility that, when filtering with (19) and (20), we did not consider other vectors of S (for example $(1, 2, 3, 4)$) whose presence would imply a modification in the algebraic structure of the system.

Proposition 19. *Two different maximal QH systems of degree n have no common weight vectors.*

Proof. Let S and T be two maximal QH systems of degree n that share the weight vector $\mathbf{w} = (s_1, s_2, s_3, d)$. Let $F_\mathsf{S}(\lambda, \mu)$ and $F_\mathsf{T}(\lambda, \mu)$ be the vector families to which $\mathbf{w}$ belongs in S and T respectively. We will show that S and T are the same system, proving that any monomial of S is contained in T and vice versa.

Let (12) be a monomial of S. Proposition 1 assures that $\mathbf{w}$ verifies the equation of the right side of (8). Since, by Theorem 15, all the weight vectors of T are those of $F_\mathsf{T}(\lambda, \mu)$, so being of the form $(rs_1, rs_2, rs_3, r(d-1)+1)$ with r a rational number, it is trivial that all of them also verify the equation (8). Therefore, due again to Proposition 1, the monomial (12) must be present in T. In the same way, we can show that all the monomials of T are in S, so S and T match. □

It should be noted that if we vary the degree n of the system, then the weight vectors could be repeated, and therefore also the families. For example, the weight vector $\mathbf{w} = (2, 2, 1, 2)$ and their corresponding family appears in the following maximal system of degree 2

$$\begin{aligned}\dot{x} &= xz + yz,\\ \dot{y} &= xz + yz,\\ \dot{z} &= z^2 + x + y,\end{aligned}$$

and also in the following maximal system of degree 3

$$\begin{aligned}\dot{x} &= z^3 + xz + yz,\\ \dot{y} &= z^3 + xz + yz,\\ \dot{z} &= z^2 + x + y.\end{aligned}$$

Remark 20. As a consequence of the previous results we have that, given a maximal QH system S of degree n, its minimum weight vector $\mathbf{w}_m$ always exists and is a unique identifier of S within the set of maximal QH systems of degree n.

6. Constructing the set of maximal inhomogeneous QH systems

By Corollary 16, we know that every inhomogeneous maximal QH system contains a minimum of three bricks. In addition, Theorem 15 assures that the system has a single family of weight vectors. Therefore given an n-degree system S of this type, we can always choose three bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$ in such a way that:

(a) At least one of them is of the same degree as S, but they do not have all the same degree, i.e., $1 \le m < n$, $1 \le k \le n$.
(b) Their respective associated equations form a system of rank 3, and therefore its solution space in $\mathbb{R}^4$ has dimension 1.
(c) The bricks are compatible. That is, some of the solutions of (b) verify the constraints (22), (23), (24) and (25).

When three bricks of S satisfy these requirements we say that they form a *seed of* S. With requirement (a) we force that the three bricks belong to an inhomogeneous system of degree n. From conditions (b) and (c) it follows that the compatibility between the three bricks is reduced to a single family of weight vectors. Consequently two distinct maximal systems cannot share a common seed, although they may share three or more bricks that do not form a seed.

A maximal system always has at least one seed, although it usually owns more. In a later theorem we will show that if we have a seed of a system S, the rest of the system and its corresponding family of vectors are totally determined. Thus, all the information about the algebraic structure of a maximal inhomogeneous QH system is encoded in any of its seeds. So we will start by identifying them in the following result.

Theorem 21. *The bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$, with $1 \le m < n$ and $1 \le k \le n$, form a seed of some n-degree maximal inhomogeneous QH system* S *if and only if the following three conditions hold:*

(i) $T_1 \cdot T_2 \le 0$ *and* $|T_1| \le |T_2|$,
(ii) $T_2 \begin{bmatrix} Y_1 & Y_2 \\ T_1 & T_2 \end{bmatrix} > 0$,
(iii) $\dfrac{Y_1 T_2 - Y_2 T_1}{(n-m)\,T_2} \ge \dfrac{x_2 T_1 - x_1 T_2}{(n-1)\,T_2}$,

where $Y_i = y_i - x_i$, $T_i = (k-m)\,x_i + (n-k)\,y_i + (m-n)\,z_i$, *for* $i = 1, 2$.

Proof. The equations associated with these three bricks form the homogeneous linear system

$$\begin{aligned} x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} &= 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} &= 0, \\ z_1\bar{s_1} + z_2\bar{s_2} + k\bar{s_3} - \bar{d} &= 0. \end{aligned} \tag{31}$$

After Gaussian transformations the system can be represented in its matrix form as

$$\begin{pmatrix} x_1 & x_2 & n & -1 \\ Y_1 & Y_2 & m-n & 0 \\ T_1 & T_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{s_1} \\ \bar{s_2} \\ \bar{s_3} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

The set of solutions obtained solving this system is

$$\left(T_2\alpha,\ -T_1\alpha,\ \frac{Y_1T_2 - Y_2T_1}{n-m}\alpha,\ \left(\frac{n\,(Y_1T_2 - Y_2T_1)}{n-m} + x_1T_2 - x_2T_1 \right)\alpha \right), \tag{33}$$

where $\alpha \in \mathbb{R}$.

In order to prove the necessary condition ($\Rightarrow$), we suppose that the three bricks form a seed. Then, by (c), some of the solutions of the set (33) verify (22), (23), (24) and (25). Let $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ be one of these solutions, i.e., a transformed vector obtained by setting a particular $\bar{\alpha} \in \mathbb{R}$ in (33). Consequently, $\bar{s_1} \geq \bar{s_2} \geq 0$ and $\bar{s_1} > 0$, so (i) must be verified. Also, $\bar{s_1}$ and $\bar{s_3}$ have the same sign, so being $n - m > 0$, we get (ii). Finally, $\bar{d} \geq \bar{s_3}$, and then

$$\bar{\alpha}\left(\frac{n}{n-m}\left(Y_1T_2 - Y_2T_1\right) + x_1T_2 - x_2T_1\right) \geq \frac{\bar{\alpha}}{n-m}\left(Y_1T_2 - Y_2T_1\right).$$

As $\bar{\alpha}$ and T_2 are not null and have the same sign (because $\bar{s_1} > 0$), the previous inequality is equivalent to

$$\frac{n}{n-m} \cdot \frac{Y_1T_2 - Y_2T_1}{T_2} + \frac{x_1T_2 - x_2T_1}{T_2} \geq \frac{Y_1T_2 - Y_2T_1}{(n-m)\,T_2},$$

and from this by simple algebraic operations we obtain

$$\frac{Y_1T_2 - Y_2T_1}{(n-m)\,T_2} \geq \frac{x_2T_1 - x_1T_2}{(n-1)\,T_2},$$

so (iii) is proved.

To prove the sufficient condition ($\Leftarrow$) we must verify (a), (b) and (c) of the seed definition. As $n > m$, (a) is fulfilled trivially. Due to $T_2 \neq 0$ and $n \neq m$, the system has rank 3, as can be seen just checking the matrix of (32). Also, as a homogeneous system of equations, it has solutions. Therefore (b) is true. Now let $T_2 > 0$ be. Setting $\alpha = n - m$ we get the following particular solution of the set (33):

$$\bar{s_1} = T_2\,(n-m),$$
$$\bar{s_2} = -T_1\,(n-m),$$
$$\bar{s_3} = Y_1T_2 - Y_2T_1,$$
$$\bar{d} = n\,(Y_1T_2 - Y_2T_1) + (n-m)\,(x_1T_2 - x_2T_1).$$

This solution verifies (22), (23), (24) and (25), then (c) holds. If $T_2 < 0$ we set $\alpha = m - n$ with the same conclusion. □

Theorem 22. *If the bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$, with $1 \leq m < n$ and $1 \leq k \leq n$, form a seed of an n-degree maximal inhomogeneous QH system* S*, then*

(i) The brick $[t_1, t_2; l]$ with $1 \leq l \leq n$, belongs to the system S *if and only if $T_1R_2 = T_2R_1$*

(ii) The minimum weight vector of S *is $\mathbf{w}_m = \left(\frac{\hat{s_1}}{G}, \frac{\hat{s_2}}{G}, \frac{\hat{s_3}}{G}, \frac{\tilde{d}}{G} + 1\right)$*

where

$Y_i = y_i - x_i$, *for* $i = 1, 2$,
$T_i = (k-m)\,x_i + (n-k)\,y_i + (m-n)\,z_i$, *for* $i = 1, 2$,
$R_i = (l-m)\,x_i + (n-l)\,y_i + (m-n)\,t_i$, *for* $i = 1, 2$,
$\hat{s_1} = |Y_1T_2 - Y_2T_1| + (n-m)\,|T_2|$,
$\hat{s_2} = |Y_1T_2 - Y_2T_1| + (n-m)\,|T_1|$,
$\hat{s_3} = |Y_1T_2 - Y_2T_1|$,
$\tilde{d} = (n-1)\,|Y_1T_2 - Y_2T_1| + \delta\,(n-m)\,(x_1T_2 - x_2T_1)$,
$\delta = \operatorname{sgn}(T_2)$,
$G = \gcd\left(\hat{s_1}, \hat{s_2}, \hat{s_3}\right)$.

Proof. (i) Since $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$ form a seed of S, we know that its associated system of equations (31) has rank 3. A fourth brick $[t_1, t_2; l]$ also belongs to S if and only if the increased system

$$\begin{aligned} x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} = 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} = 0, \\ z_1\bar{s_1} + z_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0, \\ t_1\bar{s_1} + t_2\bar{s_2} + l\bar{s_3} - \bar{d} = 0, \end{aligned} \tag{34}$$

which includes the equation associated to $[t_1, t_2; l]$ maintains the rank 3, because of the rank of (34) goes up to 4 the system becomes compatible-determined with unique solution $(0, 0, 0, 0)$, and in this case S would not be QH because it would not have weight vectors. Then the matrix expression of system (34), after Gaussian transformations, adopts the form

$$\begin{pmatrix} x_1 & x_2 & n & -1 \\ Y_1 & Y_2 & m-n & 0 \\ T_1 & T_2 & 0 & 0 \\ R_1 & R_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{s_1} \\ \bar{s_2} \\ \bar{s_3} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{35}$$

with Y_i, T_i and R_i being the values of the statement for $i = 1, 2$. Finally the matrix of (35) has rank 3 if and only if $T_1 R_2 = T_2 R_1$.

(ii) As seen in (i) the solution space of all the equations associated with bricks of S is the same as the solution space of (31). That is, it is the set defined in (33). Within (33) there exists the transformed vectors of S. Applying to (33) the inverse transformation t^{-1} given in (26), we obtain the solutions corresponding to the equations in the variables s_1, s_2, s_3, and d, which are

$$\begin{aligned} s_1 &= \alpha\left(\frac{Y_1T_2 - Y_2T_1}{n-m} + T_2\right), \\ s_2 &= \alpha\left(\frac{Y_1T_2 - Y_2T_1}{n-m} - T_1\right), \\ s_3 &= \alpha\left(\frac{Y_1T_2 - Y_2T_1}{n-m}\right), \\ d &= \alpha\left((n-1)\frac{Y_1T_2 - Y_2T_1}{n-m} + x_1T_2 - x_2T_1\right) + 1, \end{aligned} \tag{36}$$

with $\alpha \in \mathbb{R}$ and Y_i, T_i being the values of the statement for $i = 1, 2$. The elements of (36) which are formed by positive integers are the weight vectors of the maximal QH system S. We now obtain a particular weight vector. For this we can take $\alpha = \operatorname{sgn}(T_2)(n-m)$. Taking into account that by Theorem 21 we know that $Y_1T_2 - Y_2T_1$ has the same sign of T_2, and also that when T_1 is not null it has the opposite sign of T_2, we obtain the weight vector of S, $\mathbf{w} = (\hat{s_1}, \hat{s_2}, \hat{s_3}, \hat{d})$, where

$$\begin{aligned} \hat{s_1} &= |Y_1T_2 - Y_2T_1| + (n-m)\,|T_2|, \\ \hat{s_2} &= |Y_1T_2 - Y_2T_1| + (n-m)\,|T_1|, \\ \hat{s_3} &= |Y_1T_2 - Y_2T_1|, \\ \hat{d} &= (n-1)\,|Y_1T_2 - Y_2T_1| + \delta\,(n-m)\,(x_1T_2 - x_2T_1) + 1, \end{aligned}$$

where $\delta = \pm 1$ with the same sign as T_2, i.e., $\delta = \operatorname{sgn}(T_2)$. By Proposition 5 and Corollary 18 we know how to obtain the minimum weight vector $\mathbf{w}_m$ of a maximal system from one of its weight vectors $\mathbf{w} = (\hat{s_1}, \hat{s_2}, \hat{s_3}, \hat{d})$, namely

$$\mathbf{w}_m = \left(\frac{\hat{s_1}}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})}, \frac{\hat{s_2}}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})}, \frac{\hat{s_3}}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})}, \frac{\hat{d} - 1}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})} + 1\right).$$

In this way the result is proved by setting $\tilde{d} = \hat{d} - 1$. □

Remark 23. Note that if the minimum weight vector of a system S is known, we have an alternative way other than Theorem 22 (*i*) to determine whether a brick $[t_1, t_2; l]$ is in S or not. It is enough to check if the minimum weight vector of S verifies the equation associated with the brick, that is

$$t_1 s_1 + t_2 s_2 + (l - t_1 - t_2 - 1)s_3 = d - 1.$$

In the same way that happens with the seeds, if we fix the degree of S the whole information about the algebraic structure of S is in its minimum weight vector. Then when the minimum weight vector has previously been calculated this is a faster method than the one proposed in Theorem 22 (*i*).

7. The algorithm

The objective of our algorithm is to find all the maximal QH systems of a certain degree n. The only homogeneous maximal system is H_n, so the algorithm is focused on the inhomogeneous. The algorithm is mainly based on Theorem 21, Theorem 22, and Remark 23. Briefly, any system of this type must have among its constituent bricks one or more seeds. Detected a seed, all information about the structure of the system can be obtained by using Theorem 22 and Remark 23. Therefore, our first goal is to get a list of all possible seeds of n-degree systems, for which we will use Theorem 21 with the help of Propositions 11 and 12. In this way we will obtain all the wanted systems, although there can be repetitions because two different seeds can generate the same system. We will avoid these repetitions using the minimum vector of each system as its unique identifier.

It is an algorithm that requires quite computation, which also grows notably when the degree n of the required systems increases. Because of this, a basic principle of design has been to avoid unnecessary calculations. For this reason we have taken different steps that will be discussed later.

Algorithm 1: Determines whether two bricks are compatible.

Input: Two bricks: $B_i = [x_1^i, x_2^i; k^i]$, $B_j = [x_1^j, x_2^j; k^j]$
Output: *true* if the bricks are compatible, *false* otherwise

1 **Function** AreCompat(B_i, B_j)
2 $\quad Y_1 \leftarrow x_1^j - x_1^i$; $Y_2 \leftarrow x_2^j - x_2^i$
3 $\quad$ **if** $(k^i \neq k^j)$ *and* $(Y_1 > 0$ *or* $Y_1 + Y_2 > 0)$ **then**
4 $\quad\quad$ **return** *true*
5 $\quad$ **else if** $(k^i = k^j)$ *and* $(Y_1 = 0$ *or* $-Y_2/Y_1 \geq 1)$ **then**
6 $\quad\quad$ **return** *true*
7 $\quad$ **else**
8 $\quad\quad$ **return** *false*

The algorithm has a modular structure, and is formed by a main process together with four auxiliary functions. The criterion that we followed to extract computation from the main body to the functions has been to isolate those calculations that are repeatedly executed.

In order to facilitate its practical implementation, but at the same time to provide a tool as general as possible, we present the algorithm in pseudocode. The structure is highly detailed so that their later translation to any programming language will be simple.

The first function, AreCompat (B_i, B_j), receives two bricks and determines whether they are compatible or not. We try to avoid unnecessary computation by leaving the function as soon as possible. We do not consider the special cases of incompatibility stated in Remark 13, since these would fulfill their function of filtering in a very limited number of executions of the function, and in return we would have a remarkable computational cost. Besides, the few incompatible cases that are allowed to pass are subsequently filtered into other functions. We make use of Proposition 11 (Line 3) if both bricks are of different degree, and of Proposition 12 (Line 5) if the bricks are of the same degree. Note that in case of bricks of different degrees it is important the order of the inputs: the highest degree brick must be the first one.

Algorithm 2: Determines whether three bricks form a seed.

Input: Three bricks: $B_i = [x_1^i, x_2^i; k^i]$, $B_j = [x_1^j, x_2^j; k^j]$, $B_p = [x_1^p, x_2^p; k^p]$
Output: *true* if the bricks form a seed, *false* otherwise

1 **Function** areSeed(B_i, B_j, B_p)
2 $\quad T_2 \leftarrow (k^p - k^j)x_2^i + (k^i - k^p)x_2^j + (k^j - k^i)x_2^p$
3 $\quad$ **if** $T_2 = 0$ **then**
4 $\quad\quad$ **return** *false*
5 $\quad T_1 \leftarrow (k^p - k^j)x_1^i + (k^i - k^p)x_1^j + (k^j - k^i)x_1^p$
6 $\quad$ **if** $T_1 \cdot T_2 > 0$ *or* $|T_1| > |T_2|$ **then**
7 $\quad\quad$ **return** *false*
8 $\quad Y_1 \leftarrow x_1^j - x_1^i$; $Y_2 \leftarrow x_2^j - x_2^i$
9 $\quad$ **if** $T_2 \cdot (Y_1 T_2 - Y_2 T_1) \leq 0$ **then**
10 $\quad\quad$ **return** *false*
11 $\quad$ **if** $\dfrac{Y_1 T_2 - Y_2 T_1}{(k^i - k^j)\, T_2} < \dfrac{x_2 T_1 - x_1 T_2}{(k^i - 1)\, T_2}$ **then**
12 $\quad\quad$ **return** *false*
13 $\quad$ **return** *true*

Our second function, areSeed(B_i, B_j, B_p), determines whether the three bricks B_i, B_j, and B_p, form a seed or not. The function verifies the requirements stated in Theorem 21. It avoids unnecessary computation by exiting the function at the moment in which one of these conditions is not verified. The calculation of the values of Y_1, Y_2, T_1 and T_2 only takes place when it is indispensable. Condition (i) of Theorem 21 is checked in Lines 3 and 6; condition (ii) in Line 9; and condition (iii) in Line 11 of the function.

On the other hand, calculateWm(B_i, B_j, B_p) function accurately reproduces the assertion (ii) of Theorem 22. That is, it receives three bricks, which are known to form a seed of a maximal QH system S, and calculates the corresponding minimum weight vector of S.

Algorithm 3: Finds the minimum weight vector of a QH system.

Input: Three bricks that form a seed: $B_i = [x_1^i, x_2^i; k^i]$, $B_j = [x_1^j, x_2^j; k^j]$, $B_p = [x_1^p, x_2^p; k^p]$
Output: The minimum weight vector $\mathbf{w}_m$ corresponding to that seed

Function calculateWm(B_i, B_j, B_p)
$\quad Y_1 \leftarrow x_1^j - x_1^i$
$\quad Y_2 \leftarrow x_2^j - x_2^i$
$\quad T_1 \leftarrow (k^p - k^j)x_1^i + (k^i - k^p)x_1^j + (k^j - k^i)x_1^p$
$\quad T_2 \leftarrow (k^p - k^j)x_2^i + (k^i - k^p)x_2^j + (k^j - k^i)x_2^p$
$\quad \delta \leftarrow sign(T_2)$
$\quad \hat{s_1} \leftarrow |Y_1 T_2 - Y_2 T_1| + (k^i - k^j)|T_2|$
$\quad \hat{s_2} \leftarrow |Y_1 T_2 - Y_2 T_1| + (k^i - k^j)|T_1|$
$\quad \hat{s_3} \leftarrow |Y_1 T_2 - Y_2 T_1|$
$\quad \tilde{d} \leftarrow (k^i - 1)|Y_1 T_2 - Y_2 T_1| + \delta(k^i - k^j)(x_1^i T_2 - x_2^i T_1)$
$\quad G \leftarrow gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})$
$\quad \mathbf{w}_m \leftarrow (\dfrac{\hat{s_1}}{G}, \dfrac{\hat{s_2}}{G}, \dfrac{\hat{s_3}}{G}, \dfrac{\tilde{d}}{G} + 1)$
$\quad$ **return** $\mathbf{w}_m$

The last of the auxiliary functions is isBrickInSystem$(\mathbf{w}_m, B_q)$. As we have seen, the minimum vector of a maximal QH system stores all information about the algebraic structure of the system. This function receives the minimum vector $\mathbf{w}_m$ of a system S together with a brick B_q, and decides if B_q is present in S or not (Line 2). It is based on the result stated in Remark 23, because when we have the minimum vector, this method requires much less computation than the one exposed in assertion (i) of Theorem 22. This brief function has the last word in the process of construction of the maximal QH systems.

Algorithm 4: Determines if a brick is in a maximal QH system with a given minimum vector.

Input: The minimum weight vector $\mathbf{w}_m = (s_1, s_2, s_3, d)$ of a QH system and a brick $B_q = [x_1^q, x_2^q; k^q]$
Output: *true* if the brick is in the system, *false* otherwise

Function isBrickInSystem($\mathbf{w}_m$, B_q)
    if $x_1^q s_1 + x_2^q s_2 + (k^q - x_1^q - x_2^q - 1)s_3 = d - 1$ **then**
        return *true*
    else
        return *false*

The main process of the algorithm (see Algorithm 5) starts by asking the user for the only necessary input datum, that is, the degree n of the maximal QH systems that must be listed. With this datum we apply Proposition 10 to obtain two values: N (Line 1) is the total number of n-degree available bricks, that is, the cardinal of B_n; and T (Line 2) is the total number of bricks of degree less than or equal to n, that is, the cardinal of $\{\mathsf{B}_k\}_{k=1}^{n}$. To avoid repeating systems, we will store in the matrix Aux the minimum weight vector of each new system that we find.

Algorithm 5: Provides all QH systems of a given degree.

Input: Degree of the systems (n)
Output: List of all QH systems of degree n

1 $N \leftarrow \frac{(k+1)(k+6)}{2}$
2 $T \leftarrow \frac{n^3}{6} + 2n^2 + \frac{29}{6}n$
3 $Aux \leftarrow$ empty matrix of 4 columns
4 create ordered list of bricks $\{B_i\}_{i=1}^{T}$, where $B_i = \left[x_1^i, x_2^i; k^i\right]$
5 **for** $i \leftarrow 1$ **to** N **do**
6   **for** $j \leftarrow N+1$ **to** T **do**
7   **if** areCompat(B_i, B_j) **then**
8   **for** $p \leftarrow i+1$ **to** $j-1$ **do**
9   **if** areCompat(B_i, B_p) *and* areCompat(B_p, B_j) **then**
10   **if** areSeed(B_i, B_j, B_p) **then**
11                         $\mathbf{w}_m \leftarrow$ calculateWm(B_i, B_j, B_p)
12   **if** $\mathbf{w}_m$ *is not a row of Aux* **then**
13   add $\mathbf{w}_m$ as a new row of Aux
14   **for** $q \leftarrow 1$ **to** T **do**
15   **if** isBrickInSystem($\mathbf{w}_m$, B_q) **then**
16   add B_q to system S
17   output system S

The ordered list of the T available bricks is made in Line 4. These must be ordered so that we can go over them one by one. The only condition for this order is to place first the N bricks of maximum degree. Once this is done, the rest of the ordination details are irrelevant to the operation of the algorithm.

Subsequently, there is a series of nested *for* loops (Lines 5, 6 and 8) which are intended to run, without repetitions, all possible trios of bricks made up of at least one brick of degree n and at least one brick of degree less than n. Thus we assure two things: the degree of the systems obtained from these trios is n, and the systems are inhomogeneous.

Obviously not all brick's trios are a seed, so we check each of them by calling the function areSeed$\left(B_i, B_j, B_p\right)$ (Line 10). In fact, most of trios do not form a seed. So with the aim of saving computation of unnecessary calls to this expensive function, we include several conditional structures (Lines 7 and 9) in which we check every required compatibilities of pairs of bricks. This is done by means of function areCompat (B_i, B_j).

For each detected seed, we know that there is a maximal inhomogeneous QH system. But two different seeds can give rise to the same system, thus producing repetitions. To avoid them, we obtain the minimum weight vector associated with the seed by calling (Line 11) the function calculateWm (B_i, B_j, B_p). The easiest way to identify this type of systems is through its minimum weight vectors. If the minimum weight vector obtained is already in the matrix Aux, it means that this system has already been found before, in which case we discard it. Otherwise we insert the vector into Aux and go to the process of building the system.

The function isBrickInSystem $(\mathbf{w}_m, B_q)$ is called for each of the T available Bricks (Line 15), even for the three bricks which, by forming the seed, we know that they are in the system. It is a function with little computation, so establishing filters to avoid calling it in these three particular cases would have more computational cost than leaving it that way.

Every time the function isBrickInSystem $(\mathbf{w}_m, B_q)$ returns *true* implies that the brick $B_q = [x_1^q, x_2^q; k^q]$ belongs to the system S we are building. It follows that (see Line 16)

(i) The monomial $a_{x_1^q+1, x_2^q, k^q-x_1^q-x_2^q-1} x^{x_1^q+1} y^{x_2^q} z^{k^q-x_1^q-x_2^q-1}$ is added to the P component of S.

(ii) The monomial $b_{x_1^q, x_2^q+1, k^q-x_1^q-x_2^q-1} x^{x_1^q} y^{x_2^q+1} z^{k^q-x_1^q-x_2^q-1}$ is added to the Q component of S.

(iii) The monomial $c_{x_1^q, x_2^q, k^q-x_1^q-x_2^q} x^{x_1^q} y^{x_2^q} z^{k^q-x_1^q-x_2^q}$ is added to the R component of S.

We must take into account here the cases in which a brick contributes with a single monomial, studied previously in this work.

Finally being the construction process finished, the algorithm returns S in the format that is considered most appropriated (Line 17). This task is done recursively until the trios of bricks, and therefore the seeds and the possibility of forming new systems of these characteristics, are exhausted.

8. QH systems of degree 2

The following is the list of normal forms of 3-dimensional maximal inhomogeneous QH systems of degree 2, obtained by the algorithm described in the previous section. The minimum weight vector $\mathbf{w}_m$ of each system is also provided. Being $\mathbf{w}_m$ the generator of the unique weight vector family of each system, it is easy to obtain from it the remaining weight vectors. Since these are maximal systems, the coefficients a, b and c of the systems in the list can take any complex value other than zero.

$$\mathrm{S}_1: \begin{array}{l} \dot{x} = a_{020}y^2 + a_{011}yz + a_{002}z^2 + a_{100}x \\ \dot{y} = b_{010}y + b_{001}z \\ \dot{z} = c_{010}y + c_{001}z \\ \mathbf{w}_m = (2, 1, 1, 1) \end{array}$$

$$\mathrm{S}_2: \begin{array}{l} \dot{x} = a_{002}z^2 + a_{100}x + a_{010}y \\ \dot{y} = b_{002}z^2 + b_{100}x + b_{010}y \\ \dot{z} = c_{001}z \\ \mathbf{w}_m = (2, 2, 1, 1) \end{array}$$

$$\mathrm{S}_3: \begin{array}{l} \dot{x} = a_{020}y^2 + a_{011}yz + a_{002}z^2 \\ \dot{y} = b_{100}x \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (3, 2, 2, 2) \end{array}$$

$$\mathrm{S}_4: \begin{array}{l} \dot{x} = a_{002}z^2 \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{100}x + c_{010}y \\ \mathbf{w}_m = (3, 3, 2, 2) \end{array}$$

$$\mathrm{S}_5: \begin{array}{l} \dot{x} = a_{002}z^2 \\ \dot{y} = b_{100}x \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (5, 4, 3, 2) \end{array}$$

$$\mathrm{S}_6: \begin{array}{l} \dot{x} = a_{011}yz + a_{100}x \\ \dot{y} = b_{002}z^2 + b_{010}y \\ \dot{z} = c_{001}z \\ \mathbf{w}_m = (3, 2, 1, 1) \end{array}$$

$$\mathrm{S}_7: \begin{array}{l} \dot{x} = a_{020}y^2 + a_{100}x \\ \dot{y} = b_{002}z^2 + b_{010}y \\ \dot{z} = c_{001}z \\ \mathbf{w}_m = (4, 2, 1, 1) \end{array}$$

$$\mathrm{S}_8: \begin{array}{l} \dot{x} = a_{011}yz \\ \dot{y} = b_{002}z^2 + b_{100}x \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (4, 3, 2, 2) \end{array}$$

$$\mathsf{S}_9: \begin{array}{l} \dot{x} = a_{110}xy + a_{101}xz \\ \dot{y} = b_{020}y^2 + b_{011}yz + b_{002}z^2 + b_{100}x \\ \dot{z} = c_{020}y^2 + c_{011}yz + c_{002}z^2 + c_{100}x \\ \mathbf{w}_m = (2,1,1,2) \end{array}$$

$$\mathsf{S}_{10}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{002}z^2 + b_{100}x \\ \dot{z} = 0 \\ \mathbf{w}_m = (6,4,3,3) \end{array}$$

$$\mathsf{S}_{11}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (5,3,2,2) \end{array}$$

$$\mathsf{S}_{12}: \begin{array}{l} \dot{x} = a_{011}yz \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (5,4,3,3) \end{array}$$

$$\mathsf{S}_{13}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (7,5,4,4) \end{array}$$

$$\mathsf{S}_{14}: \begin{array}{l} \dot{x} = a_{101}xz + a_{011}yz \\ \dot{y} = b_{101}xz + b_{011}yz \\ \dot{z} = c_{002}z^2 + c_{100}x + c_{010}y \\ \mathbf{w}_m = (2,2,1,2) \end{array}$$

$$\mathsf{S}_{15}: \begin{array}{l} \dot{x} = a_{101}xz + a_{020}y^2 \\ \dot{y} = b_{011}yz + b_{100}x \\ \dot{z} = c_{002}z^2 + c_{010}y \\ \mathbf{w}_m = (3,2,1,2) \end{array}$$

$$\mathsf{S}_{16}: \begin{array}{l} \dot{x} = a_{101}xz + a_{020}y^2 \\ \dot{y} = b_{011}yz \\ \dot{z} = c_{002}z^2 + c_{100}x \\ \mathbf{w}_m = (4,3,2,3) \end{array}$$

$$\mathsf{S}_{17}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{101}xz \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (4,3,1,3) \end{array}$$

$$\mathsf{S}_{18}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{101}xz \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (5,4,2,4) \end{array}$$

$$\mathsf{S}_{19}: \begin{array}{l} \dot{x} = a_{110}xy \\ \dot{y} = b_{020}y^2 + b_{101}xz \\ \dot{z} = c_{011}yz + c_{100}x \\ \mathbf{w}_m = (3,2,1,3) \end{array}$$

$$\mathsf{S}_{20}: \begin{array}{l} \dot{x} = 0 \\ \dot{y} = b_{101}xz \\ \dot{z} = c_{020}y^2 + c_{100}x \\ \mathbf{w}_m = (4,2,1,4) \end{array}$$

We could add to this list of inhomogeneous systems the only homogeneous maximal QH system of degree 2, H_2, having $\mathbf{w}_m = (1,1,1,2)$. Thus, we obtain the complete list of maximal systems of degree 2.

In particular non-maximal QH systems are also represented in the above list. They are obtained from the maximals just canceling some of the coefficients, taking into account that we must always leave at least a monomial of degree 2, and that if we additionally want the system to be inhomogeneous, we must also leave some monomials of degree 1.

Another aspect to consider is that, as we have seen, in order to simplify our algorithm this provides only those systems whose weight vector family satisfies the restriction (19). This is just one of the six possible restrictions of this type that we can establish. But the remaining QH systems are again easy to obtain from the list we have. We could add to the 20 inhomogeneous systems of the list those systems obtained from making permutations of the variables x, y, and z. In this way we arrive at a total of 102 maximal inhomogeneous systems of degree 2, and not to 120 as one would expect of multiplying the number of systems by 3!. This is because there are systems, such as S_1 or S_3, which are symmetrical with respect to two of their variables, and therefore there are permutations that provide repeated systems. Thus, by adding the unique homogeneous system we have a total of 103 maximal QH systems of degree 2.

The number of systems increases notably with the degree. There are 137 inhomogeneous systems of degree 3 verifying the restriction (19), and we have found 643 systems of degree 4 and 2119 systems of degree 5 making use of our algorithm.

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3.3. Analytic integrability of quasi-homogeneous systems via the Yoshida method

Como se ha referido anteriormente, los estudios dedicados a los sistemas quasi-homogéneos en dimensiones superiores a 2, y en particular aquellos centrados en el tema de la integrabilidad, son escasos. Partiendo del artículo anterior (Sección 3.2), en el que se facilita un método para obtener todos los sistemas quasi-homogéneos de dimensión 3, con este trabajo se alcanzó un doble objetivo. Por una parte, se realizó un análisis exhaustivo de uno de los procedimientos disponibles para la integración de sistemas quasi-homogéneos, el llamado método de Yoshida, aplicable este a sistemas de dimensión arbitraria n. Posteriormente, se empleó dicho método para estudiar las integrales primeras analíticas de todos los sistemas quasi-homogéneos de grado 2 en $\mathbb{R}^3$, utilizando para ello la lista de formas normales obtenidas en el artículo de la Sección 3.2.

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Analytic integrability of quasi-homogeneous systems via the Yoshida method

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ABSTRACT

The objective of this paper is twofold. First we do a survey on what we call the Yoshida method for studying the analytic first integrals of the quasi-homogeneous polynomial differential systems. After, we apply the Yoshida method for studying the analytic first integrals of all the quasi-homogeneous polynomial differential systems in $\mathbb{R}^3$ of degree 2.

1. Introduction

In 1983 Haruo Yoshida publishes a number of interesting results (Yoshida, 1983a,b) that establish conditions for the integrability of some classes of differential systems and provide a way of finding first integrals for such systems. Later on several authors, such as Bessis (1990), Furta (1996), Goriely (1996), Llibre and Zhang (2002), Kasperczuk (2007), Liu et al. (2006) and Maciejewski and Przybylska (2017) have continued to develop his ideas to form what we now call the *Yoshida method*.

In essence the method is based on the correspondence between certain characteristic values of the first integrals and others inherent to the differential system (the so-called *Kowalevskaya exponents*), which are calculable in a finite number of steps.

The main purpose of this work is to analyze the capabilities of the Yoshida method as a tool for the integration of quasi-homogeneous differential systems in the space $\mathbb{R}^3$, a class of differential systems

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on which these results have been little exploited so far. Additional to this analysis of the Yoshida method, another of our objectives is to carry out a compilation of the main results on this class of quasi-homogeneous differential systems published to date on the subject.

Consider an n-dimensional autonomous polynomial differential system of the form

$$\frac{dx_i}{dt} = \dot{x}_i = P_i(\mathbf{x}), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \qquad i = 1, \ldots, n, \tag{1}$$

where P_i belongs to the polynomial ring over $\mathbb{R}$ in the real variables $x_1, \ldots, x_n$, for $i = 1, \ldots, n$. As usual the degree of the system is $h = \max\{h_1, \ldots, h_n\}$, where h_i is the degree of P_i.

System (1) is called *quasi-homogeneous* (in the following, simply *QH*) of *weight degree* $d \in \mathbb{Z}^+$ with *weight exponents* $s_1, \ldots, s_n \in \mathbb{Z}^+$ when for any $\alpha \in \mathbb{R}^+$ the following condition is satisfied for $i = 1, \ldots, n$:

$$P_i\left(\alpha^{s_1} x_1, \ldots, \alpha^{s_n} x_n\right) = \alpha^{s_i - 1 + d} P_i(x_1, \ldots, x_n). \tag{2}$$

Any vector of positive integers $\mathbf{v} = (s_1, \ldots, s_n, d)$ for which (2) holds is called *weight vector* of the QH system. Every QH system has an infinite number of weight vectors, because $\mathbf{v} = (s_1, \ldots, s_n, d)$ is a weight vector if and only if $\mathbf{w} = (ks_1, \ldots, ks_n, k(d-1)+1)$ is a weight vector for any $k \in \mathbb{Z}^+$. Then the weight vectors of a QH system can be grouped into families (see García et al. (2019)): the set of weight vectors that verify $s_1 = \lambda_i s_i$ for $i = 2, \ldots, n$ forms the family of ratio $(\lambda_2, \ldots, \lambda_n) \in \left(\mathbb{Q}^+\right)^{n-1}$. We will use as the representative vector of a family of weight vectors the one that verifies $\gcd(s_1, \ldots, s_n) = 1$. Besides, according to García et al. (2019), in the set of weight vectors of a QH system it is possible to define a partial order relation as follows: given two weight vectors, $\mathbf{v} = (a_1, \ldots, a_{n+1})$ and $\mathbf{w} = (b_1, \ldots, b_{n+1})$, we say that $\mathbf{v} \leq \mathbf{w}$ when $a_i \leq b_i$ for $i = 1, \ldots, n+1$. If there does exists a weight vector $\mathbf{v_m}$ verifying that $\mathbf{v_m} \leq \mathbf{w}$ for any other weight vector $\mathbf{w}$, we say that $\mathbf{v_m}$ is the *minimum weight vector* of the QH system.

A QH system is called *maximal* if any new monomial added to its structure maintaining the degree of the system prevents it being QH. Knowing the maximal systems, we can determine all the QH systems (García et al., 2019).

The QH systems constitute a set within the polynomial differential systems, which includes homogeneous systems as a particular case. For a given degree h the homogeneous systems coincide with those QH having the weight vector $(1, \ldots, 1, h)$ among their weight vectors. Starting from the idea of homogeneity and introducing different weights for the variables, the concept of QH is reached in a natural way. Therefore these last years the QH systems have been the subject of research by many authors, especially the planar QH systems and their integrability, see for instance García et al. (2013) and the references cited therein. However the study of the first integrals of the QH systems in dimensions higher than 2, such as the ones that we are dealing with in this paper, is in general a difficult problem that has received little attention, see for instance Llibre et al. (2015).

In order to determine the set of normal forms of these systems for a given degree, algorithms have been developed for both the plane (García et al., 2013, 2018) and the 3-dimensional space (García et al., 2019).

In some of the preliminary works, Yoshida (1983a) and others (Goriely, 1996) do not strictly deal with QH systems, but with other types of differential equations called *similarity invariants*. This set is characterized by the existence of certain rational numbers g_1, ..., g_n such that

$$P_i\left(\varepsilon^{g_1} x_1, \ldots, \varepsilon^{g_n} x_n\right) = \varepsilon^{g_i + 1} P_i(x_1, \ldots, x_n) \tag{3}$$

is verified for any nonzero real ε and for $i = 1, \ldots, n$. Similarity invariants have two fundamental properties. First, they are invariant under the transformation

$$t \to \varepsilon^{-1} t, \; x_1 \to \varepsilon^{g_1} x_1, \ldots, \; x_n \to \varepsilon^{g_n} x_n,$$

for any constant $\varepsilon \in \mathbb{R} \setminus \{0\}$. Moreover, if the system of equations

$$P_i(c_1, \ldots, c_n) = -g_i c_i, \qquad i = 1, \ldots, n,$$

has some nonzero solution, then the differential system has a solution of the form

$$\varphi(t) = \left(c_1 t^{-g_1}, \ldots, c_n t^{-g_n}\right).$$

We remark that conditions (2) and (3) are equivalent by simply setting $g_i = s_i/(d-1)$ and taking $\varepsilon = \alpha^{d-1}$. Consequently, a QH system belongs to the set of similarity invariant differential systems as long as it has some weight vector $(s_1, \ldots, s_n, d)$ with $d \neq 1$. In these cases, $(g_1, \ldots, g_n)$ represents all weight vectors of the family of $(s_1, \ldots, s_n, d)$. From now on when we talk about QH systems, we will understand that they are within the similarity invariant type.

This work is organized as follows. Section 2 is devoted to the theoretical bases of the Yoshida method, including the most relevant theorems of the publications referring to the topic, as well as several original concepts and results (Definition 2, Propositions 3, 4, 5, 7 and 9) which add new knowledge and clarity to the method. In Section 3, in order to simplify subsequent calculations, the canonical forms of the 20 existing QH systems of degree 2 in dimension 3 are obtained. Finally, in Section 4 we use the results of Section 2 for studying the analytical integrability of the canonical forms obtained in Section 3.

2. Yoshida method

2.1. Yoshida first integrals

The main goal of the Yoshida method is the calculation of certain type of polynomial first integrals of a differential system. Given an n-dimensional differential system (1), a non-constant real function $H(\mathbf{x})$ is a *first integral* of system (1) on an open subset $\Omega \subseteq \mathbb{R}^n$ if H is constant over all solution curves $\varphi(t) = (x_1(t), \ldots, x_n(t))$ of system (1) contained in Ω. In case that $H \in C^1(\Omega)$, then the previous definition is equivalent to

$$\sum_{i=1}^{n} P_i(\mathbf{x}) \frac{\partial H}{\partial x_i}(\mathbf{x}) \equiv 0 \tag{4}$$

for all $\mathbf{x} \in \Omega$. A first integral is *global* when Ω matches the system's domain, and it is called *polynomial* (resp. *analytic*, resp. *algebraic*) if H is a polynomial (resp. analytic, resp. algebraic) function.

A set of real functions $H_1, \ldots, H_r$ is *functionally independent* in $\mathbb{R}^n$ if the rank of the $r \times n$ matrix

$$\begin{pmatrix} \dfrac{\partial H_1}{\partial x_1} & \cdots & \dfrac{\partial H_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial H_r}{\partial x_1} & \cdots & \dfrac{\partial H_r}{\partial x_n} \end{pmatrix}(\mathbf{x})$$

is r at all points $\mathbf{x} \in \mathbb{R}^n$ where the matrix is defined, with the exception of a zero Lebesgue-measure subset. Otherwise the functions are *functionally dependent*, which geometrically implies that the gradients $\nabla H_i(\mathbf{x})$ are linearly dependent at almost all the points $\mathbf{x} \in \mathbb{R}^n$.

Several notions of integrability appear in the literature. We will work on this concept through analytical functions. It is said that system (1) is *completely integrable* if there exist $n-1$ functionally independent analytical first integrals, and *partially integrable* if the number of independent analytical first integrals is less than $n-1$. On the other hand authors as Yoshida (1983b) and Goriely (1996) focus on the algebraic first integrals. They refer to the differential system (1) as *algebraically integrable in the weak sense* if there exist k $(1 \le k \le n-1)$ functionally independent algebraic first integrals $H_i(\mathbf{x})$, $i = 1, \ldots, k$, (which define an $(n-k)$-dimensional algebraic variety $\mathcal{L}$) and other $n-1-k$ independent first integrals given by the integral of closed 1-form defined on $\mathcal{L}$,

$$H_i(\mathbf{x}) = \sum_{j=1}^{n-k} \int^{x_j} \phi_{ij}(\mathbf{x})\, dx_j, \quad i = 1, \ldots, n-1-k,$$

where ϕ_{ij} are algebraic functions of $\mathbf{x}$. Finally their stronger definition of *algebraically integrable* system is equivalent to the weak one but setting $k = n - 1$.

The interest in the search for first integrals lies in the fact that in case we can prove that (1) is completely integrable, with first integrals $H_1,...,H_{n-1}$, then the orbits of the system are determined by intersecting the invariant set $\mathcal{F}_i = \left\{H_i^{-1}(x) \mid x \in \mathbb{R}\right\}$, $i = 1, ..., n-1$. Even the case of partial integrability is interesting, because the knowledge of a first integral implies the knowledge of $n-1$-dimension surfaces in which the orbits live, whose behavior can be studied over these surfaces with the advantage of reducing one dimension.

QH systems should not be mistaken with quasi-homogeneous polynomials. A polynomial $P(x_1, ..., x_n)$ is *quasi-homogeneous* with *weight exponents* $\mathbf{s} = (s_1, ..., s_n) \in \left(\mathbb{Z}^+\right)^n$ and *weight degree* $k \in \mathbb{Z}^+$ when for any $\alpha \in \mathbb{R}^+$,

$$P\left(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n\right) = \alpha^k P\left(x_1, ..., x_n\right). \tag{5}$$

To simplify we will call $\mathbf{s}$-*type* all those quasi-homogeneous polynomials with weight exponents $\mathbf{s}$, whatever their weight degree k be. It is obvious that a $\mathbf{s}$-type polynomial with weight degree k is also of $\boldsymbol{ps}$-type with weight degree pk for every $p \in \mathbb{Z}^+$, so we can assume $\gcd(s_1, ..., s_n) = 1$.

The set of $\mathbf{s}$-*type* quasi-homogeneous polynomials with weight degree k is constituted by the functions of the form

$$\sum_{(e_1,...,e_n)\in D} A x_1^{e_1} ... x_n^{e_n},$$

where $A \in \mathbb{R}$ are arbitrary coefficients and D is the collection of non-negative solutions of the diophantine equation $s_1 e_1 + \cdots + s_n e_n = k$.

Given an arbitrary set of weight exponents $\mathbf{s} = (s_1, ..., s_n) \in \left(\mathbb{Z}^+\right)^n$, any monomial $x_1^{\beta_1} ... x_n^{\beta_n}$ is $\mathbf{s}$-type of degree $k = \sum_{i=1}^{n} s_i \beta_i$. As a consequence, given an analytical function $H(x_1, ..., x_n)$, it is possible to split H in a unique form $H = \sum_k P^k$, where every P^k is a $\mathbf{s}$-type polynomial of weight degree k, that is $P^k\left(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n\right) = \alpha^k P^k(x_1, ..., x_n)$.

The following result is proved in Llibre and Zhang (2002) for polynomial first integrals, although the proof for analytical first integrals is the same.

Proposition 1. *Let* (1) *be a QH differential system with weight exponents* $\mathbf{s} = (s_1, ..., s_n) \in \left(\mathbb{Z}^+\right)^n$, *and let* H *be an analytic function whose decomposition into* $\mathbf{s}$-*type polynomials is* $H = \sum_k P^k$. *Then* H *is a first integral of system* (1) *if and only if each polynomial* P^k *is a first integral of system* (1).

From Proposition 1 if we want to study the analytical first integrals of a QH system, it is enough to know those first integrals that are of $\mathbf{s}$-type, $\mathbf{s}$ being the weight exponent of the system. All other analytical integrals can be built using these.

Definition 2. Let (1) be a QH differential system with weight exponents $\mathbf{s} = (s_1, ..., s_n) \in \left(\mathbb{Z}^+\right)^n$. Any $\mathbf{s}$-type first integral of system (1) is called *Yoshida first integral (YFI)* of system (1).

With respect to the number of independent YFIs, we have the following result, which although intuitive, has not been proven as far as we know.

Proposition 3. *Let* (1) *be a QH differential system. The number of functionally independent analytical first integrals of system* (1) *matches with the number of functionally independent YFIs.*

Proof. Of course any set of functionally independent polynomial first integrals is in particular a set of analytical independent functions. Then let $\{H_1(\mathbf{x}), ..., H_r(\mathbf{x})\}$ be a set of r functionally independent analytical first integrals. So if we conveniently choose the variables $x_1, ..., x_r$, the value of the determinant

$$D = \begin{bmatrix} \frac{\partial H_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial H_1}{\partial x_r}(\mathbf{x}) \\ \vdots & \cdots & \vdots \\ \frac{\partial H_r}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial H_r}{\partial x_r}(\mathbf{x}) \end{bmatrix}$$

is different from zero for all values of $\mathbf{x} \in \mathbb{R}^n$ except at most for a subset of zero Lebesgue measure. Since the first integrals $H_i(\mathbf{x})$, $i = 1, ..., r$, are analytic functions, they can be expressed as sums of YFIs,

$$H_i(\mathbf{x}) = \sum_{k_i} P_i^{k_i},\ i = 1, ..., r,$$

and taking into account the properties of the determinants, we reach

$$D = \sum_{k_1} \cdots \sum_{k_r} \begin{bmatrix} \frac{\partial P_1^{k_1}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial P_1^{k_1}}{\partial x_r}(\mathbf{x}) \\ \vdots & \cdots & \vdots \\ \frac{\partial P_r^{k_r}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial P_r^{k_r}}{\partial x_r}(\mathbf{x}) \end{bmatrix} \neq 0.$$

At least one of the determinants involved in the previous summation must be non-zero. Let be the one in which the indexes $k_1, ..., k_r$ take the particular values $k_{1_0}, ..., k_{r_0}$, which means that there must be r YFIs $\left\{P_1^{k_{1_0}}, ..., P_r^{k_{r_0}}\right\}$ that are functionally independent. □

Returning to the objectives of the Yoshida method, it is now understood why it focuses on the search for polynomial first integrals, and in particular on YFIs: there may be many other analytical first integrals, and even others that are also $\mathbf{s}^*$-type for different weight exponents $\mathbf{s}^*$. However, for the purposes of integrability of the system, what is of interest is to obtain functionally independent sets of first integrals with the greatest possible number of first integrals, and this can be achieved by studying exclusively YFIs.

If the problem is approached from the point of view of algebraic integrability, there are similar results to the previous ones: based on preliminary works of Bruns (1887), Yoshida (1983a) proved that every algebraic first integral is built from rational quasi-homogeneous first integrals. On the other hand, Goriely (1996) proved that the highest number of functionally independent algebraic first integrals is reached within the subset of the quasi-homogeneous rationals. The latter are therefore the equivalent in algebraic integrability to what the YFI mean in the field of analytic integrability.

2.2. Balances and Kowalevskaya exponents

When Haruo Yoshida (1983a, 1983b) established the bases of the integration method we are discussing here, he was recovering the works that Sofia Kowalevskaya (1889) had published at the end of the 19th century. The ideas of the Russian mathematician, although not lacking in controversy at the time of its publication, had made a remarkable contribution to the study of integrability in the classical rigid body problem. However, and perhaps because eventually the method developed by Kowalevskaya were not appropriate to be applied to other physical phenomena, the fact is that her advances remained forgotten from the First World War until the beginning of the 80's, when its theoretical aspects are recovered for the field of integrability.

Now we suppose that system (1) is a QH polynomial differential system of weight degree d with respect to the weight exponents $\mathbf{s} = (s_1, ..., s_n)$. As we advanced before, we define the vector $\mathbf{g} = \mathbf{s}/(d-1)$. Then any non-trivial solution $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the polynomial system of equations

$$P_i(c_1, ..., c_n) + g_i c_i = 0, \qquad i = 1, ..., n \tag{6}$$

is called a *balance* of system (1). It is clear that the idea of balance makes no sense if $d = 1$. The balances take different denominations in the literature, and for example they are called *Darboux points* in works of Maciejewski and Przybylska (2017), or *way directions* in Furta (1996).

Each balance provides a particular solution of the differential system (1), called *scale-invariant solution*, of the form

$$\varphi_{\mathbf{c}}(t) = \left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right), \tag{7}$$

which is deduced trivially from the fact that $\mathbf{c}$ is a solution of (6) and that, being system (1) QH, because

$$P_i\left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right) = t^{-g_i - 1} P_i(c_1, ..., c_n), \quad i = 1, ..., n.$$

Proposition 4. *Let (1) be a QH differential system with weight degree $d \neq 1$. If H is a YFI and $\mathbf{c} = (c_1, ..., c_n)$ is a balance of the system, then $H(\mathbf{c}) = 0$.*

Proof. Let k be the weight degree of H, and (7) the scale-invariant solution linked to the balance $\mathbf{c}$. If H is a YFI, then by (5) setting $\alpha = t^{1/1-d}$ we get

$$H(\varphi_{\mathbf{c}}(t)) = H\left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right) = t^{k/1-d} H(c_1, ..., c_n).$$

On the other hand, we know that H is constant along the solution $\varphi_{\mathbf{c}}(t)$. Since $k/1-d \neq 0$, we know that $t^{k/1-d}$ is not constant, and thus we obtain that $H(c_1, ..., c_n) = 0$. □

From the proof of Proposition 4 it follows that, if there is more than one balance, and therefore more than one scale-invariant solution, all of them live on the same level surface of every YFI. So we have proved the next result.

Corollary 5. *Given a YFI H of a QH differential system with weight degree $d \neq 1$, any scale-invariant solution $\varphi_{\mathbf{c}}(t)$ is on the level surface determined by $H(x_1, ..., x_n) = 0$.*

For each balance $\mathbf{c}$ the $n \times n$ matrix defined by

$$K(\mathbf{c}) = (K_{ij}(\mathbf{c})) = \left(\frac{\partial P_i}{\partial x_j}(\mathbf{c}) + \delta_{ij} g_j\right), \quad i, j = 1, ..., n,$$

is called the *Kowalevskaya matrix* of the QH differential system associated to the balance $\mathbf{c}$, and the eigenvalues of $K(\mathbf{c})$ are the *Kowalevskaya exponents* of the balance $\mathbf{c}$. These exponents, calculable in a finite number of steps, are the values on which all the theory of the Yoshida method for the search of the first integrals of QH systems is based. It is known that every Kowalevskaya matrix has -1 as eigenvalue. However this Kowalevskaya exponent does not have practical utility in the Yoshida method. This result has been proven in Furta (1996).

Proposition 6. *Let (1) be a QH differential system with weight degree $d \neq 1$ and let $\mathbf{c} = (c_1, ..., c_n)$ be a balance of this system. Then -1 is a Kowalevskaya exponent of $\mathbf{c}$ and $\mathbf{gc} = (g_1 c_1, ..., g_n c_n)$ is its corresponding eigenvector.*

We will name the eigenvector $\lambda_n = -1$ as the *trivial* Kowalevskaya exponent, and the rest of the eigenvalues $(\lambda_1, ..., \lambda_{n-1})$ as *non-trivial* Kowalevskaya exponents.

2.2.1. Some notes on the number of balances and Kowalevskaya exponents

In a QH system each family of weight vectors determines a single vector **g**, so the number of balances depends on the number of families of the system. Note that the only QH systems with more than one family of weight vectors are special types of homogeneous QH (García et al., 2018). Having only one family of weight vectors in the system, the polynomial character of (6) and Bezout's Theorem guarantee that the amount of balances cannot exceed $\prod_{i=1}^{n} h_i$, being h_i the degree of P_i (always provided that $d \neq 1$). Obviously it may also be the case that the system lacks them.

We note that the balances are grouped into equivalence classes whose cardinal depends on the weight degree d of the corresponding weight vector. The relevant fact about this is that all balances of a certain equivalence class provide the same essential information, that is, the same Kowalevskaya exponents:

Proposition 7. *Let (1) be a QH differential system with weight vector* $\mathbf{v} = (s_1, \ldots, s_n, d)$, $d \neq 1$, *and let* $\mathbf{c} = (c_1, \ldots, c_n)$ *be a balance. Then all the elements of*

$$B_{\mathbf{c}}^{\mathbf{v}} = \left\{ \left(r_p^{s_1} c_1, \ldots, r_p^{s_n} c_n \right) \mid p = 1, \ldots, d-1 \right\}$$

are balances of system (1), where r_p *runs in* $G_d = \left\{ r \in \mathbb{C} \mid r^{d-1} = 1 \right\}$.

Proof. Let $r \in \mathbb{C}$ be such that $r^{d-1} = 1$. Since **v** is a weight vector, then

$$P_i \left(r^{s_1} c_1, \ldots, r^{s_n} c_n \right) = r^{s_i - 1 + d} P_i \left(c_1, \ldots, c_n \right), \qquad i = 1, \ldots, n. \tag{8}$$

Since **c** is a balance by (6) and (8) we have

$$P_i \left(r^{s_1} c_1, \ldots, r^{s_n} c_n \right) = r^{s_i - 1 + d} \left(-g_i c_i \right) = -g_i \left(r^{s_i} c_i \right), \qquad i = 1, \ldots, n,$$

and as a consequence $\left(r^{s_1} c_1, \ldots, r^{s_n} c_n \right)$ is a balance. □

Remark 8. Note that if **v** and **w** are two weight vectors of the same family, the sets $B_{\mathbf{c}}^{\mathbf{v}}$ and $B_{\mathbf{c}}^{\mathbf{w}}$ coincide: if $\mathbf{v}_m = (s_1, \ldots, s_n, d)$ is the representative vector of the family ($\gcd(s_1, \ldots, s_n) = 1$), any other vector of it has the form $(k s_1, \ldots, k s_n, k(d-1) + 1)$, being $k \in \mathbb{Z}^+$. Therefore the set of complex values $\left\{ r^{k s_i} \in \mathbb{C} \mid r^{k(d-1)} = 1 \right\}$ trivially matches the set $\left\{ r^{s_i} \in \mathbb{C} \mid r^{d-1} = 1 \right\}$.

Proposition 9. *Under the assumptions of Proposition 7 all balances of the set* $B_{\mathbf{c}}^{\mathbf{v}}$ *give rise to the same Kowalevskaya exponents.*

Proof. Let $K(\mathbf{c})$ be the Kowalevskaya matrix of balance **c**, i.e.

$$K(\mathbf{c}) = \left(K_{ij}(\mathbf{c}) \right) = \left(\frac{\partial P_i}{\partial x_j} (\mathbf{c}) + \delta_{ij} g_j \right), \quad i, j = 1, \ldots, n,$$

and, fixed $r \in G_d$, let $K\left(r^{\mathbf{s}} \mathbf{c} \right)$ be the Kowalevskaya matrix of balance $r^{\mathbf{s}} \mathbf{c} = \left(r^{s_1} c_1, \ldots, r^{s_n} c_n \right)$, that is

$$K_{ij}\left(r^{\mathbf{s}} \mathbf{c} \right) = \frac{\partial P_i}{\partial x_j} \left(r^{\mathbf{s}} \mathbf{c} \right) + \delta_{ij} g_j, \quad i, j = 1, \ldots, n.$$

We define the functions $f_i(x_1, \ldots, x_n) = P_i\left(r^{s_1} x_1, \ldots, r^{s_n} x_n \right)$ for $i = 1, \ldots, n$, with which we have that $f_i = P_i \circ g$ for $i = 1, \ldots, n$, being $g(x_1, \ldots, x_n) = \left(r^{s_1} x_1, \ldots, r^{s_n} x_n \right)$. Since $(s_1, \ldots, s_n, d)$ is a weight vector and $r^{d-1} = 1$, it follows that $f_i(x_1, \ldots, x_n) = r^{s_i} P_i(x_1, \ldots, x_n)$, so

$$\nabla f_i(x_1, \ldots, x_n) = r^{s_i} \nabla P_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \tag{9}$$

On the other hand, the chain rule sets that $\nabla f_i(x_1, \ldots, x_n) = \nabla P_i(g(x_1, \ldots, x_n)) \cdot J_g(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$, and since $J_g(x_1, \ldots, x_n) = diag\left(r^{s_1}, \ldots, r^{s_n} \right)$ is a regular matrix with inverse $J_g^{-1}(x_1, \ldots, x_n) = \operatorname{diag}\left(r^{-s_1}, \ldots, r^{-s_n} \right)$, we can write

$$\nabla P_i\left(g\left(x_1, \ldots, x_n\right)\right)=\nabla f_i\left(x_1, \ldots, x_n\right) \cdot J_g^{-1}\left(x_1, \ldots, x_n\right), \quad i=1, \ldots, n. \tag{10}$$

This implies, due to (9) and (10), that

$$\frac{\partial P_i}{\partial x_j}\left(r^{s_1} x_1, \ldots, r^{s_n} x_n\right)=r^{s_i-s_j} \frac{\partial P_i}{\partial x_j}\left(x_1, \ldots, x_n\right), \quad i, j=1, \ldots, n,$$

and consequently, setting $(x_1, \ldots, x_n)=\mathbf{c}$ and taking into account that on the main diagonal the powers of r are equal to 1, we obtain

$$K_{ij}\left(r^{\mathbf{s}} \mathbf{c}\right)=r^{s_i-s_j} K_{ij}(\mathbf{c}), \quad i, j=1, \ldots, n.$$

Therefore we only have to take the matrix $J_g(x_1, \ldots, x_n)$ as the change matrix to see that the matrices $K(\mathbf{c})$ and $K\left(r^{\mathbf{s}}\mathbf{c}\right)$ are similar. Therefore they have the same spectrum. □

2.3. Main results of the Yoshida method

Since the early 80's, when the first outcomes of what could be called the Yoshida method are published, until today, contributions to the subject have been copious. A large group of researchers have expanded, not without obstacles neither difficulties, the collection of useful theorems. In this section we will present in chronological order of publication those that in our opinion are the most relevant results, together with some comments on how the weaknesses found in the theory have been partially solved.

Some of these theorems were published focusing on the search for algebraic first integrals, but as YFIs are algebraic, they are also valid results for the study of analytic first integrals.

The following theorem, due to Yoshida (1983a), constitutes the origin of the whole theory and relates the Kowalevskaya exponents of a system with the weight degrees of its potential first integrals. In its original publication, intended for algebraic first integrals, it included the additional condition that $\nabla H(\mathbf{c})$ be finite, which is unnecessary for polynomial first integrals. Additionally, Yoshida exactly equates weight degrees with Kowalevskaya exponents, because he takes as the weight exponent of H the vector $\mathbf{g}=\mathbf{s}/(d-1)$ instead of $\mathbf{s}$.

Theorem 10. *Let (1) be a QH differential system with weight degree $d \neq 1$ and let $\mathbf{c}$ be a balance whose non-trivial Kowalevskaya exponents are $\{\lambda_1, \ldots, \lambda_{n-1}\}$. If H is a YFI verifying $\nabla H(\mathbf{c}) \neq \mathbf{0}$, then its weight degree is $\lambda_j(d-1)$ for some $j \in \{1, \ldots, n-1\}$.*

The practical use of this result is powerful. Because knowing the Kowalevskaya exponents of a balance, we limit the search for YFIs to those whose weight degrees adjust to certain values. But this idea has an important weakness, only partially solved in subsequent publications: any YFI that does not verify $\nabla H(\mathbf{c}) \neq \mathbf{0}$ will be "hidden", outside our search radius. For example, any power H^p of a YFI H will remain hidden, because from Proposition 4, $\nabla H^p(\mathbf{c})=pH^{p-1}(\mathbf{c}) \cdot \nabla H(\mathbf{c})=\mathbf{0}$. This is not relevant for the purposes of studying the integrability of the system, because H^p is functionally dependent on H, but unfortunately there exist YFIs that are functionally independent of the rest and whose gradient is null on all balances:

Example 11. The differential system

$$\dot{x}=x^2+3z^2, \quad \dot{y}=2xu, \quad \dot{z}=-2xz-y^2-u^2, \quad \dot{u}=-2xy$$

is QH with weight vector $(s_1, s_2, s_3, s_4, d)=(1,1,1,1,2)$ and has the following balances:

$$\mathbf{c}_1=(-1,0,0,0), \quad \mathbf{c}_2=(1/2, 0, i/2, 0), \quad \mathbf{c}_3=(1/2, 0, -i/2, 0).$$

The non-trivial Kowalevskaya exponents corresponding to $\mathbf{c}_1$ are $\{3, 1+2i, 1-2i\}$ while those corresponding to $\mathbf{c}_2$ and $\mathbf{c}_3$ are $\{3, 1+i, 1-i\}$. The YFI $H_1(x, y, z, u)=x^2z+xy^2+xu^2+z^3$ of weight degree 3, concerns the Kowalevskaya exponent $\lambda = 3(d-1) = 3$, which appears in the three balances. However there is also the YFI $H_2(x, y, z, u) = y^2 + u^2$ of weight degree 2. We note that the Kowalevskaya exponent $\lambda = 2(d-1) = 2$ does not appear in any of the three balances, as would be expected, and this is because the aforementioned condition is not met:

$$\nabla H_2(\mathbf{c}_i) = \mathbf{0}, \quad i = 1, 2, 3.$$

Furthermore it is easy to verify that H_1 and H_2 are functionally independent, so if we do not detect H_2 we are missing relevant information regarding integrability.

Despite this problem Theorem 10 is very useful because it serves as an orientation regarding possible weight degrees of existing YFIs. A similar result but focused on the search for sets of independent YFIs with the same weight degree, is as follows (Yoshida, 1983a):

Theorem 12. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, \ldots, \lambda_{n-1}\}$. *Let* $H_1, \ldots, H_r$ *be functionally independent YFI with the same weight degree* k. *If the vectors* $\nabla H_1(\mathbf{c}), \ldots, \nabla H_r(\mathbf{c})$ *are linearly independent, then the common weight degree is* $k = \lambda_j(d-1)$ *for some* $j \in \{1, \ldots, n-1\}$, *with* λ_j *being a Kowalevskaya exponent of multiplicity at least* r.

The following theorem provides a necessary condition for complete integrability. It has experienced many modifications. It was originally published by Yoshida (1983b) based on the weak conception of algebraic integrability, although González Gascón (1988) warned that the proof was actually only valid for systems in the plane. Later on Bessis (1990) shows, providing counterexamples, that this theorem is false in dimensions higher than 2. Finally Goriely (1996) proves that the result is valid as long as the idea of integrability used is the strong algebraic one (or equivalently, our complete integrability with analytical functions).

Theorem 13. *Let* (1) *be a completely integrable QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, \ldots, \lambda_{n-1}\}$. *Then* $\lambda_j \in \mathbb{Q}$ *for all* $j \in \{1, \ldots, n-1\}$.

Let X be either $\mathbb{N}$ or $\mathbb{Z}$. A set of complex values $\lambda_1, \ldots, \lambda_n$ is X-*independent* when

$$\sum_{j=1}^{n} \alpha_j \lambda_j = 0, \qquad \alpha_j \in X \quad \forall j \in \{1, \ldots, n\},$$

implies that $\alpha_j = 0 \quad \forall j \in \{1, \ldots, n\}$. Otherwise the set is called X-*dependent*.

The next theorem, due to Furta (1996), is useful to determine the partial integrability when we already have a YFI.

Theorem 14. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, \ldots, \lambda_{n-1}\}$. *Let* H *be a YFI verifying* $\nabla H(\mathbf{c}) \neq \mathbf{0}$ *whose weight degree is* $\lambda_j(d-1)$ *for some* $j \in \{1, \ldots, n-1\}$. *If the set* $\left\{-1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n\right\}$ *is* $\mathbb{Z}$-*independent, then any other first integral* H' *of system* (1) *is a function of* H, *i.e.* $H' = \mathcal{F}(H)$, *where* $\mathcal{F}$ *is a smooth function.*

The following result and its corollary once again relate the Kowalevskaya exponents with the weight degrees of YFIs, but with an interesting novelty: $\nabla H(\mathbf{c}) \neq \mathbf{0}$ is no longer required from the first integrals. Therefore, this theorem captures all YFIs, leaving no "hidden" ones. The credit goes to Goriely (1996), although after it has been improved by Llibre and Zhang (2002), and finally by Liu et al. (2006).

Theorem 15. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, \ldots, \lambda_{n-1}\}$. *Let* H *be a YFI whose weight degree is* k. *Then there exist natural numbers* $\alpha_1, \ldots, \alpha_{n-1}$ *verifying* $0 < \alpha_1 + \cdots + \alpha_{n-1} \leq k$ *such that*

$$\sum_{j=1}^{n-1} \alpha_j \lambda_j = \frac{k}{d-1}.$$

Corollary 16. *Let (1) be a QH differential system with weight degree $d \neq 1$ and let* **c** *be a balance. If the Kowalevskaya exponents of* **c** *are $\mathbb{N}$-independent, then there is no analytical first integral of system (1).*

Finally this contribution by Maciejewski and Przybylska (2017), which links the weight vector of the system with the Kowalevskaya exponents of each existing balances, is useful when there are computational difficulties in obtaining the Kowalevskaya matrix.

Theorem 17. *Let (1) be a QH differential system with weight exponent $\mathbf{s} = (s_1, \ldots, s_n) \in \left(\mathbb{Z}^+\right)^n$ and weight degree $d \neq 1$, and let $\{\mathbf{c}_i\}_{i\in I}$ be the set of balances of the system, whose non-trivial Kowalevskaya exponents are, respectively, $\left\{\lambda_{i1}, \ldots, \lambda_{i(n-1)}\right\}$. Then*

$$\sum_{i\in I} \frac{\left(\lambda_{i1} + \cdots + \lambda_{i(n-1)}\right)^j}{\lambda_{i1} \cdot \ldots \cdot \lambda_{i(n-1)}} = \frac{(g_1 + \cdots + g_n + 1)^j}{g_1 \cdot \ldots \cdot g_n}, \qquad j = 0, \ldots n-1,$$

being $g_i = s_i/(d-1)$, $i = 1, \ldots, n$.

3. Canonical forms

In García et al. (2019) an algorithm was published to determine, given a fixed degree, all QH systems in dimension 3. Such algorithm provides all normal forms of maximal systems. Then all other quasi-homogeneous systems can be trivially deduced from the maximal set, because they can be considered as particular cases of the maximal normal forms in which some monomials are zero. Some interesting properties of the 3-dimensional maximal systems are proved in García et al. (2019), like the fact that a maximal QH system has a unique weight vector family, or that a maximal QH system always has a minimum weight vector. In fact, the minimum vector has a simple characterization in the unique-family case: a weight vector (s_1, s_2, s_3, d) is the minimum weight vector if and only if $\gcd(s_1, s_2, s_3, d) = 1$. The existence of the minimum vector $\mathbf{v_m}$, added to the fact that two different maximal QH systems of the same degree n have no common weight vectors, confers to $\mathbf{v_m}$ the character of unique identifier within the set of maximal systems of degree n.

The simplest type of maximal 3-dimensional QH system is that of degree 2. The algorithm published in García et al. (2019) provides 20 normal forms of this type, to which the trivial case of the homogeneous system should be added. They are the following, accompanied by their respective minimum vectors:

$$\mathrm{N}_1: \begin{array}{l} \dot{x} = ay^2 + byz + cz^2 + dx \\ \dot{y} = ey + fz \\ \dot{z} = gy + hz \\ \mathbf{w}_m = (2,1,1,1) \end{array} \qquad \mathrm{N}_4: \begin{array}{l} \dot{x} = az^2 \\ \dot{y} = bz^2 \\ \dot{z} = cx + dy \\ \mathbf{w}_m = (3,3,2,2) \end{array}$$

$$\mathrm{N}_2: \begin{array}{l} \dot{x} = az^2 + bx + cy \\ \dot{y} = dz^2 + ex + fy \\ \dot{z} = gz \\ \mathbf{w}_m = (2,2,1,1) \end{array} \qquad \mathrm{N}_5: \begin{array}{l} \dot{x} = az^2 \\ \dot{y} = bx \\ \dot{z} = cy \\ \mathbf{w}_m = (5,4,3,2) \end{array}$$

$$\mathrm{N}_3: \begin{array}{l} \dot{x} = ay^2 + byz + cz^2 \\ \dot{y} = dx \\ \dot{z} = ex \\ \mathbf{w}_m = (3,2,2,2) \end{array} \qquad \mathrm{N}_6: \begin{array}{l} \dot{x} = ayz + bx \\ \dot{y} = cz^2 + dy \\ \dot{z} = ez \\ \mathbf{w}_m = (3,2,1,1) \end{array}$$

$$\mathsf{N}_7: \quad \begin{aligned} \dot{x} &= ay^2 + bx \\ \dot{y} &= cz^2 + dy \\ \dot{z} &= ez \\ \mathbf{w}_m &= (4, 2, 1, 1) \end{aligned}$$

$$\mathsf{N}_8: \quad \begin{aligned} \dot{x} &= ayz \\ \dot{y} &= bz^2 + cx \\ \dot{z} &= dy \\ \mathbf{w}_m &= (4, 3, 2, 2) \end{aligned}$$

$$\mathsf{N}_9: \quad \begin{aligned} \dot{x} &= axy + bxz \\ \dot{y} &= cy^2 + dyz + ez^2 + fx \\ \dot{z} &= gy^2 + hyz + iz^2 + jx \\ \mathbf{w}_m &= (2, 1, 1, 2) \end{aligned}$$

$$\mathsf{N}_{10}: \quad \begin{aligned} \dot{x} &= ay^2 \\ \dot{y} &= bz^2 + cx \\ \dot{z} &= 0 \\ \mathbf{w}_m &= (6, 4, 3, 3) \end{aligned}$$

$$\mathsf{N}_{11}: \quad \begin{aligned} \dot{x} &= ay^2 \\ \dot{y} &= bz^2 \\ \dot{z} &= cy \\ \mathbf{w}_m &= (5, 3, 2, 2) \end{aligned}$$

$$\mathsf{N}_{12}: \quad \begin{aligned} \dot{x} &= ayz \\ \dot{y} &= bz^2 \\ \dot{z} &= cx \\ \mathbf{w}_m &= (5, 4, 3, 3) \end{aligned}$$

$$\mathsf{N}_{13}: \quad \begin{aligned} \dot{x} &= ay^2 \\ \dot{y} &= bz^2 \\ \dot{z} &= cx \\ \mathbf{w}_m &= (7, 5, 4, 4) \end{aligned}$$

$$\mathsf{N}_{14}: \quad \begin{aligned} \dot{x} &= axz + byz \\ \dot{y} &= cxz + dyz \\ \dot{z} &= ez^2 + fx + gy \\ \mathbf{w}_m &= (2, 2, 1, 2) \end{aligned}$$

$$\mathsf{N}_{15}: \quad \begin{aligned} \dot{x} &= axz + by^2 \\ \dot{y} &= cyz + dx \\ \dot{z} &= ez^2 + fy \\ \mathbf{w}_m &= (3, 2, 1, 2) \end{aligned}$$

$$\mathsf{N}_{16}: \quad \begin{aligned} \dot{x} &= axz + by^2 \\ \dot{y} &= cyz \\ \dot{z} &= dz^2 + ex \\ \mathbf{w}_m &= (4, 3, 2, 3) \end{aligned}$$

$$\mathsf{N}_{17}: \quad \begin{aligned} \dot{x} &= ay^2 \\ \dot{y} &= bxz \\ \dot{z} &= cy \\ \mathbf{w}_m &= (4, 3, 1, 3) \end{aligned}$$

$$\mathsf{N}_{18}: \quad \begin{aligned} \dot{x} &= ay^2 \\ \dot{y} &= bxz \\ \dot{z} &= cx \\ \mathbf{w}_m &= (5, 4, 2, 4) \end{aligned}$$

$$\mathsf{N}_{19}: \quad \begin{aligned} \dot{x} &= axy \\ \dot{y} &= by^2 + cxz \\ \dot{z} &= dyz + ex \\ \mathbf{w}_m &= (3, 2, 1, 3) \end{aligned}$$

$$\mathsf{N}_{20}: \quad \begin{aligned} \dot{x} &= 0 \\ \dot{y} &= axz \\ \dot{z} &= by^2 + cx \\ \mathbf{w}_m &= (4, 2, 1, 4) \end{aligned}$$

Since these are maximal systems, their coefficients a, b, ... i, j can take any real value other than zero.

Our intention is to use these families of systems to test the capabilities of the Yoshida method on three-dimensional QH systems. In order to simplify the calculations as much as possible, we will carry out a reduction on the number of parameters. Doing a linear change of variables we will preserve their topological equivalence and they will remain in the class of the QH systems.

Theorem 18. *Any maximal quadratic QH system in $\mathbb{R}^3$ can be written, after a rescaling of the variables and time, as one of the following systems:*

$$\mathsf{S}_1: \quad \begin{aligned} \dot{x} &= y^2 + yz + pz^2 + x \\ \dot{y} &= qy + rz \\ \dot{z} &= sy + uz \\ \mathbf{w}_m &= (2, 1, 1, 1) \end{aligned}$$

$$\mathsf{S}_2: \quad \begin{aligned} \dot{x} &= z^2 + x + y \\ \dot{y} &= pz^2 + qx + ry \\ \dot{z} &= sz \\ \mathbf{w}_m &= (2, 2, 1, 1) \end{aligned}$$

$$\mathsf{S}_3: \begin{array}{l} \dot{x} = py^2 + qyz + z^2 \\ \dot{y} = x \\ \dot{z} = x \\ \mathbf{w}_m = (3, 2, 2, 2) \end{array}$$

$$\mathsf{S}_4: \begin{array}{l} \dot{x} = pz^2 \\ \dot{y} = z^2 \\ \dot{z} = x + y \\ \mathbf{w}_m = (3, 3, 2, 2) \end{array}$$

$$\mathsf{S}_5: \begin{array}{l} \dot{x} = z^2 \\ \dot{y} = x \\ \dot{z} = y \\ \mathbf{w}_m = (5, 4, 3, 2) \end{array}$$

$$\mathsf{S}_6: \begin{array}{l} \dot{x} = yz + px \\ \dot{y} = z^2 + qy \\ \dot{z} = z \\ \mathbf{w}_m = (3, 2, 1, 1) \end{array}$$

$$\mathsf{S}_7: \begin{array}{l} \dot{x} = y^2 + x \\ \dot{y} = \pm z^2 + py \\ \dot{z} = qz \\ \mathbf{w}_m = (4, 2, 1, 1) \end{array}$$

$$\mathsf{S}_8: \begin{array}{l} \dot{x} = pyz \\ \dot{y} = z^2 + x \\ \dot{z} = y \\ \mathbf{w}_m = (4, 3, 2, 2) \end{array}$$

$$\mathsf{S}_9: \begin{array}{l} \dot{x} = xy + xz \\ \dot{y} = py^2 + qyz + rz^2 + x \\ \dot{z} = sy^2 + uyz + vz^2 + wx \\ \mathbf{w}_m = (2, 1, 1, 2) \end{array}$$

$$\mathsf{S}_{10}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = \pm z^2 + x \\ \dot{z} = 0 \\ \mathbf{w}_m = (6, 4, 3, 3) \end{array}$$

$$\mathsf{S}_{11}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = z^2 \\ \dot{z} = y \\ \mathbf{w}_m = (5, 3, 2, 2) \end{array}$$

$$\mathsf{S}_{12}: \begin{array}{l} \dot{x} = yz \\ \dot{y} = z^2 \\ \dot{z} = x \\ \mathbf{w}_m = (5, 4, 3, 3) \end{array}$$

$$\mathsf{S}_{13}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = z^2 \\ \dot{z} = x \\ \mathbf{w}_m = (7, 5, 4, 4) \end{array}$$

$$\mathsf{S}_{14}: \begin{array}{l} \dot{x} = pxz + qyz \\ \dot{y} = rxz + syz \\ \dot{z} = z^2 + x + y \\ \mathbf{w}_m = (2, 2, 1, 2) \end{array}$$

$$\mathsf{S}_{15}: \begin{array}{l} \dot{x} = xz + y^2 \\ \dot{y} = pyz + x \\ \dot{z} = qz^2 + ry \\ \mathbf{w}_m = (3, 2, 1, 2) \end{array}$$

$$\mathsf{S}_{16}: \begin{array}{l} \dot{x} = xz \pm y^2 \\ \dot{y} = pyz \\ \dot{z} = qz^2 + x \\ \mathbf{w}_m = (4, 3, 2, 3) \end{array}$$

$$\mathsf{S}_{17}: \begin{array}{l} \dot{x} = \pm y^2 \\ \dot{y} = \pm xz \\ \dot{z} = \pm y \\ \mathbf{w}_m = (4, 3, 1, 3) \end{array}$$

$$\mathsf{S}_{18}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = xz \\ \dot{z} = x \\ \mathbf{w}_m = (5, 4, 2, 4) \end{array}$$

$$\mathsf{S}_{19}: \begin{array}{l} \dot{x} = pxy \\ \dot{y} = qy^2 \pm xz \\ \dot{z} = yz \pm x \\ \mathbf{w}_m = (3, 2, 1, 3) \end{array}$$

$$\mathsf{S}_{20}: \begin{array}{l} \dot{x} = 0 \\ \dot{y} = xz \\ \dot{z} = y^2 + x \\ \mathbf{w}_m = (4, 2, 1, 4) \end{array}$$

where the coefficients p, q, r, s, u, v, w are nonzero real numbers in all systems.

Proof. As stated any maximal QH system in $\mathbb{R}^3$ can be expressed as one of the normal forms N_m. We denote its components by

$$P_i(x, y, z) = \sum_{k=1}^{n_i} h_i^k x^{A_i^k} y^{B_i^k} z^{C_i^k}, \qquad i = 1, 2, 3, \tag{11}$$

where n_i is the number of monomials of the polynomial P_i, $h_i^k \in \{a, b, c, ..., j\} \subseteq \mathbb{R} \setminus \{0\}$ is the coefficient of the k-th monomial of P_i, and A_i^k, B_i^k, C_i^k are respectively the exponents of the variables x, y, z in the k-th monomial of P_i, for k between 1 and n_i.

To reduce the greatest possible number of parameters, while still obtaining systems that are topologically equivalent to the starting ones, we make the following linear change of variables:

$$x = \alpha X, \quad y = \beta Y, \quad z = \gamma Z, \quad t = \delta T, \tag{12}$$

where α, β, γ, δ are nonzero real numbers to be determined. With these new variables, system (1) with $n = 3$ is transformed into

$$\frac{dX}{dT} = \bar{P}_1(X, Y, Z), \quad \frac{dY}{dT} = \bar{P}_2(X, Y, Z), \quad \frac{dZ}{dT} = \bar{P}_3(X, Y, Z), \tag{13}$$

where

$$\bar{P}_i(X, Y, Z) = \frac{\delta}{g(i)} P_j(\alpha X, \beta Y, \gamma Z), \qquad i = 1, 2, 3, \tag{14}$$

with $g(1) = \alpha$, $g(2) = \beta$, $g(3) = \gamma$. Taking into account (11) and (14) we get

$$\bar{P}_i(x, y, z) = \sum_{k=1}^{n_i} \frac{\delta}{g(i)} h_i^k \alpha^{A_i^k} \beta^{B_i^k} \gamma^{C_i^k} X^{A_i^k} Y^{B_i^k} Z^{C_i^k}, \qquad i = 1, 2, 3.$$

Now we look for the values of α, β, γ, δ based on $h_i^k \in \{a, b, c, ..., j\}$ coefficients, so that the change (12) provides a system (13) as simplified as possible, that is, with the largest number of unitary coefficients. We therefore propose the system of $n_1 + n_2 + n_3$ equations in the variables α, β, γ, δ:

$$\left\{ \frac{\delta}{g(i)} h_i^k \alpha^{A_i^k} \beta^{B_i^k} \gamma^{C_i^k} = 1 \text{ such that } 1 \le k \le n_i \,;\; i = 1, 2, 3 \right\}.$$

In case that this system is compatible only for certain values of the h_i^k, we will successively eliminate equations from it until it has a solution. Each equation eliminated means a parameter that we cannot convert to ± 1. Finally, we get the solution $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, valid for any h_i^k, which provides the optimal change (12): $x = \alpha_0 X$, $y = \beta_0 Y$, $z = \gamma_0 Z$, $t = \delta_0 T$. Note that this solution always exists if we reduce equations conveniently, because h_i^k is nonzero and the equation

$$\frac{\delta}{g(i)} h_i^k \alpha^{A_i^k} \beta^{B_i^k} \gamma^{C_i^k} = 1$$

has the solution $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\left(g(i)/h_i^k \right)^{1/A_i^k}, 1, 1, 1 \right)$.

Following this procedure with the 20 normal forms $\mathsf{N_m}$ it is possible to reduce a maximum of three parameters in each of them. Next, for each normal form, we provide the optimal change found and the values taken by the parameters of the canonical forms $\mathsf{S_m}$ that have not been possible to reduce to unit values:

S1 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{a}{d}, 1, \frac{a}{b}, \frac{1}{d} \right)$ $p = \frac{ac}{b^2}$, $q = \frac{e}{d}$, $r = \frac{af}{bd}$, $s = \frac{bg}{ad}$, $u = \frac{h}{d}$

S2 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{a}{b}, \frac{a}{c}, 1, \frac{1}{b} \right)$ $p = \frac{cd}{ab}$, $q = \frac{ce}{b^2}$, $r = \frac{f}{b}$, $s = \frac{g}{b}$

S3 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{ce^2}, \frac{d}{ce^2}, \frac{1}{ce}, 1 \right)$ $p = \frac{ad^2}{ce^2}$, $q = \frac{bd}{ce}$

S4 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{bcd}, \frac{1}{bd^2}, \frac{1}{bd}, 1 \right)$ $p = \frac{ac}{bd}$

S5 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{ab^2c^2}, \frac{1}{abc^2}, \frac{1}{abc}, 1 \right)$

S6 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{ac}{e^2}, \frac{c}{e}, 1, \frac{1}{e}\right)$ $p = \frac{b}{e}$, $q = \frac{d}{e}$

S7 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{a}{b}, 1, \sqrt{|\frac{b}{c}|}, \frac{1}{b}\right)$ $p = \frac{d}{b}$, $q = \frac{e}{b}$

S8 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{bcd^2}, \frac{1}{bd^2}, \frac{1}{bd}, 1\right)$ $p = \frac{ac}{bd}$

S9 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{af}, \frac{1}{a}, \frac{1}{b}, 1\right)$ $p = \frac{c}{a}$, $q = \frac{d}{b}$, $r = \frac{ae}{b^2}$, $s = \frac{bg}{a^2}$, $u = \frac{h}{a}$, $v = \frac{i}{b}$, $w = \frac{bj}{af}$

S10 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{ac^2}, \frac{1}{ac}, \frac{1}{\sqrt{|abc|}}, 1\right)$

S11 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{a}{b^{1/3}c^{2/3}}, 1, \frac{c^{1/3}}{b^{1/3}}, \frac{1}{b^{1/3}c^{2/3}}\right)$

S12 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{a^{1/3}b^{1/3}}{c^{2/3}}, \frac{b^{2/3}}{a^{1/3}c^{1/3}}, 1, \frac{1}{a^{1/3}b^{1/3}c^{1/3}}\right)$

S13 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{a^{1/3}b^{2/3}c^{4/3}}, \frac{1}{a^{2/3}b^{1/3}c^{2/3}}, \frac{1}{a^{1/3}b^{2/3}c^{1/3}}, 1\right)$

S14 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{ef}, \frac{1}{eg}, \frac{1}{e}, 1\right)$ $p = \frac{a}{e}$, $q = \frac{bf}{eg}$, $r = \frac{cg}{ef}$, $s = \frac{d}{e}$

S15 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{bd^2}, \frac{1}{bd}, \frac{1}{a}, 1\right)$ $p = \frac{c}{a}$, $q = \frac{e}{a}$, $r = \frac{af}{bd}$

S16 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{ae}, \frac{1}{\sqrt{|abe|}}, \frac{1}{a}, 1\right)$ $p = \frac{c}{a}$, $q = \frac{d}{a}$

S17 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{bc}, \frac{1}{\sqrt{|abc|}}, \sqrt{|\frac{c}{ab}|}, 1\right)$

S18 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{a^{1/3}b^{2/3}c^{2/3}}, \frac{1}{a^{2/3}b^{1/3}c^{1/3}}, \frac{c^{1/3}}{a^{1/3}b^{2/3}}, 1\right)$

S19 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{\sqrt{|cde|}}, \frac{1}{d}, \sqrt{|\frac{e}{cd}|}, 1\right)$ $p = \frac{a}{d}$, $q = \frac{b}{d}$

S20 $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{a^{2/3}b^{1/3}c^{1/3}}, \frac{c^{1/3}}{a^{1/3}b^{2/3}}, \frac{c^{2/3}}{a^{2/3}b^{1/3}}, 1\right)$

On the other hand, since the systems of each canonical form $\mathrm{S_m}$ are a subset of the normal form $\mathrm{N_m}$, their corresponding minimum vector does not change. □

Example 19. As a sample of the procedure followed in the proof of the previous theorem, detailed calculations for the reduction of parameters in the normal form $\mathrm{N_3}$ are given below. The maximal QH differential system $\mathrm{N_3}$ is

$$\dot{x} = ay^2 + byz + cz^2, \quad \dot{y} = dx, \quad \dot{z} = ex.$$

Applying the change of variables (12) the system is transformed into

$$\dot{X} = \frac{\beta^2\delta}{\alpha}aY^2 + \frac{\beta\gamma\delta}{\alpha}bYZ + \frac{\gamma^2\delta}{\alpha}cZ^2, \quad \dot{Y} = \frac{\alpha\delta}{\beta}dX, \quad \dot{Z} = \frac{\alpha\delta}{\gamma}eX. \tag{15}$$

We are now looking for values for α, β, γ, δ based on a, b, c, d, e so that the change (12) provides a system (15) as simplified as possible; that is, with the highest number of unitary coefficients. Therefore we consider the system of equations

$$\frac{\beta^2\delta}{\alpha}a = 1, \quad \frac{\beta\gamma\delta}{\alpha}b = 1, \quad \frac{\gamma^2\delta}{\alpha}c = 1, \quad \frac{\alpha\delta}{\beta}d = 1, \quad \frac{\alpha\delta}{\gamma}e = 1.$$

Not all values of a, b, c, d, e give rise to a compatible system in the variables α, β, γ, δ. If we eliminate one of the five equations, this situation remains. But eliminating two equations we have compatible systems. We should get rid of equations in such a way that the solutions obtained afterwards have no roots, which leads to some of the parameters a, b, c, d, e having to be positive. We remember that the only condition for these values is that they are not null. Thereby if we eliminate the first two equations then we obtain a compatible system whose δ-based solutions are

$$\alpha = \frac{1}{ce^2\delta^3}, \quad \beta = \frac{d}{ce^2\delta^2}, \quad \gamma = \frac{1}{ce\delta^2}.$$

With this result, whatever value $\delta \neq 0$ we take, and in particular for $\delta = 1$, the normal form N_3 is transformed into (we have returned, in order to simplicity, to the variables x, y, z, t):

$$\mathsf{S}_3 : \begin{array}{l} \dot{x} = py^2 + qyz + z^2 \\ \dot{y} = x \\ \dot{z} = x \end{array}$$

where $p = \dfrac{ad^2}{ce^2} \in \mathbb{R} \setminus \{0\}$, $q = \dfrac{bd}{ce} \in \mathbb{R} \setminus \{0\}$. In short we have eliminated three of the five parameters and found a canonical form.

4. Application of the method to 3-dimensional QH systems of degree 2

The study of the integrability, that is, of the existence of first integrals, in a differential system depending on parameters, is generally a difficult problem. Except for some simple cases, this task is very hard and there are no completely satisfactory methods to solve it. One of the procedures available for this, provided we treat with QH systems, is the Yoshida method discussed in this article. With the intention of evaluating the capabilities of this method, without the support of any other integration tool, we will apply it to every 3-dimensional QH of degree 2, a wide set of systems whose canonical forms were obtained in the previous section. Note that in this paper and in dimension 3 a system is considered completely integrable when there are two functionally independent analytic first integrals, and from Proposition 3 this is equivalent to the existence of two functionally independent YFIs.

Theorem 20. *The differential systems corresponding to the canonical form S_i are completely integrable, and respectively have the functionally independent YFIs H_i and G_i indicated below, for $i = 3, 4, 8, 10, 18$.*

(a) $H_3(x, y, z) = y - z$
$G_3(x, y, z) = 3x^2 - (1 + 2p)\, y^3 + 3y^2z - 3(1 + q)\, yz^2 + (q - 1)\, z^3$
(b) $H_4(x, y, z) = x - py$
$G_4(x, y, z) = 3x^2 + 3py^2 - 2pz^3$
(c) $H_8(x, y, z) = 2x - pz^2$
$G_8(x, y, z) = 6xz - 3y^2 + (2 - 2p)\, z^3$
(d) $H_{10}(x, y, z) = z$
$G_{10}(x, y, z) = 3x^2 + 6pxz^2 - 2y^3$
(e) $H_{18}(x, y, z) = 2y - z^2$
$G_{18}(x, y, z) = 15x^2 - 30y^2z + 20yz^3 - 4z^5$

Proof. (a) The only balance of the S_3 canonical form is

$$\mathbf{c} = (-12, 6, 6) \,/\, (1 + p + q),$$

and its corresponding Kowalevskaya matrix is

$$K(\mathbf{c}) = \begin{pmatrix} 3 & \dfrac{12p+6q}{p+q+1} & \dfrac{12+6q}{p+q+1} \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

which provides the non-trivial Kowalevskaya exponents $\lambda_1 = 2$, $\lambda_2 = 6$. Note that when $p + q = -1$ there is no balance, and therefore no possibility of applying the method.

Following Theorem 10, and taking into account that $d = 2$, these exponents could provide YFIs of weight degrees, respectively, $k_1 = 2$ and $k_2 = 6$. We will now work with the first of them. We denote by H_3 a possible YFI of S_3 with weight degree $k_1 = 2$. In this case both H_3 and its monomials must verify (5) for $\mathbf{s} = (3, 2, 2)$. That is, if $Ax^a y^b z^c$ is a monomial of H_3, with $A \in \mathbb{R}$ and $a, b, c \in \mathbb{N}$, it must be verified that

$$A\alpha^{3a+2b+2c} x^a y^b z^c = \alpha^2 A x^a y^b z^c,$$

for any $\alpha \in \mathbb{R}^+$. Then the solutions in the domain of the natural numbers of the diophantine equation $3a + 2b + 2c = 2$ will provide the exponents of the potential monomials of H_3. The only solutions are $(0, 1, 0)$ and $(0, 0, 1)$, so to be quasi-homogeneous with weight vector $(3, 2, 2, 2)$, the polynomial must be of the form $H_3(x, y, z) = Ay + Bz$, being A and B real coefficients. Furthermore H_3 must meet (4) to be a first integral, this implies that $x(A + B) \equiv 0$ for all $x \in \mathbb{R}$. Taking $A = 1$, $B = -1$, we obtain the YFI

$$H_3(x, y, z) = y - z.$$

Now let G_3 be a possible YFI of S_3 with weight degree $k_2 = 6$. If $Ax^a y^b z^c$ is a monomial of G_3, it must verify $A\alpha^{3a+2b+2c} x^a y^b z^c = \alpha^6 A x^a y^b z^c$ for any $\alpha \in \mathbb{R}^+$, so we need the natural solutions of $3a + 2b + 2c = 6$. These are $(2, 0, 0)$, $(0, 3, 0)$, $(0, 2, 1)$, $(0, 1, 2)$ and $(0, 0, 3)$, therefore the polynomial must be of the form $G_3(x, y, z) = A^2 + By^3 + Cy^2 z + Dyz^2 + Ez^3$, being A, B, C, D and E real coefficients. It is easy to prove that for G_3 to also meet (4), its coefficients must verify the following system of equations:

$$2A + D + 3E = 0, \quad 2qA + 2C + 2D = 0, \quad 2pA + 3B + C = 0.$$

The solution $(A, B, C, D, E) = (3, -1 - 2p, 3, -3 - 3q, q - 1)$ provides the YFI

$$G_3(x, y, z) = 3x^2 - (1 + 2p)\, y^3 + 3y^2 z - 3(1 + q)\, yz^2 + (q - 1)\, z^3.$$

Regarding the case $p + q = -1$, where as we have pointed out, the method cannot be applied, we observe that the two YFIs obtained are also valid in this case.

Finally the YFIs H_3 and G_3 are functionally independent, because the matrix

$$\begin{pmatrix} \nabla H_3(x, y, z) \\ \nabla G_3(x, y, z) \end{pmatrix} = \begin{pmatrix} 0 & 6x \\ 1 & -3(1+2p)\, y^2 + 6yz - 3(1+q)\, z^2 \\ -1 & 3y^2 - 6(1+q)\, yz + 3(q-1)\, z^2 \end{pmatrix}^t$$

has rank 2 at all points except at most for the null measure set $x = 0$.

To obtain the YFIs of the rest of the canonical forms, the same procedure must be followed. The main data in the calculations of each case are the following:

(b) The balance is $\mathbf{c} = \left(-12p/(1+p)^2, -12/(1+p)^2, 6/(1+p)\right)$, which provides the non-trivial Kowalevskaya exponents $\lambda_1 = 3$ and $\lambda_2 = 6$ and, as $d = 2$, those are also the possible weight degrees for the YFIs. With the weight degree 3 we have H_4 and with 6 we obtain G_4, both being functionally independent. When $p = -1$ the system given has no balances, but the two YFIs achieved for $p \neq -1$ are also valid.

(c) The balance is $\mathbf{c} = \left(72p/(2+p)^2, -24/(2+p), 12/(2+p)\right)$, which provides the non-trivial Kowalevskaya exponents $\lambda_1 = 4$ and $\lambda_2 = 6$ and, as $d = 2$, those are also the possible weight degrees for the YFIs. With the weight degree 4 we have H_8 and with 6 we obtain G_8, both being functionally independent. When $p = 2$ the system has no balances, but the two YFIs achieved for $p \neq -2$ are also valid.

(d) Since the weight degree of this canonical form is $d = 3$, two equivalent balances could be expected for each equivalence class (Proposition 7). However, because the third component is canceled, both coincide at $\mathbf{c} = (-12, 6, 0)$. The Kowalevskaya exponents are $\lambda_1 = 3/2$ and $\lambda_2 = 6$, and therefore the potential weight degrees for the YFIs are 3 and 12. From the first we obtain H_{10} and from the second G_{10}, both functionally independent.

(e) Now $d = 4$ and three equivalent complex balances appear:

$$\mathbf{c_1} = \left(\frac{-4\gamma}{\delta^5}, \frac{2\gamma^2}{\delta^4}, \frac{2\gamma}{\delta^2}\right), \ \mathbf{c_2} = \left(\frac{4\alpha}{\delta^5}, \frac{2\alpha^2}{\delta^4}, \frac{-2\alpha}{\delta^2}\right),$$
$$\mathbf{c_3} = \left(\frac{-4\beta^2\gamma}{\delta^5}, \frac{-2\beta\gamma^2}{\delta^4}, \frac{2\beta^2\gamma}{\delta^2}\right),$$

where $\alpha = \sqrt[3]{-5}$, $\beta = \sqrt[3]{-1}$, $\gamma = \sqrt[3]{5}$ and $\delta = \sqrt[3]{3}$. The common Kowalevskaya exponents are $\lambda_1 = 4/3$ and $\lambda_2 = 10/3$, so the possible weight degrees for the YFIs are 4 and 10. From the first we obtain H_{18} and from the second G_{18}, both functionally independent. □

In all cases of Theorem 20 the Yoshida method provides the maximum level of information. All Kowalevskaya exponents give rise to a YFI, and therefore there can be no “hidden” analytic first integrals that are functionally independent of the rest. Based on Theorem 10 it is evident that for this to happen, the gradients of these YFIs cannot be zero on the balances. This is a matter that, as a mere confirmation, we can verify later. As an example,

$$\nabla H_{18}(\mathbf{c_1}) = \left(0, 2, -\frac{4\sqrt[3]{5}}{\left(\sqrt[3]{3}\right)^2}\right) \neq \mathbf{0}.$$

Theorem 21. *The differential systems corresponding to the canonical form* S_i *are not completely integrable for* $i = 5, 12, 13, 17$.

Proof. The only balance of the S_5 canonical form is $\mathbf{c} = (-720, 180, -60)$, and its corresponding Kowalevskaya matrix is

$$K(\mathbf{c}) = \begin{pmatrix} 5 & 0 & -120 \\ 1 & 4 & 0 \\ 0 & 1 & 3 \end{pmatrix},$$

which provides the non-trivial Kowalevskaya exponents $\lambda_{1,2} = \dfrac{13 \pm \sqrt{71}i}{2} \notin \mathbb{Q}$. The proof for this canonical form concludes by applying Theorem 13.

The proof is identical for the canonical forms S_{12}, S_{13} and S_{17}, which have non-rational Kowalevskaya exponents, respectively, $\dfrac{7 \pm \sqrt{11}i}{2}$, $\dfrac{19 \pm \sqrt{199}i}{6}$ and $\dfrac{5 \pm \sqrt{13}}{2}$. □

We are mainly interested in determining if there is complete integrability or not, because only in this case all the trajectories of the system can be controlled. However we can ask ourselves if any of these four canonical forms is partially integrable. That is, if they have one and only one functionally independent YFI. Obviously, if it exists it must be “hidden” because it does not fulfill the gradient condition. We are going to study, as an example, the canonical form S_5: first, we exclude using Corollary 16 to prove the nonexistence of analytic first integrals, because their conditions are not met

$(\lambda_1 + \lambda_2 - 13 = 0)$. Therefore without imposing the gradient condition, we are left with Theorem 15: if there is a YFI of weight degree $k \in \mathbb{Z}^+$, there must be $\alpha_1,\ \alpha_2 \in \mathbb{N}$ such that

$$\alpha_1 \left(\frac{13 + \sqrt{71}i}{2} \right) + \alpha_2 \left(\frac{13 - \sqrt{71}i}{2} \right) = \frac{k}{2-1}, \qquad 0 < \alpha_1 + \alpha_2 \leq k.$$

Then $\sqrt{71}i\,(\alpha_1 - \alpha_2) = 2k - 13\,(\alpha_1 + \alpha_2) \in \mathbb{Z}$, and as a consequence $\alpha_1 = \alpha_2 = \alpha \in \mathbb{Z}^+$. We conclude that $k = 13\alpha$, and this means that the only possible weight degrees k would be the multiples of 13.

Following a procedure of construction of first integrals identical to that carried out in the proof of Theorem 20, and with the help of software of computational algebra, we have verified that there are no YFI of weight degree less than or equal to 104 for this canonical form.

We carried out the same study with the forms S_{12}, S_{13} and S_{17} and it is proved that, if they are partially integrable, the YFI's weight degrees would be multiples of 14, 19 and 10, respectively. And based on this, also as in the previous case we have to discard the existence of first integrals up to 112, 114 and 100 weight degrees, respectively.

In the same way as in the canonical forms of Theorem 20, we can affirm that the Yoshida method provides a powerful answer in the cases covered by the previous Theorem: there is no complete integrability. However the rest of the canonical forms present difficulties of various kinds for the method, which we classify below.

4.1. Problematic cases

i) Not similarity invariant systems (d=1). When all the weight degrees of a QH system verify $d = 1$, it is not a similarity invariant system. In this case the vector $\mathbf{g}$ cannot be defined, and therefore none of the results of the Yoshida method can be applied. We are in this situation with the canonical forms S_1, S_2, S_6 and S_7.

ii) There are no balances. When the system (6) has only the trivial solution, there are no balances, and then there is no possibility of applying the method either. Within our family of canonical forms, this situation occurs in S_{20}, whose system (6) is as follows and only has the solution $\mathbf{0}$:

$$\frac{4x}{3} = 0, \qquad \frac{2y}{3} + xz = 0, \qquad x + y^2 + \frac{z}{3} = 0.$$

It is noteworthy that this canonical form has a very evident YFI of weight degree 4, $H_{20}(x, y, z) = x$. Furthermore, the YFI $G_{20}(x, y, z) = 6xy - 3xz^2 + 2y^3$, of weight degree 6, can be easily found. As both first integrals are functionally independent, this canonical form is completely integrable. This shows that the Yoshida method can even “fail” in very trivial cases.

iii) Not all Kovalevskaya exponents provide a YFI. The canonical form S_{11} has a unique balance $\mathbf{c} = (-144/5, -12, 6)$, which in turn provides the non-trivial Kowalevskaya exponents $\lambda_1 = 5$ and $\lambda_2 = 6$. From the second of them we obtain, using Theorem 10, the YFI $H_{11}(x, y, z) = 3y^2 - 2z^3$. However the Kowalevskaya exponent $\lambda_1 = 5$ does not provide any first integral, which does not exclude the existence of another independent one. In fact, using alternative integration methods to that of Yoshida, we know that it exists, because the divergence of S_{11} is null and we can apply Corollary 6 of Llibre et al. (2015). It is evident that, if analytical, this hidden first integral G_{11} would verify $\nabla G_{11}\,(\mathbf{c}) = \mathbf{0}$. However G_{11} has been searched using software running all its possible weight degrees up to 20, and the only YFI found are the powers of H_{11}. We conjecture that G_{11} is not analytical.

iv) Kowalevskaya exponents depend on parameters. The Yoshida method does not seem efficient to study the integrability of these kind of canonical forms, which moreover coincide in our study group with those in which there are degree 2 monomials in the three components $\dot{x}$, $\dot{y}$, $\dot{z}$. These are cases S_9, S_{14}, S_{15}, S_{16} and S_{19}. The first problem with these canonical forms arises with balances. In all five cases, very complicated parameter-dependent balances appear, obviously obtained through the use of software. As they are made up of too many lines each, we cannot reproduce them here. However in all forms except S_9 there are also simple balances that allow to work with them. In summary:

- Form S_9 has five unrepeatable balances.
- Form S_{14} has three balances. Two of them are unrepeatable and a third is $\mathbf{c}_{14} = (0, 0, -1)$.
- Form S_{15} has three balances. Two of them are unrepeatable and a third is $\mathbf{c}_{15} = (0, 0, -1/q)$.
- Form S_{16} has six balances, although they are really reduced to three, because the weight degree is $d = 3$ (see Propositions 7 and 9). Of these three, one is unrepeatable and the other two are $\mathbf{c}^1_{16} = (0, 0, -1/r)$ and $\mathbf{c}^2_{16} = (2 - 4r, 0, -2)$.
- Form S_{19} has four balances, although they are really reduced to two, because the weight degree of the form is $d = 3$. Of these one is unrepeatable and the other is $\mathbf{c}_{19} = (0, -1/q, 0)$.

In any case a treatable Kowalevskaya exponent is obtained from one of those unrepeatable balances (except trivial -1, which always appears), reaching up to 50 pages in some cases. From the simple balances, existing in the forms S_{14}, S_{15}, S_{16} and S_{19}, simple exponents do emerge, although all of them with the common characteristic of being dependent on the parameters of the canonical form. It is noteworthy that when there are parameters of type ± 1 (S_{16} and S_{19} forms), the choice made does not affect the Kowalevskaya exponents that are obtained. Summarizing:

- Form S_{14} has the non-trivial Kowalevskaya exponents, obtained from $\mathbf{c}_{14}$,

$$\lambda_{1,2} = \left(4 - p - s \pm \sqrt{p + 4qr - 2ps + s^2}\right)/2.$$

- Form S_{15} has the non-trivial Kowalevskaya exponents, obtained from $\mathbf{c}_{15}$,

$$\lambda_1 = 3 - {}^1\!/_q, \quad \lambda_2 = 2 - {}^p\!/_q.$$

- Form S_{16} has two balances. From $\mathbf{c}^1_{16}$ are obtained

$$\lambda_1 = 2 - {}^1\!/_r, \quad \lambda_2 = {}^3\!/_2 - {}^q\!/_r$$

  and from $\mathbf{c}^2_{16}$ we have

$$\lambda_1 = {}^3\!/_2 - 2q, \quad \lambda_2 = 2 - 4r.$$

- Form S_{19} has the non-trivial Kowalevskaya exponents, obtained from $\mathbf{c}_{19}$,

$$\lambda_1 = {}^1\!/_2 - {}^1\!/_q, \quad \lambda_2 = {}^3\!/_2 - {}^p\!/_q.$$

In general the information that we can extract from these exponents is very scarce. Based on Theorem 13 we can establish the most relevant result, which is applied to three of the canonical forms:

Proposition 22. *The differential systems corresponding to the canonical forms* S_{15}, S_{16} *and* S_{19} *are not completely integrable when any of their parameters is irrational.*

But the rest of the theory provides little more: with Theorems 10 and 12 we can establish conditions for potential weight degrees, but the “hidden by the gradient” YFIs will always be left out. For example it is trivial that a necessary condition for the canonical form S_{15} to have two not hidden YFIs is that its parameters p, q verify

$$(p, q) \in \{({}^m\!/_n, -{}^1\!/_n) \mid m, n \in \mathbb{Z} \setminus \{0\}\,;\ -1 \le m;\ -2 \le n\}\,.$$

Similar conditions can be obtained for S_{16} and S_{19}. Another requirement, which only adds information to the above if there are non-rational parameters, can be extracted from Theorem 15:

Proposition 23. *A necessary condition so that the differential systems corresponding to the canonical form* S_{15} *(respectively* S_{16}*, respectively* S_{19}*) can have some analytic first integral is that q be an affine function of p,* $q = Ap + B$*, being A and B rationals (respectively r of q, respectively q of p).*

Proof. To be an analytic first integral of S_{15} there must be a YFI of a certain weight degree $k \in \mathbb{Z}^+$. In such a case there must exist $\alpha_1, \alpha_2 \in \mathbb{N}$ verifying $0 < \alpha_1 + \alpha_2 \leq k$ and $\alpha_1 (3 - 1/q) + \alpha_2 (2 - p/q) = k$. This is equivalent to have $q = Ap + B$ with $A = \alpha_2 / 3\alpha_1 + 2\alpha_2 - k$, $B = \alpha_1 / 3\alpha_1 + 2\alpha_2 - k$. The cases S_{16} and S_{19} are identical. □

We can know little about the canonical forms S_9 and S_{14}. If we disregard the method and test with possible low weight degrees, we observe for example that S_9 has a YFI of weight degree 1 if and only if the components $\dot{y}$ and $\dot{z}$ are proportional, i.e., if $s/p = u/q = v/r = w$. This first integral is $wy - z$. Some other partial results can be obtained.

In summary, the problem with these canonical forms when applying the method is that their Kowalevskaya exponents depend on the parameters of the form, in addition to being, in many cases, intractable due to their complexity, even using *Mathematica* software. In those cases the only knowledge that the Yoshida method provides are some situations of non-integrability, along with particular results that have little value and that we do not reproduce.

4.2. Conclusions

The Yoshida method has traditionally been undervalued as mere theory with no practical application. This is because, in principle, it does not provide information about the expression of the searched first integrals, but only about their weight degrees. Furthermore, the main result (Theorem 10) presents the mentioned problem of "hidden" YFIs. Thus, until recently, almost all articles devoted to the topic were purely theoretical works which did not carry out the effective calculation of first integrals. But more recently, works like Llibre et al. (2015) do contribute with practical applications. A technique is developed that, based on the knowledge of the weight exponents that all YFI must have, is capable of constructing the first integral through the resolution of diophantine equations. However, in order to use this technique, it is necessary to clarify the theory, which gives it a formal structure and which proves some results that are being used as true. We hope to have achieved that clarification in subsection 2.1, mainly contributing with the concept of Yoshida First Integral (YFI) and Proposition 3, which equates the number of analytic integrals of interest with that of YFIs.

From these considerations it is necessary to evaluate the usefulness of the method as a tool for the integration of quasi-homogeneous systems. In our study we have obtained conclusive results for 9 out of 20 analyzed forms. For this reason, we can conclude that the Yoshida method is useful in the practical study of analytical integrability, but that, like any other tool dedicated to this complex task, it has notable limitations that make it necessary to complement it with other techniques.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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3.4. Factor de impacto de las publicaciones

Se recoge en esta sección los índices principales que acreditan la calidad de las revistas en que se han publicado los trabajos que forman parte de esta memoria. Se reflejarán los índices más importantes derivados del *Journal of Citation Reports* de *Web of Science* (índice de impacto, índice de impacto a 5 años e índice de influencia) y dos criterios alternativos, basados en los datos de *Scopus* y utilizados cada vez de manera más general, como son el índice *CiteScore* y el *Scimago Journal Rank (SJR)*.

- El artículo de la Sección 3.1 fue publicado en la revista Qualitative Theory of Dynamical Systems en el año 2018 y los indicadores correspondientes a ese año son:
  - **JCR:**
    Índice de Impacto: **0.986**. Cuartil **Q2** (Tercil **T1**) en Mathematics. Posición: 95 de 314.
    Indice de impacto de 5 años: **0.923**. Cuartil **Q2** en Mathematics.
    Índice de Influencia: **0.395**. Cuartil **Q3** en Mathematics.
  - **SCOPUS**:
    CiteScore: **1.7**. Cuartil **Q2** en Discrete Mathematics and Combinatory. Posición: 23 de 71.
    SJR: **0.44**. Cuartil **Q2** en Applied Mathematics. Posición: 257 de 552.
- Los artículos de las Secciones 3.2 y 3.3 fueron publicados en la revista Journal of Symbolic Computation en 2019 y 2021. Por razones obvias sólo están disponibles los indicadores de 2019 que reflejamos a continuación:
  - **JCR**:
    Índice de Impacto: **0.673**. Cuartil **Q4** en Mathematics, Applied. Posición: 215 de 261.
    Indice de impacto de 5 años: **0.890**. Cuartil **Q3** en Mathematics, Applied.
    Índice de Influencia: **0.559**. Cuartil **Q2** en Computer Science, Theory and Methods.
  - **SCOPUS**:
    CiteScore: **2.4**. Cuartil **Q1** en Algebra and Number Theory. Posición: 12 de 99.
    SJR: **0.488**. Cuartil **Q2** en Computational Mathematics. Posición: 76 de 186.

Capítulo 4

Discusión de resultados, conclusiones y líneas futuras

4.1. Discusión de resultados y conclusiones

Se considera, como conclusión general, que los resultados obtenidos a lo largo del desarrollo de esta tesis tienen un alto grado de conexión con los objetivos fijados inicialmente (ver Capítulo 2). Los frutos de esta investigación han permitido:

- Diseñar e implementar los algoritmos que proporcionan los sistemas quasi-homogéneos para cada grado fijado en dimensiones 2 y 3, lo que constituye un recurso muy útil para los investigadores que trabajan con dichos sistemas.
- Obtener el número de formas canónicas de los sistemas quasi-homogéneos en el plano, utilizando para ello la función indicatriz de Euler, una herramienta importante de la Teoría de Números.
- Describir y delimitar los sistemas quasi-homogéneos maximales, lo que permitió una simplificación importante en la clasificación en el caso de dimensión 3 y se espera que lo sea también en dimensiones superiores.
- Identificar las propiedades más importantes de los sistemas quasi-homogéneos, profundizando especialmente en la estructura de sus vectores peso y las relaciones entre sus coeficientes.
- Analizar con detalle el método de Yoshida para la obtención de integrales primeras polinomiales, así como las aportaciones realizadas posteriormente por otros autores para el perfeccionamiento del método.
- Obtener, para el caso de sistemas quasi-homogéneos de dimensión 3 y grado 2, todas las integrales primeras que proporciona el método

de Yoshida cuando este resulta efectivo, y poner de manifiesto las fortalezas y debilidades del método cuando se aplica en este contexto.

4.2. Líneas futuras de trabajo

Los próximos objetivos se basan en el desarrollo con más amplitud de las lineas generales desarrolladas en esta memoria, así como en la introducción de una nueva linea en el campo de los sistemas diferenciales fraccionarios. En particular, se pretende:

- Obtener un algoritmo para la generación de los sistemas quasi-homogéneos en una dimensión n arbitraria.
- Estudiar la dinámica global de aquellos sistemas quasi-homogéneos de grado 2 en dimensión 3 que sean completamente integrables a través de integrales primeras polinomiales.
- Utilizar métodos alternativos para la obtención de integrales primeras polinomiales en los casos en los que el método de Yoshida no resulta eficiente.
- Extender el concepto de sistema quasi-homogéneo al ámbito del Cálculo Fraccionario.

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