Article

A Note on Some Reduction Formulas for the Incomplete Beta Function and the Lerch Transcendent

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Abstract: We derive new reduction formulas for the incomplete beta function $B(\nu, 0, z)$ and the Lerch transcendent $\Phi(z, 1, \nu)$ in terms of elementary functions when ν is rational and z is complex. As an application, we calculate some new integrals. Additionally, we use these reduction formulas to test the performance of the algorithms devoted to the numerical evaluation of the incomplete beta function.

1. Introduction

The origin of the beta function $B(\nu, \mu)$ goes back to Wallis' attempt of the calculation of π [1]. For this purpose, he evaluated the integral

$$B(\nu, \mu) = \int_0^1 t^{\nu-1}(1-t)^{\mu-1}dt, \tag{1}$$

where ν and μ are integers or $\mu = 1$ and ν is rational. Moreover, Wallis suggested that [2] (p. 4)

$$\frac{\pi}{4} = \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{1/2}dx = \frac{1}{4}\lim_{n\to\infty}\left(\frac{2\cdot 4\cdot 6\,\cdots\,2n}{1\cdot 3\cdot 5\,\cdots\,(2n-1)}\frac{1}{\sqrt{n}}\right)^2.$$

This result may have led Euler to consider the integral (1) for ν and μ not necessarily integers and its relation to the gamma function. In fact, Euler derived the following relation between the beta and gamma functions [2] (Equation 1.1.13):

$$B(\nu, \mu) = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}.$$

A natural generalization of the beta function is the incomplete beta function, defined as [3] (Equation 8.17.1)

$$B(\nu, \mu, z) = \int_0^z t^{\nu-1}(1-t)^{\mu-1}dt, \qquad 0 \le z \le 1,\ \nu, \mu > 0, \tag{2}$$

where it is straightforward to continue analytically to complex values of ν, μ, and z.

Many applications have been developed over time regarding the $B(\nu, \mu, z)$ function. For instance, in statistics it is used extensively as the probability integral of the beta distribution [4] (pp. 210–275). Additionally, it appears in statistical mechanics for Monte Carlo sampling [5], in the analysis of packings of granular objects [6], and in growth formulas in cosmology [7]. Therefore, to evaluate the $B(\nu, \mu, z)$ function, it is quite interesting to have reduction formulas to simplify its computation, both symbolically and numerically. For

instance, when $\mu = m + 1$ is a positive integer (i.e., $m = 0, 1, 2, \ldots$), we have the following reduction formula in terms of elementary functions [8] (Equation 58:4:3)

$$\mathrm{B}(\nu, m + 1, z) = z^{\nu} \sum_{k=0}^{m} \binom{m}{k} \frac{(-z)^k}{k + \nu}. \tag{3}$$

However, when $\mu = 0$, the incomplete beta function is given in terms of the Lerch transcendent [8] (Equation 58:4:4)

$$\mathrm{B}(\nu, 0, z) = z^{\nu} \Phi(z, 1, \nu), \quad \nu > 0, \tag{4}$$

where the Lerch transcendent is defined as [9] (Equation 1.11(1))

$$\Phi(z, s, \nu) = \sum_{k=0}^{\infty} \frac{z^k}{(k + \nu)^s}, \quad |z| < 1, \ \nu \neq 0, -1, -2, \ldots \tag{5}$$

It is worth noting that (3) can be proved by induction from (4) and (5), applying the connection formula [8] (Equation 58:5:3):

$$\mathrm{B}(\nu, \mu, z) = \mathrm{B}(\nu + 1, \mu, z) + \mathrm{B}(\nu, \mu + 1, z).$$

Nevertheless, reduction formulas for $\mathrm{B}(\nu, 0, z)$ when ν is a rational number do not seem to be reported in the most common literature. The aim of this note is just to provide such reduction formulas in terms of elementary functions. As an application, we will calculate some new integrals in terms of elementary functions. Additionally, we will check that the numerical evaluation of the incomplete beta function is improved with these reduction formulas.

This paper is organized as follows: In Section 2 we derive reduction formulas for $\mathrm{B}(\nu, 0, z)$, for ν both positive rational and negative rational. Particular cases of the reduction formulas for $\nu = n$ and $\nu = n + 1/2$ (where n is a non-negative integer) are also considered. In Section 3, we apply the reduction formulas derived in Section 2 to calculate some integrals which do not seem to be reported in the most common literature. Furthermore, for particular values of the parameters, the symbolic computation of these integrals is quite accelerated using the aforementioned reduction formulas. Moreover, we use these reduction formulas to numerically test the performance of the algorithm provided in MATHEMATICA$^{\mathrm{TM}}$ to compute the incomplete beta function.

2. Main Results

First, note that according to (4) and (5),

$$\mathrm{B}(\nu, 0, z) = \sum_{k=0}^{\infty} \frac{z^{k+\nu}}{k + \nu}. \tag{6}$$

The series in Equation (6) is divergent for non-positive integral values of ν. Therefore, we will consider two separate cases in this section: $\nu \in \mathbb{Q}^+$ and $\nu \in \mathbb{Q}^- \backslash \{-1, -2, \ldots\}$. Here $\mathbb{Q}^+$ and $\mathbb{Q}^-$ denote the sets of positive and negative rational numbers, respectively.

2.1. Case $\nu \in \mathbb{Q}^+$

Consider $\nu = n + r > 0$ where $n = \lfloor \nu \rfloor \geq 0$ is the integer part of ν and $0 \leq r \leq 1$. From (6), we have

$$\begin{aligned}
\mathrm{B}(n + r, 0, z) &= \sum_{k=0}^{\infty} \frac{z^{k+n+r}}{k + n + r} = \sum_{k=n}^{\infty} \frac{z^{k+r}}{k + r} \\
&= \sum_{k=0}^{\infty} \frac{z^{k+r}}{k + r} - \sum_{k=0}^{n-1} \frac{z^{k+r}}{k + r}.
\end{aligned} \tag{7}$$

Set $r = 1$ in (7) and then apply the Taylor expansion [3] (Equation 4.6.1)

$$\log(1+z) = -\sum_{k=1}^{\infty} \frac{(-z)^k}{k},$$

to obtain

$$B(n+1,0,z) = -\log(1-z) - \sum_{k=1}^{n} \frac{z^k}{k}, \quad n = 0,1,2,\ldots \tag{8}$$

Furthermore, set $r = 1/2$ in (7) and apply the Taylor expansion [3] (Equation 4.38.5)

$$\tanh^{-1} z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}, \tag{9}$$

to obtain

$$B\left(n+\frac{1}{2},0,z\right) = 2\left(\tanh^{-1}\sqrt{z} - \sum_{k=0}^{n-1} \frac{z^{k+1/2}}{2k+1}\right), \quad n = 0,1,2,\ldots \tag{10}$$

More generally, set $r = p/q \in \mathbb{Q}$ in (7) with p,q coprimes. Then,

$$B\left(n+\frac{p}{q},0,z\right) = z^{p/q}\sum_{k=0}^{\infty} \frac{z^k}{k+p/q} - \sum_{k=0}^{n-1} \frac{z^{k+p/q}}{k+p/q}, \tag{11}$$

Rewrite the first sum of (11) as a hypergeometric function (see [2] (p. 61–62)),

$$\sum_{k=0}^{\infty} \frac{z^k}{k+p/q} = \frac{1}{p/q} \, {}_2F_1\left(\begin{array}{c} 1, p/q \\ 1+p/q \end{array}\Big| z\right). \tag{12}$$

Apply now the reduction formula [10] (Equation 7.3.1.131)

$$\begin{aligned} &{}_2F_1\left(\begin{array}{c} 1, p/q \\ 1+p/q \end{array}\Big| z\right) \\ &= -\frac{p}{q} z^{-p/q} \sum_{k=0}^{q-1} \exp\left(-\frac{2\pi i p k}{q}\right) \log\left(1 - z^{1/q}\exp\left(\frac{2\pi i k}{q}\right)\right), \\ &p, q = 1,2,\ldots; \; p \le q. \end{aligned} \tag{13}$$

Therefore, taking into account (12) and (13), rewrite (11) as the following result.

Theorem 1. *For $\nu = n + \frac{p}{q} \in \mathbb{Q}^+$, with $n = \lfloor \nu \rfloor$ and p,q coprimes, the reduction formula*

$$B(\nu,0,z) = z^{\nu}\Phi(z,1,\nu) \tag{14}$$

$$= -\sum_{k=0}^{q-1} \exp\left(-\frac{2\pi i p k}{q}\right) \log\left(1 - z^{1/q}\exp\left(\frac{2\pi i k}{q}\right)\right) - \sum_{k=0}^{n-1} \frac{z^{k+p/q}}{k+p/q}.$$

holds true.

Remark 1. *Notice that the reduction formula (10) is included in (14), but not (8), which is a singular case.*

2.2. Case $v \in \mathbb{Q}^- \backslash \{-1, -2, \ldots\}$

Consider $v = -n + r < 0$ where $n = \lfloor |v - 1| \rfloor \geq 0$, and $0 < r < 1$. From (6), we have

$$
\begin{aligned}
\mathrm{B}(-n + r, 0, z) &= \sum_{k=0}^{\infty} \frac{z^{k-n+r}}{k - n + r} = \sum_{k=-n}^{\infty} \frac{z^{k+r}}{k + r} \\
&= \sum_{k=0}^{\infty} \frac{z^{k+r}}{k + r} + \sum_{k=1}^{n} \frac{z^{-k+r}}{-k + r}.
\end{aligned}
\tag{15}
$$

Taking $r = 1/2$ and applying again (9), we have

$$
\mathrm{B}\left(-n + \frac{1}{2}, 0, z\right) = 2 \left(\tanh^{-1} \sqrt{z} - \sum_{k=1}^{n} \frac{z^{-k+1/2}}{2k - 1} \right).
\tag{16}
$$

More generally, take $r = p/q \in \mathbb{Q}$ with p, q coprimes in (15), and apply (12) to obtain the following result.

Theorem 2. *For $v = -n + \frac{p}{q} \in \mathbb{Q}^-$, with $n = \lfloor |v - 1| \rfloor$ and p, q coprimes, the reduction formula*

$$
\mathrm{B}(v, 0, z) \tag{17}
$$
$$
= -\sum_{k=0}^{q-1} \exp\left(-\frac{2\pi i p k}{q}\right) \log\left(1 - z^{1/q} \exp\left(\frac{2\pi i k}{q}\right)\right) + \sum_{k=1}^{n} \frac{z^{p/q-k}}{p/q - k}.
$$

holds true.

Remark 2. *Notice that (16) is included in (17) as a particular case. Additionally, in (17), $\mathrm{B}(v, 0, z) \neq z^v \Phi(z, 1, v)$ since (4) does not hold true for $v < 0$.*

3. Applications

In this section, we apply the reduction formulas obtained in Section 2 to express certain integrals in terms of elementary functions and evaluate the incomplete beta function with some specified arguments. Additionally, we will use these reduction formulas as a benchmark for the computation of the incomplete beta function.

3.1. Calculation of Integrals

Straightforward from the definition of the incomplete beta function given in (2), we obtain the following integral representation:

$$
\mathrm{B}(v, \mu, z) = z^v \int_0^1 t^{v-1} (1 - zt)^{\mu-1} dt.
\tag{18}
$$

Additionally, an integral representation of the Lerch transcendent is [9] (Equation 1.11(3))

$$
\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt, \quad \mathrm{Re}\, v > 0.
\tag{19}
$$

Notice that from (4), (18) and (19), we have

$$
I(v, z) := \int_0^1 \frac{t^{v-1}}{1 - zt} dt = \int_0^{\infty} \frac{e^{-(v-1)t}}{e^t - z} dt = z^{-v} \mathrm{B}(v, 0, z), \quad \mathrm{Re}\, v > 0.
\tag{20}
$$

Therefore, from (14) and (20), we have for $\nu = n + \frac{p}{q} \in \mathbb{Q}^+$, with $n = \lfloor \nu \rfloor$ and p, q coprimes,

$$
\begin{aligned}
I(\nu, z) & \\
= \; -z^{-n-p/q} \sum_{k=0}^{q-1} \exp\left(-\frac{2\pi ipk}{q}\right) & \log\left(1 - z^{1/q} \exp\left(\frac{2\pi ik}{q}\right)\right) - \sum_{k=0}^{n-1} \frac{z^{k-n}}{k+p/q}.
\end{aligned}
$$
(21)

As another example, in the literature we found [8] (Equation 58:14:7)

$$
\int_0^z \tanh^{2\lambda-1} t \; dt = \frac{1}{2} \mathrm{B}\left(\lambda, 0, \tanh^2 z\right), \quad \mathrm{Re}\,\lambda > 0.
$$
(22)

Therefore, from (14) and (22), we have for $\lambda = n + \frac{p}{q} \in \mathbb{Q}^+$, with $n = \lfloor \lambda \rfloor$ and p, q coprimes,

$$
\begin{aligned}
\int_0^z \tanh^{2\lambda-1} t \; dt & \\
= \; -\frac{1}{2} \sum_{k=0}^{q-1} \exp\left(-\frac{2\pi ipk}{q}\right) & \log\left(1 - (\tanh z)^{2/q} \exp\left(\frac{2\pi ik}{q}\right)\right) \\
& -\frac{1}{2} \sum_{k=0}^{n-1} \frac{(\tanh z)^{2(k+p/q)}}{k+p/q}.
\end{aligned}
$$
(23)

The integral given in (23) generalizes the results found in the literature for $\lambda = n + 1$ and $\lambda = n + \frac{1}{2}$ with $n = 0, 1, 2, \ldots$ [11] (Eqns. 2.424.2–3).

3.2. Numerical Evaluation

From a numerical point of view, the reduction Formulas (14) and (17) are quite useful to plot $\mathrm{B}(\nu, 0, z)$ as a function of ν in the real domain. However, for some real values of ν and z, we obtain a complex value for $\mathrm{B}(\nu, 0, z)$. In these cases, the imaginary part of $\mathrm{B}(\nu, 0, z)$ is not always easy to compute. Figure 1 shows the plot of $\mathrm{Im}(\mathrm{B}(\nu, 0, z))$ as a function of z for $\nu = 12.3$. The reduction formula (14) shows the correct answer, i.e., $\mathrm{Im}(\mathrm{B}(\nu, 0, z)) = -\pi$, meanwhile the numerical evaluation of $\mathrm{Im}(\mathrm{B}(\nu, 0, z))$ with MATHEMATICA$^{\mathrm{TM}}$ diverges from this result. A similar feature is observed using (17) and a negative value for ν. It is worth noting that the equivalent numerical evaluation of $\mathrm{Im}(z^\nu \Phi(z, 1, \nu))$ with MATHEMATICA$^{\mathrm{TM}}$ yields also $-\pi$.
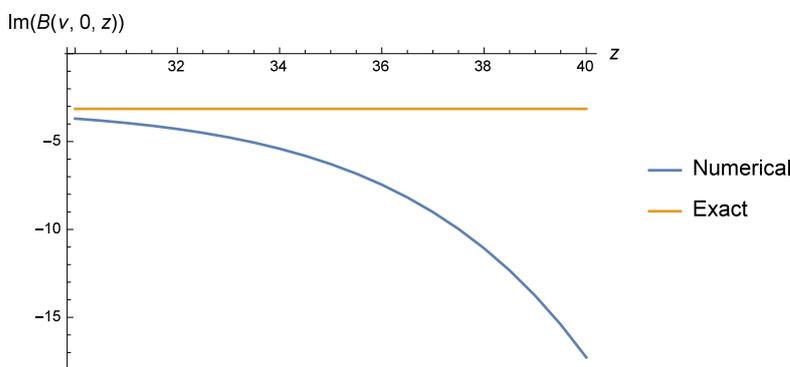
Figure 1. Evaluation of $\mathrm{Im}(\mathrm{B}(\nu, 0, z))$ with MATHEMATICA$^{\mathrm{TM}}$ and (14) with $\nu = 12.3$.

4. Conclusions

On the one hand, we have derived in (14) and (17) new expressions for the incomplete beta function $\mathrm{B}(\nu, 0, z)$ and the Lerch transcendent $\Phi(z, 1, \nu)$ in terms of elementary functions when ν is rational and z is complex. Particular formulas for non-negative integers values of ν and for half-integer values of ν are given in (8), (10) and (16) respectively.

On the other hand, we have calculated the integrals given (20) from the reduction Formulas (14) and (17) and the integral representation of the incomplete beta function and the Lerch transcendent. Additionally, in (23), the integral $\int_0^z \tanh^\alpha t \, dt$ is calculated in terms of elementary functions for $\alpha \in \mathbb{Q}$ and $\alpha > -1$. It is worth noting that (23) generalizes the results found in the literature, which are restricted to $\alpha = n + 1$ and $\alpha = n + 1/2$ with $n = 0, 1, 2, \ldots$.

Finally, with the aid of the reduction Formulas (14) and (17), we have tested that the numerical algorithm provided by MATHEMATICA$^{\text{TM}}$ sometimes fails to compute the imaginary part of B$(\nu, 0, z)$. Additionally, the reduction Formulas (14) and (17) are numerically useful to plot B$(\nu, 0, z)$ as a function of ν in the real domain.

All the results presented in this paper have been implemented in MATHEMATICA$^{\text{TM}}$ and can be downloaded from https://bit.ly/2XT7UjK, (accessed on 24 June 2021).

References

1. Wallis, J. *Arithmetica Infinitorum*; Leon Lichfield Academia Typographi: Oxford, UK, 1656.
2. Andrews, G.E.; Askey, R.; Roy, R.; Roy, R.; Askey, R. *Special Functions*; Cambridge University Press: Cambridge, UK, 1999; Volume 71.
3. Olver, F.W.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. *NIST Handbook of Mathematical Functions Hardback and CD-ROM*; Cambridge University Press: Cambridge, UK, 2010.
4. Johnson, N.L.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*; John Wiley & Sons: Hoboken, NJ, USA, 1995; Volume 2.
5. Kofke, D.A. Comment on "The incomplete beta function law for parallel tempering sampling of classical canonical systems" [J. Chem. Phys. 120, 4119 (2004)]. *J. Chem. Phys.* **2004**, *121*, 1167. [CrossRef] [PubMed]
6. Prellberg, T.; Owczarek, A.L. Stacking models of vesicles and compact clusters. *J. Stat. Phys.* **1995**, *80*, 755–779. [CrossRef]
7. Hamilton, A. Formulae for growth factors in expanding universes containing matter and a cosmological constant. *Mon. Not. R. Astron. Soc.* **2001**, *322*, 419–425. [CrossRef]
8. Oldham, K.B.; Myland, J.; Spanier, J. *An atlas of Functions: With Equator, the Atlas Function Calculator*; Springer Science & Business Media: Berlin, Germany, 2010.
9. Bateman, H. *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA, 1953; Volume 1.
10. Prudnikov, A.; Brychkov, Y.A.; Marichev, O. *Integrals and Series Volume 3: More Special Functions*; Taylor and Francis: Oxford, UK, 2003.
11. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*; Academic Press: Cambridge, MA, USA, 2014.