

Two species of semilocal cosmic strings in Abelian gauged $\mathbb{C}\mathbb{P}^2$ -sigma models coupled to gravity

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We study a semilocal Abelian Higgs sigma model with $\mathbb{C}\mathbb{P}^2$ -valued scalar fields coupled to gravity. We establish the conditions needed for self-duality in this system and obtain cosmic strings, both of self-dual character and out of the self-dual point, in the reference chart of $\mathbb{C}\mathbb{P}^2$. In any of the other two charts of a minimum atlas, the transition functions break the global $SU(2)$ symmetry to the Abelian subgroup $e^{i\sigma^3\alpha}$. In this context, using the transition functions, a new self-dual structure can be constructed, which defines a second species of cosmic strings. Because of the presence of a potential energy for the scalar fields, each species takes values in a different bounded region of the target space and is most naturally described using a particular chart of $\mathbb{C}\mathbb{P}^2$.

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I. INTRODUCTION

Topological defects [1] are long lasting solutions of spontaneously broken field theories whose stability is due to the existence of topologically disconnected sectors in the field configuration space, a property inherited from the nontriviality of certain homotopy group of the vacuum manifold \mathcal{V} , depending on the dimension of the space and the symmetry of the defect. In many important cases, e.g., grand unified theories (GUTs), the vacuum manifold is a homogeneous coset space $\mathcal{V} = \frac{G}{H}$ where G is the symmetry group of the system and H the subgroup of G surviving symmetry breaking. Prominent among these topological defects are strings or vortices in four-dimensional (4D) space-time that grow themselves along a spacelike curve. Thus, the sources of string defects are point defects arising when $\Pi_1(\mathcal{V}) \neq 0$ and according to the exact homotopy sequence are classified by $\Pi_0(H)$. The core of the string is a filament of false vacuum which carries a mass per unit length μ of order v^2 , where v is the scale of symmetry breaking which, for grand unified theories, is of order 10^{16} GeV. Thus, cosmic strings formed during the GUT phase transition are highly massive, $\mu \simeq 10^{22}$ g/cm, and they have a major influence on cosmology and galaxy

formation [2]. Given that the great mass density is accompanied by a similar tension, curved parts of the string rapidly contract and disappear, so that, finally, the strings settle themselves in straight line configurations. In principle, the existence of string solutions is possible in theories with only global symmetries, although in this case the Goldstone mode which comes together with symmetry breaking makes the mass of the string diverge logarithmically with the distance to the center. If the symmetry is local, however, the gauge fields neutralize the Goldstone mode and give rise to strings with finite mass per unit length, and whose core, apart from false vacuum, contains also trapped magnetic flux. Gravitational effects are also very different in the global and local cases. In the former, there is a gravitational force which decreases inversely with the distance from the center, while in the latter the gravitational field far from the string is effectively null. This is due to the exponential decay of the fields with distance, and to the fact that the string enjoys both translational and boost invariance along its direction, implying that the gravitational pulls of mass density and tension exactly cancel. There is only one remnant in the form of a deficit angle, such that the string appears from long distance as a conic singularity in space. The most important physical effects revealing the presence of a cosmic string are thus gravitational lensing, discontinuities in the Doppler effect for particle moving toward the string, and the distortions created by the conic geometry in the gravitational or electromagnetic fields of test particles [1].

The most simple model for understanding the structure of a cosmic string is, apart from the string with Dirac delta

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profile of [3], the exact solution of Gott [4]. In this solution, the plane transverse to the string direction is divided into two regions: the core, with a finite radius and uniform mass density, and the exterior zone, in which the energy-momentum tensor vanishes. Thus, the geometry of this plane changes abruptly at the borderline: the string interior conforms itself as a spherical cap, while the exterior is a flat cone. There is a critical value μ_{crit} for the mass density such that if $\mu < \mu_{\text{crit}}$ the cap is smaller than a hemisphere and the deficit angle of the exterior region is lower than 2π , while for $\mu > \mu_{\text{crit}}$ the transition between the spherical and conical geometries occurs above the equator, the deficit angle exceeds 2π , and the exterior cone is inverted like a dunce cap. This latter case corresponds to the so-called supermassive strings and, when it occurs, the exterior solution closes itself at finite distance and the transverse plane never reaches infinity. Instead, the vertex of the dunce cap can be thought of as a new string of zero radius located in front of the original one. Finally, when $\mu = 2\mu_{\text{crit}}$, both strings merge, the geometry closes into a complete sphere and there is not a string exterior.

The basic field theory which shows how cosmic strings with finite mass per unit length and a regular profile can arise in the context of realistic particle physics is the Abelian Higgs model (AHM)[5]. This theory describes the interaction between a complex scalar field and a $U(1)$ gauge field and displays an interesting phenomenology. When gravity is disregarded, the model constitutes a relativistic Ginzburg-Landau theory of superconductivity, with the dynamics depending on the balance between two nondimensional parameters, the gauge coupling e and the Higgs self-coupling λ . Superconductivity is thus of type I if $\beta = \frac{\lambda}{e^2} < 1$ and of type II if $\beta > 1$, while in the so-called self-dual limit $\beta = 1$ special features such as Bogomolny equations and Bogomolny-Prasad-Sommerfield (BPS) states emerge. $\Pi_1(\mathcal{V}) = \mathbb{Z}$ in the AHM and, therefore, the theory supports vortices which are classified by an integer n . In multivortex configurations, the forces among vortices are attractive if $\beta < 1$ and repulsive when $\beta > 1$ [6], whereas for $\beta = 1$ the vortices remain in equilibrium irrespectively of the distances among them. These results indicate that radially symmetric vortices with $n > 1$ tend to disaggregate when $\beta > 1$ but keep bound otherwise. Once gravity is included in the picture, the solutions become cosmic strings. Cosmic strings in the Abelian Higgs model were discovered by Garfinkle [7] and their stability was established by Gregory [8]. The self-dual case was subsequently investigated by Linet [9,10] and Comtet and Gibbons [11]. The AHM cosmic strings with nonvanishing cosmological constant were introduced in [12,13] and, in particular, the de Sitter case has been studied in [14] and the anti-de Sitter one in [15]. Along with the strings decaying asymptotically to Minkowski space-time plus a deficit angle, the AHM has also room for cosmic strings approaching a Kasner-Melvin geometry in the exterior region, and

also in this case there are the normal and supermassive regimes: for low μ , the length of circles around the string decreases with distance, whereas that when μ is high enough a circle of infinite perimeter is attained for a finite radius [16]. Finally, let us mention that, although in this case on a three-dimensional space-time, the AHM can be replaced by other closely related systems like Maxwell-Chern-Simons-Higgs or pure Chern-Simons-Higgs theories which also harbor gravitating vortices, and these solutions have been also investigated [17].

Along the years, the original Abelian Higgs model has been generalized in a variety of forms that conserve however some of its most salient properties, namely, the existence of type I and II superconductivities and of a self-dual regime. In particular, it has been considered the inclusion of a dielectric function depending on the scalar field [18], the incorporation of function multiplying the covariant derivatives, which plays the role of a metric in the scalar field target space [19], or the combined effect of both factors [20]. Also, the noncommutative version of the AHM has been studied, both in its original formulation [21] and with a dielectric function in place [22]. These modifications made it possible to ponder variants of the AHM in which the Higgs field is valued on a compact manifold like a sphere [23,24], and situations of this type have been also studied in the cosmic string context [25]. Another direction in which the Abelian Higgs model has been extended is the addition of more scalar fields, giving rise to the so-called semilocal models. The insertion of a new complex scalar field into the AHM enlarges the symmetry group to a tensor product of a global $SU(2)$ times a gauge $U(1)$ groups and makes the model portray the limit of the bosonic sector of electroweak theory for Weinberg angle $\theta_W = \frac{\pi}{2}$. The vacuum manifold is in this model simply connected, $\mathcal{V} = \mathbb{S}^3$, but there are nonetheless vortices, which are dynamically stable in the type I superconductivity domain, but whose magnetic field tends to disperse into broad lumps when superconductivity changes to type II [26–28]. In the self-dual regime, there is neutral equilibrium among confined vortices and lumps, and the widening of flux is thus a flat direction in moduli space. Models of this type have been recently studied coupled to gravity [29,30]. Here, as in the usual AHM, it is interesting to consider generalized semilocal models with dielectric functions or with scalar fields valued on compact manifolds, although in this case to attain self-duality there is the requirement that the metric in field space has to be Kähler. It should be noted that generalizations of this kind are well motivated physically and deserve a detailed analysis. In fact, as it is well known, self-duality has its roots in supersymmetry, but a likely role of supersymmetric field models in physics is to be low-energy effective theories describing the dynamics of other, more fundamental, degrees of freedom. From this perspective, while the ordinary semilocal model could be interpreted as the bosonic sector of a fundamental supersymmetric theory,

if we allow for nonrenormalizable interactions and try to build a low-energy effective theory we are forced to introduce a Kähler potential and a dielectric function [31], and are thus led to the type of generalized systems mentioned above.

In particular, with several scalar fields, the simplest case is a semilocal sigma model with target space $\mathbb{C}\mathbb{P}^2$. This model was studied in [23] focusing on the analysis and description of the two species of vortices found in the system. The central theme of the present article is to extend the treatment given in [23] to the gravitating case, finding thus $\mathbb{C}\mathbb{P}^2$ -valued semilocal cosmic strings of two species. We organize the paper as follows. Section II defines the model to be studied and fixes the field equations and boundary conditions obeyed by the cosmic strings encompassed in it in the reference chart. In Sec. III, self-duality of the model is established and radially symmetric solutions representing self-dual cosmic strings found. Non-self-dual cosmic strings are the subject of Sec. IV. In Sec. V, we shift the playground to the second chart of $\mathbb{C}\mathbb{P}^2$ and describe the self-dual cosmic strings arising in a new type of semilocal theory where the global symmetry group is also $U(1)$. Some final comments are offered in Sec. VI.

Throughout this work, we follow the seminal papers [7,9,10], and the excellent book [1], adapting the treatment given there for the gravitating Abelian Higgs model to the case of the sigma $\mathbb{C}\mathbb{P}^2$ theory. Apart from the original paper [26] in which they appeared for the first time, another very useful references to deal with semilocal models in Minkowski space-time are [27,28].

II. THE SEMILOCAL NONLINEAR $\mathbb{C}\mathbb{P}^2$ -SIGMA MODEL COUPLED TO GRAVITY

We shall deal with a version of the Abelian Higgs model in which the complex scalar fields are maps from Minkowski space-time to the complex projective space $\mathbb{C}\mathbb{P}^2$. This complex manifold is the quotient manifold of $\mathbb{C}^3 - \{(0,0,0)\}$ modulo proportionality, i.e., modulo the equivalence relation $(z_1, z_2, z_3) \simeq (z'_1, z'_2, z'_3)$ if $z'_a = w z_a$, $a = 1, 2, 3$ for some complex number $w \neq 0$. This manifold has complex dimension two and a minimum atlas is formed by three charts V_1, V_2 , and V_3 , such that $z_a \neq 0$ in V_a and the chart is equipped with a pair of complex inhomogeneous coordinates by dividing the three-tuples of each equivalence class by their z_a element. $\mathbb{C}\mathbb{P}^2$ is endowed with a Kähler structure in a natural way such that the metric tensor, the Fubini-Study metric is, in the reference chart with inhomogeneous coordinates (ψ_1, ψ_2) , of the form

$$ds^2_{\mathbb{C}\mathbb{P}^2} = h_{p\bar{q}}(\psi, \psi^*) d\psi_p d\psi_q^*, \quad h_{p\bar{q}} = \frac{\partial^2 K}{\partial \psi_p \partial \psi_q^*},$$

where the Kähler potential is

$$K(\psi, \psi^*) = 4v^2 \log \left(1 + \frac{|\psi_1|^2 + |\psi_2|^2}{v^2} \right) \equiv 4v^2 \log \mathcal{D}.$$

Explicitly,

$$h_{1\bar{1}} = \frac{4}{\mathcal{D}} - \frac{4|\psi_1|^2}{v^2 \mathcal{D}^2} \quad h_{1\bar{2}} = -\frac{4\psi_1^* \psi_2}{v^2 \mathcal{D}^2}$$

$$h_{2\bar{1}} = -\frac{4\psi_2^* \psi_1}{v^2 \mathcal{D}^2} \quad h_{2\bar{2}} = \frac{4}{\mathcal{D}} - \frac{4|\psi_2|^2}{v^2 \mathcal{D}^2}.$$

Notice that our conventions are such that if we pass from $\mathbb{C}\mathbb{P}^2$ to $\mathbb{C}\mathbb{P}^1 \simeq \mathbb{S}^2$ by taking one of the two inhomogeneous coordinates, say ψ_2 , to vanish, the Fubini-Study metric reduces to the standard round metric on \mathbb{S}^2 , with v the radius of the sphere and $\psi_1 = v \frac{y_1 + iy_2}{v - y_3}$, where (y_1, y_2, y_3) are real Euclidean coordinates in target space and $y_1^2 + y_2^2 + y_3^2 = v^2$. Since $h_{1\bar{1}}$ and $h_{2\bar{2}}$ are real and $h_{1\bar{2}} = h_{2\bar{1}}^*$, the metric is Hermitian and $h_{p\bar{q}} M_{p\bar{q}} \in \mathbb{R}$ if $M_{p\bar{q}}$ is a Hermitian matrix. Besides, the form of the Kähler potential makes it obvious that the Fubini-Study metric is isometric under $SU(2)$ transformations of the form $\chi_p = U_{pi} \psi_i$, $U \in SU(2)$.

We next consider physical models in which the inhomogeneous coordinates are promoted to complex scalar fields $\psi_1(x)$ and $\psi_2(x)$ living on curved four-dimensional space-time and taking values on $\mathbb{C}\mathbb{P}^2$. The action is of the form

$$S = \int d^4x \sqrt{-g} \mathcal{L}, \quad (1)$$

where we choose a space-time metric $g_{\mu\nu}$ of mostly plus signature and the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} \mathcal{E}(|\psi|) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} h_{p\bar{q}}(\psi, \psi^*) D_\mu \psi_p D^\mu \psi_q^* - \frac{1}{2} W^2(|\psi|). \quad (2)$$

We include a dielectric function \mathcal{E} and, for later convenience, we write the scalar potential as the square of a function W . Both \mathcal{E} and W depend only on $|\psi|^2 = |\psi_1|^2 + |\psi_2|^2$ in order to respect the $SU(2)$ isometry of the target manifold, and the covariant derivative is defined as $D_\mu \psi_p = \partial_\mu \psi_p - ie A_\mu \psi_p$. The Lagrangian thus exhibits an interplay between global $SU(2)$ and gauge $U(1)$ symmetries, fitting into the paradigm of semilocal theories. In the natural system of units where the Planck constant is 2π and the speed of light in vacuum is 1, the physical dimensions of fields and parameters in terms of mass M are $[A_\mu] = [\psi_p] = M$, $[e] = 1$, $[W] = M^2$, and $[\mathcal{E}] = [h_{p\bar{q}}] = 1$. The Kähler potential in turn has dimension $[K] = M^2$, since $[D] = 1$. A remark on notation: for space-time indices, we use the standard Einstein convention with upper and lower indices for, respectively, contravariant and covariant tensorial components, but for internal $\mathbb{C}\mathbb{P}^2$ indices, we think preferable to keep all indices down in order to reduce

cluttering. Repeated internal indices are summed; nevertheless, in some formulas, we will write explicitly summation symbols if it is convenient for clarity.

The Euler-Lagrange equations coming from (1) and (2) are

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\mathcal{E}(|\psi|)F^{\mu\nu}) = \frac{i}{2}eh_{p\bar{q}}(D^\nu\psi_p\psi_q^* - \psi_pD^\nu\psi_q^*), \quad (3)$$

$$\begin{aligned} & \frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g}h_{p\bar{q}}D^\mu\psi_p) \\ &= \frac{1}{2}\frac{\partial\mathcal{E}(|\psi|)}{\partial\psi_q^*}F_{\alpha\beta}F^{\alpha\beta} + \frac{\partial h_{p\bar{l}}}{\partial\psi_q^*}D_\lambda\psi_pD^\lambda\psi_l^* \\ &+ 2W(|\psi|)\frac{\partial W(|\psi|)}{\partial\psi_q^*} \end{aligned} \quad (4)$$

and must be solved in conjunction with the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}. \quad (5)$$

The energy-momentum tensor $T_{\mu\nu} = g_{\mu\nu}\mathcal{L} - 2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}$ takes the manifestly real and symmetric form

$$\begin{aligned} T_{\mu\nu} &= \mathcal{E}(|\psi|)F_{\mu\lambda}F_\nu^\lambda - \frac{1}{4}\mathcal{E}(|\psi|)g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}W^2(|\psi|) \\ &+ \frac{1}{2}(h_{p\bar{q}}D_\mu\psi_pD_\nu\psi_q^* + h_{q\bar{p}}D_\nu\psi_qD_\mu\psi_p^* \\ &- g_{\mu\nu}h_{p\bar{q}}D_\alpha\psi_pD^\alpha\psi_q^*). \end{aligned} \quad (6)$$

We are interested in configurations independent of the time $x^0 \equiv t$ and of the third spatial coordinate $x^3 \equiv z$, and whose energy per unit length along the z axis is finite. We choose thus the Weyl $A_0 = 0$ and axial $A_3 = 0$ gauges, with the consequence that the corresponding covariant derivatives and gauge field strengths identically vanish: $D_t\psi_p = D_z\psi_p = F_{tz} = F_{tk} = F_{zk} = 0$, where we denote indices 0 and 3 directly as t and z and use Latin indices i, j, k, \dots for the coordinates x^1 and x^2 in the plane perpendicular to the z axis. Accordingly with the situation that we are going to study, we will take for the metric the ansatz, [1,10],

$$ds^2 = e^{A(x^1, x^2)}(-dt^2 + dz^2) + \gamma_{ij}(x^1, x^2)dx^i dx^j, \quad (7)$$

with all coefficients depending only on the coordinates on the normal plane to z . This form of the metric gives a block-diagonal Einstein tensor with $G_{tz} = G_{tk} = G_{zk} = 0$ and requires boost invariance in the t - z plane, $G_{tt} = -G_{zz}$. This is consistent with the Einstein equations (5): using $g_{tk} = g_{zk} = g_{tz} = 0$ in (6), we see that $T_{tz} = T_{tk} = T_{zk} = 0$, whereas the nonvanishing elements are

$$\begin{aligned} T_{tt} &= -T_{zz} \\ &= e^A \left[\frac{1}{4}\mathcal{E}(|\psi|)F_{ij}F^{ij} + \frac{1}{2}h_{p\bar{q}}(\psi, \psi^*)D_k\psi_pD^k\psi_q^* \right. \\ &\quad \left. + \frac{1}{2}W^2(|\psi|) \right], \end{aligned} \quad (8)$$

$$\begin{aligned} T_{ij} &= \mathcal{E}(|\psi|)F_{ik}F_j^k - \frac{1}{4}\mathcal{E}(|\psi|)\gamma_{ij}F_{lm}F^{lm} - \frac{1}{2}\gamma_{ij}W^2(|\psi|) \\ &+ \frac{1}{2}(h_{p\bar{q}}D_i\psi_pD_j\psi_q^* + h_{q\bar{p}}D_j\psi_qD_i\psi_p^* \\ &- \gamma_{ij}h_{p\bar{q}}D_k\psi_pD^k\psi_q^*). \end{aligned} \quad (9)$$

Thus, the action per unit length along the z -axis for configurations independent of t and z is

$$S = \int dt d^2x e^A \sqrt{\gamma} \mathcal{L} = - \int dt d^2x \sqrt{\gamma} T_{tt},$$

and both members of the Euler-Lagrange equations (3) with $\nu = t$ or $\nu = z$ vanish identically, while in Eq. (4) for the scalar field there are only contributions coming from space-time indices 1 and 2. An important quantity is the energy per unit length along z -axis, which is given by

$$\begin{aligned} \mu &= - \int d^2x \sqrt{\gamma} T^t_t \\ &= \int d^2x \sqrt{\gamma} \left\{ \frac{1}{4}\mathcal{E}(|\psi|)F_{ij}F^{ij} + \frac{1}{2}h_{p\bar{q}}(\psi, \psi^*)D_k\psi_pD^k\psi_q^* \right. \\ &\quad \left. + \frac{1}{2}W^2(|\psi|) \right\} \end{aligned} \quad (10)$$

and coincides with minus the action per unit length and time only if $A = 0$.

We are interested in the solutions of the system which are cylindrically symmetric, the seeds indeed of the cosmic strings living in the model. Cylindrical symmetry makes it convenient to use polar coordinates $x^1 = \rho = \sqrt{x^2 + y^2}$ and $x^2 = \phi = \arctan \frac{y}{x}$ in the perpendicular plane to the z -axis and to assume for the fields an ansatz of the form

$$\begin{aligned} eA_\rho &= 0 & eA_\phi &= n - a(\rho), \\ \psi_p &= v f_p(\rho) e^{i\varphi_p(\phi)} \quad (\text{no sum}), \end{aligned} \quad (11)$$

with $p = 1, 2$, and

$$\varphi_1(\phi) = n_1\phi \quad \varphi_2(\phi) = n_2\phi + \omega, \quad (12)$$

where n, n_1 , and n_2 are integers and f_1 and f_2 are real functions. To fix the behavior of the fields at the center of the string and at infinity, we shall take for granted a fact to be justified later, namely, that the vacuum expectation value (vev) of the scalar fields is the same energy scale v

which determines the Kähler structure of $\mathbb{C}\mathbb{P}^2$, i.e., we shall assume that $W(v) = 0$. Also, we make use of the global $SU(2)$ symmetry to choose ψ_1 as the field component taking a nonzero vacuum expectation value at infinity, a choice which implies that the vorticity n_1 coincides with n . With these stipulations, regularity of the fields at $\rho = 0$ and finiteness of the energy per unit length of the string require the boundary conditions

$$f_1(0) = 0 \quad f_2(0) = \mathcal{H}_0 \delta_{n_2,0} \quad a(0) = n, \quad (13)$$

$$f_1(\infty) = 1 \quad f_2(\infty) = 0 \quad a(\infty) = 0, \quad (14)$$

where $\mathcal{H}_0 \in \mathbb{R}$. To complete the ansatz, we specify the form of the metric as in [7]

$$A = A(\rho) \quad \gamma_{ij} dx^i dx^j = d\rho^2 + H^2(\rho) d\phi^2. \quad (15)$$

Therefore, in order to forbid conic singularities at the origin and to ensure that at large distances the space-time is flat, with the coordinate t measuring the proper time of an asymptotic observer at rest, the boundary conditions

$$A'(0) = 0 \quad A(\infty) = 0, \quad (16)$$

$$H(0) = 0 \quad H'(0) = 1 \quad (17)$$

ought to be imposed, the prime denoting derivation with respect to ρ . Notice that by fixing the condition $A(\infty) = 0$ we discard from the beginning solutions leading to the Kasner-Melvin vacuum far from the origin and focus instead on a Minkowskian asymptotic space with a deficit angle $\Delta\phi$. This deficit angle is measured by the derivative of $H(\rho)$ at infinity according to the formula

$$\frac{\Delta\phi}{2\pi} = 1 - H'(\infty).$$

Moreover, cylindrically symmetric solutions will carry a magnetic field and a quantized magnetic flux given by

$$B = {}_*F^{tz} = \frac{1}{He^A} F_{\rho\phi}, \quad \Phi_M = \int d^2x H e^A B = \frac{2\pi n}{e} \quad (18)$$

as a consequence of the finiteness of μ . Notice that the quantization of the magnetic flux comes from the quantized vorticity of the vector field; see (11).

The metric (15) brings along a diagonal Einstein tensor with $G_{\rho\phi} = 0$, such that consistence with the Einstein equations (5) requires that the energy-momentum tensor is also diagonal. We can check that this is the case by plugging the ansatz (11), (15) into (9), verifying in this way that the only nonvanishing components of $T^\mu{}_\nu$ are

$$T^t{}_t = -\frac{1}{2}(\varepsilon_a + \varepsilon_f + \varepsilon_{af} + u)$$

$$T^z{}_z = -\frac{1}{2}(\varepsilon_a + \varepsilon_f + \varepsilon_{af} + u)$$

$$T^\rho{}_\rho = \frac{1}{2}(\varepsilon_a + \varepsilon_f - \varepsilon_{af} - u)$$

$$T^\phi{}_\phi = \frac{1}{2}(\varepsilon_a - \varepsilon_f + \varepsilon_{af} - u),$$

where we split the elements of the components of the energy-momentum tensor in the four terms used in the second paper of Ref. [16], which in our case are

$$\varepsilon_a = \frac{\mathcal{E}(|\psi|)}{e^2 H^2} \left(\frac{da}{d\rho} \right)^2$$

$$\varepsilon_f = v^2 \sum_{p=1}^2 \sum_{q=1}^2 h_{p\bar{q}} \frac{df_p}{d\rho} \frac{df_q}{d\rho} e^{i(\varphi_p - \varphi_q)}$$

$$\varepsilon_{af} = v^2 \sum_{p=1}^2 \sum_{q=1}^2 \frac{h_{p\bar{q}}}{H^2} \mathcal{A}_p \mathcal{A}_q f_p f_q e^{i(\varphi_p - \varphi_q)}$$

$$u = W^2(|\psi|), \quad \mathcal{A}_p(\rho) = a(\rho) + n_p - n.$$

We stress that although \mathcal{A}_p and φ_p carry an internal label, they are in fact $\mathbb{C}\mathbb{P}^2$ scalars.

Substitution of the cylindrically symmetric ansatz for the fields and the metric in the Euler-Lagrange equations (3) and (4) yields the second-order ordinary differential equation (ODE) system,

$$\begin{aligned} & \frac{d}{d\rho} \left(\frac{e^A}{H} \mathcal{E}(|\psi|) \frac{da}{d\rho} \right) \\ &= \frac{1}{2} e^2 v^2 \sum_{p=1}^2 \sum_{q=1}^2 \frac{e^A}{H} (\mathcal{A}_p + \mathcal{A}_q) h_{p\bar{q}} f_p f_q e^{i(\varphi_p - \varphi_q)} \quad (19) \end{aligned}$$

$$\begin{aligned} & \frac{v}{He^A} \sum_{p=1}^2 \frac{d}{d\rho} \left(He^A h_{p\bar{q}} \frac{df_p}{d\rho} e^{i\varphi_p} \right) - \frac{v}{H^2} \sum_{p=1}^2 h_{p\bar{q}} \mathcal{A}_p^2 f_p e^{i\varphi_p} - \frac{v^2}{H^2} \sum_{p=1}^2 \sum_{l=1}^2 \frac{\partial h_{p\bar{q}}}{\partial \psi_l} \mathcal{A}_p \mathcal{A}_l f_p f_l e^{i(\varphi_l + \varphi_p)} \\ &+ \frac{v^2}{H^2} \sum_{p=1}^2 \sum_{l=1}^2 \frac{\partial h_{p\bar{q}}}{\partial \psi_l^*} \mathcal{A}_p \mathcal{A}_l f_p f_l e^{i(-\varphi_l + \varphi_p)} \\ &= \frac{1}{e^2 H^2} \frac{\partial \mathcal{E}(|\psi|)}{\partial \psi_q^*} \left(\frac{da}{d\rho} \right)^2 + 2W(|\psi|) \frac{\partial W(|\psi|)}{\partial \psi_q^*} + v^2 \sum_{p=1}^2 \sum_{l=1}^2 \frac{\partial h_{p\bar{l}}}{\partial \psi_q^*} \left[\frac{df_p}{d\rho} \frac{df_l}{d\rho} + \frac{\mathcal{A}_p \mathcal{A}_l}{H^2} f_p f_l \right] e^{i(\varphi_p - \varphi_l)}, \quad (20) \end{aligned}$$

whereas regarding the Einstein equations, there is an argument given by Garfinkle for the AHM, which depends only on the regularity of the fields at $\rho = 0$ and energy-momentum conservation and it is therefore valid also for our system, which shows that only two of the three possible ones are independent. The independent equations can be expressed in different forms [1], and we choose here to write them as

$$\frac{1}{H} \frac{d}{d\rho} \left[\frac{dH}{d\rho} + \frac{1}{2} \left(H \frac{dA}{d\rho} \right) \right] + \frac{1}{4} \left(\frac{dA}{d\rho} \right)^2 = 8\pi G T'_t, \quad (21)$$

$$\frac{d^2 A}{d\rho^2} + \frac{3}{4} \left(\frac{dA}{d\rho} \right)^2 = 8\pi G T^\phi_\phi. \quad (22)$$

In particular, integration in ρ of Eq. (21) gives the relation between the deficit angle and the energy per unit length as [7]

$$\Delta\phi = 8\pi G\mu + \frac{\pi}{2} \int_0^\infty d\rho H \left(\frac{dA}{d\rho} \right)^2. \quad (23)$$

Thus, the problem of finding semilocal $\mathbb{C}\mathbb{P}^2$ cosmic strings consists in solving the second-order ODE system (19)–(22) with the boundary conditions (13), (14), (16), and (17). We will come back to this system later, but first we study in the next section a self-dual regime which makes the problem easier.

III. SELF-DUALITY IN THE SEMILOCAL $\mathbb{C}\mathbb{P}^2$ -SIGMA MODEL COUPLED TO GRAVITY

A. Self-duality and first-order field equations

The dynamics of the gravitating semilocal $\mathbb{C}\mathbb{P}^2$ -sigma model is governed by a system of quite complicated second-order differential equations allowing cosmic strings. Let us now search for a self-dual version of the model which simplifies matters by making room to first-order field ODE equations, to be satisfied not only by cylindrically symmetric configurations. In order to do so, we will assume from the start that $A = 0$ in the metric, such that the energy and action per unit length along z are proportional. This assumption simplifies also greatly the Einstein tensor, because it implies that the only nonvanishing Christoffel symbols are those with three Latin indices. Thus, the four-dimensional curvature scalar R coincides with the two-dimensional one $R^{(\gamma)}$ coming from γ_{ij} , whereas this metric, being two-dimensional, gives a trivial Einstein tensor, namely,

$$G_{tt} = -G_{zz} = \frac{1}{2} R^{(\gamma)} \quad G_{tk} = G_{zk} = G_{ij} = 0,$$

where $R^{(\gamma)}$ is the scalar curvature in the plane perpendicular to the string. This is consistent with $T_{tt} = -T_{zz}$, which, as

we have seen, is true in our model, but requires also $T_{ij} = 0$, the old Poincare stability criterion [32], a fact that will be necessary to check after the self-duality equations will be found. An important ingredient to establish this first-order ODE system is the Levi-Civita tensor of the γ_{ij} metric

$$\begin{aligned} \varepsilon_{ij} &= \sqrt{\gamma} \tilde{\varepsilon}_{ij} & \varepsilon^{ij} &= \frac{1}{\sqrt{\gamma}} \tilde{\varepsilon}^{ij} \\ \tilde{\varepsilon}_{12} &= -\tilde{\varepsilon}_{21} = \tilde{\varepsilon}^{12} = -\tilde{\varepsilon}^{21} = 1, \end{aligned}$$

which has some useful properties like

$$\begin{aligned} \varepsilon_{ij} \varepsilon^{ik} &= \delta_j^k & \varepsilon_{ij} \varepsilon^{ij} &= 2 \\ \varepsilon_{ij} \varepsilon_{kl} \gamma^{jl} &= \gamma_{ik} & \varepsilon^{ij} \varepsilon^{kl} \gamma_{jl} &= \gamma^{ik}. \end{aligned}$$

To proceed, we focus our attention on two quadratic expressions which are as follows:

(1) The first one, denoted X_F^2 , is given by

$$\begin{aligned} X_F^2 &= \left(\sqrt{\mathcal{E}(|\psi|)} F_{ij} \pm \varepsilon_{ij} W(|\psi|) \right) \\ &\quad \times \left(\sqrt{\mathcal{E}(|\psi|)} F^{ij} \pm \varepsilon^{ij} W(|\psi|) \right) \end{aligned}$$

and can be directly expanded into an expression containing two terms appearing in the energy density,

$$\begin{aligned} X_F^2 &= \mathcal{E}(|\psi|) F_{ij} F^{ij} + 2W^2(|\psi|) \\ &\quad \pm \frac{4}{\sqrt{\gamma}} \sqrt{\mathcal{E}(|\psi|)} W(|\psi|) F_{12}. \end{aligned}$$

(2) The second one, denoted as $|X_D|^2$, is

$$|X_D|^2 = h_{p\bar{q}} (D_i \psi_p \pm i \varepsilon_{ik} D^k \psi_p) (D^i \psi_q^* \mp i \varepsilon^{il} D_l \psi_q^*).$$

It may be recast as

$$\begin{aligned} |X_D|^2 &= 2h_{p\bar{q}} D_k \psi_p D^k \psi_q^* \mp \frac{4}{\sqrt{\gamma}} \mathcal{R} \quad \text{with} \\ \mathcal{R} &= \frac{i}{2} \sqrt{\gamma} h_{p\bar{q}} \varepsilon^{ij} D_i \psi_p D_j \psi_q^*. \end{aligned}$$

In the modulus-argument form of complex functions $\psi_p = |\psi_p| e^{i\chi_p}$ (no sum), the covariant derivatives read $D_i \psi_p = \psi_p (\partial_i \log |\psi_p| - i e V_{ip})$ (no sum) with $V_{ip} = A_i - \frac{1}{e} \partial_i \chi_p$. Introduce now the real symmetric matrix $M_{pq} = \frac{4}{D} (|\psi_p|^2 \delta_{pq} - \frac{|\psi_p|^2 |\psi_q|^2}{v^2 D})$ (no sum) to find

$$\mathcal{R} = e\sqrt{\gamma}\varepsilon^{ij}V_{ip}S_{jp} \quad \text{where}$$

$$S_{jp} = M_{pq}\partial_j \log |\psi_q| = 2\partial_j \frac{|\psi_p|^2}{\mathcal{D}}.$$

This implies

$$\mathcal{R} = 2e\partial_j \left(\sqrt{\gamma}\varepsilon^{ij}V_{ip} \frac{|\psi_p|^2}{\mathcal{D}} \right) + 2e \frac{|\psi_1|^2 + |\psi_2|^2}{\mathcal{D}} F_{12},$$

allowing to write finally $|X_D|^2$ as

$$|X_D|^2 = 2h_{p\bar{q}}D_k\psi_p D^k\psi_q^* \mp \frac{8e}{\sqrt{\gamma}}\partial_j \left(\sqrt{\gamma}\varepsilon^{ij}V_{ip} \frac{|\psi_p|^2}{\mathcal{D}} \right) \mp \frac{4e}{\sqrt{\gamma}}P(|\psi|)F_{12},$$

where

$$P(|\psi|) = 2 \frac{|\psi_1|^2 + |\psi_2|^2}{\mathcal{D}}. \quad (24)$$

The point of introducing X_F^2 and $|X_D|^2$ is that we can use them to write the energy per unit length (10) as

$$\mu = \int d^2x \sqrt{\gamma} \left\{ \frac{1}{4}X_F^2 + \frac{1}{4}|X_D|^2 \mp \frac{1}{\sqrt{\gamma}}\sqrt{\mathcal{E}(|\psi|)}W(|\psi|)F_{12} \pm \frac{2e}{\sqrt{\gamma}}\partial_j \left(\sqrt{\gamma}\varepsilon^{ij}V_{ip} \frac{|\psi_p|^2}{\mathcal{D}} \right) \pm \frac{e}{\sqrt{\gamma}}P(|\psi|)F_{12} \right\}. \quad (25)$$

Therefore, the choice of the ‘‘superpotential’’ $W(|\psi|)$ and the dielectric function $\mathcal{E}(|\psi|)$ such that

$$\sqrt{\mathcal{E}(|\psi|)}W(|\psi|) = e(P(|\psi|) - v^2) \quad (26)$$

shows μ as the sum of two squares plus a topological term proportional to the magnetic flux,

$$\mu = \int d^2x \sqrt{\gamma} \left\{ \frac{1}{4}X_F^2 + \frac{1}{4}|X_D|^2 \right\} \pm ev^2\Phi_M. \quad (27)$$

The choice (26) fixes the generic dependence of $W(|\psi|)$ on $\mathcal{E}(|\psi|)$,

$$U(|\psi|) \equiv \frac{1}{2}W^2(|\psi|) = \frac{e^2v^4}{2\mathcal{E}(|\psi|)} \left(\frac{|\psi_1|^2 + |\psi_2|^2 - v^2}{|\psi_1|^2 + |\psi_2|^2 + v^2} \right)^2, \quad (28)$$

which determines the self-dual character of the system. By this statement, we mean that the zeroes of X_F and X_D , the solutions of the first-order ODE system

$$F_{12} = \pm ev^2\sqrt{\gamma} \frac{v^2 - |\psi_1|^2 - |\psi_2|^2}{\mathcal{E}(|\psi|)(v^2 + |\psi_1|^2 + |\psi_2|^2)}, \quad (29)$$

$$D_1\psi_p \pm i\sqrt{\gamma}D^2\psi_p = 0 \quad (30)$$

are absolute minima of μ . The string tension of any solution of the first-order equations with the appropriate boundary condition is thus

$$\mu = 2\pi|n|v^2,$$

which coincides with the nongravitating $\mathbb{C}\mathbb{P}^2$ string tensions unveiled in Ref. [23] if we set $v^2 = \frac{a^2}{2}$ as the proportionality between the parameters introduced in the present work and [23].

Moreover, it is clear from the form of $U(|\psi|)$ that the set of its zeroes is the set of constant scalar fields such that $|\psi_1^{(v)}|^2 + |\psi_2^{(v)}|^2 = v^2$. Therefore, the vacuum orbit is \mathbb{S}^3 if $\mathcal{E}(|\psi|)$ has no poles in $\mathbb{C}\mathbb{P}^2$ and the vacuum expectation value of the scalar field is any point in \mathbb{S}^3 , e.g., $\psi_1^{(v)} = v$, $\psi_2^{(v)} = 0$, as announced in the previous section. A subtle point is hidden in the selection of the Kähler metric and the vacuum orbit in terms of a unique parameter. On one hand, $g^2 = \frac{1}{v^2}$ is the (dimensionful) coupling constant appearing in the nonlinear sigma model. On the other hand, v also sets the energy scale of the symmetry breaking. The dual role of v reveals a very economic structure of the space of parameters.

Configurations of finite energy density tend thus to \mathbb{S}^3 -vacuum orbit $|\psi_1^{(v)}|^2 + |\psi_2^{(v)}|^2 = v^2$ in the circle at spatial infinity in the plane perpendicular to the string, e.g., $|\psi_1| = v$, $|\psi_2| = 0$ for $|x| \rightarrow \infty$. As recently mentioned, the flux quantization condition (18) and the structure of (27) imply that the bound

$$\mu \geq ev^2|\Phi_M| = 2\pi v^2|n| \quad (31)$$

is saturated if the Bogomolny equations (29) and (30) are satisfied. The upper sign gives $F_{12} > 0$ ever that $|\psi_1|^2 + |\psi_2|^2 < v^2$ and corresponds thus to $\Phi_M > 0$, i.e., to positive vorticity n . Given the proportionality between μ and the action per unit length along the z axis, and the fact that solutions of the Bogomolny equations give absolute minima of μ , these solutions are also extrema of the actions and thus solutions of the Euler-Lagrange equations. It remains to check that solutions of (29) and (30) have vanishing T_{ij} . Let us write $T_{ij} = T_{ij}^{(F)} + T_{ij}^{(D)}$ with $T_{ij}^{(F)}$ and $T_{ij}^{(D)}$, respectively, the first and second lines of (9). Then, using (29), one finds

$$\begin{aligned} T_{ij}^{(F)} &= \gamma^{kl}\mathcal{E}(|\psi|)F_{ik}F_{jl} - \frac{1}{4}\gamma_{ij}(\mathcal{E}(|\psi|)F_{mn}F^{mn} + 2W^2(|\psi|)) \\ &= \gamma^{kl}\varepsilon_{ik}\varepsilon_{jl}W^2(|\psi|) - \frac{1}{4}\gamma_{ij}(\varepsilon_{mn}\varepsilon^{mn}W^2(|\psi|) \\ &\quad + 2W^2(|\psi|)) = 0. \end{aligned}$$

By means of (30) in turn, we unveil the identity

$$D_k \psi_p D^k \psi_q^* = \pm i \epsilon^{lm} D_l \psi_p D_m \psi_q^*,$$

from which we derive the formula

$$D_i \psi_p D_j \psi_q^* + D_j \psi_p D_i \psi_q^* = \pm i \gamma_{ij} \epsilon^{lm} D_l \psi_p D_m \psi_q^*,$$

making clear that $T_{ij}^{(D)} = 0$. Thus, the Bogomolny equations and the Einstein equations involving latin indices are compatible. Finally, we must consider together with (29) and (30) the only remaining Einstein equation

$$R^{(\gamma)} = 16\pi G T_{tt}. \quad (32)$$

We express the tt -component of the energy-momentum tensor as

$$T_{tt} = \frac{1}{2} h_{p\bar{q}} D_i \psi_p D^i \psi_q^* + \frac{e^2 v^4}{\mathcal{E}(|\psi_1|^2, |\psi_2|^2)} \left(\frac{|\psi_1|^2 + |\psi_2|^2 - v^2}{|\psi_1|^2 + |\psi_2|^2 + v^2} \right)^2,$$

using again the Bogomolny equation. We arrive thus to the complete system of equations for obtaining self-dual cosmic strings in the $\mathbb{C}P^2$ -sigma model.

In sum, self-duality fixes $W(|\psi|)$ in terms of $\mathcal{E}(|\psi|)$, but leaves still freedom to choose the dielectric function at will. In what follows, we will focus on the minimal and most natural choice $\mathcal{E}(|\psi|) = 1$. There are, however, other interesting possibilities. We mention two of them. First, by choosing

$$\mathcal{E}(|\psi|) = \frac{2}{\mathcal{D}} = \frac{2v^2}{v^2 + |\psi_1|^2 + |\psi_2|^2}, \quad (33)$$

the Bogomolny equations take the form

$$F_{12} = \pm \frac{e}{2} \sqrt{\gamma} (v^2 - |\psi_1|^2 - |\psi_2|^2), \quad (34)$$

$$D_1 \psi_p \pm i \sqrt{\gamma} D^2 \psi_p = 0, \quad (35)$$

which is exactly that of the Bogomolny equations of the standard gravitating semilocal model, even though we are now working with scalar fields taking values on $\mathbb{C}P^2$ rather than in \mathbb{C} . The difference is encoded in the Einstein equation (32), where the kinetic term in the tt -component of the energy-momentum tensor includes the metric of $\mathbb{C}P^2$, and thus the solutions of (34) and (35) curve the

space-time manifold in a different way than standard semilocal cosmic strings would do.

A second example is

$$\mathcal{E}(|\psi|) = \frac{\mu^2}{4e^2} \frac{\mathcal{D}^2}{|\psi_1|^2 + |\psi_2|^2} = \frac{\mu^2}{4e^2 v^4} \frac{(|\psi_1|^2 + |\psi_2|^2 + v^2)^2}{|\psi_1|^2 + |\psi_2|^2}. \quad (36)$$

In the nongravitational case, and working in $2+1$ dimensions, this dielectric function is the appropriate for obtaining self-dual Chern-Simons-Higgs vortices with target $\mathbb{C}P^2$ and Chern-Simons coupling μ . Thus, (36) gives self-dual gravitating vortices which converge to Chern-Simons vortices when the Newton constant tends to zero: $G \rightarrow 0$. They are not true Chern-Simons gravitating solitons, however: these carry not only magnetic, but also electric field, and thus angular momentum, which requires a metric with nondiagonal tk components instead of (7). Nevertheless, when Gv^2 is small, the role of these nondiagonal components in the solutions of [17] is subdominant, and the vortices obtained by means of (36) are good approximations to the solutions of the full Chern-Simons gravitating model. In particular, employing (36), one can extend to the gravitational case some of the phenomenology of Chern-Simons vortices, like the presence of solitons of both topological and nontopological nature whose existence is due to the fact that the vacuum orbit in this case includes also the points $\psi^{(v)} = 0$ and $\psi^{(v)} = +\infty$. Models with dielectric functions interpolating between (33) and (36) have also been investigated [22,33].

To end this subsection, let us finally compare our system with other very rich nonlinear field theory with $\mathbb{C}P^N$ as the target space: the $\mathbb{C}P^N$ model studied, e.g., in the paper [34] or [35] and reviewed in Sec. 4.5 of [36], where other relevant references can be found. These models are formulated in two-dimensional Euclidean space and include $N+1$ complex scalar fields arranged into a $N+1$ tuple $\vec{n}(x)$ subjected to the constraint $\vec{n}^*(x) \cdot \vec{n}(x) = 1$ and the $U(1)$ gauge identification $\vec{n}(x) \equiv \vec{n} e^{i\Lambda(x)}$. This system enjoys a very interesting dynamics and displays some of the typical features of non-Abelian gauge theories, like asymptotic freedom and the presence of instantons. In fact, the model can be described in terms of an Abelian gauge field minimally coupled to the scalars, and the instantons obey first-order equations, saturate a Bogomolny bound for the action, and are classified by a topological number proportional to the magnetic flux of the gauge field. In that sense, there are some remarkable similarities with the theory we are studying, but there are also important differences. In particular, in $\mathbb{C}P^N$ models, the gauge field is effective and does not represent any independent degrees of freedom, and thus the Maxwell term is

absent.¹ Also, like pure Yang-Mills theory in $(3+1)D$ Minkowski space-time, the $\mathbb{C}\mathbb{P}^N$ model is scale invariant, so that a potential like the one we have engineered is forbidden. In fact, if we ignore gravity and consider our x_1 - x_2 plane as the two-dimensional Euclidean space where a $\mathbb{C}\mathbb{P}^2$ model lives, it turns out that the self-duality equations for instantons, when expressed in any chart, are simply the Cauchy-Riemann conditions in the complex coordinate $z = x_1 + ix_2$ for the fields ψ_1 and ψ_2 . Thus, they coincide with our Bogomolny equations only if the coupling e , and thus the potential, vanishes. It seems therefore that our approach and the usual $\mathbb{C}\mathbb{P}^N$ models are in some sense complementary: because the scale of instantons ranges from zero to infinity, they disappear from the spectrum in our system. The two ingredients, gauge coupling and potential energy density, required to accommodate cosmic strings or vortices break scale invariance. This complementarity manifests itself also in that regular $\mathbb{C}\mathbb{P}^N$ -instantons embedded in our model have an infinite energy cost: e.g., the instanton with topological number one is in our coordinates

$$\psi_1(z) = \alpha \frac{z - z_0}{\omega}, \quad \psi_2(z) = \beta \frac{z - z_0}{\omega}, \quad (37)$$

where α and β are complex constants such that $|\alpha|^2 + |\beta|^2 = 1$ and z_0 and ω are, respectively, complex and real arbitrary constants reflecting the translational and scale invariance of the pure $\mathbb{C}\mathbb{P}^2$ model. Thus, for $|z| \rightarrow \infty$, the fields diverge in our nonscale-invariant system: they do not approach the vacuum orbit, the zeroes of the potential, at spatial infinity.

B. Self-dual semilocal $\mathbb{C}\mathbb{P}^2$ cosmic strings

Let us now specialize to cylindrically symmetric solutions. Thus, we take $x^1 = \rho$ and $x^2 = \phi$ and use the ansatz (11), (12) for the fields and the metric (15), working with $\mathcal{E}(|\psi|) = 1$. Plugging these expressions into the Bogomolny equations and the energy-momentum tensor, using that the curvature scalar of (15) is $R^{(\gamma)} = -\frac{2}{H} \frac{d^2 H}{d\rho^2}$, and turning to nondimensional quantities by means of the redefinitions

$$x = ev\rho \quad h(x) = evH(\rho) \quad 8\pi Gv^2 = \kappa^2,$$

we arrive to the ODE system which are the self-duality equations for cylindrically symmetric configurations,

$$\frac{da}{dx} = h \frac{f_1^2 + f_2^2 - 1}{f_1^2 + f_2^2 + 1}, \quad (38)$$

$$\frac{df_1}{dx} = \frac{a}{h} f_1, \quad (39)$$

$$\frac{df_2}{dx} = \frac{a + l - n}{h} f_2, \quad (40)$$

$$\frac{1}{h} \frac{d^2 h}{dx^2} = -\kappa^2 \left[m_{pq} \frac{df_p}{dx} \frac{df_q}{dx} + \left(\frac{f_1^2 + f_2^2 - 1}{f_1^2 + f_2^2 + 1} \right)^2 \right]. \quad (41)$$

For definiteness, we work with the upper sign of (29) and (30), we put $n_2 = l$, and define m_{pq} as the real symmetric matrix of elements

$$m_{pq} = \frac{4}{1 + f_1^2 + f_2^2} \left(\delta_{pq} - \frac{f_p f_q}{1 + f_1^2 + f_2^2} \right).$$

The boundary conditions are (13), (14), (16), and (17), with the understanding that the prime means now a derivative with respect to x . Besides, the nondimensional magnetic field $\mathcal{B} = \frac{B}{e^2 v^2}$ and energy density $\epsilon(x)$, defined by $\mu = 2\pi v^2 \int_0^\infty dx \epsilon(x)$, are as follows:

$$e\mathcal{B} = -\frac{1}{h} \frac{da}{dx},$$

$$\epsilon(x) = h(x) \left[m_{pq} \frac{df_p}{dx} \frac{df_q}{dx} + \left(\frac{f_1^2 + f_2^2 - 1}{f_1^2 + f_2^2 + 1} \right)^2 \right]. \quad (42)$$

The standard procedure in the search of self-dual cosmic string solutions is the following three-step shooting approach:

- (1) First, we solve analytically the system of ODE's (41) near the origin. If $x \simeq 0$, we assume the following form of the solutions:

$$f_1(x) \simeq \alpha_1 x^{r_1}, \quad f_2(x) \simeq \delta_{l0} \mathcal{H}_0 + \alpha_2 x^{r_2},$$

$$a(x) \simeq n - \delta x^s \quad (43)$$

at leading order. Here α_1 , α_2 , r_1 , r_2 , \mathcal{H}_0 , δ , and s are constants to be determined by solving the linearized system. For instance, in the case $l = 0$ and $\mathcal{H}_0 \neq 0$, Eq. (39) implies $r_1 = n$, Eq. (38) fixes $s = 2$ and $\delta = \frac{1 - \mathcal{H}_0^2}{2(1 + \mathcal{H}_0^2)}$, whereas (40) gives $r_2 = 2$ and $\alpha_2 = -\frac{1}{2} \delta \mathcal{H}_0$. If $l \neq 0$, we find again $r_1 = n$ and $s = 2$, but now (38) gives $\delta = \frac{1}{2}$ and (40) implies $r_2 = l$. In both cases, for $x \simeq 0$, (41) implies $h(x) \simeq x$ at leading order. To sum up, for x near zero, the leading order approximations for the fields and the metric are

¹A Maxwell term and a mass for scalars are generated by radiative corrections in the large N limit. In the effective one-loop theory, $U(1)$ is unbroken and there is confinement, but there are not instantons [35].

$$\begin{aligned}
f_1(x) &\simeq \alpha_1 x^n \\
f_2(x) &\simeq \delta_{l0} \mathcal{H}_0 \left[1 - \frac{1}{4} \frac{1 - \mathcal{H}_0^2}{1 + \mathcal{H}_0^2} x^2 \right] + (1 - \delta_{l0}) \alpha_2 x^l \\
a(x) &\simeq n - \frac{1 - \delta_{l0} \mathcal{H}_0^2}{2(1 + \delta_{l0} \mathcal{H}_0^2)} x^2 \quad h(x) \simeq x. \quad (44)
\end{aligned}$$

- (2) Second, we solve the system (41) for $x \rightarrow \infty$. A first consequence of the boundary conditions at infinity and (40) is that l is limited to the values $l = 0, 1, 2, \dots, n - 1$. Given that the energy-momentum vanishes in this region, (41) implies a linear profile for $h(x)$ and we approximate the metric very far from the origin in the form $h(x) \simeq c + rx$ where r and c are constants. Then, combining (39) and (40), it follows that $f_2(x) \simeq C x^{\frac{l-n}{r}} f_1(x)$ for large x at leading order and some constant C . If we put $f_1(x) = 1 - g(x)$ with $g(x)$ small, from (39) $a = -rx \frac{dg}{dx}$ and then from (38) we obtain the equation

$$x^2 \frac{d^2 g}{dx^2} + x \frac{dg}{dx} - x^2 g = -\frac{C^2}{2} x^{2\frac{r+l-n}{r}},$$

with solution $g(x) = \frac{C^2}{2} x^{2\frac{l-n}{r}}$ for $x \rightarrow \infty$. Thus, very far from the origin, the dominant behavior is

$$\begin{aligned}
f_1(x) &\simeq 1 - \frac{C^2}{2x^{2\frac{r+l-n}{r}}} & f_2(x) &\simeq \frac{C}{x^{\frac{n-l}{r}}} \\
a(x) &\simeq (n-l) \frac{C^2}{x^{2\frac{r+l-n}{r}}} & h(x) &\simeq rx. \quad (45)
\end{aligned}$$

Knowledge of the solution of the cosmic string gravitational field at infinity suffices to determine the deficit angle. Since $A = 0$ in the self-dual case, (23) and the saturation of the bound (31) imply $\Delta\phi = 2\pi\kappa^2 n$, both here and in the standard AHM. Thus, because $r = 1 - \frac{\Delta\phi}{2\pi}$, this parameter measures the relation $r = 1 - \frac{\kappa^2}{\kappa_{\text{crit}}^2}$ between the gravitational coupling and the critical coupling $\kappa_{\text{crit}} = \frac{1}{\sqrt{n}}$, which is the threshold for the formation of super-massive cosmic strings.

- (3) Third, and last step, we develop a numerical method to find a solution interpolating between the solution close to the origin with the solution near infinity. We integrate the self-duality equations (41) by means of an explicit Runge-Kutta method, starting from a small $x_0 = 10^{-6}$ value of the nondimensional radial coordinate and seeking for the value of α_1 in (44) which gives the best convergence of the fields to the boundary conditions at large x .

We show some figures with results for the scalar fields $f_1(x)$ and $f_2(x)$, the magnetic field, the energy density, and the metric coefficient $h(x)$. Figures 1 and 2 refer to the case with vorticity $n = 1$ and a value of $\mathcal{H}_0 = 0.4$, taking for

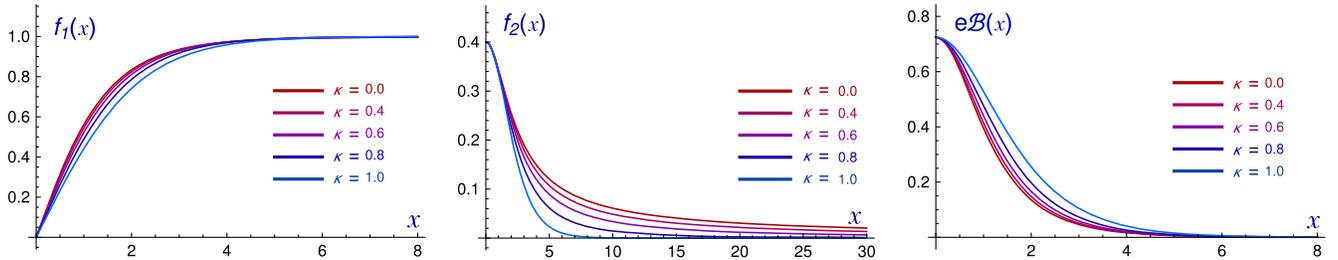


FIG. 1. Profiles of f_1 , f_2 and the magnetic field for $n = 1$ and $\mathcal{H}_0 = 0.4$ and several strengths of the gravitational interaction.

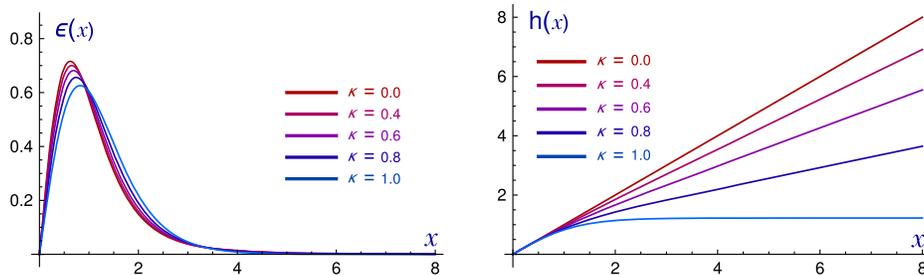


FIG. 2. Profiles of the energy density and the metric coefficient h for $n = 1$ and $\mathcal{H}_0 = 0.4$ and several strengths of the gravitational interaction.

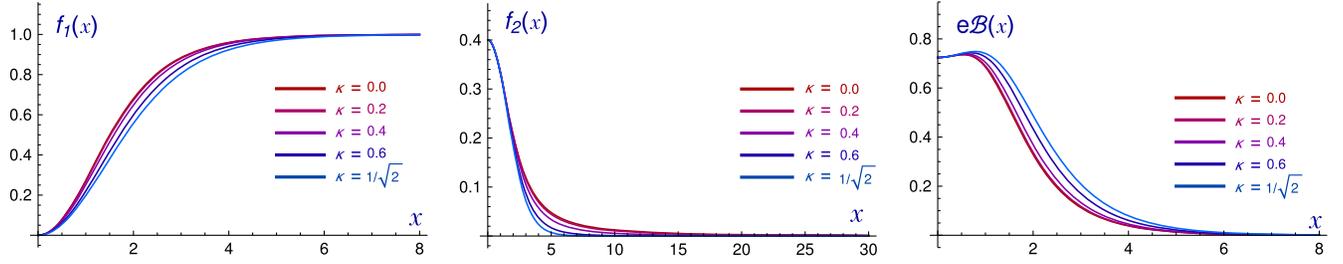


FIG. 3. Profiles of f_1 , f_2 and the magnetic field for $n = 2$, $l = 0$, and $\mathcal{H}_0 = 0.4$ and several strengths of the gravitational interaction.

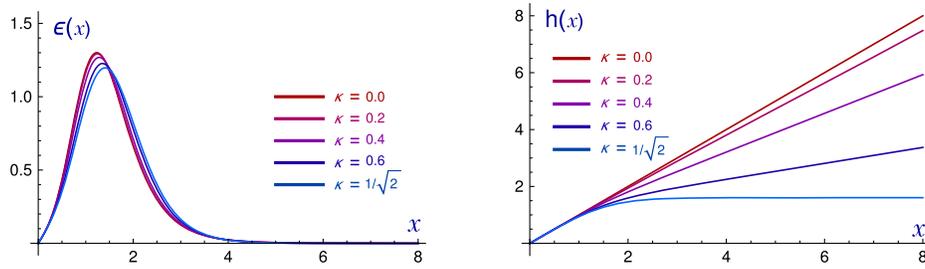


FIG. 4. Profiles of the energy density and the metric coefficient h for $n = 2$, $l = 0$, and $\mathcal{H}_0 = 0.4$ and several strengths of the gravitational interaction.

the gravitational coupling five values between $\kappa = 0$ and the critical value $\kappa_{\text{crit}} = 1$ at the frontier of supermassive behavior. In Figs. 3 and 4, we consider vorticity $n = 2$ with $l = 0$ and the same value of \mathcal{H}_0 than before, also with five values of the gravitational coupling growing up to the critical value $\kappa_{\text{crit}} = \frac{1}{\sqrt{2}}$ corresponding to that vorticity. Finally, Figs. 5 and 6 present also results for the $n = 2$

case, but now with $l = 1$ and $\alpha_2 = 0.4$ in (44). As we can see from the figures, for each x , the scalar fields are greater for smaller gravitational coupling, while the magnetic field shows the opposite behavior. The field f_1 grows monotonically with x , but it has nonzero slope at the center of the string only for vorticity $n = 1$, being the initial growing softer for higher n as a consequence of (44). For both

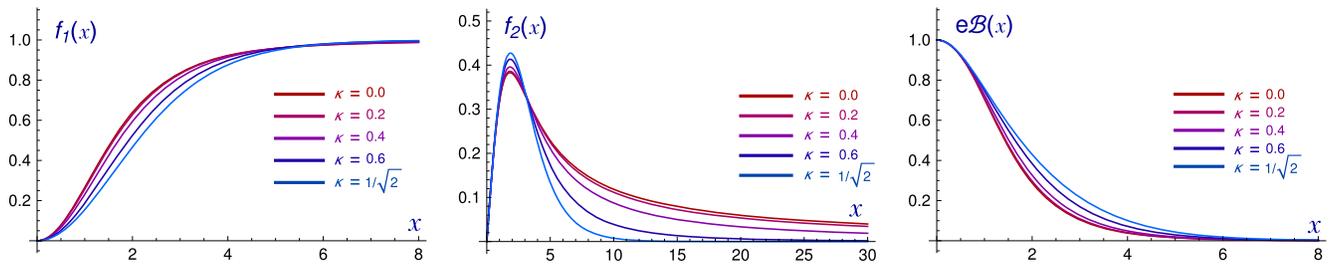


FIG. 5. Profiles of f_1 , f_2 and the magnetic field for $n = 2$, $l = 1$ and $\alpha_2 = 0.4$ and several strengths of the gravitational interaction.

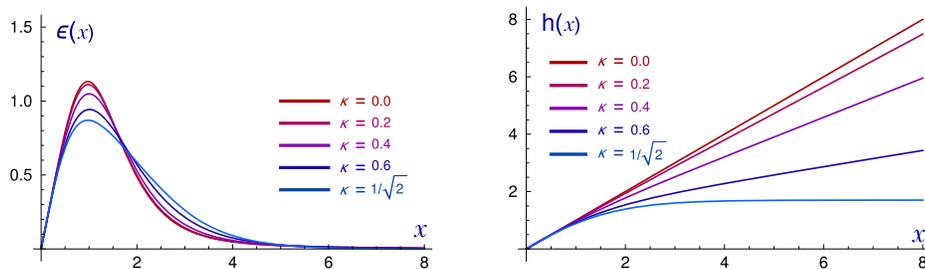


FIG. 6. Profiles of the energy density and the metric coefficient h for $n = 2$, $l = 1$ and $\alpha_2 = 0.4$ and several strengths of the gravitational interaction.

vorticities, f_2 decreases monotonically if $l = 0$, but when $n = 2$ and $l = 1$, f_2 vanishes at the center of the string and grows for small values of x , reaching a maximum for a range of values of x comparable to the range in which f_1 begins to be close to its vacuum expectation value. The magnetic field is maximum at $x = 0$ for $n = 1$ and $n = 2$ with $l = 1$, but in the case $n = 2$ with $l = 0$, the highest value of the magnetic field is displaced from the origin and configures a narrow annulus around the string center. In all cases, the energy density is zero at $x = 0$ and grows up to a maximum, which is higher for lower gravitational coupling; for $n = 1$ and $n = 2$, $l = 0$, the maximum is appreciably closer to the center for smaller κ , but when $n = 2$, $l = 1$, the distance from the maximum to the center is very similar for all values of κ . Also, in all cases, the metric coefficient $h(x)$ grows with x , reaching a clearly visible linear behavior quite soon, even before than the fields reach their asymptotic values. The slope of $h(x)$ decreases, and thus the deficit angle increases with κ , being 1 for $\kappa = 0$ and 0 for $\kappa = \kappa_{\text{crit}}$.

Although all the physical properties of the self-dual cosmic string solutions in the reference chart are precisely described in the formulas and figures in this subsection, we emphasize three qualitative features that distinguish them as belonging to a first species of string defects in this system:

- (1) All the energy lumps associated to these solutions are concentrated in the spatial plane perpendicular to the string, either decaying to their vacuum values exponentially when $f_2(\rho) = 0$ or more spread throughout the spatial plane if $f_2(\rho) \neq 0$. The crux of the matter is that all of them are mapped into the subset of the reference chart of \mathbb{CP}^2 comprised between the point $(f_1(0) = 0, f_2(0) = \delta_{10}\mathcal{H}_0)$ and the \mathbb{S}^3 -vacuum orbit.
- (2) In the case $f_2(x) = 0$, the situation is easier to visualize because the system reduces to target in a \mathbb{CP}^1 submanifold of \mathbb{CP}^2 , the vacuum orbit becomes the parallel $|\psi_1^{(v)}|^2 = v^2$, and the solution is mapped into the spherical cap bounded by this parallel and the South Pole $|\psi_1| = 0$.
- (3) Any attempt to prolongate these cosmic strings to the North Hemisphere would demand infinite energy because the fields should trespass the vacuum orbit.

\mathbb{CP}^2 instantons appear in our model when we decouple gravity ($G = 0$) and the gauge field ($e = 0$). They are similar to the solutions found in this subsection in that in both cases there is a modulus ruling their spatial size, the ω parameter of (37) for instantons and \mathcal{H}_0 for cosmic strings, although the origin of the instanton modulus is scale invariance and this symmetry is lost for cosmic strings. Objects of both types show however appreciable differences in the range that they cover on target space: the scalar fields of cosmic strings take values on a region of \mathbb{CP}^2 bounded by the vacuum orbit, whilst instantons

spread over all \mathbb{CP}^2 . As commented before, this makes instantons infinitely massive when we take $e > 0$ to obtain cosmic strings.

C. Multicenter self-dual cosmic strings

A generalization of the Atiyah-Singer index theorem led E. Weinberg in [37] to show that the moduli space of self-dual vortices in the AHM has dimensions $2n$ where n is the vorticity. This means that there is freedom of placing the centers of the flux tubes with a quantum of magnetic flux throughout the plane. The explicit zero modes corresponding to this freedom were found and explicitly constructed in Ref. [38] for the AHM. Identical task was performed in Ref. [39] about the vortex zero modes in the nonlinear \mathbb{CP}^1 -sigma model. In both papers, we started from the cylindrically symmetric multivortex solutions to find their deformations respecting self-duality. The index theorem showing that the dimension of the moduli space of self-dual vortices in the semilocal AHM is $4n$ was developed in Refs. [40,41]. The construction of the general self-dual vortices in the semilocal AHM in terms of the n -centers of the magnetic flux tubes plus other n complex parameters characterizing the behavior of the ψ_2 field was achieved in Ref. [28]. It is natural to explore how much this scheme survives the coupling to gravity. Thus, let us now briefly consider the existence of self-dual solutions beyond those with cylindrical symmetry. For this purpose, it is convenient to adopt for the metric on the transverse plane the conformal gauge $\gamma_{ij} = e^{-4V(x)}\delta_{ij}$ [10]. This gauge is advantageous in that the Bogomolny equation (30) takes in it the same form $D_1\psi_p + iD_2\psi_p = 0$ than in Euclidean coordinates, and thus we can use the Poincaré $\bar{\partial}$ -lemma to guarantee that the solution of (30) for $p = 1$ has exactly n zeros on the plane, counted with multiplicity [42]. Let us denote these zeros by z_1, z_2, \dots, z_n , where we are now using a complex coordinate $z = x^1 + ix^2$. Also, (30) for $p = 2$ implies that $\psi_2 = w(z)\psi_1$, where [28]

$$w(z) = \frac{Q_n(z)}{P_n(z)}, \quad P_n(z) = \prod_{s=1}^n (z - z_s),$$

$$Q_n(z) = q_{n-1}z^{n-1} + \dots + q_1z + q_0.$$

Using these expressions, the remaining Bogomolny equation (29) may be rewritten in the form

$$\Delta \log |\psi_1| = e^2 v^2 e^{-4V} \frac{|\psi_1|^2(1 + |w|^2) - v^2}{|\psi_1|^2(1 + |w|^2) + v^2} + 2\pi \sum_{s=1}^n \delta^{(2)}(z - z_s), \quad (46)$$

where Δ denotes here the Euclidean Laplacian, while the Einstein equation (32) is

$$\Delta \left[V - 4\pi G v^2 \left(\log \mathcal{D} - \frac{1}{2} \log \frac{|\psi_1|^2}{\prod_{s=1}^n |z - z_s|^2} \right) \right] = 0.$$

This means that the expression into the brackets is harmonic and bounded, and hence a constant. Thus, we obtain for the conformal factor exponent $V(x^1, x^2)$,

$$V = 4\pi G v^2 \left[\log(v^2 + |\psi_1|^2(1 + |w|^2)) - \frac{1}{2} \log \frac{|\psi_1|^2}{\prod_{s=1}^n |z - z_s|^2} \right] + \text{constant}. \quad (47)$$

Therefore, a solution of (46) with $V(x^1, x^2)$ given by (47) and such that $\psi_1 \rightarrow v$ for $|z| \rightarrow \infty$ and $\psi_1 = 0$ at the n zeros z_s represents a system of separated and parallel cosmic strings which are centered around the positions $z = z_s$. Of course, to show rigorously that (46) with these boundary conditions has a unique solution is a difficult problem in functional analysis which is beyond the scope of this work, but we can nevertheless argue through the heuristic physical arguments pointed in [10] that this is the case. We saw in previous subsection that out of the core of the vortex the metric converges very quickly to the Minkowskian form with a deficit angle, meaning that well-separated strings at rest do not feel any mutual gravitational forces. Therefore, the existence of a state of equilibrium for such a system depends only on the balance between the $U(1)$ and scalar interactions among the strings, and given that gravity does not enter in this balance, we can study it in the limit in which the gravitational coupling G vanishes. $G = 0$ implies $V = 0$ and with the change of variables

$$\xi = \log(|f_1|^2(1 + |w|^2)) - u_1,$$

$$u_1 = \log(|P_n(z)|^2 + |Q_n(z)|^2) - \sum_{s=1}^n \log(\nu + |z - z_s|^2),$$

where ν is an arbitrary constant, Eq. (46) becomes

$$\Delta \xi = g_0 + \tanh \frac{u_1 + \xi}{2} \quad g_0 = 4 \sum_{s=1}^n \frac{\nu}{\nu + |z - z_s|^2}. \quad (48)$$

This is the variational equation of the functional

$$a(\xi) = \int d^2z \left\{ \frac{1}{2} |\nabla \xi|^2 + g_0 \xi + 2 \log \left(\cosh \frac{u_1 + \xi}{2} \right) \right\} \quad (49)$$

and the proof that (48) has a unique solution with $\xi \rightarrow 0$ for $|z| \rightarrow \infty$ and $\xi \rightarrow -\infty$ at the n zeros z_s proceeds through the analysis of this functional along the lines developed by Taubes [42] for the AHM. Notice, however, that for $|z| \rightarrow \infty$ and at the points z_s , (49) and the functional corresponding to standard semilocal vortices coincide. It is

thus to be expected that the result of the analysis is the same in this case than for the semilocal model [28], leading also for the $\mathbb{C}\mathbb{P}^2$ model to the existence of multivortex solutions. In fact, as explained in [28], the crucial points of the proof are to show that $a(\xi)$ is strictly convex and that its radial derivative $(D_\xi a)(\xi)$ is positive for a sufficiently large ball in the space of ξ functions. To check the first point, it is convenient to adopt the notation of [42] to write

$$a(\xi) = \frac{1}{2} \|\nabla \xi\|^2 + \langle g_0, \xi \rangle + 2 \left\langle 1, \log \left(\cosh \frac{u_1 + \xi}{2} \right) \right\rangle,$$

with $L^2(\mathbb{C})$ norm and scalar product. Then, given that the first term is quadratic, hence strictly convex, the second is linear and the third has positive second derivative respect ξ , and it is thus also convex, it follows that $a(\xi)$ is strictly convex. With regard to the second point, the radial derivative of $a(\xi)$ is

$$(D_\xi a)(\xi) = \|\nabla \xi\|^2 + M(\xi) \\ M(\xi) = \left\langle \frac{\xi}{e^{u_1 + \xi} + 1}, g_0 - 1 + e^{u_1 + \xi}(1 + g_0) \right\rangle$$

and has the same structure than in the standard semilocal case, in which

$$M_{\text{SML}}(\xi) = \langle \xi, g_0 - 1 + e^{u_1 + \xi} \rangle.$$

Now, by taking ν large enough, it is easy to see that negative contributions to $M(\xi)$ are always less important than those appearing in $M_{\text{SML}}(\xi)$. Thus, like in the standard semilocal case, one can prove that the radial derivative is positive and there is a unique solution of (46).

IV. TYPE I AND TYPE II VERSUS SELF-DUAL COSMIC STRINGS

We come back in this section to the second-order ODE system obtained at the end of Sec. II, taking, as in Sec. III, $\mathcal{E}(|\psi|) = 1$ for the dielectric function but now the scalar potential is noncritically coupled, i.e.,

$$U(|\psi|) \equiv \frac{1}{2} W^2(|\psi|) = \frac{\lambda v^4}{2} \left(\frac{|\psi_1|^2 + |\psi_2|^2 - v^2}{|\psi_1|^2 + |\psi_2|^2 + v^2} \right)^2, \quad (50)$$

with $\lambda \neq e^2$. Substitution of the explicit form of the Fubini-Study metric in the Euler-Lagrange equations (19) and (20) brings about the cancellation of all phases and then, passing to nondimensional variables, the equations become

$$\frac{d}{dx} \left(\frac{e^A}{h} \frac{da}{dx} \right) = 2 \frac{e^A}{h} \left[\frac{2}{\mathfrak{D}} \sum_{p=1}^2 \mathcal{A}_p f_p^2 - \frac{1}{\mathfrak{D}^2} \sum_{p=1}^2 \sum_{l=1}^2 (\mathcal{A}_p + \mathcal{A}_l) f_p^2 f_l^2 \right], \quad (51)$$

$$\begin{aligned}
& \frac{1}{he^A} \frac{d}{dx} \left\{ he^A \left[\frac{1}{\mathfrak{D}} \frac{df_q}{dx} - \frac{f_q}{\mathfrak{D}^2} \sum_{l=1}^2 f_l \frac{df_l}{dx} \right] \right\} \\
& - \frac{\mathcal{A}_q^2 f_q}{h^2 \mathfrak{D}} + \frac{\mathcal{A}_q f_q}{h^2 \mathfrak{D}^2} \sum_{l=1}^2 \mathcal{A}_l f_l^2 \\
& = -\frac{1}{\mathfrak{D}^2} \left[\sum_{l=1}^2 \mathcal{X}_{ll} f_q + \sum_{l=1}^2 \mathcal{X}_{ql} f_l - \frac{2}{\mathfrak{D}} \sum_{p=1}^2 \sum_{l=1}^2 \mathcal{X}_{pl} f_p f_l f_q \right] \\
& + \beta \frac{\mathfrak{D}-2}{\mathfrak{D}^3} f_q, \tag{52}
\end{aligned}$$

where there is no sum on repeated q indices, $\beta = \frac{\lambda}{e^2}$, and

$$\begin{aligned}
\mathfrak{D} &= 1 + f_1^2 + f_2^2 \\
\mathcal{X}_{pl} &= \frac{df_p}{dx} \frac{df_l}{dx} + \frac{\mathcal{A}_p \mathcal{A}_l}{h^2} f_p f_l \text{ (no sum)}.
\end{aligned}$$

Also, in nondimensional variables, the Einstein equations (21) and (22) are

$$\begin{aligned}
& \frac{1}{h} \left[\frac{d^2 h}{dx^2} + \frac{1}{2} \frac{d}{dx} \left(h \frac{da}{dx} \right) \right] + \frac{1}{4} \left(\frac{dA}{dx} \right)^2 \\
& = -\frac{\kappa^2}{2} (\varepsilon_a + \varepsilon_f + \varepsilon_{af} + u), \tag{53}
\end{aligned}$$

$$\frac{d^2 A}{dx^2} + \frac{3}{4} \left(\frac{dA}{dx} \right)^2 = \frac{\kappa^2}{2} (\varepsilon_a - \varepsilon_f + \varepsilon_{af} - u), \tag{54}$$

where the different contributions to the energy-momentum tensor turn out to be

$$\begin{aligned}
\varepsilon_a &= \frac{1}{h^2} \left(\frac{da}{dx} \right)^2 \\
\varepsilon_f &= \frac{4}{\mathfrak{D}} \left[\sum_{l=1}^2 \left(\frac{df_l}{dx} \right)^2 - \frac{1}{\mathfrak{D}} \left(\sum_{l=1}^2 f_l \frac{df_l}{dx} \right)^2 \right] \\
\varepsilon_{af} &= \frac{4}{h^2 \mathfrak{D}} \left[\sum_{l=1}^2 \mathcal{A}_l^2 f_l^2 - \frac{1}{\mathfrak{D}} \left(\sum_{l=1}^2 \mathcal{A}_l f_l^2 \right)^2 \right] \\
u &= \beta \left(\frac{\mathfrak{D}-2}{\mathfrak{D}} \right)^2.
\end{aligned}$$

Thus, to find non-self-dual cosmic strings, we have to solve the system (51)–(54) with boundary conditions (13), (14), (16), and (17), but given that the equations are invariant under the shift of $A(x)$ by a constant, it is more convenient for the shooting method to change (16) by the initial condition

$$A(0) = 0 \quad A'(0) = 0, \tag{55}$$

since for a solution approaching Minkowski space with a deficit angle at large distances, Einstein equations ensure

that $A(x)$ will be constant at infinity and thus, subtracting this constant from the solution obtained with (55), we recover the behavior (16). We will concentrate on the case with $l=0$, so that from the boundary conditions at the origin we can assume that for very small x the fields are of the form

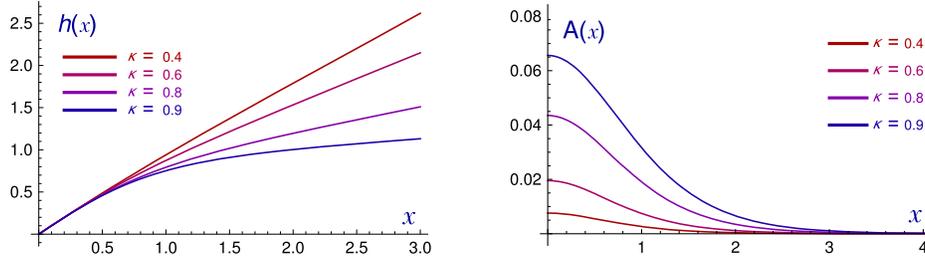
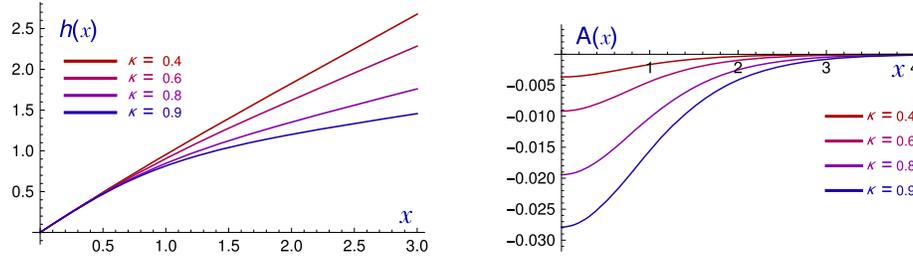
$$\begin{aligned}
f_1(x) &\simeq \alpha_1 x^{r_1} & f_2(x) &\simeq \mathcal{H}_0 + \alpha_2 x^{r_2} & a(x) &\simeq n - \delta x^s \\
h(x) &\simeq x & A(x) &\simeq \eta x^t,
\end{aligned}$$

with positive exponents and $t \geq 2$. The difference with the self-dual case is that now the equations are second order and we need to fix not only the field values, but also their derivatives, for some x_0 very close to zero, so that although $l=0$ we have to retain the second term in $f_2(x)$. Substituting these expressions in (51)–(54) and working to leading order, we find $r_1 = n$ and $r_2 = s = 2$ like in the self-dual situation, and also

$$\begin{aligned}
t = 2 \quad \alpha_2 &= -\frac{1}{4} \frac{1 - \mathcal{H}_0^2}{1 + \mathcal{H}_0^2} \mathcal{H}_0 \beta \\
\eta &= \kappa^2 \left[\delta^2 - \frac{1}{4} \left(\frac{1 - \mathcal{H}_0^2}{1 + \mathcal{H}_0^2} \right)^2 \beta \right].
\end{aligned}$$

We have thus two free parameters, α_1 and δ , and we shall proceed by a shooting procedure in which we will take $x_0 = 10^{-6}$ and will try to adjust the values of these parameters in such a way that for large x the fields behave consistently with the boundary conditions at infinity. In particular, an interesting case to be considered is that of $\mathcal{H}_0 = 0$. In that situation, the target manifold of the scalar fields reduces to $\mathbb{C}\mathbb{P}^1$, or \mathbb{S}^2 , and we have thus cosmic strings with values on the sphere. Vortices and cosmic strings on the sphere were previously studied by means of the Bogomolny equations in [23–25] and we will here revisit this problem to provide some new results for the non-self-dual case, where the main novelty is the presence of a nonvanishing function $A(x)$. Thus, we show in Figs. 7 and 8 the profile of this function, and also of the other metric coefficient $h(x)$, where we have considered the effect of several values of the gravitational coupling for scalar potentials of types I and II, namely, with $\beta = 0.8$ and $\beta = 1.5$. As one can see from the figures, the space-time of the cosmic string becomes Minkowskian at distances quite close to the string center and, for fixed gravitational coupling, the slope of $h(x)$ becomes smaller, and thus the deficit angle greater, as β increases. As for $A(x)$, we see that for values of the coupling β in the range studied, the departure of this function with respect to the self-dual regime $A(x) = 0$ is rather small, and that $A(x)$ decreases monotonically from positive values to zero when $\beta > 1$, whereas its performance is just the opposite for $\beta < 1$.

We have also computed the values of the energy per unit length and the deficit angle for cosmic strings on the sphere

FIG. 7. The metric coefficients $A(x)$ and $h(x)$ for $\mathcal{H}_0 = 0$, $\beta = 1.5$ and several strengths of the gravitational interaction.FIG. 8. The metric coefficients $A(x)$ and $h(x)$ for $\mathcal{H}_0 = 0$, $\beta = 0.8$ and several strengths of the gravitational interaction.

in several cases, and we show the results in Tables I and II. As we can see, for each fixed value of the coupling κ , μ and $\Delta\phi$ increase with β , a behavior that in the second case was also clearly observed in the Figs. 7 and 8. Moreover, while the deficit angle always increases as the gravitational coupling grows, the energy per unit length increases or decreases with κ depending on whether $\beta > 1$ or $\beta < 1$.

Finally, in Table III, we present the values of the energy per unit length of non-self-dual vortices on the sphere with $\kappa = 0$ for several vorticities. We observe that for $\beta > 1$ the energy of a vortex with vorticity n is higher than n times the energy of the vortex of unit vorticity, while in the case $\beta < 1$ we obtain the opposite behavior; see Fig. 9.

TABLE I. Energy per unit length normalized to 2π for $\mathcal{H}_0 = 0$ and several values of β and κ .

κ	Energy per unit length ($\frac{\mu}{2\pi}$)					
	0	0.2	0.4	0.6	0.8	0.9
$\beta = 0.8$	0.963	0.962	0.961	0.958	0.954	0.951
$\beta = 1$	1.000	1.000	1.000	1.000	1.000	1.000
$\beta = 1.5$	1.071	1.072	1.075	1.081	1.092	1.100

TABLE II. Deficit angle normalized to 2π for $\mathcal{H}_0 = 0$ and several values of β and κ .

κ	Deficit angle ($\frac{\Delta\phi}{2\pi}$)					
	0	0.2	0.4	0.6	0.8	0.9
$\beta = 0.8$	0.000	0.038	0.154	0.345	0.611	0.770
$\beta = 1$	0.000	0.040	0.160	0.360	0.640	0.810
$\beta = 1.5$	0.000	0.043	0.172	0.389	0.699	0.891

This confirms the character of superconductivity in this system in the usual way: for $\beta > 1$, the forces between vortices are repulsive and the superconductivity is of type II, while forces are attractive for $\beta < 1$ giving rise to type-I superconductivity.

When $\mathcal{H}_0 \neq 0$, it is more difficult to make the shooting converge properly at large distances, because in these cases the fields approach their asymptotic values as inverse powers of x instead of exponentially. Typically, by fixing the parameters α_1 and δ with a precision of nine decimal digits, one obtains a solution which behaves correctly for x up to around 10 or 12, but to extend the solution beyond that range requires much more precision and computer time. In general, the rates of convergence of the scalar and gauge fields, and that of the metric tensor coefficient $h(x)$, are quite satisfactory, while $A(x)$ meets the expected behavior at infinity much poorly. To palliate this problem, the pure shooting method has to be supplemented by an estimation of the fields at large distances. In fact, as one can check using Eqs. (51) and (52), for $\beta \neq 1$, the dominant contributions to the fields at high x have the same form (45) than in the self-dual case, and bearing this in mind we can

TABLE III. Energy per unit length normalized to 2π for vortices on the sphere of several vorticities without gravitational coupling.

n	Energy per unit length ($\frac{\mu}{2\pi}$)			
	1	2	3	4
$\beta = 0.8$	0.963	1.897	2.821	3.740
$\beta = 1$	1.000	2.000	3.000	4.000
$\beta = 1.5$	1.071	2.203	3.357	4.522

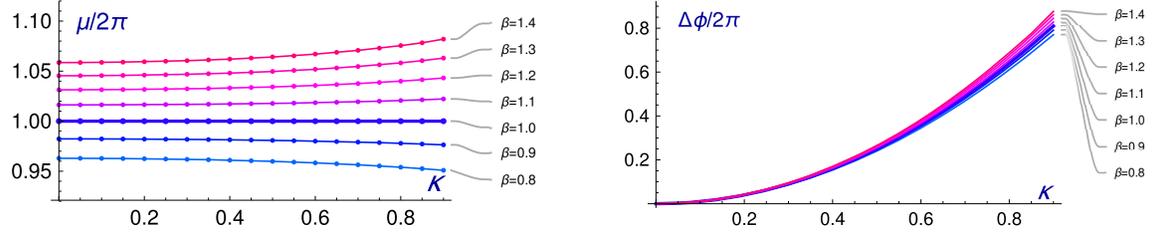


FIG. 9. Energy per unit length (left) and deficit angle (right) normalized to 2π as a function of κ and several values of β for the case $\mathcal{H}_0 = 0$.

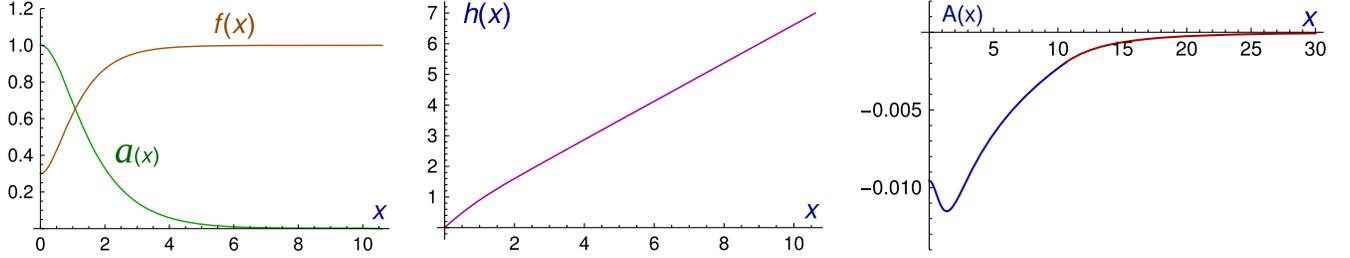


FIG. 10. Profiles of $a(x)$, $f(x)$, $h(x)$, and $A(x)$ for the case $n = 1$, $\mathcal{H}_0 = 0.3$, $\beta = 1.2$, and $\kappa = 0.6$.

approximate the component T^ϕ_ϕ of the energy-momentum tensor for large x as

$$T^\phi_\phi \simeq -\frac{1}{2} \frac{C^2 n^2}{r^2} x^{-2(\frac{n}{\beta}+1)}$$

at leading order. Using this result in Eq. (54), we obtain for $A(x)$, also at leading order and far from the origin,

$$A(x) \simeq -\frac{\kappa^2 C^2 n^2}{4n(2n+r)} x^{-\frac{2n}{\beta}} + \text{constant}, \quad (56)$$

an expression that can be employed to extend the results obtained by shooting to longer x when necessary. As an example, we show in Fig. 10 the results of the shooting method with α_1 and δ adjusted to nine decimal figures for the case $n = 1$, $\mathcal{H}_0 = 0.3$, $\beta = 1.2$, and $\kappa = 0.6$. As we can see from the figure, the convergence of $f(x) = \sqrt{f_1^2(x) + f_2^2(x)}$, $a(x)$, and $h(x)$ is quite convincing, but $A(x)$ does not attain so quickly its constant asymptotic value: the blue part of the profile, obtained by the shooting, shows still a slope of 0.00059, for an absolute value of 0.0076, even when the fields have already

reached very close values to their vevs. This can be repaired by means of (56), which has been used to draw the red part of the graphic. Thus, while for small values of \mathcal{H}_0 , we expect a form of $A(x)$ similar to those found for cosmic strings on the sphere, when \mathcal{H}_0 is higher the pattern is different: in the case at hand, $A(x)$ begins decreasing near $x = 0$, but it arrives to a minimum for finite x and then increases asymptotically toward zero, keeping constant negative sign for all x . As some further examples of the behavior of the system for $\mathcal{H}_0 \neq 0$, we present several results for the case $\mathcal{H}_0 = 0.3$ in Tables IV and V. To obtain the values shown in Table IV, we have taken into account that (45) implies that $T^t_t(x) \propto x^{-\frac{2n}{\beta}-2}$ at leading order for large x . Thus, if we define $\hat{\mu}(y) = -\int_0^y dx h(x) T^t_t(x)$, we can estimate

$$\frac{\mu}{2\pi} = \hat{\mu}(y) + \frac{1}{2} r \hat{\mu}'(y) y,$$

where for y we substitute the larger x value for which the shooting method achieves a good convergence, and $\hat{\mu}'(y)$ is the corresponding value of the integrand. In the case of Table V, we use the known behaviors of $T^t_t(x)$ and $A(x)$ for

TABLE IV. Energy per unit length normalized to 2π for several values of β and κ for the case with $\mathcal{H}_0 = 0.3$.

κ	Energy per unit length ($\frac{\mu}{2\pi}$)					
	0.0	0.2	0.3	0.4	0.5	0.6
$\beta = 0.8$	0.974	0.974	0.975	0.977	0.980	0.984
$\beta = 1$	1.000	1.000	1.000	1.000	1.000	1.000
$\beta = 1.2$	1.037	1.037	1.040	1.042	1.047	1.055

TABLE V. Deficit angle normalized to 2π for several values of β and κ for the case with $\mathcal{H}_0 = 0.3$.

κ	Deficit angle ($\frac{\Delta\phi}{2\pi}$)					
	0.0	0.2	0.3	0.4	0.5	0.6
$\beta = 0.8$	0.000	0.039	0.087	0.156	0.243	0.352
$\beta = 1$	0.000	0.040	0.090	0.160	0.250	0.360
$\beta = 1.2$	0.000	0.042	0.094	0.167	0.263	0.382

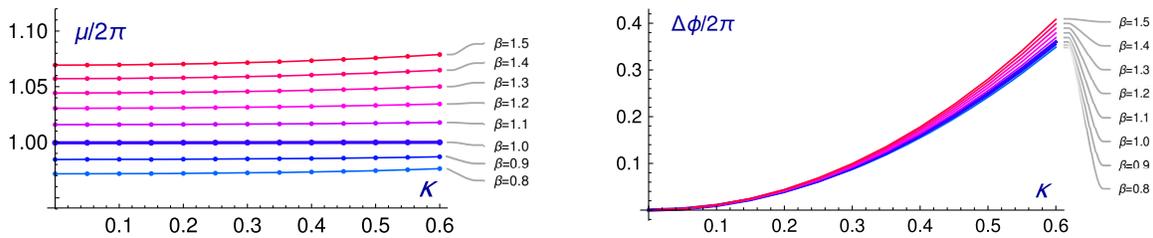


FIG. 11. Energy per unit length (left) and deficit angle (right) normalized to 2π as a function of κ for several values of β with $\mathcal{H}_0 = 0.3$.

large x and Eq. (53) to approach $h'(x) \simeq r - (\frac{2}{r} - 1)bx$, with some constant b , in that zone, and from this it follows the estimate

$$\frac{\Delta\phi}{2\pi} = 1 - \frac{2h'(y)}{2 - h''(y)y},$$

with the same y than before. At any rate, we find a similar pattern than that described previously; see Fig. 11.

Let us finally comment on the issue of stability. Viewed as solutions on the sphere, cosmic strings with $\mathcal{H}_0 = 0$ are stable by topological reasons, although for type II superconductivity those with $n > 1$ tend to break up into $n = 1$ components due to repulsive forces. Their stability status, however, is different when they are embedded in the full $\mathbb{C}\mathbb{P}^2$ model. For the standard semilocal model without gravity, the analysis of the stability of Nielsen-Olesen vortices (i.e., with $\psi_2 = 0$) of topological number $n = 1$ was carried out in [27], and that paper offers also an explanation of the results based on the following heuristic argument. When we perturb a Nielsen-Olesen vortex adding a non-null ψ_2 component, new terms in the energy per unit length appear; those coming from the covariant derivatives are always positive, giving rise to an increase in energy, but the presence of a $\psi_2 \neq 0$ component makes the potential energy

$$U(|\psi|) = \frac{\lambda v^4}{2} (|\psi_1|^2 + |\psi_2|^2 - v^2)^2$$

to go down. If the coupling λ in the potential is great enough, the effect of the potential prevails, the energy per unit length decreases, and the Nielsen-Olesen solution turns out to be unstable. On the other hand, when $\lambda \rightarrow 0$, the potential is very small and the most important contribution comes from gradient energy, making the Nielsen-Olesen solution stable. The boundary between both regimes is precisely the self-dual limit $\lambda = e^2$, because in this case we know from the Bogomolny analysis that \mathcal{H}_0 is a modulus and the energy is the same for ψ_2 null or different from zero: all vortices are in neutral equilibrium by saturating the lower bound $\mu = 2\pi v^2$. In the $\mathbb{C}\mathbb{P}^2$ model, the terms coming from covariant derivatives are more complicated, but the behavior of the potential energy (50) with regards to the addition of a ψ_2 component to the Nielsen-Olesen

solution is exactly the same than in the standard semilocal case. Therefore, like there, the $\mathbb{C}\mathbb{P}^2$ solutions with $\mathcal{H}_0 = 0$ and high β have an unstable mode toward developing a second scalar component, with the accompanying dispersion of the magnetic flux. Also, like there, this behavior has to cease when $\beta = 1$ due to the saturation of the Bogomolny bound. Thus, we expect that the self-dual limit of the $\mathbb{C}\mathbb{P}^2$ theory is again the borderline between unstable and stable Nielsen-Olesen solutions for, respectively, type II and type I superconductivity. This reasoning does not depend on the presence or absence of gravity and, in any case, the qualitative behavior without gravity backreaction should still be valid for small gravitational coupling. Although a complete justification of these expectations requires a detailed analysis of the Hessian spectrum, the results shown in Figs. 9 and 11 seem to point in the correct direction. For instance, we can see in these figures that the type I solution with $\beta = 0.8$ and $\kappa = 0.6$ displays a lower energy per unit length when $\mathcal{H}_0 = 0$ than for $\mathcal{H}_0 = 0.3$, while in the case of the type II solution with $\beta = 1.4$ and the same κ the situation is the opposite.

At any event, it is stressed in [27] that the instability of type II Nielsen-Olesen vortices does not preclude the existence of some metastable semilocal vortex solution with $\beta > 1$ in which the magnetic flux disperses throughout some broad but finite core. From this perspective, a detailed study of type II solutions with $\mathcal{H}_0 \neq 0$ is interesting.

V. CHANGING CHARTS: A SECOND SPECIES OF SEMILOCAL $\mathbb{C}\mathbb{P}^2$ COSMIC STRINGS

In this section, we investigate cosmic strings living in a second chart of the minimum atlas of $\mathbb{C}\mathbb{P}^2$: the structure of the cosmic string manifold in the third chart is identical to the structure in the second chart, i.e., cosmic strings of the second species live both in the second and third charts. We choose the transition functions from the previous chart V_ψ to the new chart V_ϕ in their intersection $V_\psi \cap V_\phi$ as follows:

$$\phi_1 = \frac{v^2}{\psi_1^*}, \quad \phi_2 = v \frac{\psi_2^*}{\psi_1^*}. \quad (57)$$

Note that we use in the definition of the transition function the conjugates of the original fields. The reason is that we

want the quanta to carry the same electric charge in all the charts. The Kähler potential looks slightly different,

$$K = 4v^2 \left[\log \left(1 + \frac{|\phi_1|^2 + |\phi_2|^2}{v^2} \right) - \log \frac{\phi_1 \phi_1^*}{v^2} \right],$$

but gives a Kähler metric formally identical to that in V_ψ ,

$$\begin{aligned} h_{1\bar{1}} &= \frac{4}{\tilde{D}} - \frac{4|\phi_1|^2}{v^2 \tilde{D}^2} & h_{1\bar{2}} &= -\frac{4\phi_1^* \phi_2}{v^2 \tilde{D}^2} \\ h_{2\bar{1}} &= -\frac{4\phi_2^* \phi_1}{v^2 \tilde{D}^2} & h_{2\bar{2}} &= \frac{4}{\tilde{D}} - \frac{4|\phi_2|^2}{v^2 \tilde{D}^2}, \end{aligned}$$

where $\tilde{D} = 1 + \frac{|\phi_1|^2 + |\phi_2|^2}{v^2}$. The vacuum orbit, however, changes from \mathbb{S}^3 to the three-dimensional solid torus $\mathbb{D}^2 \times \mathbb{S}^1$, the direct product of the 2D disk times a circle,

$$|\psi_1^{(v)}|^2 + |\psi_2^{(v)}|^2 = v^2 \Rightarrow |\phi_1^{(v)}|^2 - |\phi_2^{(v)}|^2 = v^2.$$

The explanation of this statement is clear if we write the algebraic equation defining the vacuum orbit in the form $|a|^2 - |b|^2 = 1$ where $a = \frac{\phi_1^{(v)}}{|v|}$ and $b = \frac{\phi_2^{(v)}}{|v|}$. A point in the solid torus corresponds thus to $(\frac{b}{a}, \frac{a}{|a|})$ since $\frac{b}{a}$ parametrizes the open disk \mathbb{D}^2 . The closure of \mathbb{D}^2 is achieved adding the infinity points $|a| = \infty$, $|b| = \infty$. It is of note that the solid torus is homeomorphic to the $SU(1, 1)$ Lie group formed by the set of matrices $M = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$.

As a consequence, the $G = U(1)_{\text{gauge}} \times SU(2)_{\text{global}}$ symmetry exhibited in the reference chart

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow e^{ie\chi(x)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow e^{i \sum_{a=1}^3 \alpha_a \frac{\sigma_a}{2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where σ^a are the Pauli matrices and α_a the parameters of the $SU(2)$ group, collapses to the subgroup $U(1)_{\text{gauge}} \times U(1)_{\text{global}}$,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{ie\chi(x)} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow e^{i\alpha_3 \frac{\sigma_3}{2}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

in the new chart, whereas the $U(1)_{\text{gauge}}$ subgroup only acts on the ϕ_1 scalar field. Note that this $U(1)_{\text{global}}$ is also the subgroup $(\begin{smallmatrix} e^{i\alpha_3} & 0 \\ 0 & e^{-i\alpha_3} \end{smallmatrix})$ of $SU(1, 1)$.

Regarding the covariant derivatives, in the second chart they are related with their counterparts in the reference chart as follows:

$$\begin{aligned} D_\mu \psi_1 &= -\frac{v^2}{\phi_1^{*2}} (\partial_\mu \phi_1^* + ieA_\mu \phi_1^*), \\ D_\mu \psi_2 &= -\frac{v\phi_2^*}{\phi_1^{*2}} (\partial_\mu \phi_1^* + ieA_\mu \phi_1^*) + \frac{v}{\phi_1^*} \partial_\mu \phi_2^*. \end{aligned}$$

We observe again that the field ϕ_2 , coupled to the gauge field only from the Kähler potential, is neutral. The transition functions neatly show that ϕ_2 is invariant with respect to $U(1)$ gauge transformations. ϕ_1 , however, carries charge e , compelling us to define the covariant derivatives, separately for ϕ_1 and ϕ_2 , in the compact form

$$\begin{aligned} D_\mu \phi_p &= \partial_\mu \phi_p - ieN_p A_\mu \phi_p, \quad p = 1, 2, \\ N_1 &= +1, \quad N_2 = 0. \end{aligned} \quad (58)$$

Armed with this tool, we write the action in identical form in the V_ϕ -chart to the action in the V_ψ -chart introducing a new superpotential to be adjusted by requiring self-duality,

$$\begin{aligned} S &= \int dx^4 \sqrt{-g} \mathcal{L}, \\ \mathcal{L} &= -\frac{1}{4} \mathcal{E}(|\phi_1|, |\phi_2|) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} h_{p\bar{q}}(\phi, \phi^*) D_\mu \phi_p D^\mu \phi_q^* \\ &\quad - \frac{1}{2} \tilde{W}^2(\phi, \phi^*), \end{aligned} \quad (59)$$

where $\phi = (\phi_1, \phi_2)$. Now all the steps starting in formula (2) and ending in formula (26) are almost identical with some minor differences, e.g., in the behavior of the ϕ_2 field. We skip to repeat all this stuff here, but remark that there is an important novelty: the function $P(\psi)$ reduces in the new chart to

$$\tilde{P}(|\phi_1|, |\phi_2|) = 2 \frac{|\phi_1|^2}{\tilde{D}},$$

because ϕ_2 is neutral. Therefore, self-duality is possible if

$$\begin{aligned} \sqrt{\mathcal{E}(|\phi_1|, |\phi_2|)} \tilde{W}(|\phi_1|, |\phi_2|) &= e(\tilde{P}(|\phi_1|, |\phi_2|) - v^2) \\ \Rightarrow \tilde{U}(|\phi_1|, |\phi_2|) &= \frac{e^2 v^4}{2\mathcal{E}(|\phi_1|, |\phi_2|)} \left(\frac{|\phi_1|^2 - |\phi_2|^2 - v^2}{|\phi_1|^2 + |\phi_2|^2 + v^2} \right)^2. \end{aligned} \quad (60)$$

Thus, we see that self-duality and the change of charts work consistently and, in fact, the potential (60) might be also directly computed applying the transition function to the self-dual potential working in the V_ψ reference chart. Then, one easily derives the self-duality first-order equations,

$$F_{12} = \pm \frac{ev^2 \sqrt{\gamma}}{\mathcal{E}(|\phi_1|, |\phi_2|)} \left(\frac{|\phi_1|^2 - |\phi_2|^2 - v^2}{|\phi_1|^2 + |\phi_2|^2 + v^2} \right), \quad (61)$$

$$D_1\phi_p \pm i\sqrt{\gamma}D^2\phi_p = 0, \quad p = 1, 2. \quad (62)$$

We omit that calculation, but the proof that the solutions of (61) and (62) have null the planar components of the energy momentum tensor, $T_{ij} = 0$, runs in parallel to the same demonstration for self-dual cosmic strings of the first species. Finally, we must solve together with (61) and (62) the Einstein equation

$$R^{(\gamma)} = 16\pi GT_{tt}. \quad (63)$$

As it is described in the subsection 3.3 for the reference chart, Eq. (62) in the conformal gauge, $\gamma_{ij} = e^{-4\tilde{V}(x^1, x^2)}\delta_{ij}$, becomes $D_1\phi_p \pm D_2\phi_p = 0$, i.e., is identical to the akin equations in the Euclidean plane $\bar{\partial}_A\phi_p = 0$. Therefore, for the field $\phi_2(x_1, x_2)$, (62) is the Cauchy-Riemann equation in this conformal gauge. The ϕ_2 -solutions are accordingly holomorphic/antiholomorphic functions $\phi_2 = f(x^1 \pm ix^2)$. Together with the finite energy conditions, we shall keep only constant functions as bona fide solutions. The boundary condition $\lim_{z \rightarrow +\infty} \phi_2(z) = w_2 \in \mathbb{C}$ then sets $\phi_2(z) = w_2$ as the solutions throughout the whole plane. To reach the vacuum orbit at infinity, the limit of the other field at the boundary $\lim_{z \rightarrow +\infty} \phi_1(z) = w_1 \in \mathbb{C}$ is forced to satisfy the equation $|w_1|^2 - |w_2|^2 = v^2$. We stress that the string tension of self-dual solutions in this chart is also $\mu = 2\pi|n|v^2$ like the tension of cosmic strings in the reference chart. The tension of this species of cosmic strings fits with the tension of nongravitating strings in the analogous chart described in [23] if we identify $v^2 = \rho^2$ where ρ is the parameter introduced in [23].

The vortex equation (61) may be rewritten *à la* Jaffe-Taubes [42],

$$\begin{aligned} \Delta \log |\phi_1| &= e^2 v^2 e^{-4\tilde{V}} \frac{|\phi_1(z, \bar{z})|^2 - |w_2|^2 - v^2}{|\phi_1(z, \bar{z})|^2 + |w_2|^2 + v^2} \\ &+ 2\pi \sum_{s=1}^n \delta^{(2)}(z - z_s). \end{aligned} \quad (64)$$

The dimension of the moduli space of solutions of Eq. (64) is $2n + 2$ due to the freedom of moving the centers of the cosmic strings z_s and the freedom of choosing a point in $\mathbb{D}^2 \times \mathbb{S}^1$ by changing w_2 . In particular, the choice $\phi_1^{(v)} = v$, $\phi_2^{(v)} = 0$ selects as the self-dual cosmic strings in this chart identical solutions to the self-dual cosmic strings in the $\mathbb{C}\mathbb{P}^1$ -sigma model.

It remains to solve the Einstein equation (63). In the conformal gauge, this equation becomes

$$\Delta \left[\tilde{V} - 4\pi G v^2 \left(\log \tilde{D} - \frac{1}{2} \log \frac{|\phi_1(z, \bar{z})|^2}{\prod_{s=1}^n |z - z_s|^2} \right) \right] = 0. \quad (65)$$

Therefore, the quantity between brackets is harmonic and the solution for the metric is

$$\begin{aligned} \tilde{V}(z, \bar{z}) &= 4\pi G v^2 \left(\log \frac{v^2 + |\phi_1(z, \bar{z})|^2 + |w_2|^2}{v^2} \right. \\ &\left. - \frac{1}{2} \log \frac{|\phi_1(z, \bar{z})|^2}{\prod_{s=1}^n |z - z_s|^2} \right) + \text{constant}. \end{aligned} \quad (66)$$

The search for cylindrically symmetric cosmic strings of the second species proceeds along similar lines to the task performed on the first species. Thus, we stop here to avoid unnecessary repetitions. Nevertheless, we emphasize the main features of this second species of self-dual cosmic strings which are as follows:

- (1) Like the cosmic strings of the first species, the cosmic strings of the second species are concentrated in the plane perpendicular to the string. Contrarily to the first species, they take values in the subset of V_ϕ , instead of V_ψ , comprised between the point $(\phi_1 = 0, \phi_2 = w_2)$ and the vacuum orbit $\mathbb{D}^2 \times \mathbb{S}^1$. This is a region of $\mathbb{C}\mathbb{P}^2$ disjoint to the region in which cosmic strings of the first species are evaluated. In particular, if $w_2 = 0$, the cosmic strings are mapped into the spherical cap of $\mathbb{C}\mathbb{P}^1$ complementary to the spherical cap in V_ψ where the cosmic strings of the first species live.
- (2) Solutions of both species can be described using any chart. Nevertheless, at the center of a cosmic string of the first species $\psi_1 = 0$, so that $\phi_1 = \infty$ and the string looks as a singular configuration from the point of view of V_ϕ . The same is true for the second species as seen from the reference chart. Thus, the most natural treatment is to use V_ψ and V_ϕ for, respectively, first and second species.
- (3) Since there is a potential energy for $\mathbb{C}\mathbb{P}^2$ fields, solutions taking values in different bounded regions of $\mathbb{C}\mathbb{P}^2$ come from different effective dynamics, which can therefore lead to different physical properties. Nevertheless, cosmic strings of both species have in our model the same energy per unit length.
- (4) One might think of a configuration encompassing a cosmic string of the first species plus another string of the second species, both of vorticity n . Since the total string tension adds to $4\pi|n|v^2$ and this is precisely the degree of a map from \mathbb{S}^2 to $\mathbb{C}\mathbb{P}^N$ that characterizes the $\mathbb{C}\mathbb{P}^N$ -instantons, this combination of two strings of different species could be interpreted as a partonic (or meronic) structure of the instanton. A caveat: even BPS vortices suffer a low-energy dynamics, see, e.g., [43] and references quoted therein. Thus, the double cosmic string configuration is not solution of the static field equations. Only in the limit $G = 0$, $e^2 = 0$ the instanton partonic structure can survive.

VI. FINAL COMMENTS

The main theme of this investigation has been the cosmic strings, type I, type II, and self-dual, existing in Abelian gauged nonlinear $\mathbb{C}\mathbb{P}^2$ -sigma model coupled to gravity. Thus, we address a concrete case in the general framework of research about topological defects in gravitational scenarios. This is a fertile subject with many branches, some more theoretical, see, e.g., a former work [44] centered around kinks interacting with the gravitational field created by themselves in Jackiw-Teitelboim $(1 + 1)$ -dimensional pseudo-Riemannian universes, and some more directly connected to realistic cosmological problems, like in the present case. Namely, we have studied topological defects and the gravitational fields created by them in the context of Abelian gauged nonlinear sigma models coupled to Einstein gravity in $(3 + 1)$ -dimensional pseudo-Riemannian universes.

A first remark about the construction of the cosmic strings achieved in this work is the observation that trading the target manifold $\mathbb{C}\mathbb{P}^2$ by $\mathbb{C}\mathbb{P}^N$ the same structure arises. There is a first species of cosmic strings living in the reference chart, but identical cosmic strings of the second species appear in the remaining N charts that, together with the reference chart, form a minimum atlas of $\mathbb{C}\mathbb{P}^N$.

It is also of note that, in comparison with the non-gravitating solitonic strings of Ref. [23], here the parameter space is reduced for simplicity. From ρ , the radius of the sphere which determines the Fubini-Study metric, and a , which sets the scale of the gauge symmetry breaking,

in [23], in this work there is a reduction to a single parameter v which determines both the Kähler metric and the vacuum expectation value of the scalar fields. As a consequence, the differences between the cosmic strings of the two species are less evident, but the parameter space used in [23] could be easily translated to the gravitating case dealt with in the present work.

Another possible route to extend this work is the choice of more general dielectric functions. In Ref. [45], Bazeia *et al.* proposed an interesting family of dielectric functions susceptible of supporting self-dual topological vortices in shrewd generalizations of the Abelian Higgs model. The main novelty is the existence of vacuum orbits formed by several connected components. The challenge is to promote these structures to nonlinear gauged sigma models coupled to gravity.

Finally, we comment about the possibility of introducing another Kähler metrics different from the Fubini-Study metric. The existence of self-dual cosmic strings when the target manifold is endowed with general Kähler metrics is to be expected because these metrics permit the extension of Abelian gauged nonlinear sigma models to their $\mathcal{N} = 2$ supersymmetric partners [46].

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