Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

CEDYA CMA 2020 GIJÓN

SēMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

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Universidad de Oviedo
Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
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servipub@uniovi.es

ISBN: 978-84-18482-21-2

Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Asymptotic analysis of the behavior of a viscous fluid between two very close mobile surfaces

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Abstract

The aim of this work is the study of the behavior of an incompressible viscous fluid moving between two closely spaced surfaces, also in motion. To carry out this work we use the asymptotic expansion method that allows us to formally justify two different models starting from the same initial problem: a lubrication model and a shallow water model. The type of model that yields depends on whether the fluid is"pressure dominated" and on the boundary conditions imposed. We discuss in detail under what conditions each of the models would be applicable.

1. Introduction

In this work, we are interested, in a first step, in justifying using the asymptotic development technique, a lubrication model in a thin domain with curved mean surface.

The asymptotic analysis method is a mathematical tool that has been widely used to obtain and justify reduced models, both in solid and fluid mechanics, when one or two of the dimensions of the domain are much smaller than the others. In particular, in fluid mechanics, these technique has been applied to derive lubrication models, shallow water models, tube flow models, etc. (see, for example, [1]- [9], [11]- [12], and many others). Here, we follow the steps of [2], but changing the starting point.

We consider a three-dimensional thin domain, Ω_t^ε, filled by a fluid, that varies with time $t \in [0, T]$, given by

$$\begin{aligned}\Omega_t^\varepsilon \;=\; & \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \mathbb{R}^3 : x_i(\xi_1,\xi_2,t) \le x_i^\varepsilon \le x_i(\xi_1,\xi_2,t) + h^\varepsilon(\xi_1,\xi_2,t) N_i(\xi_1,\xi_2,t),\right. \\ & \left.(i=1,2,3),\ (\xi_1,\xi_2) \in D \subset \mathbb{R}^2\right\} \end{aligned} \tag{1.1}$$

where $\vec{X}_t(\xi_1,\xi_2) = \vec{X}(\xi_1,\xi_2,t) = (x_1(\xi_1,\xi_2,t), x_2(\xi_1,\xi_2,t), x_3(\xi_1,\xi_2,t))$ is the lower bound surface parametrization, $h^\varepsilon(\xi_1,\xi_2,t)$ is the gap between the two surfaces in motion, and $\vec{N}(\xi_1,\xi_2,t)$ is the unit normal vector:

$$\vec{N}(\xi_1,\xi_2,t) = \frac{\dfrac{\partial \vec{X}}{\partial \xi_1} \times \dfrac{\partial \vec{X}}{\partial \xi_2}}{\left\| \dfrac{\partial \vec{X}}{\partial \xi_1} \times \dfrac{\partial \vec{X}}{\partial \xi_2} \right\|} \tag{1.2}$$

The lower bound surface is assumed to be regular and the gap is assumed to be small with regard to the dimension of the bound surfaces. We take into account that the fluid film between the surfaces is thin by introducing a small non-dimensional parameter ε, and setting that

$$h^\varepsilon(\xi_1,\xi_2,t) = \varepsilon h(\xi_1,\xi_2,t) \tag{1.3}$$

where

$$h(\xi_1,\xi_2,t) \ge h_0 > 0, \quad \forall\, (\xi_1,\xi_2) \in D \subset \mathbb{R}^2,\ \forall\, t \in [0,T]. \tag{1.4}$$

Let us suppose that the fluid motion is governed by Navier-Stokes equations since we consider that it is an incompressible newtonian fluid,

$$\rho_0 \left(\frac{\partial u_i^\varepsilon}{\partial t^\varepsilon} + \frac{\partial u_i^\varepsilon}{\partial x_j^\varepsilon} u_j^\varepsilon \right) = -\frac{\partial p^\varepsilon}{\partial x_i^\varepsilon} + \mu \left(\frac{\partial^2 u_i^\varepsilon}{\partial (x_1^\varepsilon)^2} + \frac{\partial^2 u_i^\varepsilon}{\partial (x_2^\varepsilon)^2} + \frac{\partial^2 u_i^\varepsilon}{\partial (x_3^\varepsilon)^2} \right) + \rho_0 f_i^\varepsilon, \quad (i=1,2,3) \tag{1.5}$$

$$\frac{\partial u_j^\varepsilon}{\partial x_j^\varepsilon} = 0 \tag{1.6}$$

where repeated indices indicate summation (j takes values from 1 to 3), ρ_0 is the fluid density, $\vec{u}^\varepsilon$ is the velocity, p^ε is the pressure and $\vec{f}^\varepsilon$ is the density of applied volume forces.

 ISBN 978-84-18482-21-2

Now, we consider a reference domain independent of ε and t

$$\Omega = D \times [0,1] \tag{1.7}$$

related to Ω_t^ε by the following change of variable:

$$t^\varepsilon = t \tag{1.8}$$
$$x_i^\varepsilon = x_i(\xi_1,\xi_2,t) + \varepsilon\xi_3 h(\xi_1,\xi_2,t)N_i(\xi_1,\xi_2,t) \tag{1.9}$$

where $(\xi_1,\xi_2) \in D$ and $\xi_3 \in [0,1]$, and we make a change of basis to a new basis $\{\vec{a}_1,\vec{a}_2,\vec{a}_3\}$, where

$$\vec{a}_1(\xi_1,\xi_2,t) = \frac{\partial \vec{X}(\xi_1,\xi_2,t)}{\partial \xi_1} \tag{1.10}$$
$$\vec{a}_2(\xi_1,\xi_2,t) = \frac{\partial \vec{X}(\xi_1,\xi_2,t)}{\partial \xi_2} \tag{1.11}$$
$$\vec{a}_3(\xi_1,\xi_2,t) = \vec{N}(\xi_1,\xi_2,t) \tag{1.12}$$

The details about the change of variable and basis can be found in [13].

The velocity and the applied forces $(\vec{u}^\varepsilon, \vec{f}^\varepsilon)$ are written in the new basis (1.10)-(1.12) as follows:

$$\vec{u}^\varepsilon = u_i^\varepsilon \vec{e}_i = u_k(\varepsilon)\vec{a}_k \tag{1.13}$$
$$\vec{f}^\varepsilon = f_i^\varepsilon \vec{e}_i = f_k(\varepsilon)\vec{a}_k \tag{1.14}$$

so

$$u_i^\varepsilon = (u_k(\varepsilon)\vec{a}_k)\cdot\vec{e}_i = u_k(\varepsilon)a_{ki} \tag{1.15}$$
$$f_i^\varepsilon = (f_k(\varepsilon)\vec{a}_k)\cdot\vec{e}_i = f_k(\varepsilon)a_{ki} \tag{1.16}$$

where $a_{ki} = \vec{a}_k \cdot \vec{e}_i$, and we assume that the velocity, the pressure and the applied forces can be developed in powers of ε as in [2], [1], [6], [11] and [12]:

$$u_i(\varepsilon) = u_i^0 + \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \cdots \quad (i=1,2,3) \tag{1.17}$$
$$p(\varepsilon) = \varepsilon^{-2}p^{-2} + \varepsilon^{-1}p^{-1} + p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots \tag{1.18}$$
$$f_i(\varepsilon) = f_i^0 + \varepsilon f_i^1 + \varepsilon^2 f_i^2 + \cdots \quad (i=1,2,3) \tag{1.19}$$

Taking into account (1.15)-(1.16), equations (1.5)-(1.6) yield ($i = 1,2,3$):

$$\rho_0\left(\frac{\partial(u_k(\varepsilon)a_{ki})}{\partial t^\varepsilon} + \frac{\partial(u_k(\varepsilon)a_{ki})}{\partial x_j^\varepsilon}(u_k(\varepsilon)a_{kj})\right) = -\frac{\partial p(\varepsilon)}{\partial x_i^\varepsilon}$$
$$+\mu\left(\frac{\partial^2(u_k(\varepsilon)a_{ki})}{\partial(x_1^\varepsilon)^2} + \frac{\partial^2(u_k(\varepsilon)a_{ki})}{\partial(x_2^\varepsilon)^2} + \frac{\partial^2(u_k(\varepsilon)a_{ki})}{\partial(x_3^\varepsilon)^2}\right) + \rho_0 f_k(\varepsilon)a_{ki} \tag{1.20}$$
$$\frac{\partial(u_k(\varepsilon)a_{kj})}{\partial x_j^\varepsilon} = 0 \tag{1.21}$$

Next, we substitute developments (1.17)-(1.19) in Navier-Stokes equations written in the reference domain ((1.20)-(1.21)) and we identify the terms multiplied by the same power of ε. In this way we obtain a series of equations that will allow us to determine the terms of the previous developments.

In the next two sections we summarize the results obtained in [13].

2. A new generalized lubrication model

If we assume that the fluid slips at the lower surface ($\xi_3 = 0$), and at the upper surface ($\xi_3 = 1$), but there is continuity in the normal direction, so the tangential velocities at the lower and upper surfaces are known, and the normal velocity of each of them must match the fluid velocity, we derive the following generalized lubrication equation:

$$\frac{1}{\sqrt{A^0}}\mathrm{div}\left(\frac{h^3}{\sqrt{A^0}}M\nabla p^{-2}\right) = 12\mu\frac{\partial h}{\partial t} + 12\mu\frac{hA^1}{A^0}\left(\frac{\partial\vec{X}}{\partial t}\cdot\vec{N}\right)$$
$$-6\mu\nabla h\cdot(\vec{W}^0 - \vec{V}^0) + \frac{6\mu h}{\sqrt{A^0}}\mathrm{div}(\sqrt{A^0}(\vec{W}^0 + \vec{V}^0)) \tag{2.1}$$

where the pressure is approximated by $\varepsilon^{-2}p^{-2}$

$$A^0 = EG - F^2 \quad (2.2)$$

$$A^1 = -eG - gE + 2fF \quad (2.3)$$

$$M = \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \quad (2.4)$$

and $E, F, G,\ e, f, g$ are the coefficients of the first and second fundamental forms, respectively, of the surface parametrized by $\vec{X}$. $\vec{V}^0 = (V_1^0, V_2^0)$ and $\vec{W}^0 = (W_1^0, W_2^0)$ are the approximations of order 0 on ε of the tangential velocity at the lower and upper surfaces respectively.

Once p^{-2} is calculated we have the following approximation of the three components of the velocity:

$$u_1^0 = \frac{h^2(\xi_3^2 - \xi_3)}{2\mu A^0}\left(G\frac{\partial p^{-2}}{\partial \xi_1} - F\frac{\partial p^{-2}}{\partial \xi_2}\right) + \xi_3(W_1^0 - V_1^0) + V_1^0 \quad (2.5)$$

$$u_2^0 = \frac{h^2(\xi_3^2 - \xi_3)}{2\mu A^0}\left(E\frac{\partial p^{-2}}{\partial \xi_2} - F\frac{\partial p^{-2}}{\partial \xi_1}\right) + \xi_3(W_2^0 - V_2^0) + V_2^0 \quad (2.6)$$

$$u_3^0 = \frac{\partial \vec{X}}{\partial t} \cdot \vec{N} \quad (2.7)$$

If we consider the classic assumptions to derive Reynolds equations (domain independent of time, $x_3 = 0$ in (1.1), upper surface fixed, lower surface moving in the x_1-direction with constant velocity), we re-obtain the classic Reynolds equation (see [10]) from (2.1).

3. A new thin fluid layer model

During this process we have observed that, depending on the boundary conditions, other models can be obtained. In this section, we change the boundary conditions that we imposed in the first case: instead of assuming that we know the velocities on the upper and lower boundaries of the domain, we assume that we know the tractions on these upper and lower boundaries. In particular, we assume that the normal component of the traction on $\xi_3 = 0$ and on $\xi_3 = 1$ are known pressures (π_0^ε and π_1^ε), and that the tangential component of the traction on these surfaces are friction forces depending on the value of the velocities on ∂D.

Under these assumptions we derive a shallow water model that allow us to determine h, V_1^0 and V_2^0:

$$\frac{\partial h}{\partial t} + \frac{h}{\sqrt{A^0}}\mathrm{div}\left(\sqrt{A^0}\vec{V}^0\right) + \frac{hA^1}{A^0}\left(\frac{\partial \vec{X}}{\partial t}\cdot \vec{N}\right) = 0 \quad (3.1)$$

$$\begin{aligned}\frac{\partial V_i^0}{\partial t} &+ \sum_{l=1}^{2}\left(V_l^0 - C_l^0\right)\frac{\partial V_i^0}{\partial \xi_l} + \sum_{k=1}^{2}\left(R_{ik}^0 + \sum_{l=1}^{2} H_{ilk}^0 V_l^0\right)V_k^0 = -\frac{1}{\rho_0}\left(\alpha_i^0\frac{\partial \pi_0^0}{\partial \xi_1} + \beta_i^0\frac{\partial \pi_0^0}{\partial \xi_2}\right)\\ &+ \nu\left\{\sum_{m=1}^{2}\sum_{l=1}^{2}\frac{\partial^2 V_i^0}{\partial \xi_m \partial \xi_l}J_{lm}^0 + \sum_{k=1}^{2}\sum_{l=1}^{2}\frac{\partial V_k^0}{\partial \xi_l}(L_{kli}^0 + \psi(h)_{ikl}^0)\right.\\ &\left.+ \sum_{k=1}^{2} V_k^0(S_{ik}^0 + \chi(h)_{ik}^0) + \hat{\kappa}(h)_i^0\right\} + F_i^0(h) - Q_{i3}^0\left(\frac{\partial \vec{X}}{\partial t}\cdot\vec{N}\right) \quad (i = 1, 2)\end{aligned} \quad (3.2)$$

where the coefficients $\alpha_i^0, \beta_i^0, C_l^0, H_{ilk}^0, J_{lm}^0, L_{kli}^0, Q_{i3}^0, R_{ik}^0, S_{ik}^0$ depend only on the lower bound surface parametrization, $\vec{X}$ while the coefficients $F_i^0(h)$, $\psi(h)_{ikl}^0$, $\chi(h)_{ik}^0$), $\hat{\kappa}(h)_i^0$ depend both on the parametrization and on the gap h. The detailed definition of these coefficients is given in [13].

Let π_0^0 be the approximation of order 0 on ε of the pressure π_0^ε. Then, we obtain the following approximations of the velocity and the pressure:

$$u_i^0 = W_i^0 = V_i^0 \quad i = 1, 2 \quad (3.3)$$

$$u_3^0 = \frac{\partial \vec{X}}{\partial t}\cdot\vec{N} \quad (3.4)$$

$$p^0 = \frac{2\mu}{h}\frac{\partial h}{\partial t} + \pi_0^0 \quad (3.5)$$

4. Conclusions

Thus, two new models that can not be found in the literature, as far as we know, are presented here. Both models have been derived starting from the same initial problem, an incompressible viscous fluid moving between two closely spaced surfaces.

The method used to justify them allows us to answer the question of when each of them is applicable. We reach the conclusion that the magnitude of the pressure differences at the lateral boundary of the domain is key when deciding which of the two models best describes the fluid behavior.

Boundary conditions tell us which of the two models should be used when simulating the flow of a thin fluid layer between two surfaces: if the fluid pressure is dominant (that is, it is of order $O(\varepsilon^{-2})$), and the fluid velocity is known on the upper and lower surfaces, we must use the lubrication model; if the fluid pressure is not dominant (that is, it is of order $O(1)$), and the tractions are known on the upper and lower surfaces, we must use the shallow water model. In the first case we will say that the fluid is "driven by the pressure" and in the second that it is "driven by the velocity".

Acknowledgements

This work has been partially supported by Ministerio de Economía y Competitividad of Spain, under grant MTM2016-78718-P with the participation of FEDER, and the European Union's Horizon 2020 Research and Innovation Programme, under the Marie Sklodowska-Curie Grant Agreement No 823731 CONMECH.

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