# Proceedings

## of the

# XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
# XVI Congreso de Matemática Aplicada

**Gijón (Asturias), Spain**

**June 14-18, 2021**

CEDYA CMA 2020 GIJÓN

SeMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

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Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
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servipub@uniovi.es

ISBN: 978-84-18482-21-2

## Foreword

It is with great pleasure that we present the Proceedings of the 26$^{th}$ Congress of Differential Equations and Applications / 16$^{th}$ Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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# Contents

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
*Gijón, 14-18 junio 2021*
(pp. 258–264)

# Minimal complexity of subharmonics in a class of planar periodic predator-prey models

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**Abstract**

This contribution analyzes the existence of $nT$-periodic coexistence states, for $n \geq 1$, in two classes of non-autonomous predator-prey Volterra systems with periodic coefficients. In the first place, when the model is non-degenerate it is shown that the Poincaré–Birkhoff twist theorem can be applied to get the existence of subharmonics of arbitrary order. In the second place, it will be analyzed a *degenerate* predator-prey model introduced in [9] and [5] and, then, deeply studied in [7]. By analyzing the iterates of the Poincaré map of the system, it is shown that it admits nontrivial $nT$-periodic coexistence states for every $n \geq 2$.

## 1. Introduction

In this contribution, we study the existence of positive subharmonics of arbitrary order ($nT$-periodic coexistence states) of the periodic Volterra predator-prey model

$$\begin{cases} u' = \lambda\alpha(t)u(1-v), \\ v' = \lambda\beta(t)v(-1+u), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is regarded as a parameter, and, for some $T > 0$, $\alpha(t)$ and $\beta(t)$ are $T$-periodic real continuous functions. Throughout this note, we set

$$A := \int_0^T \alpha(s)ds \quad \text{and} \quad B := \int_0^T \beta(s)ds.$$

Two different cases can arise according to whether, or not, the following condition holds

$$\text{supp } \alpha \cap \text{supp } \beta \neq \emptyset. \tag{1.2}$$

In this *non-degenerate* situation, which has been sketched in Figure 1, the existence of subharmonics of arbitrary order, for sufficiently large $\lambda$, can be obtained through an updated version of the celebrated Poincaré–Birkhoff twist theorem.

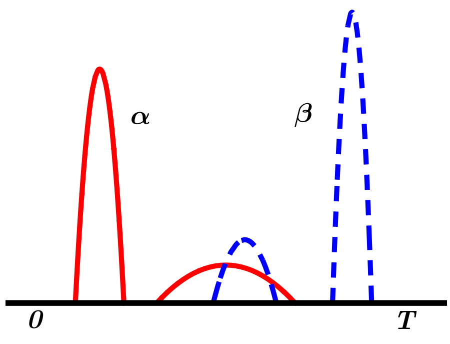


**Fig. 1** $\alpha$ (continuous line) and $\beta$ (dashed line) satisfying (1.2).

However, in the *degenerate* case when, instead of (1.2), the next condition holds

$$\text{supp } \alpha \cap \text{supp } \beta = \emptyset, \tag{1.3}$$

then the Poincaré–Birkhoff theorem is unable to provide, in general, with subharmonics of arbitrary order, unless $\alpha(t)$ and $\beta(t)$ have some special nodal structure. An admissible distribution of $\alpha$ and $\beta$ is sketched in Figure 2.

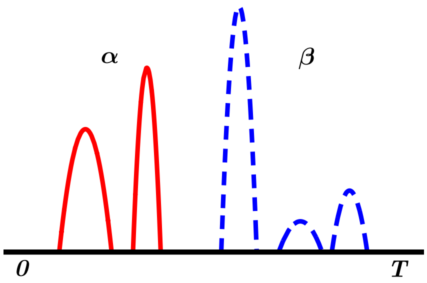


**Fig. 2** $\alpha$ (continuous line) and $\beta$ (dashed line) satisfying (1.3).

## 2. The non-degenerate case

The non-degenerate case when (1.2) holds has been recently analyzed in [8] by adapting, in a sophisticated way, some original ideas going back to [3] (later revised and applied in [2] and [10]), where a Poincaré–Birkhoff version for Hamiltonian systems was delivered. Note that the change of variables

$$x = \log u, \qquad y = \log v,$$

transforms (1.1) into the planar Hamiltonian system

$$\begin{cases} x' = -\lambda\alpha(t)(e^y - 1), \\ y' = \lambda\beta(t)(e^x - 1). \end{cases} \tag{2.1}$$

The updated version of the Poincaré–Birkhoff twist theorem that will be used reads as follows:

**Theorem 2.1 (Poincaré–Birkhoff)** *Assume that there exist* $0 < \varrho_0 < \varrho_1$ *and a positive integer* $\omega$ *such that*

$$\mathrm{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] > \omega \quad \textit{and} \quad \mathrm{rot}_{\varrho_1}[(x_0, y_0); [0, nT]] < \omega,$$

*where*

$$\mathrm{rot}_{\rho}[(x_0, y_0); [0, nT]] = \frac{\theta(nT) - \theta(0)}{2\pi}$$

*with* $||(x_0, y_0)|| = \rho$; $\theta(t)$ *being the angular polar coordinate of the solution starting at* $(x_0, y_0)$*, say* $(x(t), y(t))$*. Then,* (2.1) *admits, at least, two* $nT$*-periodic solutions lying in different periodicity classes with rotation number* $\omega$*.*

As a consequence of Theorem 2.1, the following result holds.

**Theorem 2.2** *Assume* (1.2)*. Then, for every positive integers* $\omega$ *and* $n$*, there exists* $\lambda_n^\omega > 0$ *such that* (2.1) *possesses, at least, two* $nT$*-periodic solutions with rotation number* $\omega$ *for every* $\lambda > \lambda_n^\omega$*.*

**Proof** Firstly, attention will be focused into the small solutions of (2.1). Obviously, there exists $\varepsilon > 0$ such that

$$(e^\xi - 1)\xi \geq \frac{\xi^2}{2} \quad \text{if} \ |\xi| < \varepsilon. \tag{2.2}$$

Choose $(x_0, y_0)$ sufficiently close to $(0, 0)$, say $|(x_0, y_0)| \leq \varrho_0$, so that the solution of (2.1) with $(x(0), y(0)) = (x_0, y_0)$, say $(x(t), y(t))$, satisfy $|(x(t), y(t))| < \varepsilon$ for all $t \in [0, nT]$. This is possible by continuous dependence on the initial conditions.

According to (1.2), there are $\tau \in (0, T)$ and $\delta > 0$ such that $\alpha(t)\beta(t) > 0$ for every $t \in [\tau - \delta, \tau + \delta] \subsetneq [0, T]$. Thus,

$$\zeta := \min_{t\in[\tau-\delta,\tau+\delta]} \{\alpha(t), \beta(t)\} > 0.$$

Consequently, due to (2.1) and (2.2), we obtain that, for every $t \in [0, nT]$,

$$\theta'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t) + y^2(t)} = \frac{\lambda\beta(t)(e^{x(t)} - 1)x(t) + \lambda\alpha(t)(e^{y(t)} - 1)y(t)}{x^2(t) + y^2(t)} \geq \frac{\lambda}{2}\frac{\beta(t)x^2(t) + \alpha(t)y^2(t)}{x^2(t) + y^2(t)} \geq \frac{\lambda\zeta}{2}. \tag{2.3}$$

Hence, owing to (2.3),

$$\operatorname{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] = \frac{\theta(nT) - \theta(0)}{2\pi} = \frac{1}{2\pi} \int_0^{nT} \theta'(s) ds \geq \frac{n}{2\pi} \int_{\tau-\delta}^{\tau+\delta} \theta'(s) ds \geq \frac{n\lambda\zeta 2\delta}{2\pi}.$$

Therefore,

$$\operatorname{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] > \omega \quad \text{if} \quad \lambda > \frac{\pi\omega}{n\zeta\delta} =: \lambda_n^{\omega}.$$

On the other hand, sufficiently large solutions do not rotate. Indeed, arguing by contradiction, assume that, for some solution $(x(t), y(t))$, we have that $\operatorname{rot}_{\varrho}[(x_0, y_0); [0, nT]] \geq 1$. Then, e.g., it must cross entirely the third quadrant. So, there exists $[\tau_1, \tau_2] \subset [0, nT]$ such that $y(\tau_1) = 0, x(\tau_1) < 0, y(\tau_2) < 0, x(\tau_2) = 0$, and $x(t) < 0$ and $y(t) < 0$ for all $t \in (\tau_1, \tau_2)$. Hence, for every $t \in [\tau_1, \tau_2]$, we find that

$$|x(t)| = |\lambda \int_t^{\tau_2} \alpha(s)(e^{y(s)} - 1) ds| \leq \lambda \int_0^{nT} \alpha(s) ds = \lambda n A,$$

$$|y(t)| = |\lambda \int_{\tau_1}^{t} \beta(s)(e^{x(s)} - 1) ds| \leq \lambda \int_0^{nT} \beta(s) ds = \lambda n B.$$

Therefore, if there exists $\tau_0 \in [0, nT]$ such that $(x(\tau_0), y(\tau_0))$ lies in the third quadrant and $x^2(\tau_0) + y^2(\tau_0) > R_1^2 := \lambda^2 n^2 (A^2 + B^2)$, then $(x(t), y(t))$ cannot cross the entire third quadrant. Similarly, since $|e^{x(t)} - 1|$ (resp. $|e^{y(t)} - 1|$) are bounded in the second (resp. fourth) quadrant, there exists $R_2 > 0$ (resp. $R_3 > 0$) such that $x^2(t) + y^2(t) < R_2^2$ (resp. $x^2(t) + y^2(t) < R_3^2$) if the solution crosses the second (resp. fourth) quadrant. Therefore, taking $R := \max\{R_1, R_2, R_3\}$, if $(x(\hat{t}), y(\hat{t}))$ lies in the second, third or fourth quadrants and $x(\hat{t})^2 + y(\hat{t})^2 > R$ for some $\hat{t} \in [0, nT]$, then, the solution $(x(t), y(t))$ cannot cross the corresponding quadrant.

Finally, let $s_0 \in [0, nT]$ be such that $x(s_0) = 0$ and $0 < y(s_0) \leq R$, and consider the maximal interval $[s_1, s_0] \subset [0, s_0]$ such that $x(t), y(t) \geq 0$ for all $t \in [s_1, s_0]$. By (2.1), $y(t)$ is non-decreasing in $[s_1, s_0]$ and, hence, $0 \leq y(t) \leq R$ for all $t \in [s_1, s_0]$. Since $y(t)$ is bounded, $x(t)$ must be bounded too. Thus, there exists a constant $R_* \geq R > 0$ such that if $x^2(\tilde{t}) + y^2(\tilde{t}) > R_*^2$ for some $\tilde{t} \in [0, nT]$ with $(x(\tilde{t}), y(\tilde{t}))$ lying in the first quadrant, then the solution $(x(t), y(t))$ cannot cross neither the second, nor the third and fourth quadrants. Therefore, $x(0)^2 + y(0)^2 = \varrho_1^2 > R_*^2$ implies that $\operatorname{rot}_{\varrho_1}[(x(0), y(0)); [0, nT]] < 1$ and hence, the hypothesis of Theorem 2.1 holds for every $\lambda > \lambda_n^{\omega}$, which ends the proof. □

**Remark 2.3** Although Theorem 2.2 has a counterpart for a more general class of Hamiltonian systems of the type

$$\begin{cases} x' = -\lambda\alpha(t) f(y), \\ y' = \lambda\beta(t) g(x), \end{cases}$$

where $f$ and $g$ satisfy certain boundedness and sign conditions (see [8, Sec. 2]), in this note we are focusing our attention into the predator-prey model (1.1). Thus, we restrict ourselves to consider $f$ and $g$ as they appear in (2.1).

## 3. The degenerate case

To analyze the problem (1.1) under the condition (1.3), we suppose that either

$$\operatorname{supp} \alpha \subset [t_0^1, t_1^1] \quad \text{and} \quad \operatorname{supp} \beta \subset [t_0^2, t_1^2], \tag{3.1}$$

or else

$$\operatorname{supp} \beta \subset [t_0^1, t_1^1] \quad \text{and} \quad \operatorname{supp} \alpha \subset [t_0^2, t_1^2], \tag{3.2}$$

for some partition

$$0 \leq t_0^1 < t_1^1 \leq t_0^2 < t_1^2 \leq T.$$

By (1.3), the system (1.1) can be integrated. Thus, in case (3.1) we have that, for every $t \in [0, T]$,

$$u(t) = u_0 e^{(1-v_0)\lambda \int_0^t \alpha(s) ds}, \quad v(t) = v_0 e^{(u(T)-1)\lambda \int_0^t \beta(s) ds},$$

whereas, in case (3.2),

$$u(t) = u_0 e^{(1-v(T))\lambda \int_0^t \alpha(s) ds}, \quad v(t) = v_0 e^{(u_0-1)\lambda \int_0^t \beta(s) ds},$$

for all $t \in [0, T]$. Hence, in case (3.1), the $T$-time Poincaré map is

$$(u_1, v_1) := \mathcal{P}_1(u_0, v_0) := (u(T), v(T)) = (u_0 e^{(1-v_0)\lambda A}, v_0 e^{(u_1-1)\lambda B}).$$

while, in case (3.2), is given through

$$(u_1, v_1) := \mathcal{P}_1(u_0, v_0) := (u(T), v(T)) = (u_0 e^{(1-v_1)\lambda A}, v_0 e^{(u_0-1)\lambda B}).$$

Consequently, iterating $n$ times these maps, it becomes apparent that either

$$\begin{aligned}(u_n, v_n) &:= \mathcal{P}_n(u_0, v_0) = \mathcal{P}_1^n(u_0, v_0) := (u(nT), v(nT)) = (u_{n-1} e^{(1-v_{n-1})\lambda A}, v_{n-1} e^{(u_n-1)\lambda B}) \\ &= (u_0 e^{(n-v_0-v_1-\cdots-v_{n-1})\lambda A}, v_0 e^{(u_1+u_2+\cdots+u_n-n)\lambda B})\end{aligned} \tag{3.3}$$

under condition (3.1), or

$$\begin{aligned}(u_n, v_n) &:= \mathcal{P}_n(u_0, v_0) = \mathcal{P}_1^n(u_0, v_0) := (u(nT), v(nT)) = (u_{n-1} e^{(1-v_n)\lambda A}, v_{n-1} e^{(u_{n-1}-1)\lambda B}) \\ &= (u_0 e^{(n-v_1-v_2-\cdots-v_n)\lambda A}, v_0 e^{(u_0+u_1+\cdots+u_{n-1}-n)\lambda B})\end{aligned} \tag{3.4}$$

under condition (3.2). By the uniqueness for the underlying Cauchy problem, the $nT$-periodic coexistence states of (1.1) are given by the positive fixed points of $\mathcal{P}_n$. Thus, by (3.3) and (3.4), we are driven to solve the system

$$\begin{cases} n = u_0 + u_1 + \cdots + u_{n-1}, \\ n = v_0 + v_1 + \cdots + v_{n-1}. \end{cases} \tag{3.5}$$

Naturally, the $u_i's$ and the $v_i's$ are different depending on (3.1) or (3.2). Our next result deals with the $T$-periodic and $2T$-periodic cases.

**Theorem 3.1** *Assume* (3.1) *or* (3.2). *Then,* (1.1) *does not admit any non-trivial $T$-periodic coexistence state. Moreover,* (1.1) *possesses exactly two non-trivial $2T$-periodic coexistence states if, and only if,*

$$\lambda > \frac{2}{\sqrt{AB}}. \tag{3.6}$$

**Proof** First, suppose (3.1). Then, by (3.3), $(u_1, v_1) = \mathcal{P}_1(u_0, v_0) = (u_0, v_0)$ if, and only if, $v_0 = 1$ and $u_0 = u_1 = 1$. Thus, $(u(t), v(t))$ is a $T$-periodic coexistence state if, and only if, $(u(t), v(t)) = (1, 1)$, which is the equilibrium of the system (1.1). Similarly,

$$(u_2, v_2) = \mathcal{P}_2(u_0, v_0) = (u_0 e^{(2-v_0-v_1)\lambda A}, v_0 e^{(u_1+u_2-2)\lambda B}) = (u_0, v_0)$$

if, and only if,

$$2 = v_0 + v_1 \quad \text{and} \quad 2 = u_1 + u_2 = u_0 + u_1,$$

or, equivalently,

$$2 = v_0 + v_0 e^{(u_1-1)\lambda B} = v_0 + v_0 e^{(1-u_0)\lambda B} \quad \text{and} \quad 2 = u_0 + u_0 e^{(1-v_0)\lambda A}. \tag{3.7}$$

Hence,

$$u_0 = \frac{2}{1 + e^{(1-v_0)\lambda A}}.$$

Setting $x := v_0$ and substituting $u_0$ in the first equation of (3.7) it is apparent that the $2T$-periodic coexistence states are given by the zeros of the map

$$\varphi(x) = x[e^{\frac{e^{(1-x)\lambda A}-1}{e^{(1-x)\lambda A}+1}\lambda B} + 1] - 2.$$

It is easily seen that

$$\varphi(x) < 0 \ \text{ if } \ x \le 0, \quad \varphi(1) = 0, \quad \varphi(x) > 0 \ \text{ if } \ x \ge 2, \quad \text{and} \ \ \varphi'(1) = 2 - \lambda^2 \frac{AB}{2}.$$

By (3.6), we find that $\varphi'(1) < 0$. Thus, there are $0 < x_1 < 1 < x_2 < 2$ such that $\varphi(x_1) = \varphi(x_2) = 0$, i.e., (1.1) has two non-trivial $2T$-periodic coexistence states. The uniqueness follows by analyzing $\varphi''$, much like in the proof of [7, Th. 2.1]. Similarly, one can derive the necessity of (3.6). This ends the proof when (3.1) is satisfied.

Now, assume (3.2). Then, by (3.4) and arguing as above, we find that

$$(u_1, v_1) = \mathcal{P}_1(u_0, v_0) = (u_0, v_0)$$

if, and only if, $(u(t), v(t)) = (1, 1)$. Moreover,

$$(u_2, v_2) = \mathcal{P}_2(u_0, v_0) = (u_0 e^{(2-v_1-v_2)\lambda A}, v_0 e^{(u_0+u_1-2)\lambda B}).$$

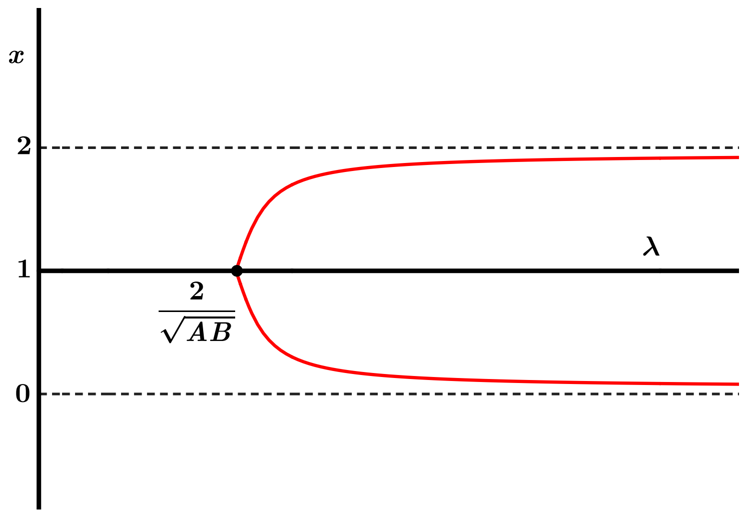


**Fig. 3** Global bifurcation diagram to $2T$-periodic coexistence states.

Thus, in this occasion, the $2T$-periodic coexistence states of (1.1) are given by the zeros of the map

$$\psi(x) = x[e^{\frac{1-e^{(x-1)\lambda A}}{1+e^{(x-1)\lambda A}}\lambda B} + 1] - 2.$$

Those with $x \neq 1$ provide us with the non-trivial $2T$-periodic coexistence states of (1.1). Adapting the previous argument, it readily follows the same result as before. This concludes the proof. □

Figure 3 shows the global bifurcation diagram of $2T$-periodic coexistence states of (1.1) in each of the cases (3.1), or (3.2). In both cases, they bifurcate supercritically from the equilibrium $(1, 1)$ at $\lambda = \frac{2}{\sqrt{AB}}$.

Subsequently, we will make explicit the dependence of the functions $\varphi$ and $\psi$ defined in the proof of Theorem 3.1 on the variables $x$ and $\lambda$. Since

$$\varphi(x, \lambda) = \psi(x, -\lambda),$$

dealing with the case when $\lambda > 0$ under condition (3.1) is the same as dealing with the case when $\lambda < 0$ under (3.2), in the sense that the $2T$-periodic coexistence states of (1.1) in each of these cases must coincide. From a biological point of view, this is rather natural. Actually, it is equivalent to inter-exchanging the role of the prey and the predator in the model.

Our last result provides us with the $nT$-periodic coexistence states of (1.1) when $n \geq 2$ in case (3.1). To get it, we must impose the following condition

$$A = B \quad \text{and} \quad u_0 = v_0 = x. \tag{3.8}$$

**Theorem 3.2** *Assume* (3.8). *Then, for every* $\lambda > \frac{2}{A}$, (1.1) *admits, at least, n coexistence states with period nT if n is even, and n* $- 1$ *coexistence states with period nT if n is odd.*

**Proof** First, we set $E_0(x) = 1$, and

$$E_n(x) := \begin{cases} e^{[\frac{n+1}{2} - x(E_0(x)+E_2(x)+\cdots+E_{n-1}(x))]\lambda A} & \text{if} \quad n \in 2\mathbb{N}+1, \\ e^{[x(E_1(x)+E_3(x)+\cdots+E_{n-1}(x)) - \frac{n}{2}]\lambda A} & \text{if} \quad n \in 2\mathbb{N}. \end{cases} \tag{3.9}$$

By (3.8), it turns out that

$$\varphi_n(x) = \varphi_{n-1}(x) - 1 + E_{n-1}(x),$$

where $\varphi_1(x) = x - 1$, is the map whose zeros provide us with the $nT$-periodic coexistence states of (1.1). As the analysis of these maps is fraught with a number of serious technical difficulties, in order to obtain some information concerning the $nT$-periodic coexistence states of (1.1), we are driven to analyze the variational equations of these maps at the trivial curve $(\lambda, 1)$,

$$p_n(\lambda) := \frac{\partial \varphi_n}{\partial x}(\lambda, 1).$$

It is easy to prove that $p_n(\lambda)$ is a sequence of polynomials in the indeterminate $\lambda$ that satisfy the recursive formula

$$p_n(\lambda) = [2 - (-1)^n A\lambda] p_{n-1}(\lambda) - p_{n-2}(\lambda),$$

where $p_1(\lambda) = 1$ and $p_2(\lambda) = 2 - A\lambda$. From this recursive formula, it can be shown that any root of $p_n$ is real and algebraically simple. Thanks to these features, for any given $r \in p_n^{-1}(0)$, the transversality condition

$$\frac{dp_n(r)}{d\lambda}(N[p_n(r)]) \oplus R[p_n(r)] = \mathbb{R}$$

holds, where $N$ and $R$ stand for the null space and the rank, respectively, of the underlying one-dimensional operators. Thus, for any given $r \in p_n^{-1}(0)$, the algebraic multiplicity of Esquinas and López-Gómez [4] equals one at every point $(r, 1)$. So, according to Crandall and Rabinowitz [1, Th. 1.7], a local bifurcation occurs at every point $(r, 1)$. Moreover, by the unilateral theorem of López-Gómez [6, Th. 6.4.3], the underlying subcomponents of $nT$-periodic coexistence states are unbounded in $\lambda$, in agrement with Rabinowitz [11]. As the number of positive roots of $p_n(\lambda)$ equals $\frac{n}{2}$ if $n$ is even and $\frac{n-1}{2}$ if $n$ is odd, the result holds. This ends the proof. $\square$

Figure 4 shows the global bifurcation diagram provided by Theorem 3.2. It is an ideal global bifurcation diagram as the local bifurcation directions and the eventual secondary bifurcations have not been analyzed yet. According to [7, Th. 6.1], the local bifurcations of the $3T$-periodic component is transcritical, while the $4T$-periodic component bifurcates subcritically from the trivial curve $(\lambda, 1)$.

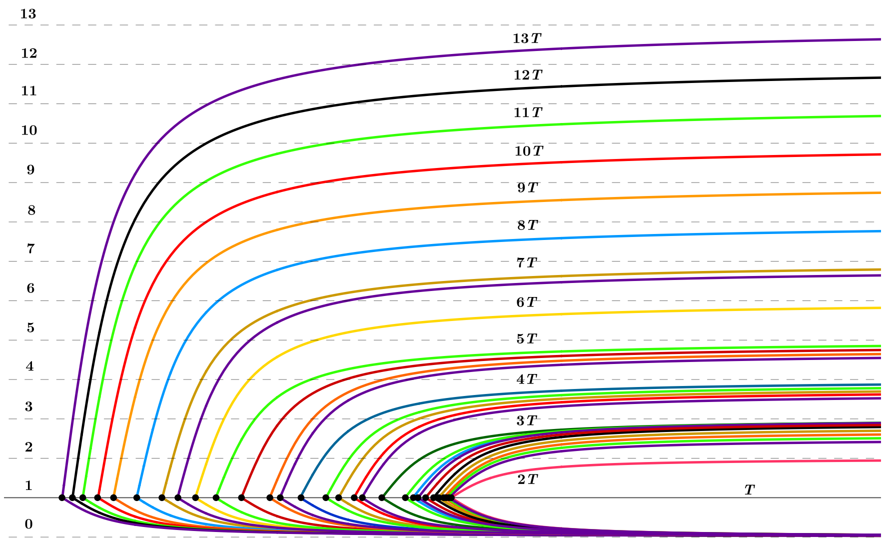


**Fig. 4** Bifurcation diagram of (1.1) under condition (3.8).

## Acknowledgements

This paper has been written under the joint auspices of the IMI of Complutense University and the Ministry of Science, Technology and Universities of Spain, under Research Grant PGC2018-097104-B-100 (first author), and the INDAM-GNAMPA under the research project SiDiA (Dynamical Systems and Applications) of the University of Udine (third author). The second author has been as well supported by the Contract CT42/18-CT43/18 of Complutense University of Madrid.

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