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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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On iterative schemes for matrix equations

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Abstract

In this work we focus on solving quadratic matrix equations. We start by transforming the quadratic matrix equation into a fixed point equation. From this transformation, we propose an iterative scheme of stable successive approximations. We study the global convergence of this iterative scheme. In addition, we obtain a result of restricted global convergence to the well-known Picard method using a technique of auxiliary points. From the results obtained, we analyze the location and separation of the solutions of the quadratic matrix equation considered. Finally, we build a hybrid iterative scheme, predictor-corrector, which allow us to approximate a solution of the quadratic matrix equation more efficiently.

1. Introduction

The study of quadratic matrix equation is motivated by the great variety of problems where appears. Quadratic matrix equation arises in many areas of scientific computing and engineering applications. For instance, algebraic Riccati equations arising in control theory [8]. Another important class of quadratic matrix equations is motivated by noisy Wiener-Hopf problems for Markov chains [9].

Although some algebraic Riccati equations are quadratic matrix equation, and vice versa, the two classes of equations require different techniques for analysis and solution in general.

In this study we are interested in the simplest quadratic matrix equation:

$$\mathbb{Q}(X) = X^2 - BX - C = 0, \qquad B, C \in \mathbb{R}^{m\times m}, \tag{1.1}$$

which occurs in a variety of applications, for example, it may arise in the well known quadratic eigenvalue problem:

$$Q(\lambda)x = \lambda^2 Ax + \lambda Bx + Cx = 0, \text{ with } A, B, C \in \mathbb{C}^{m\times m},$$

that arises in the analysis of structural systems and vibration problems [10].

The application of iterative schemes is commonly used to approximate a solution of equation (1.1). We obtain qualitative results about the equation at issue from the study of the convergence. For instance, a solution existence result is obtained for equation (1.1), with the so-called existence ball of an iterative scheme given in [1], which allows us to locate a solution. On the other hand, a result of uniqueness of the solution allows us to separate solutions [2]. Finally, the iterative scheme considered, under the convergence conditions obtained, allows us to approximate a solution of equation (1.1). This is how the three main aims of our work arise: locate, separate and approximate a solution of equation (1.1).

The paper is organized as follows. In Section 2, we present different conditions to locate and separate solutions of equation (1.1) from the study of the convergence of the Successive Approximations and Picard methods. In Section 3, we define a hybrid iterative scheme to approximate a solution of equation (1.1).

2. The Successive Approximations and Picard Methods

In what follows, we suppose that there exists X^* a fixed matrix of T with $T(X) = X$, $T : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ in $\overline{B(X^*, R)}$. In this case, we use the following modification of the Banach Fixed Point Theorem.

Theorem 2.1 *If $\Omega \subset \mathbb{R}^{m\times m}$ is convex and compact and $T : \Omega \to \Omega$ is a contraction, then T admits a unique fixed matrix in Ω and it can be approximated from $X_{n+1} = T(X_n)$, $n \geq 0$, for any X_0 given in Ω.*

So, we look for conditions on R so that the Successive Approximations Method is convergent for any starting matrix X_0 in $\overline{B(X^*, R)}$. Thus, we obtain a local convergence result.

Now, we provide a basic technical result whose proof is easily followed taking into account

$$T'(X)Y = -(X - B)^{-1}Y(X - B)^{-1}C.$$

Lemma 2.2 *Let X^* be a fixed matrix of T in $\overline{B(X^*,R)}$ and we suppose that there exists $(X^*-B)^{-1}$ with $\|(X^*-B)^{-1}\| \leq \beta$. For each $X \in \overline{B(X^*,R)}$, with $R < 1/\beta$, are satisfied*:

(i) *there exists $(X^*+t(X-X^*)-B)^{-1}$ for $t \in [0,1]$, and $\|(X^*+t(X-X^*)-B)^{-1}\| \leq f_R(t)$, where $f_R(t) = \dfrac{\beta}{1-t\beta R}$,*

(ii) $\|T'(X^*+t(X-X^*))\| \leq f_R(t)^2\|C\|$,

(iii) $\|T'(X^*+t(X-X^*)-T'(X^*))(X-X^*)\| \leq (f_R(t)^2 + f_R(0)^2)\|X-X^*\|\|C\|$.

Now, to apply the modification of the Banach Fixed Point Theorem to operator T, restricted to $\Omega = \overline{B(X^*,R)}$ with $R > 0$, T must be a contraction map of Ω into itself. That happens if $R < \dfrac{1}{\beta} - \sqrt{c}$, where we denote $\|C\| = c$. Notice that, in this case, this implies that $R < \dfrac{1}{\beta}$ and we can prove the following local result.

Theorem 2.3 *Let X^* be a fixed point of T and we suppose that there exists $(X^*-B)^{-1}$ with $\|(X^*-B)^{-1}\| \leq \beta$. If $\beta^2 c < 1$, with $c = \|C\|$, then, the Successive Approximations Method*

$$X_0 \text{ given in } \mathbb{R}^{m\times m}, \quad X_{n+1} = T(X_n), \quad n \geq 0, \tag{2.1}$$

is convergent to X^*, from any starting matrix $X_0 \in \overline{B(X^*,R)}$, with $R \in \left(0, \dfrac{1}{\beta} - \sqrt{c}\right)$. Moreover, X^* is the unique fixed matrix of the operator T in $\overline{B(X^*,R)}$.

We observe that if X^* is a fixed matrix of the operator T, such that there exists $(X^*-B)^{-1}$ with $\|(X^*-B)^{-1}\| \leq \beta$,it follows

$$\|X^*\| = \|T(X^*)\| \leq \|(X^*-B)^{-1}\|\|C\| \leq \beta c.$$

Therefore, we have $X^* \in \overline{B(0,\beta c)}$, where we denote by 0 the null matrix in $\mathbb{R}^{m\times m}$. So, the domain $\overline{B(0,R)}$, with $R \geq \beta c$, can be a convenient domain where to ensure the convergence of the Successive Approximations Method.

Theorem 2.4 *Let $X^* \in \overline{B(0,R)}$ be a fixed matrix of T and we suppose that there exists $(X^*-B)^{-1}$ with $\|(X^*-B)^{-1}\| \leq \beta$. If $\beta^2 c < \dfrac{1}{8}$, and*

$$R \in \begin{cases} \left[\dfrac{1-\sqrt{1-8\beta^2 c}}{4\beta}, \dfrac{1}{2}\left(\dfrac{1}{\beta} - \sqrt{c}\right)\right) & \text{if } \beta^2 c \in \left(0, \dfrac{1}{9}\right), \\ \left[\dfrac{1-\sqrt{1-8\beta^2 c}}{4\beta}, \dfrac{1+\sqrt{1-8\beta^2 c}}{4\beta}\right] & \text{if } \beta^2 c \in \left[\dfrac{1}{9}, \dfrac{1}{8}\right]. \end{cases},$$

Then, from any starting matrix $X_0 \in \overline{B(0,R)}$, the Successive Approximations Method is convergent to X^*. Moreover, X^* is the unique fixed matrix of T in $\overline{B(0,R)}$.

Notice that if there exists B^{-1}, then for $X \in \overline{B(0,R)}$, it follows

$$\|I - (-B^{-1})(X-B)\| \leq \|B^{-1}\|\|X\| \leq \alpha R,$$

with $\|B^{-1}\| \leq \alpha$ and $X \in \overline{B(0,R)}$. Therefore, if $R < 1/\alpha$, then there exists $(X-B)^{-1}$ and $\|(X-B)^{-1}\| \leq \dfrac{\alpha}{1-\alpha R}$ by the perturbation lemma in matrix analysis. From this, we obtain the following restricted global convergence result.

Theorem 2.5 *Suppose that there exists B^{-1}, with $\|B^{-1}\| \leq \alpha$, and $\alpha^2 c \leq 1/4$. Then, from any starting matrix $X_0 \in \overline{B(0,R)}$, with $R \in \left[\dfrac{1-\sqrt{1-4\alpha^2 c}}{2\alpha}, \dfrac{1}{\alpha} - \sqrt{c}\right)$, the Successive Approximations Method is convergent to the fixed matrix X^* of the operator T. Moreover, X^* is the unique fixed matrix of T in $\overline{B(0,R)}$.*

Next, we illustrate the theoretical results obtained above with some examples. Firstly, we examine a simple academic case, where the technique developed can be applied. We consider the particular (QME) with:

$$B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2\epsilon(\epsilon-1) & \epsilon(2-3\epsilon) \\ -\epsilon(1+3\epsilon) & \epsilon(2+5\epsilon) \end{pmatrix}, \tag{2.2}$$

where the parameter ϵ is not zero. We find that it has the solution

$$X^* = \begin{pmatrix} \epsilon & -\epsilon \\ -\epsilon & 2\epsilon \end{pmatrix}.$$

For the value $\epsilon = 0.04$, we have $\beta^2 c = 0.16299 < 1$ and, then result of Theorem 2.3 follows immediately. The Successive Approximations Method is convergent to X^*, from any starting matrix $X_0 \in \overline{B(X^*, R)}$, with $R \in (0, 0.564271)$. Moreover, X^* is the unique fixed matrix of the operator T in $\overline{B(X^*, R)}$.

On the other hand, the results of Theorem 2.4 and 2.5 follows for smaller values of the parameter ϵ. For instance, if $\epsilon = 0.025$, then it follows that $\beta^2 c = 0.105551 < 1/8$. Thus, the local result given in Theorem 2.4 states that from any starting matrix $X_0 \in \overline{B(0, R)}$, with $R \in [0.140379, 0.313011]$, the Successive Approximations Method is convergent to X^* and is the unique fixed matrix of T in $\overline{B(0, R)}$. While, in this case, if $\alpha^2 c = 0.113448 < 1/4$, the semilocal result given in Theorem 2.4 states from any starting matrix $X_0 \in \overline{B(0, R)}$, with $R \in [0.116697, 0.296583)$, the Successive Approximations Method is convergent to the fixed matrix X^* of the operator T. Moreover, X^* is the unique fixed matrix of T in $\overline{B(0, R)}$.

It is clear that, from Theorem 2.3 we separate the solution X^* successfully from other possible solutions, despite its poor location. However, the local result obtained in Theorem 2.4 shows a better separation of the solution. On the other hand, the semilocal convergence result obtained in Theorem 2.5 is more applicable than the local result, since it does not need to know X^*. And moreover, the semilocal result is the one that best locates the aforesaid solution.

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Next, we try to smooth the results obtained. For this, we consider the Picard method:

$$X_0 \text{ given in } \mathbb{R}^{m\times m}, \quad X_{n+1} = P(X_n) = X_n - F(X_n), \quad n \geq 0, \tag{2.3}$$

where $F(X) = (I - T)(X)$, $F : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, with $F(X) = X - (X - B)^{-1}C$. Notice that, the iterations obtained by the Picard method are the same as those obtained by the Successive Approximations Method. Both methods are equivalent.

To obtain a global convergence result to the Picard method, we use auxiliary matrices. Moreover, we can establish both semilocal and local convergence results for the Picard method.

Theorem 2.6 *Let $\tilde{X} \in \mathbb{R}^{m\times m}$ such that there exists $(\tilde{X} - B)^{-1}$ with $\|(\tilde{X} - B)^{-1}\| \leq \tilde{\beta}$. We suppose that $\|F(\tilde{X})\| \leq \dfrac{1 + \tilde{\beta}^2 c - 2\tilde{\beta}\sqrt{c}}{\tilde{\beta}}$, with $c = \|C\|$, and $\tilde{\beta}^2 c < 1$. Then, from any starting matrix $X_0 \in B(\tilde{X}, R)$, the Picard method* (2.3) *converges to a solution X^* of equation* (1.1). *The solution X^* and the iterates X_n belong to $\overline{B(\tilde{X}, R)}$, for $n \geqslant 0$, where*

$$R \in \left[\frac{1 - \tilde{\beta}^2 c + \tilde{\beta}\|F(\tilde{X})\| - \sqrt{\Delta}}{2\tilde{\beta}}, \min\left\{\frac{1}{\tilde{\beta}} - \sqrt{c}, \frac{1 - \tilde{\beta}^2 c + \tilde{\beta}\|F(\tilde{X})\| + \sqrt{\Delta}}{2\tilde{\beta}}\right\}\right), \tag{2.4}$$

with $\Delta = (1 - \tilde{\beta}^2 c + \tilde{\beta}\|F(\tilde{X})\|)^2 - 4\tilde{\beta}\|F(\tilde{X})\|$. Moreover, X^* is the unique solution of equation (1.1) *in $B(\tilde{X}, \frac{1}{\tilde{\beta}} - \sqrt{c})$.*

Corollary 2.7 *Let X^* be a solution of equation* (1.1) *such that exists $(X^* - B)^{-1}$ with $\|(X^* - B)^{-1}\| \leq \beta$ and $\beta^2 c < 1$. Then, the Picard method* (2.3)*, from any starting at $X_0 \in B(X^*, R)$ converges to X^*, where $R \in \left(0, \dfrac{1}{\beta} - \sqrt{c}\right)$.*

Moreover, X^* is unique in $B\left(X^*, \dfrac{1}{\beta} - \sqrt{c}\right)$.

Next, to obtain a semilocal convergence result for the Picard method, we consider $\tilde{X} = X_0$ from Theorem 2.6.

Corollary 2.8 *Let $X_0 \in \mathbb{R}^{m\times m}$ be such that exists $(X_0 - B)^{-1}$ with $\|(X_0 - B)^{-1}\| \leq \beta_0$. Suppose that $\|F(X_0)\| \leq \dfrac{1+\beta_0^2 c - 2\beta_0\sqrt{c}}{\beta_0}$, with $c = \|C\|$, and $\beta_0^2 c < 1$. Then, the Picard method* (2.3) *converges to a solution X^* of equation equation* (1.1) *. The solution X^* and the iterates X_n belong to $\overline{B(\tilde{X}, R)}$, for $n \geqslant 0$, where*

$$R \in \left[\frac{1-\tilde{\beta}^2 c + \tilde{\beta}\|F(X_0)\| - \sqrt{\Delta}}{2\beta_0}, \min\left\{\frac{1}{\beta_0} - \sqrt{c}, \frac{1-\beta_0^2 c + \beta_0\|F(X_0)\| + \sqrt{\Delta}}{2\beta_0}\right\}\right), \tag{2.5}$$

with $\Delta = (1 - \beta_0^2 c + \beta_0\|F(X_0)\|)^2 - 4\beta_0\|F(X_0)\|$. Moreover, the solution X^* is the unique solution of the equation $F(X) = 0$ in $B\left(X_0, \dfrac{1}{\beta_0} - \sqrt{c}\right)$.

Next, we provide another semilocal convergence result for the Picard method.

Theorem 2.9 *Let $X_0 \in \mathbb{R}^{m\times m}$ such that there exists $(X_0 - B)^{-1}$ with $\|(X_0 - B)^{-1}\| \leq \beta_0$ and $\|F(X_0)\| \leq \eta_0$. We suppose that the scalar equation*

$$\left(1 + \frac{\beta_0^2 c(1-\beta_0 t)}{1 - \beta_0^2 c - 2\beta_0 t + \beta_0^2 t^2}\right)\eta_0 = t \tag{2.6}$$

has at least one positive solution and we denote by R the smallest positive root. If $R < \dfrac{1}{\beta_0} - \sqrt{c}$ and $\beta_0^2 c < 1$, then, starting at X_0, the Picard method (2.3) *converges to X^* a solution of equation* (1.1)*. Moreover, $X_n, X^* \in \overline{B(X_0, R)}$, for all $n \geq 0$, and X^* is unique in $B\left(X_0, \dfrac{1}{\beta_0} - \sqrt{c}\right)$.*

Next, we illustrate the theoretical results obtained for the Picard method, considering the simple numerical example given in (2.2).

Taking the value $\epsilon = 0.025$, and

$$\tilde{X} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix},$$

then there exists $(\tilde{X}-B)^{-1}$ with $\|(\tilde{X}-B)^{-1}\| \leq 1.09917$ and conditions of Theorem 2.6, $\|F(\tilde{X})\| \leq \dfrac{1+\tilde{\beta}^2 c - 2\tilde{\beta}\sqrt{c}}{\tilde{\beta}}$ and $\tilde{\beta}^2 c < 1$, are satisfied. Thus, from any starting matrix $X_0 \in \overline{B(\tilde{X}, R)}$ with $R \in [0.0699064, 0.608512]$, the Picard method (2.3) converges to a solution X^* of equation (2.2). The solution X^* and X_n belong to $\overline{B(\tilde{X}, R)}$, for $n \geqslant 0$. Moreover, X^* is the unique solution of (QME) in $B(\tilde{X}, 0.608512)$.

In general, both Theorem 2.6 and Theorem 2.9 provide a more precise location of the solution X^* than that obtained by the Successive Approximations Method for which we always obtain balls centered in the null matrix. However, in these results, locating a starting matrix X_0 satisfying the indicated conditions, we locate the solution in a ball centered in the aforesaid X_0.

Now, we choose the starting matrix

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

to compare the results obtained by means of the Successive Approximations and the Picard methods, in Corollary 2.8 and in Theorem 2.9,

So, the hypotheses of Corollary 2.8 with $\|(X_0-B)^{-1}\| \leq 1.11803$, $\|F(X_0)\| = 0.0895976 \leq \dfrac{1+\beta_0^2 c - 2\beta_0\sqrt{c}}{\beta_0} = 0.393375$, and $\beta_0^2 c = 0.113448 < 1$ are satisfied. Thus, the Picard method converges to a solution X^* of equation (2.2). The solution X^* and the iterates X_n belong to $\overline{B(X_0, R)}$, for $n \geqslant 0$, with $R \in [0.104128, 0.593166)$ and X^* is unique in $B(\tilde{X}, 0.593166)$.

On the other hand, equation (2.6) has at least one positive solution and the smallest positive root is $R = 0.103032$, such that, $R < \dfrac{1}{\beta_0} - \sqrt{c} = 0.593166$. Thus, starting at X_0, the Picard method converges to X^* a solution of equation (2.2). Moreover, $X_n, X^* \in \overline{B(X_0, 0.10303)}$, for all $n \geq 0$.

As we can observe, we have considered the null matrix as the starting point, the same matrix when applying the Successive Approximations Method. Notice that, the location and the separation of solutions given by the existence and uniqueness ball, respectively, is improved when we apply the Picard method. Namely, both Corollary 2.8 and Theorem 2.9 improve the results obtained to the Successive Approximations Method in Theorems 2.4 and 2.5.

3. Predictor-corrector scheme

Our next goal is the approximation of a solution of equation (1.1). Although both the Successive Approximations and the Picard methods have a linear convergence speed, their applications are useful. This is due to the fact that they have a low operational cost and good accessibility domain associated with them. Now, we propose to build a hybrid iterative scheme through a predictor-corrector method. That is, a hybrid method consisting of two stages. The idea is, firstly to apply a method which has a good accessibility and low operational cost and later, in a second stage, to apply a method that accelerates the convergence as follows:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} X_0 \in \mathbb{R}^{m\times m}, \\ X_{n+1} = \Phi(X_n), \quad n = 1, 2, \dots, N_0, \end{array}\right. \\ \left\{\begin{array}{l} Y_0 = X_{N_0+1}, \\ Y_{n+1} = \Psi(Y_n), \quad n \geqslant 0, \end{array}\right. \end{array}\right. \tag{3.1}$$

from any two one-point iterative schemes:

$$\left\{\begin{array}{l} X_0 \in \mathbb{R}^{m\times m}, \\ X_{n+1} = \Phi(X_n), \quad n \geqslant 0, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} Y_0 \in \mathbb{R}^{m\times m}, \\ Y_{n+1} = \Psi(Y_n), \quad n \geqslant 0. \end{array}\right.$$

The first iterative scheme to be applied Φ, is called the predictor iterative scheme and the second Ψ, the corrector iterative scheme. It is known that high-order iterative schemes have a reduced accessibility domain and, therefore, locating starting points for them is a difficult problem to solve. Therefore, we propose that the hybrid scheme (3.1) be convergent under the conditions that the iterative predictor scheme is. In our case we consider the hybrid iterative scheme formed by the Picard method, as a predictor, and the Newton method as a corrector iterative scheme that accelerates the convergence speed of the Picard method. Note that Newton's method is also an iterative scheme with low operational cost and quadratic convergence. Thus, we propose the following iterative scheme:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} X_0 \in \mathbb{R}^{m\times m}, \\ X_{n+1} = X_n - F(X_n), \quad n = 0, 1, 2, \dots, N_0 - 1, \end{array}\right. \\ \left\{\begin{array}{l} Y_0 = X_{N_0}, \\ Y_{n+1} = Y_n - [F'(Y_n)]^{-1}F(Y_n), \quad n \geqslant 0, \end{array}\right. \end{array}\right. \tag{3.2}$$

to approximate a solution of equation (1.1), where $F : \mathbb{R}^{m\times m} \longrightarrow \mathbb{R}^{m\times m}$, with $F(X) = X - (X - B)^{-1}C$. From now on, we use the notation $\{Z_n\}$ to refer to the hybrid method (3.2), such that

$$Z_n = \left\{\begin{array}{ll} X_n & \text{for } n = 0, 1, \dots, N_0 - 1, \\ Y_n & \text{for } n \geqslant N_0, \end{array}\right.$$

Secondly, our main is to ensure the convergence of the iterative scheme (3.2) under the same convergence conditions given for the Picard method in Theorem 2.9, locating the value of N_0. This maintains the accessibility of the Picard method for the hybrid iterative scheme (3.2).

Theorem 3.1 *Under conditions of Theorem* 2.9. *We suppose that the scalar equation Therefore, if we suppose that the scalar equation*

$$\frac{2((1-K)^2 - M(t)\eta_0)}{2(1-K)^2 - 3M(t)\eta_0} = t, \tag{3.3}$$

has at least one positive solution and we denote by $\widetilde{R}$ the smallest positive root, starting at $X_0 \in \mathbb{R}^{m\times m}$ and for

$$N_0 \geqslant 1 + \left[\max\left\{\frac{\ln\left(\frac{(1-K)^2}{2\eta_0 M(\widetilde{R})}\right)}{\ln K}, \frac{\ln\left(\frac{(1-K)(1/\beta_0 - R)}{\widetilde{R}\eta_0}\right)}{\ln K}\right\}\right], \tag{3.4}$$

where $[x]$ represents the integer part of the real number x, the hybrid iterative scheme (3.2) *converges to Z^*, a solution of equation* (1.1)*. Moreover, $Z_n, Z^* \in \overline{B(X_0, R + \delta\widetilde{R})}$, for all $n \geq 0$.*

Following the numerical example given in (2.2) we illustrate the result obtained in Theorem 3.1 for the hybrid iterative scheme (3.2) . Taking $\epsilon = 0.04$ and

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we are able to apply Theorem2.9. It's easy to see that $\widetilde{R} = 0.412888$ is the smallest positive root of scalar equation (3.3). Moreover, $N_0 \geqslant 1$ and then it is enough to iterate once with the Picard method to ensure a fast convergence with the Newton method to a solution of (QME) given in (2.2). Moreover, $Z_n, Z^* \in \overline{B(X_0, 0.160193)}$, for all $n \geq 0$.

4. Conclusions

From a fixed point type transformation of (QME), we obtain a stable iterative scheme of successive approximations. Using this scheme and the Picard method we carried out a qualitative study of (QME). We obtain domains of existence and uniqueness of solutions that allow us to locate and separate them. Moreover, we construct a hybrid method taking into account, the low operational cost and the good accessibility domain that these linear methods have associated. The numerical examples confirm that the hybrid iterative scheme improves the operational cost that involves the application of Newton's method as a corrector method, when approximating a solution of (QME).

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