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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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High-order well-balanced methods for systems of balance laws based on collocation RK ODE solvers

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Abstract

The aim of this work is to develop high-order well-balanced schemes for 1d systems of balance laws. A general methodology for developing such numerical methods was proposed by two of the authors that requires the computation, at every cell and at every time step, of the stationary solution whose cell average is equal to the numerical approximation already obtained. Since solving these problems can be difficult and expensive, our goal is to introduce a general procedure that can be applied to any one-dimensional system of balance laws based on the application of collocation RK methods to approximate the stationary solution with given cell-average.

1. Introduction

Let us consider 1d systems of balance laws of the form

$$U_t + F(U)_x = S(U)H_x, \tag{1.1}$$

where $U(x,t)$ takes value in $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$), which is supposed to be continuous. We suppose that system (1.1) is strictly hyperbolic, i.e., the Jacobian $J(U)$ of the flux function has N real distinct eigenvalues $\lambda_1(U), \cdots, \lambda_N(U)$ and associated eigenvectors $r_1(U), \cdots, r_N(U)$. Moreover, we suppose that $\lambda_i(U) \neq 0$, $i = 1, \ldots, N$.

The system (1.1) has nontrivial stationary solutions that satisfy the ODE system:

$$F(U)_x = S(U)H_x, \tag{1.2}$$

or

$$J(U)U_x = S(U)H_x. \tag{1.3}$$

A numerical method is said to be well-balanced if it preserves (in some sense) stationary solutions. This property is important when the waves generated by small perturbations of an equilibrium are to be simulated: numerical errors should not break the equilibrium. The research on the idea of constructing numerical schemes that preserve some equilibria has been developed by many authors: see, for instance, [2] , [1], [3], [11], [16], [18], [20], [21], [22], [24], [14], [8], [15], [9], [7]. See [6] and their references for a recent review on this topic.

We consider high-order finite-volume numerical methods for (1.1) of the form:

$$\frac{d\widetilde{U}_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(t) - F_{i-\frac{1}{2}}(t)\right) + \frac{1}{\Delta x}S_i, \tag{1.4}$$

where:

- $\widetilde{U}_i(t)$ is the approximation of the average of the exact solution at the ith cell, $I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$, at time t:

$$\widetilde{U}_i(t) \cong \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U(x,t)\,dx,$$

  where the length of the cells Δx is supposed to be constant for simplicity;

- $F_{i+\frac{1}{2}} = \mathbb{F}(U^{t,-}_{i+\frac{1}{2}}, U^{t,+}_{i+\frac{1}{2}})$, where $\mathbb{F}$ is a consistent numerical flux and $U^{t,\pm}_{i+\frac{1}{2}}$ are the reconstructed states at the intercells:

$$U^{t,-}_{i+\frac{1}{2}} = P^t_i(x_{i+\frac{1}{2}}), \quad U^{t,+}_{i+\frac{1}{2}} = P^t_{i+1}(x_{i+\frac{1}{2}}).$$

  Here $P^t_i(x)$ is the approximation of the solution at the ith cell given by a reconstruction operator of order p applied to the sequence of cell values $\{\widetilde{U}_i(t)\}$:

$$P^t_i(x) = P_i(x; \{\widetilde{U}_j(t)\}_{j\in\mathcal{S}_i}),$$

  where $\mathcal{S}_i$ denotes the set of indices of the cells belonging to the stencil of the cell I_i;

- finally,

$$S_i \approx \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(P_i^t(x))H_x(x)\,dx. \tag{1.5}$$

Given a function U, the following notation

$$\bar{U}_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x)\,dx, \quad \widetilde{U}_i \approx \bar{U}_i, \quad U^{i+1/2} \approx U(x_{i+1/2}), \quad \forall i,$$

is used to denote its cell averages, the approximations to its cell averages and its point values at the intercells, respectively.

2. Well-balanced numerical methods

The key point in [4], [5] is to transfer the well-balanced property to the reconstruction operator:

Definition 2.1 Given a stationary solution U^*, the reconstruction operator is said to be well-balanced for U^* if

$$P_i(x) = U^*(x), \quad \forall x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],\ \forall i, \tag{2.1}$$

where P_i is the approximation of U^* given by the reconstruction operator from the vector $\{\bar{U}_i^*\}$ of cell-averages of U^*.

One can easily prove that the numerical method (1.4) with

$$S_i = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(P_i^t(x))H_x(x)\,dx, \tag{2.2}$$

is *exactly well-balanced* if the reconstruction operator is well-balanced for every stationary solution U^*, which means that the vector of its cell-averages $\{\bar{U}_i^*\}$ (or its approximations $\{\widetilde{U}_i^*\}$ if a quadrature formula is used to compute them) is an equilibrium of (1.4).

However, in general a standard reconstruction operator is not expected to be well-balanced. The following algorithm allows us to design a well-balanced reconstruction operator P_i on the basis of a standard operator Q_i, provided that Q_i is exact for the null function (see [4]):

Algorithm 2.2 *Given a family of cell values $\{\bar{U}_i\}$, at every cell I_i:*

1. *Find, if possible, the stationary solution $U_i^*(x)$ in the stencil of cell I_i such that*:

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U_i^*(x)\,dx = \bar{U}_i. \tag{2.3}$$

   Otherwise, take $U_i^* \equiv 0$.

2. *Apply the reconstruction operator to the cell values $\{V_j\}_{j\in S_i}$ given by*

$$V_j = \bar{U}_j - \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U_i^*(x)\,dx, \quad j \in \mathcal{S}_i,$$

   to obtain:

$$Q_i(x) = Q_i(x; \{V_j\}_{j\in\mathcal{S}_i}).$$

3. *Define*

$$P_i(x) = U_i^*(x) + Q_i(x). \tag{2.4}$$

Another difficulty may come from the use of quadrature formulas to compute the cell-averages and the integral of the source term at the right-hand side of (1.4). In this case, the numerical method is still well-balanced if:

- the quadrature formula is also applied to compute the integrals appearing in the first two steps of Algorithm 2.2;

- S_i is computed as follows:

$$S_i = F\left(U_i^{t,*}(x_{i+\frac{1}{2}})\right) - F\left(U_i^{t,*}(x_{i-\frac{1}{2}})\right) + \Delta x \sum_{m=1}^{s} b_m \left(S(P_i^t(x_i^m)) - S(U_i^{t,*}(x_i^m))\right) H_x(x_i^m), \tag{2.5}$$

  where $U_i^{t,*}$ is the stationary solution found in the first step of the reconstruction procedure at the ith cell and time t, and x_i^m, b_m, $m = 1, \dots, s$ are respectively the nodes and the weights of the selected quadrature formula, whose order of accuracy is bigger or equal than p.

Notice that, at every cell and at every time step, the following nonlinear problem has to be solved:
Find U such that

$$J(U)U_x = S(U)H_x, \quad \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x)\,dx = \widetilde{U}_i, \tag{2.6}$$

where $\widetilde{U}_i$ is an approximation of the cell-average at the ith cell of the solution of (1.1) that we are looking for. In addition, once the solution U_i^* of (2.6) has been found, one has to solve two Cauchy problems in order to extend it to the cells belonging to the stencil. Specifically, (1.3) with initial condition $U(x_{i+1/2}) = U^*(x_{i+1/2})$ has to be solved forward in space and (1.3) with final condition $U(x_{i-1/2}) = U^*(x_{i-1/2})$ backward in space.

Solving these local nonlinear problems can be difficult if the analytic expression of the solutions of (1.3) are not known either in explicit or implicit form. We propose here to approximate their solutions by solving the following numerical problems:

Problem 2.3 (Local problem (LP)) *Given an index i and a state $\widetilde{U}_i \in \Omega$, find approximations*

$$U_{i,j}^{*,m},\ m = 1, \dots, s,\ j \in \mathcal{S}_i; \quad U_i^{*,i\pm 1/2};$$

of the values

$$U_i^*(x_j^m),\ m = 1, \dots, s,\ j \in \mathcal{S}_i; \quad U_i^*(x_{i\pm 1/2});$$

where U_i^* is the stationary solution that satisfies

$$\sum_{m=1}^{s} b_m U_i^*(x_i^m) = \widetilde{U}_i. \tag{2.7}$$

The numerical methods issues from this strategy are not expected to be exactly well-balanced, but they will be well-balanced according to the following definition:

Definition 2.4 The numerical method (1.4) is said to be well-balanced with order $q \geq p$ if for every stationary solution U^* of (1.1) and for every Δx, there exists an equilibrium $\{\widetilde{U}_{\Delta x,i}^*\}$ of (1.4) such that

$$\bar{U}_i^* = \widetilde{U}_{\Delta x,i}^* + O(\Delta x^q), \quad \forall i. \tag{2.8}$$

The sequence $\{\widetilde{U}_{\Delta x,i}^*\}$ is said to be a discrete stationary solution.

3. RK Collocation methods

We propose here to solve the local problems (LP) using a RK collocation method with Butcher tableau

$$\begin{array}{c|ccc} c_1 & a_{1,1} & \dots & a_{1,s} \\ c_2 & a_{2,1} & \dots & a_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s,1} & \dots & a_{s,s} \\ \hline & b_1 & \dots & b_s. \end{array}$$

Remember that, given a Cauchy problem

$$\begin{cases} U_x = G(x, U), \\ U(x_{i_0-1/2}) = U^{i_0-1/2}, \end{cases} \tag{3.1}$$

and a uniform mesh of nodes $x_{i+1/2} = x_{i-1/2} + \Delta x$, $i = i_0, i_0+1, \dots$, the numerical solutions are computed as follows:

$$U^{i+1/2} = U^{i-1/2} + \Delta x \Phi_{\Delta x}(U^{i-1/2}), \quad i = i_0, i_0+1, \dots \tag{3.2}$$

where

$$\Phi_{\Delta x}(U^{i-1/2}) = \sum_{j=1}^{s} b_j K_i^j.$$

$K_i^1, \ldots, K_i^s$ solve the nonlinear system

$$K_i^j = G\left(x_i^j, U^{i-1/2} + \Delta x \sum_{l=1}^{s} a_{j,l} K_i^l\right), \quad j = 1, \ldots, s, \tag{3.3}$$

where

$$x_i^j = x_{i-1/2} + c_j \Delta x, \quad j = 1, \ldots, s. \tag{3.4}$$

Gauss-Legendre methods will be considered here, in which $x_i^1, \ldots, x_i^s$ and $b_1, \ldots, b_s$ are respectively the quadrature points and the weights of the Gauss quadrature formula in the interval $[x_{i-1/2}, x_{i+1/2}]$. This quadrature formula will be used to compute the averages at the cells.

The key-point of collocation methods is that they can be interpreted as follows:

$$U^{i+1/2} = P_i(x_{i+1/2}),$$

where P_i is the only polynomial of degree s that satisfies:

$$\begin{cases} P_i(x_{i-1/2}) = U^{i-1/2}, \\ P_i'(x_i^j) = G(x_i^j, P_i(x_i^j)), \quad j = 1, \ldots, s. \end{cases} \tag{3.5}$$

Because of this interpretation, it can be shown that these methods are symmetric or reversible in the following sense (see [13]):

$$\Phi_{\Delta x} \circ \Phi_{-\Delta x} = Id, \quad \text{or equivalently} \quad \Phi_{\Delta x} = \Phi_{-\Delta x}^{-1}. \tag{3.6}$$

Let us describe how these methods are used to solve the local problems. Given a cell I_i, let us suppose that its stencil is

$$\mathcal{S}_i = \{i - l, \ldots, i + r\}.$$

The local problem solver based on the collocation RK methods is then as follows:

Algorithm 3.1 *Numerical solver for the local problems (LP) using collocation RK methods.*

- *Find $U^{i-1/2}$, $K_i^1, \ldots, K_i^s$ such that*

$$\begin{cases} J\left(U_i^m\right) K_i^m = S\left(U_i^m\right) H_x(x_i^m), \quad m = 1, \ldots, s, \\ \\ \displaystyle\sum_{m=1}^{s} b_m U_i^m = \widetilde{U}_i, \end{cases}$$

  where

$$U_i^m = U^{i-1/2} + \Delta x \sum_{k=1}^{s} a_{m,k} K_i^k, \quad m = 1, \ldots, s.$$

- *Compute*:

$$U^{i+1/2} = U^{i-1/2} + \Delta x \sum_{m=1}^{s} b_m K_i^m.$$

- *The approximated solution is then obtained at the rest of the stencil from the values at the intercell by applying the RK collocation method backward and forward in space.*

The output of the local solver with the notation of (LP) is then:

$$U_{i,j}^{*,m} = U_j^m, \ m = 1, \ldots, s, \ j = i - l, \ldots i + r; \quad U_i^{*,i-1/2} = U^{i-1/2}, \quad U_i^{*,i+1/2} = U^{i+1/2}.$$

It can be shown that the approximations of Cauchy problems (3.1) with $G(x, U) = J(U)^{-1} S(U) H_x$ using the RK collocation method are discrete stationary solution of the numerical schemes, what proves that they are well-balanced with order $2s$. The reversibility of RK collocation methods plays a crucial role in the proof. Notice that there are not explicit methods which have this property.

Method	Error ($i = 1$)	Error ($i = 2$)	Error ($i = 3$)
SMi	1.34E-3	2.43E-6	1.74E-8
CDWBMi	5.37E-15	5.15E-16	2.51E-14

Tab. 1 Test 1. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 200-cell mesh at time $t = 5s$.

4. Numerical experiments

The following choices have been made in order to build the well-balanced schemes introduced in this paper:

- For the first and second order well-balanced numerical schemes, the second-order 1-stage Gauss-Legendre collocation method is applied, whereas the 2-stage Gauss-Legendre collocation method is used for the third order schemes.
- The midpoint rule is considered for first and second order schemes, and the 2-point Gauss quadrature rule for third order schemes.
- The Rusanov numerical flux is considered.
- We apply the trivial reconstruction operator for the first order scheme; the MUSCL reconstruction for the second order scheme (see [23]); and the CWENO reconstruction for the third order scheme (see [17], [10]).
- First, second and third order TVD Runge-Kutta methods are used for solving the ODE system (1.4): see [12].

The following notation is introduced to denote the methods considered:

- SMi, $i = 1, 2, 3$: numerical method of order i based on the Rusanov flux and the standard, not well-balanced, reconstruction operators.
- CDWBMi, $i = 1, 2, 3$: numerical method of order i based on the Rusanov flux and the well-balanced reconstruction operator in which the discrete stationary solutions and local problems are obtained by applying the Gauss-Legendre collocation method as described in the previous section.

The numerical experiments have been performed in a computer equipped with Intel(R) Xeon(R) CPU E3-1220 v3 @ 3.10GHz with 8Mb cache using one single core.

4.1. Test 1: Burgers equation with a nonlinear source term

We consider the Burgers equation with the non-linear source term $S(U) = \sin(U)$:

$$\begin{cases} U_t + \left(\dfrac{U^2}{2}\right)_x = \sin(U), \quad x \in \mathbb{R},\ t > 0, \\ U(x, 0) = U_0(x). \end{cases} \tag{4.1}$$

We consider $x \in [-1, 1]$, $t \in [0, 5]$ and CFL= 0.9. As initial condition, we consider the stationary solution which solves the Cauchy problem

$$\begin{cases} \dfrac{dU}{dx} = \dfrac{\sin(U)}{U}, \\ U(-1) = 2. \end{cases}$$

$U(-1, t) = 2$ is imposed at $x = -1$ and free boundary conditions are considered at $x = 1$.

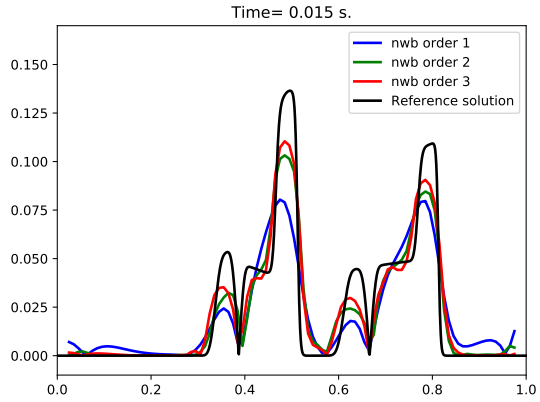
Table 1 shows the errors corresponding to SMi and CDWBMi, $i = 1, 2, 3$ respectively for a 200-cell mesh. As expected, only the well-balanced methods preserve the stationary solutions.

4.2. Test 2: shallow water equations with Manning friction

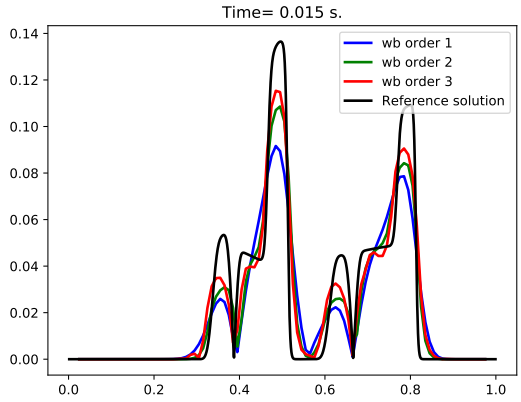
Let us consider the shallow water equations with Manning friction:

$$\begin{cases} h_t + q_x = 0, \\ q_t + \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2\right)_x = g h H_x - \dfrac{k q |q|}{h^\eta}. \end{cases} \tag{4.2}$$

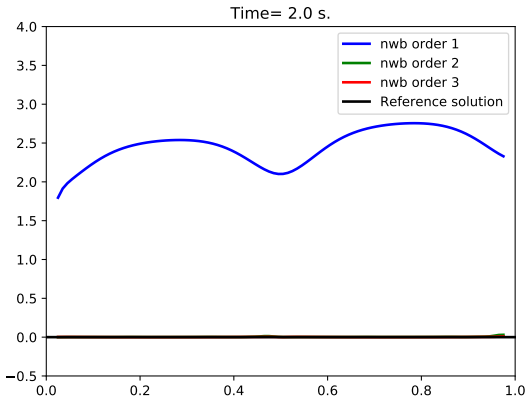
This system is used to model the flow of water in a one-dimensional channel,with a bottom that applies a friction force on the water. Here, the variable x makes reference to the axis of the channel and t is the time; $q(x, t)$ and



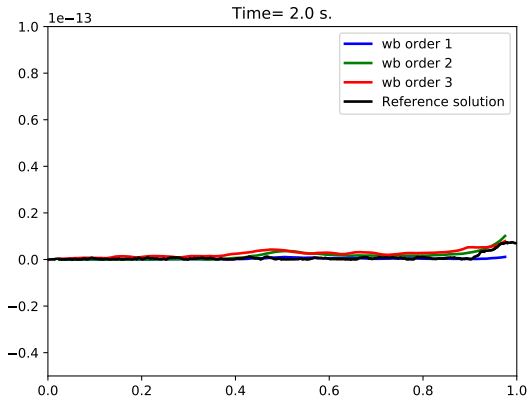
(a) SMi, $i = 1, 2, 3$. $t = 0.015s$.



(b) CDWBMi, $i = 1, 2, 3$. $t = 0.015s$.



(c) SMi, $i = 1, 2, 3..$ $t = 2s$.



(d) CDWBMi, $i = 1, 2, 3..$ $t = 2s$.

Fig. 1 Test 2. Differences between the stationary solution and the reference and numerical solutions at times $t = 0.015, 2s$ for h. Number of cells: 100.

$h(x,t)$ are the discharge and the thickness, respectively; $u = q/h$ is the depth-averaged velocity; g is the gravity; $H(x)$ is the depth function measured from a fixed reference level; k is the Manning friction coefficient; and η is a parameter equal to $\frac{7}{3}$.

Following [19], we consider $x \in [0, 1]$, $k = 0.01$ and the depth function

$$H(x) = 1 - \frac{1}{2}\frac{e^{\cos(4\pi x)} - e^{-1}}{e - e^{-1}}. \tag{4.3}$$

The initial condition $U_0(x) = [h_0(x), q_0(x)]^T$ is

$$h_0(x) = \begin{cases} h^*(x) + 0.05, & \text{if } x \in \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right], \\ h^*(x), & \text{otherwise,} \end{cases} \quad q_0(x) = \begin{cases} q^*(x) + 0.5, & \text{if } x \in \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right], \\ q^*(x), & \text{otherwise,} \end{cases} \tag{4.4}$$

where $U^*(x) = [h^*(x), q^*(x)]^T$ is the supercritical stationary solution that satisfies $q(0) = 1$ and $h(0) = 0.3$. The numerical simulation is run until $t = 2s$ using a uniform mesh with 100 cells.

Figure 1 shows the differences between the stationary solution and the numerical solutions at times $t = 0.015$ and $2s$ with SMi, $i = 1, 2, 3$ and CDWBMi, $i = 1, 2, 3$ for h (the graphs are similar for q). A reference solution has been computed with a first order well-balanced scheme on a fine mesh (1600 cells). As expected, only the well-balanced methods are able to recover the stationary solutions. This is clear in Table 2 where the errors in L^1 norm with respect to the stationary solution at time $t = 2s$ are shown for the 100-cell mesh.

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	2.42	6.12	3.57E-3	4.87E-3	1.39E-3	4.30E-4
CDWBMi	3.73E-16	3.60E-16	1.80E-15	1.99E-15	2.64E-15	8.93E-15

Tab. 2 Test 2. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 100-cell mesh at time $t = 2s$.

5. Conclusions

Following the methodology introduced in [4], we have described a general strategy in order to build a family of high-order well-balanced numerical methods that can be applied to general 1d systems of balance laws. Due to the difficulty of solving the ODE satisfied by the stationary solutions, sometimes the first step of the reconstruction procedure is required to be numerically solved: the application of the collocation RK methods to deal with these problems have been detailed in this work. The numerical methods have been applied to some systems of balance laws what allows us to check that the well-balanced property is fulfilled.

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