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Editors:
Rafael Gallego, Mariano Mateos

Universidad de Oviedo
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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Picard-type iterations for solving Fredholm integral equations

José M. Gutiérrez[1], **Miguel Á. Hernández-Verón**[1]
Universidad de La Rioja, Spain

Abstract

The theoretical solution of Fredholm integral equations involves the calculus of the inverse of an operator. However, for practical purposes, the calculus of this inverse could be not possible or very complicated. For this reason, our aim in this talk is to use iterative methods for approaching such inverse and therefore the solution of the given integral equation. In fact, we use Newton's method to obtain a method with quadratic convergence. In addition, we also use Chebyshev's method to obtain a method with cubic convergence. Next, we extend this idea to iterative methods with a given order of convergence. Finally, we propose the construction of Picard-type iterative methods that do not use derivatives or inverse operators.

1. Introduction

The goal of this work is to obtain an approximate solution of Fredholm integral equations of second kind given by

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\,dt, \quad s \in [a,b], \quad \lambda \in \mathbb{R}, \tag{1.1}$$

where $f(x) \in C[a,b]$ is a given function and the function $K(x,t)$ is a known function in $[a,b] \times [a,b]$, called kernel of the integral equation. In this equation, $y(x) \in C[a,b]$ is the unknown function to be determined.

We introduce the operator $\mathcal{K} : C[a,b] \to C[a,b]$, given by

$$[\mathcal{K}(y)](s) = \int_a^b K(x,t)y(t)\,dt, \quad s \in [a,b].$$

So, the equation (1.1) can be written in the following form

$$(\mathcal{I} - \lambda\mathcal{K})y(x) = f(x). \tag{1.2}$$

Therefore, its solution is given by obtaining $y(x)$ in (1.2):

$$y(x) = (\mathcal{I} - \lambda\mathcal{K})^{-1} f(x). \tag{1.3}$$

Formula (1.3) provides the exact solution of integral equations (1.1) in a theoretical manner. But in practice, it could be very complicated (or even impossible) the calculus of the inverse $(\mathcal{I} - \lambda\mathcal{K})^{-1}$. To avoid this difficulty, we propose the use of iterative methods for approaching this inverse and therefore the solution of the integral equation.

2. Approximating the inverse $(\mathcal{I} - \lambda\mathcal{K})^{-1}$.

Now, we consider the problem of the approximation of the inverse of the linear operator $A = \mathcal{I} - \lambda\mathcal{K}$ by means of iterative methods for solving nonlinear equations.

To do this, we introduce the following sets:

- $\mathcal{L}(C[a,b], C[a,b])$ is the set of bounded linear operators from the Banach space $C[a,b]$ on itself.
- $GL(C[a,b], C[a,b])$ is the set of invertible operators in $\mathcal{L}(C[a,b], C[a,b])$.

Given a linear operator $A \in GL(C[a,b], C[a,b])$, our target is to solve the equation

$$\mathcal{T}(H) = 0, \text{ where } \mathcal{T}(H) = H^{-1} - A \tag{2.1}$$

by means of iterative methods.

2.1. Newton's method

Our first choice is to use Newton's iteration, that in this case can be written in the following way (see [5] for details):

$$\begin{cases} N_0 \in \mathcal{L}(C[a,b], C[a,b]) \text{ given}, \\ N_{m+1} = N_m - [\mathcal{T}'(N_m)]^{-1}\mathcal{T}(N_m), \quad m \geq 0. \end{cases}$$

The application of this iterative scheme to the integral equation (3.1) allows us to construct a sequence of functions $\{y_m(x)\}_{m\geq 0}$ given by

$$\begin{cases} N_0 \in \mathcal{L}(C[a,b], C[a,b]) \text{ given}, \\ y_0(x) = N_0 f(x), \\ N_m = 2N_{m-1} - N_{m-1}AN_{m-1}, \quad m \geq 0, \\ y_m(x) = N_m f(x), \end{cases} \tag{2.2}$$

that, under adequate conditions, converge to the solution. We would like to highlight that this sequence depends on the choice of a good initial approximation H_0 for the inverse operator $A = \mathcal{I} - \lambda\mathcal{K}$ and on the function $f(x)$ that appears as "independent term" in the integral equation. In [5] we can see local and semilocal convergence results for the iterative scheme (2.2) together with some numerical examples.

2.2. Chebyshev's method

Now we consider Chebyshev's method for solving (2.1),

$$\begin{cases} C_0 \in \mathcal{L}(C[a,b], C[a,b]) \text{ given}, \\ C_{m+1} = C_m - \left[I + \frac{1}{2}L_{\mathcal{T}}(C_m)\right][\mathcal{T}'(C_m)]^{-1}\mathcal{T}(C_m), \quad m \geq 0, \end{cases}$$

where $L_{\mathcal{F}}(C_m) = [\mathcal{F}'(C_m)]^{-1}\mathcal{F}''(C_m)[\mathcal{F}'(C_m)]^{-1}\mathcal{F}(C_m)$. At a first glance, we can think that inverse operators must be used in this algorithm. But we can do the same as in Newton's method to see that Chebyshev's method does not use them (see [3] for more details). Actually, Chebyshev's iteration can be written in the form

$$\begin{cases} \mathcal{F}'(C_m)(P_m - C_m) = -\mathcal{F}(C_m), \quad k \geq 0, \\ \mathcal{F}'(C_m)(C_{m+1} - P_m) = -\frac{1}{2}\mathcal{F}''(C_m)(P_m - C_m)^2. \end{cases} \tag{2.3}$$

Then we can also avoid the use of inverse operators for approximating C_{m+1}.

In consequence, Chebyshev's method for approximating the inverse operator takes the form:

$$\begin{cases} C_0 \in \mathcal{L}(C[a,b], C[a,b]) \text{ given}, \\ C_{m+1} = 3C_m - 3C_mAC_m + C_mAC_mAC_m, \quad m \geq 0. \end{cases} \tag{2.4}$$

The application of the iterative scheme (2.4) to the integral equation (3.1) allows us to construct another sequence of functions $\{y_m(x)\}_{m\geq 0}$ given by

$$\begin{cases} C_0 \in \mathcal{L}(C[a,b], C[a,b]) \text{ given}, \\ y_0(x) = C_0 f(x), \\ C_{m+1} = 3C_m - 3C_mAC_m + C_mAC_mAC_m, \quad m \geq 0, \\ y_m(x) = C_m f(x). \end{cases} \tag{2.5}$$

As in the case of Newton's method (2.2), this sequence depends on the choice of a good initial approximation C_0 for the inverse operator $A = \mathcal{I} - \lambda\mathcal{K}$ and on the function $f(x)$. The increase in computational cost is rewarded with the increase in the order of convergence, passing from quadratic to cubic order of convergence.

2.3. Methods with a prefixed order of convergence

The next step is to generalize the iterative schemes (2.2) and (2.6), obtained from Newton's and Chebyshev's methods respectively. Our idea is to construct iterative schemes, with a prefixed order of convergence, that do not use inverse operators for approximating the inverse of an operator.

For this, we observe that both Newton's and Chebyshev's methods satisfy equalities in the form

$$\mathcal{I} - N_kA = \mathcal{I} - (2N_{k-1} - N_{k-1}AN_{k-1})L = (\mathcal{I} - N_{k-1}A)^2,$$

$$\mathcal{I} - C_k A = \mathcal{I} - (3C_{k-1} - 3C_{k-1}AC_{k-1} + C_{k-1}AC_{k-1}LC_{k-1})A = (\mathcal{I} - C_{k-1}A)^3.$$

Therefore, following the procedure developed in [1] or [6] to generate an iterative scheme with order of convergence $p \geq 2$, we can consider a sequence $T_k = \phi(T_{k-1})$ such that $\mathcal{I} - T_k A = (\mathcal{I} - T_{k-1}A)^p$, that is,

$$T_k A \quad = \quad \mathcal{I} - (\mathcal{I} - T_{k-1}A)^p = \sum_{j=0}^{p-1} \binom{p}{j+1} (-1)^j (T_{k-1}A)^j T_{k-1}A.$$

The application of the previous iterative scheme to the integral equation (3.1) allows us to construct a sequence of functions $\{y_m(x)\}_{m\geq 0}$ that generalizes the previous ones. Actually, for $p \geq 2$, it is given by

$$\begin{cases} T_0 \in \mathcal{L}(C[a,b], C[a,b]) \text{ given}, \\ y_0(x) = T_0 f(x), \\ T_m = \displaystyle\sum_{j=0}^{p-1} \binom{p}{j+1} (-1)^j (T_{m-1}A)^j T_{m-1}, \quad m \geq 1, \\ y_m(x) = T_m f(x). \end{cases} \tag{2.6}$$

Both local and semilocal convergence of these iterative schemes have been studied in [6], together with an analysis of the starting points for the application of these iterative schemes considered.

3. Picard-type iterations

Another procedure to approximate a solution of the integral equation (1.1) is to write it as a functional equation defined between two Banach spaces and to consider different iterative schemes for solving it. So, we introduce the function $F : \Omega \subseteq C[a,b] \longrightarrow C[a,b]$, where Ω is a nonempty convex domain in $C[a,b]$, with

$$F(y)(x) = y(x) - \lambda \int_0^1 K(x,t)y(t)\,dt - f(x). \tag{3.1}$$

We are interested in solving the equation $F(y) = 0$ by means of iterative schemes. Observe that a solution of this equation is a solution of equation (1.1) and vice versa. So, starting from an initial approximation of y^*, a solution of the equation $F(y) = 0$, a sequence $\{y_n\}$ of approximations is constructed such that $\lim_n y_n = y^*$. Note that the sequence $\{\|y_n - y_{n-1}\|\}$ is strictly decreasing and, at every step, a better approximation to the solution y^* is obtained.

We can obtain the sequence of approximations $\{y_n\}$ by different ways, depending on the considered iterative scheme. To approximate such a solution we can apply the well-known method of successive approximations [7], which is also known as Picard's method [4, 8] when it comes to approximate a solution of the equation $F(y) = 0$ and defined by $y_{n+1} = y_n - F(y_n)$, $n \geq 0$, with y_0 given in $\Omega \subseteq C[a,b]$. This iterative scheme has the drawback of its linear convergence, however it is an iterative scheme that does not use derivatives or inverses of operators.

A commonly used iterative scheme is the best-known is Newton's method, whose algorithm is the following:

$$\begin{cases} y_0 \text{ given in } \Omega, \\ y_{n+1} = y_n - [F'(y_n)]^{-1} F(y_n), \quad n = 0, 1, 2 \ldots \end{cases} \tag{3.2}$$

In practice it is not easy to construct the iterative scheme (3.2) for operators defined on infinite dimension spaces. The main difficulties arise, at each step, in the calculus of the inverse of the linear operator $F'(y_n)$ or, equivalently, in solving the associated linear equation. Next, we study the value of the inverse operator of $F'(y)$. So, by applying certain algebraic manipulations [2], we can obtain that:

$$[F'(y)]^{-1}\phi(\xi) = \phi(\xi) + \lambda[\mathcal{I} - \lambda\mathcal{K}]^{-1}\mathcal{K}\phi(\xi), \text{ for } \phi \in C[a,b] \text{ and } \xi \in C[a,b].$$

Now, as a consequence of the last equation, we can rewrite an iteration of the iterative scheme of Newton (2.1) as follows

$$\begin{aligned} y_{n+1}(x) &= y_n(x) - F(y_n)(x) - \lambda[\mathcal{I} - \lambda\mathcal{K}]^{-1}\mathcal{K}F(y_n)(x) \\ &= \lambda\mathcal{K}y_n(x) + f(x) - \lambda[\mathcal{I} - \lambda\mathcal{K}]^{-1}\mathcal{K}F(y_n)(x). \end{aligned}$$

Notice that, if $|\lambda|\|\mathcal{K}\| < 1$, by Banach Lemma, it follows that there exists the operator $(\mathcal{I} - \lambda\mathcal{K})^{-1}$.

Now, as in the previous section 2.1, we approximate $(\mathcal{I} - \lambda\mathcal{K})^{-1}$ by means the Newton sequence, then we define the following Ulm-type algorithm

$$\begin{cases} z_0 \text{ and } N_0 \text{ given in } \Omega \text{ and } \mathcal{L}(C[a,b], C[a,b]) \text{ respectively,} \\ z_{k+1} = \lambda \mathcal{K} z_k + f - \lambda N_k \mathcal{K} F(z_k), \\ N_{k+1} = 2N_k - N_k A N_k \quad n = 0, 1, 2 \dots \end{cases} \tag{3.3}$$

where, as Section 1, we have donoted: $\mathcal{I} - \lambda \mathcal{K} = A$.

Therefore, we present a Ulm-type iterative scheme that shares many properties of Picard iterative scheme, namely it is derivative-free and does not use inverse operators, although preserving the quadratic order of convergence that characterizes Newton's method. These features allow us to design an efficient iterative method. Actually, with a very reduced number of iterations, we can find competitive approximations to the solution of the involved Fredholm integral equation (1.1). This is one of the main targets of our research: to justify that it is enough to consider a few steps in our iterative procedure to reach a good approach to the solution.

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