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Universidad de Oviedo

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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics SĒMA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the SĒMA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier “Pancho” Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: “a mathematician is a device for turning coffee into theorems”. Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Turing instability analysis of a singular cross-diffusion problem

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Abstract

The population model of Busenberg and Travis is a paradigmatic model in ecology and tumour modelling due to its ability to capture interesting phenomena like the segregation of populations. Its singular mathematical structure enforces the consideration of regularized problems to deduce properties as fundamental as the existence of solutions. In this article we perform a weakly nonlinear stability analysis of a general class of regularized problems to study the convergence of the instability modes in the limit of the regularization parameter. We demonstrate with some specific examples that the pattern formation observed in the regularized problems, with unbounded wave numbers, is not present in the limit problem due to the amplitude decay of the oscillations. We also check the results of the stability analysis with direct finite element simulations of the problem. In this short communication, we omit the proofs of our results.

In [4], Busenberg and Travis introduced a class of singular cross-diffusion problems under the assumption that the spatial relocation of each species is due to a diffusion flow which depends on the densities of all the involved species. In the case of two species, if u_1, u_2 denote their densities, the flow, in its simplest form, may be assumed to be determined by the total population $u_1 + u_2$, and thus the conservation laws for both species lead to the system

$$\partial_t u_1 - \operatorname{div}(u_1(\nabla u_1 + \nabla u_2)) = f_1(u_1, u_2), \quad (0.1)$$

$$\partial_t u_2 - \operatorname{div}(u_2(\nabla u_1 + \nabla u_2)) = f_2(u_1, u_2). \quad (0.2)$$

The functions f_1 and f_2 capture some ecological features of the populations, such as growth, competition, etc. As usual, the equations (0.1)-(0.2) are complemented with non-negative initial data and non-flow boundary conditions.

The system (0.1)-(0.2) is called a *cross-diffusion* system because the flow of each species depend upon the densities of the other species. We call it *singular* because the resulting diffusion matrix is singular. Indeed, when rewriting (0.1)-(0.2) in matrix form, for $\mathbf{u} = (u_1, u_2)$, we get the equation

$$\partial_t \mathbf{u} - \operatorname{div}(\mathcal{A}(\mathbf{u})\nabla \mathbf{u}) = \mathbf{f}(\mathbf{u}), \quad \text{with } \mathcal{A}(\mathbf{u}) = \begin{pmatrix} u_1 & u_1 \\ u_2 & u_2 \end{pmatrix},$$

where the divergence is applied by rows. The full and singular structure of \mathcal{A} introduces serious difficulties in the mathematical analysis of the problem, as we shall comment later.

In his seminal paper [14], Turing introduced a mechanism explaining how spatially uniform equilibria may evolve, small perturbations mediating, into stable equilibria with non-trivial spatial structure. He considered a system of the type

$$\partial_t u_1 - \Delta u_1 = f_1(u_1, u_2), \quad (0.3)$$

$$\partial_t u_2 - \sigma \Delta u_2 = f_2(u_1, u_2), \quad (0.4)$$

with $\sigma > 0$, and proved that when σ is small or large enough then the stable equilibria of the dynamical system

$$\partial_t v_1 = f_1(v_1, v_2), \quad (0.5)$$

$$\partial_t v_2 = f_2(v_1, v_2), \quad (0.6)$$

are not stable for the diffusion system (0.3)-(0.4) and that, in their place, non-uniform equilibria with spatial structure become the new stable solutions. This mechanism is known as *Turing instability* or *Turing bifurcation*.

In this article we study Turing instability for the cross-diffusion singular system (0.1)-(0.2). We already know that some cross-diffusion systems, such as the paradigmatic SKT model introduced by Shigesada, Kawasaki and Teramoto [13], exhibit Turing instability when cross-diffusion coefficients are large in comparison with self-diffusion coefficients, see e.g. [10, 11]. However, the singularity of the diffusion matrix of the system (0.1)-(0.2), not present in the SKT model, introduces important mathematical difficulties to the analysis of this system.

Regarding the existence of solutions of (0.1)-(0.2), it has been proved only in some special situations: for a bounded spatial domain $\Omega \subset \mathbb{R}$ (Bertsch et al. [2]) and for $\Omega = \mathbb{R}^n$ (Bertsch et al. [3]). In their proofs, the following

observation is crucial: adding the two equations of (0.1)-(0.2) shows that if a solution of this system does exist then the total population, $u = u_1 + u_2$, satisfies the porous medium type equation

$$\partial_t u - \operatorname{div}(u \nabla u) = f(u), \tag{0.7}$$

for which the theory of existence and uniqueness of solutions is well established. In particular, if the initial data of the total population is bounded away from zero and if f is regular enough with $f(0) \geq 0$, it is known that the solution of (0.7) remains positive and smooth for all time. This allows to introduce the change of unknowns $w_i = u_i/u$, for $i = 1, 2$, into the original problem (0.1)-(0.2) to deduce the equivalent formulation

$$\partial_t u - \operatorname{div}(u \nabla u) = F_1(u, w_1), \tag{0.8}$$

$$\partial_t w_1 - \nabla u \cdot \nabla w_1 = F_2(u, w_1), \tag{0.9}$$

for certain well-behaved functions F_1 and F_2 . Being the structure of the system (0.8)-(0.9) of parabolic-hyperbolic nature, parabolic regularization of the system by adding the term $-\delta \Delta w_1$ to the left hand side of (0.9), and the consideration of the characteristics defined by the field ∇u are the main ingredients of the proofs made by Bertsch et al. [2, 3].

In [9] we followed a different approach to prove the existence of solutions of the original system (0.1)-(0.2) for a bounded domain $\Omega \subset \mathbb{R}$. We directly performed a parabolic regularization of the system by introducing a cross-diffusion perturbation term while keeping the porous medium type equation satisfied by u . More concretely, we considered the system

$$\partial_t u_1 - \operatorname{div}(u_1(\nabla u_1 + \nabla u_2)) - \frac{\delta}{2} \Delta(u_1(u_1 + u_2)) = f_1(u_1, u_2), \tag{0.10}$$

$$\partial_t u_2 - \operatorname{div}(u_2(\nabla u_1 + \nabla u_2)) - \frac{\delta}{2} \Delta(u_2(u_1 + u_2)) = f_2(u_1, u_2), \tag{0.11}$$

and then used previous results for cross-diffusion systems [6, 7, 12] to establish the existence of solutions of the approximated problems. Then, BV estimates similar to those obtained in [2] allowed to prove the convergence of the sequence $(u_1^{(\delta)}, u_2^{(\delta)})$ to a solution of the original problem. Let us finally mention that the system (0.1)-(0.2) is a limit case of a general type of problems with diffusion matrix given by

$$\mathcal{A}(\mathbf{u}) = \begin{pmatrix} a_{11}u_1 & a_{12}u_1 \\ a_{21}u_2 & a_{22}u_2 \end{pmatrix},$$

for which, if $a_{ii} > 0$, for $i = 1, 2$, and $a_{11}a_{22} > a_{12}a_{21}$ then the existence of solutions is ensured for any spatial dimension of Ω , see [8]. In addition, it has been shown that this kind of systems, when set in the whole space $\Omega = \mathbb{R}^n$, may be obtained as mean field limits [5].

Concerning Turing instability, since the diffusion matrix, $\mathcal{A}(\mathbf{u})$, corresponding to the system (0.1)-(0.2) is singular, the linearization of this system about an equilibrium of the dynamical system (0.5)-(0.6) does not provide any information on the behaviour of the equilibrium in the spatial dependent case. Thus, our approach to the investigation of Turing instability for the system (0.1)-(0.2) relies on the study of this property for approximating problems like (0.10)-(0.11) and its limit behaviour.

We prove that linear instability is always present in the limit $\delta \rightarrow 0$, which is the case when the sequence of solutions of the approximated problems (0.10)-(0.11) converges to the solution of the original problem (0.1)-(0.2). Interestingly, the linear analysis also establishes that the main instability wave number is unbounded as $\delta \rightarrow 0$.

For a clearer understanding of this convergence of a increasingly oscillating sequence of functions to a BV function (the solution of (0.1)-(0.2) ensured in [2, 9]), we perform a weakly nonlinear analysis (WNA) which allows to gain insight into the behaviour of the amplitude of the main instability mode as $\delta \rightarrow 0$. As expected, we find that the amplitude of the instability modes vanishes in the limit $\delta \rightarrow 0$ resulting, therefore, coherent with the BV convergence. In addition, this result also suggests that the uniform equilibrium is stable for the original problem. We furthermore check these analytical results by numerically comparing the WNA approximation to a FEM approximation of the nonlinear problem.

1. Main results

For simplicity, we study Turing instability for the one-dimensional spatial setting which has also the advantage of a well established existence theory for the case of a bounded domain [2, 9]. By redefining the functions f_1, f_2 , we

can fix without loss of generality $\Omega = (0, \pi)$ and then rewrite problem (0.1)-(0.2) together with the usual auxiliary conditions as

$$\partial_t u_1 - \partial_x (u_1 (\partial_x u_1 + \partial_x u_2)) = f_1(u_1, u_2) \quad \text{in } Q_T, \quad (1.1)$$

$$\partial_t u_2 - \partial_x (u_2 (\partial_x u_1 + \partial_x u_2)) = f_2(u_1, u_2) \quad \text{in } Q_T, \quad (1.2)$$

$$u_1 (\partial_x u_1 + \partial_x u_2) = u_2 (\partial_x u_1 + \partial_x u_2) = 0 \quad \text{on } \Gamma_T, \quad (1.3)$$

$$u_1(0, \cdot) = u_{10}, \quad u_2(0, \cdot) = u_{20} \quad \text{in } \Omega, \quad (1.4)$$

where $Q_T = (0, T) \times \Omega$ and the initial data u_{10}, u_{20} are non-negative functions. We assume a competitive Lotka-Volterra form for the reaction term, this is, $f_i(u_1, u_2) = u_i(\alpha_i - \beta_{i1}u_1 - \beta_{i2}u_2)$, for $i = 1, 2$, and for some non-negative parameters α_i, β_{ij} , for $i, j = 1, 2$.

In order to deal with several types of regularized problems we introduce, for positive δ and b , the uniformly parabolic cross-diffusion system

$$\partial_t u_1 - \partial_x (d_{11}^\delta(\mathbf{u}) \partial_x u_1 + d_{12}^\delta(\mathbf{u}) \partial_x u_2) = f_1^b(\mathbf{u}) \quad \text{in } Q_T, \quad (1.5)$$

$$\partial_t u_2 - \partial_x (d_{21}^\delta(\mathbf{u}) \partial_x u_1 + d_{22}^\delta(\mathbf{u}) \partial_x u_2) = f_2^b(\mathbf{u}) \quad \text{in } Q_T, \quad (1.6)$$

$$d_{11}^\delta(\mathbf{u}) \partial_x u_1 + d_{12}^\delta(\mathbf{u}) \partial_x u_2 = d_{21}^\delta(\mathbf{u}) \partial_x u_1 + d_{22}^\delta(\mathbf{u}) \partial_x u_2 = 0 \quad \text{on } \Gamma_T, \quad (1.7)$$

$$u_1(0, \cdot) = u_{10}, \quad u_2(0, \cdot) = u_{20} \quad \text{in } \Omega, \quad (1.8)$$

where the diffusion matrix $D^\delta(\mathbf{u}) = (d_{ij}^\delta(\mathbf{u}))$ and the Lotka-Volterra function $\mathbf{f}^b(\mathbf{u}) = (f_1^b(\mathbf{u}), f_2^b(\mathbf{u}))$ satisfy the assumptions H_D :

1. $D^\delta(\mathbf{u})$ is linear in \mathbf{u} and affine in δ , so that it allows the decompositions

$$D^\delta(\mathbf{u}) = D^0(\mathbf{u}) + \delta D^1(\mathbf{u}) = D^{\delta 1} u_1 + D^{\delta 2} u_2,$$

for some matrices $D^{\delta i}$ for $i = 1, 2$, being the coefficients of $D^\delta(\mathbf{u})$ given by

$$d_{ij}^\delta(\mathbf{u}) = d_{ij}^{10} u_1 + d_{ij}^{11} u_1 \delta + d_{ij}^{20} u_2 + d_{ij}^{21} u_2 \delta,$$

for some non-negative constants d_{ij}^{mn} , for $i, j, m = 1, 2$ and $n = 0, 1$.

2. We assume that $d_{ii}^\delta(\mathbf{u}) > 0$ for $i = 1, 2$, and that $\det(D^\delta(\mathbf{u}))$ is an increasing function with respect to δ satisfying $\det(D^\delta(\mathbf{u})) > 0$ if $\delta > 0$ and $\mathbf{u} \in \mathbb{R}_+^2$.
3. For $i, j = 1, 2$, $f_i^b(u_1, u_2) = u_i(\alpha_i^b - \beta_{i1}^b u_1 - \beta_{i2}^b u_2)$ for some non-negative α_i^b, β_{ij}^b , such that $\alpha_i^b \rightarrow \alpha_i$ and $\beta_{ij}^b \rightarrow \beta_{ij}$ as $b \rightarrow 0$. Moreover, using the notation $\alpha_i^0 = \alpha_i$ and $\beta_{ij}^0 = \beta_{ij}$, we assume, for $b \geq 0$,

$$\begin{aligned} \beta_{22}^b \alpha_1^b - \beta_{12}^b \alpha_2^b &> 0, \quad \beta_{11}^b \alpha_2^b - \beta_{21}^b \alpha_1^b > 0, \\ \det(B^b) &> 0, \quad \text{tr}(B^b) \geq 0, \quad \text{where } B^b = \begin{pmatrix} \beta_{ij}^b \end{pmatrix}. \end{aligned} \quad (1.9)$$

Observe that (1.9) ensures the existence of a stable coexistence equilibrium for the dynamical system (0.5)-(0.6), given by

$$\mathbf{u}^* = (u_1^*, u_2^*) = \left(\frac{\beta_{22}^b \alpha_1^b - \beta_{12}^b \alpha_2^b}{\beta_{11}^b \beta_{22}^b - \beta_{12}^b \beta_{21}^b}, \frac{\beta_{11}^b \alpha_2^b - \beta_{21}^b \alpha_1^b}{\beta_{11}^b \beta_{22}^b - \beta_{12}^b \beta_{21}^b} \right). \quad (1.10)$$

There are two examples of $D^\delta(\mathbf{u})$ in which we are specially interested. The first, due to its simplicity for the calculations. We set

$$D^\delta(\mathbf{u}) = \begin{pmatrix} (1 + \delta)u_1 & u_1 \\ u_2 & (1 + \delta)u_2 \end{pmatrix}, \quad (1.11)$$

for which $\det(D^\delta(\mathbf{u})) = \delta(2 + \delta)u_1 u_2$. According to [8], the second hypothesis of H_D guarantees the well-posedness of the problem (1.5)-(1.8) corresponding to this diffusion matrix. The second example corresponds to the approximation used in [9] for proving the existence of BV solutions of the original problem (1.1)-(1.4):

$$D^\delta(\mathbf{u}) = \begin{pmatrix} (1 + \delta)u_1 + \frac{\delta}{2}u_2 & (1 + \frac{\delta}{2})u_1 \\ (1 + \frac{\delta}{2})u_2 & \frac{\delta}{2}u_1 + (1 + \delta)u_2 \end{pmatrix}, \quad (1.12)$$

for which $\det(D^\delta(\mathbf{u})) = \frac{1}{2}\delta(1+\delta)(u_1+u_2)^2$.

The approximation of the reaction terms introduced in the system (1.5)-(1.8) is not essential. Its aim is to support the specific example we deal with in Theorem 1.3, but can be ignored ($b = 0$) in the general linear and weakly nonlinear analysis of Theorems 1.1 and 1.2. Nevertheless, we state these results taking it into account. Our first result gives conditions under which linear instability arises. The following notation is used:

$$K = D\mathbf{f}^b(\mathbf{u}^*) = \begin{pmatrix} -\beta_{11}^b u_1^* & -\beta_{12}^b u_1^* \\ -\beta_{21}^b u_2^* & -\beta_{22}^b u_2^* \end{pmatrix}.$$

Theorem 1.1 (Linear instability) *Assume H_D , with $b \geq 0$. Let \mathbf{u}^* be the coexistence equilibrium defined by (1.10). If*

$$\text{tr}(K^{-1}D^\delta(\mathbf{u}^*)) > 0 \quad \text{for all } \delta \geq 0 \quad (1.13)$$

then there exists $\delta_c > 0$ such that if $\delta < \delta_c$ then \mathbf{u}^ is a linearly unstable equilibrium for problem (1.5)-(1.8). In such situation, the wave number of the main instability mode tends to infinity as $\delta \rightarrow 0$.*

Condition (1.13) is equivalent to

$$d_{11}^\delta(\mathbf{u}^*)\beta_{22}^b u_2^* + d_{22}^\delta(\mathbf{u}^*)\beta_{11}^b u_1^* < d_{12}^\delta(\mathbf{u}^*)\beta_{21}^b u_2^* + d_{21}^\delta(\mathbf{u}^*)\beta_{12}^b u_1^* \quad (1.14)$$

and introduces a further restriction on the matrix of competence coefficients. Roughly speaking, for B^b to fulfil both (1.9) and (1.14), its elements must be such that intra-population joint competence is larger than inter-population joint competence (condition (1.9)) and one of the inter-population competence coefficients is large in comparison with the others (condition (1.14)). A numeric example we shall work with along the article is

$$B^b = \begin{pmatrix} 1 & \frac{b}{2} \\ 2 & 1 \end{pmatrix}, \quad \text{with } b \in (0, \frac{1}{2}). \quad (1.15)$$

Assuming the forms of $D^\delta(\mathbf{u}^*)$ given in Examples 1 and 2, see (1.11) and (1.12), we have that the conditions (1.9) and (1.14) on B^b are satisfied if $\delta < b/4$ (Example 1) or $\delta < bu_1^*u_2^*/(u_1^*+u_2^*)^2$ (Example 2). Therefore, the most meaningful case when δ is close to zero is satisfied by both diffusion matrices.

Our second result allows to estimate not only the instability wave numbers provided by the linear analysis but also the amplitude corresponding to these modes. The approximation of the steady state solution is obtained using a weakly nonlinear analysis (WNA) based on the method of multiple scales.

Theorem 1.2 *Assume the hypothesis of Theorem 1.1 and let $\varepsilon^2 = (\delta_c - \delta)/\delta_c$ be a small number. Then, there exist sets of data problem such that the stationary WNA approximation to the solution \mathbf{u} of problem (1.5)-(1.8) is given by*

$$\mathbf{v}(x) = \mathbf{u}^* + \varepsilon \boldsymbol{\rho} \sqrt{A_\infty} \cos(k_c x) + \varepsilon^2 A_\infty (\mathbf{v}_{20} + \mathbf{v}_{22} \cos(2k_c x)) + O(\varepsilon^3), \quad (1.16)$$

where $k_c \in \mathbb{Z}$ is the critical wave number corresponding to δ_c , A_∞ is a positive constant and $\boldsymbol{\rho}$, \mathbf{v}_{20} and \mathbf{v}_{22} are constant vectors.

Our third result focuses on the limit behaviour of the critical parameters and the amplitude when $\delta \rightarrow 0$, this is, when the solutions of the approximated problems converge to the solution of the original singular problem. For the sake of simplicity, we limit our study to the following example:

$$\partial_t u_1 - \partial_x(u_1(\partial_x u_1 + \partial_x u_2)) = u_1(1 - u_1), \quad (1.17)$$

$$\partial_t u_2 - \partial_x(u_2(\partial_x u_1 + \partial_x u_2)) = u_2(4 - (2u_1 + u_2)), \quad (1.18)$$

whose solutions we approximate by the two-parameter family of solutions of

$$\partial_t u_1 - \partial_x(u_1((1+\delta)\partial_x u_1 + \partial_x u_2)) = u_1(1 - (u_1 + \frac{b}{2}u_2)), \quad (1.19)$$

$$\partial_t u_2 - \partial_x(u_2(\partial_x u_1 + (1+\delta)\partial_x u_2)) = u_2(4 - (2u_1 + u_2)). \quad (1.20)$$

On one hand, Theorem 1.1 ensures the existence of $\delta_c > 0$ such that, for any $b \geq 0$, the equilibrium $\mathbf{u}^* = \frac{1}{1-b}(1-2b, 2)$ of (1.19)-(1.20) becomes unstable for $\delta < \delta_c$, with an associated critical wave number such that $k_c \rightarrow \infty$ as $\delta \rightarrow 0$.

On the other hand, for $\delta < b/4$ and $b \rightarrow 0$, the sequence of solutions of (1.19)-(1.20) converges to a solution of (1.17)-(1.18) in the space $BV(0, T, L^\infty(\Omega)) \cup L^\infty(0, T; BV(\Omega))$. Therefore, for the approximation (1.16) provided by the weakly nonlinear analysis to remain valid for all $\delta > 0$, the corresponding amplitude A_∞ must vanish in the limit $\delta \rightarrow 0$, making in this way compatible the increase of oscillations with its BV regularity.

Theorem 1.3 *Set $\alpha = (1, 4)$, and let $D^\delta(\mathbf{u})$ and B^b be given by (1.11) and (1.15), respectively, for $b < 1/2$ and $0 < \delta < b/4$. Then, there exists $\delta_c(b) > 0$ such that if $\delta < \delta_c(b)$ then $\mathbf{u}^* = \frac{1}{1-b}(1-2b, 2)$ is linearly unstable for problem (1.5)-(1.8). In addition,*

$$\lim_{b \rightarrow 0} \delta_c(b) = 0, \quad \lim_{b \rightarrow 0} k_c(b) = \infty,$$

and the amplitude provided by the weakly nonlinear analysis satisfies

$$\lim_{b \rightarrow 0} A_\infty(b) = 0.$$

In particular, the weakly nonlinear approximation \mathbf{v} given by (1.16) satisfies $\mathbf{v} \rightarrow \mathbf{u}^*$ uniformly in Ω as $b \rightarrow 0$.

2. Numerical experiments

In order to analyze the quality of the approximation provided by the WNA, as well as the properties stated in Theorems 1.1 to 1.3, we compare it to a numerical approximation of the evolution problem computed through the finite element method (FEM).

For the FEM approximation, we used the open source software deal.II [1] to implement a time semi-implicit scheme with a spatial linear-wise finite element discretization. For the time discretization, we take in the experiments a uniform time partition of time step $\tau = 0.01$. For the spatial discretization, we take a uniform partition of the interval $\Omega = (0, \pi)$ with spatial step depending on the predicted wave number of the pattern, see Table 1.

Let, initially, $t = t_0 = 0$ and set $(u_1^0, u_2^0) = (u_{10}, u_{20})$. For $n \geq 1$, the discrete problem is: Find $u_1^n, u_2^n \in S^h$ such that

$$\frac{1}{\tau}(u_1^n - u_1^{n-1}, \chi)^h + (d_{11}^\delta(\mathbf{u}^n) \partial_x u_1^n + d_{12}^\delta(\mathbf{u}^n) \partial_x u_2^n, \partial_x \chi)^h = (f_1^b(u_1^n, u_2^n), \chi)^h, \quad (2.1)$$

$$\frac{1}{\tau}(u_2^n - u_2^{n-1}, \chi)^h + (d_{21}^\delta(\mathbf{u}^n) \partial_x u_1^n + d_{22}^\delta(\mathbf{u}^n) \partial_x u_2^n, \partial_x \chi)^h = (f_2^b(u_1^n, u_2^n), \chi)^h, \quad (2.2)$$

for every $\chi \in S^h$, the finite element space of piecewise \mathbb{Q}_1 -elements. Here, $(\cdot, \cdot)^h$ stands for a discrete semi-inner product on $C(\bar{\Omega})$.

Since (2.1)-(2.2) is a nonlinear algebraic problem, we use a fixed point argument to approximate its solution, (u_1^n, u_2^n) , at each time slice $t = t_n$, from the previous approximation (u_1^{n-1}, u_2^{n-1}) . Let $u_1^{n,0} = u_1^{n-1}$ and $u_2^{n,0} = u_2^{n-1}$. Then, for $k \geq 1$ the linear problem to solve is: Find $(u_1^{n,k}, u_2^{n,k})$ such that for all $\chi \in S^h$

$$\begin{aligned} & \frac{1}{\tau}(u_1^{n,k} - u_1^{n-1}, \chi)^h + (d_{11}^\delta(\mathbf{u}^{n,k-1}) \partial_x u_1^{n,k} + d_{12}^\delta(\mathbf{u}^{n,k-1}) \partial_x u_2^{n,k}, \partial_x \chi)^h \\ & \quad = (u_1^{n,k} (\alpha_1^b - \beta_{11}^b u_1^{n,k-1} - \beta_{12}^b u_2^{n,k-1}), \chi)^h, \\ & \frac{1}{\tau}(u_2^{n,k} - u_2^{n-1}, \chi)^h + (d_{21}^\delta(\mathbf{u}^{n,k-1}) \partial_x u_1^{n,k} + d_{22}^\delta(\mathbf{u}^{n,k-1}) \partial_x u_2^{n,k}, \partial_x \chi)^h \\ & \quad = (u_2^{n,k} (\alpha_2^b - \beta_{21}^b u_1^{n,k-1} - \beta_{22}^b u_2^{n,k-1}), \chi)^h. \end{aligned}$$

We use the stopping criteria

$$\max(\|u_1^{n,k} - u_1^{n,k-1}\|_2, \|u_2^{n,k} - u_2^{n,k-1}\|_2) < \text{tol}_{FP},$$

for values of tol_{FP} chosen empirically, and set $(u_1^n, u_2^n) = (u_1^{n,k}, u_2^{n,k})$. Finally, we integrate in time until a numerical stationary solution, (u_1^S, u_2^S) , is achieved. This is determined by

$$\max(\|u_1^{n,1} - u_1^{n-1}\|_2, \|u_2^{n,1} - u_2^{n-1}\|_2) < \text{tol}_S,$$

where tol_S is chosen empirically too. In the following experiments we always fix $\text{tol}_{FP} = 1.e-07$ and $\text{tol}_S = 1.e-12$.

	Simulation 1	Simulation 2	Simulation 3
b	3.85e-02	9.91e-03	4.42e-03
$\delta(b)$	4.53e-05	2.94e-06	5.83e-07
$k_c(b)$	10	20	30
$A_\infty(b)$	1.21e-02	3.1e-03	1.4e-03
Number of nodes	128	256	512
Time steps to stationary	3.e+04	1.9e+05	4.4e+05
Execution time (hours)	1.67	19.26	84.77

Tab. 1 Data set for the Experiment 1. Wave numbers and times are rounded. Execution time measured for a standard laptop with i7 processor.

2.1. Experiment 1

We investigate the behaviour of the instabilities arising in the solutions of the approximated problems (1.5)-(1.8) when $\delta \rightarrow 0$. Our main aim is to check if the predictions of the weakly nonlinear analysis stated in Theorem 1.3 are captured by the FEM approximation too. Thus, we use the diffusion matrix $D^\delta(\mathbf{u})$ and the competence parameters B^b given by (1.11) and (1.15), respectively.

We run three simulations according to the choice of b , see Table 1, and fix $\delta = 0.95\delta_c(b)$ in all of them, so that \mathbf{u}^* is unstable and pattern formation follows.

In Fig. 1 we show the typical onset and transmission of disturbances found in all the experiments. In this figure and in the following we plot only the first component of the solution, being the behaviour of the second component similar. After a fast decay of the initial data towards the unstable equilibrium, a perturbation with the wave number predicted by the linear analysis grows from one side of the boundary to the rest of the domain until reaching the steady state, see Fig. 2. In the latter figure, we may check the good accordance between the FEM and the WNA approximations which, in numeric figures, have a relative difference of the order 10^{-5} .

In Fig. 3 we show three interesting behaviours of solutions when $\delta \rightarrow 0$. In the left panel, the shrinking amplitude of the stationary patterns while the wave number increases. The equilibrium has been subtracted from the solution to center the pattern in $y = 0$. The center panel shows the time evolution of the amplitude (log scale) as given by the exact solution of the Stuart-Landau equation. We readily see that the stabilization time is a decreasing function of δ . This fact together with the increment of the wave number when $\delta \rightarrow 0$ results in very high execution times, see Table 1. Finally, the third panel shows how the variation of the numerical stationary solution

$$\int_{\Omega} |\partial_x u_1(T, x)| dx$$

is an increasing function of δ and tends to zero as $\delta \rightarrow 0$, in agreement with the regularity of solutions stated by the theoretical results.

2.2. Experiment 2

We repeated Experiment 1 replacing the diffusion matrix $D^\delta(\mathbf{u})$ by that defined in (1.12) In Table 2 we show the relative differences in L^p , given by

$$RD_p(\varphi_1, \varphi_2) = \frac{\|\varphi_1 - \varphi_2\|_{L^p}}{\|\varphi_1\|_{L^p}}, \tag{2.3}$$

of the critical bifurcation parameter, δ_c , the *stationary* solution of the FEM approximation, $\mathbf{u}(T, \cdot)$, the WNA approximation, \mathbf{v} , and the pattern amplitude, A_∞ , corresponding to both approximations of the original diffusion matrix. We see that although the critical bifurcation parameter is clearly affected by the approximation scheme, the FEM and WNA approximations provided by both schemes are in a very good agreement, as well as the amplitudes of the instability patterns, suggesting that in the limit $\delta \rightarrow 0$ both sequences of approximations converge to the same limit.

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	Simulation 1	Simulation 2	Simulation 3
$RD_\infty(\delta_c^{(E1)}, \delta_c^{(E2)})$	0.136	0.117	0.113
$RD_2(\mathbf{u}^{(E1)}(T, \cdot), \mathbf{u}^{(E2)}(T, \cdot))$	3.74e-06	8.68e-07	5.41e-07
$RD_2(\mathbf{v}^{(E1)}, \mathbf{v}^{(E2)})$	3.46e-06	2.13e-07	4.19e-08
$RD_\infty(A_\infty^{(E1)}, A_\infty^{(E2)})$	2.90e-03	6.82e-04	2.99e-4

Tab. 2 Comparison between the results obtained with the approximation diffusion matrices corresponding to Example 1 (E1) and Example 2 (E2), given by (1.11) and (1.12) respectively. RD_p denotes the relative difference in L^p , see (2.3).

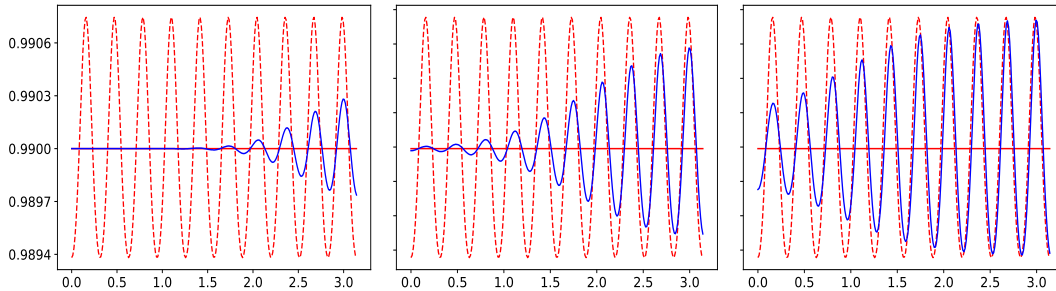


Fig. 1 Typical evolution of disturbances

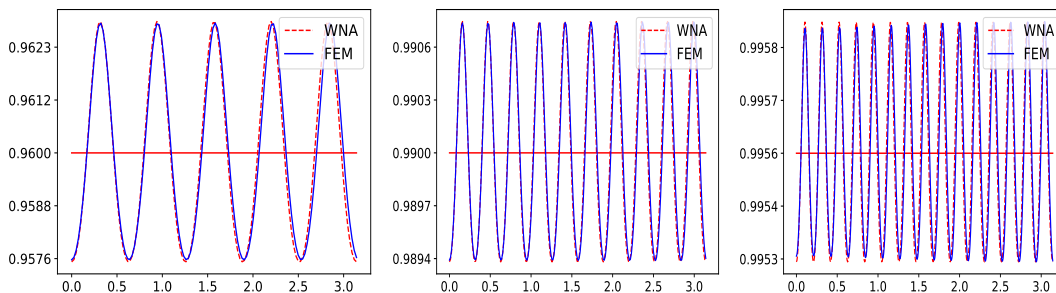


Fig. 2 Experiment 1. WNA and FEM approximations corresponding to Simulations 1 to 3 (left to right). Notice the different scales in the ordinates axis showing the decreasing amplitude of the oscillations.

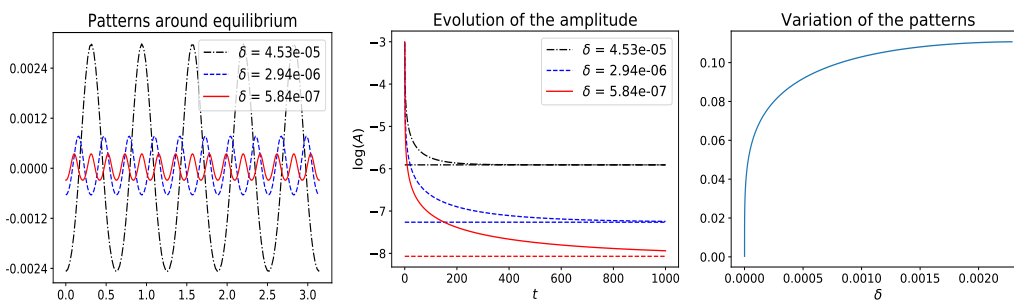


Fig. 3 Experiment 1. Behaviour of the patterns as $\delta \rightarrow 0$.

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