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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Contents

On numerical approximations to diffuse-interface tumor growth models
Acosta-Soba D., Guillén-González F. and Rodríguez-Galván J.R. **8**

An optimized sixth-order explicit RKN method to solve oscillating systems
Ahmed Demba M., Ramos H., Kumam P. and Watthayu W. **15**

The propagation of smallness property and its utility in controllability problems
Apraiz J. **23**

Theoretical and numerical results for some inverse problems for PDEs
Apraiz J., Doubova A., Fernández-Cara E. and Yamamoto M. **31**

Pricing TARN options with a stochastic local volatility model
Arregui I. and Ráfales J. **39**

XVA for American options with two stochastic factors: modelling, mathematical analysis and numerical methods
Arregui I., Salvador B., Ševčovič D. and Vázquez C. **44**

A numerical method to solve Maxwell's equations in 3D singular geometry
Assous F. and Raichik I. **51**

Analysis of a SEIRS metapopulation model with fast migration
Atienza P. and Sanz-Lorenzo L. **58**

Goal-oriented adaptive finite element methods with optimal computational complexity
Becker R., Gantner G., Innerberger M. and Praetorius D. **65**

On volume constraint problems related to the fractional Laplacian
Bellido J.C. and Ortega A. **73**

A semi-implicit Lagrange-projection-type finite volume scheme exactly well-balanced for 1D shallow-water system
Caballero-Cárdenas C., Castro M.J., Morales de Luna T. and Muñoz-Ruiz M.L. **82**

SEIRD model with nonlocal diffusion
Calvo Pereira A.N. **90**

Two-sided methods for the nonlinear eigenvalue problem
Campos C. and Roman J.E. **97**

Fractionary iterative methods for solving nonlinear problems
Candelario G., Cordero A., Torregrosa J.R. and Vassileva M.P. **105**

Well posedness and numerical solution of kinetic models for angiogenesis
Carpio A., Cebrián E. and Duro G. **109**

Variable time-step modal methods to integrate the time-dependent neutron diffusion equation
Carreño A., Vidal-Ferràndiz A., Ginestar D. and Verdú G. **114**

Homoclinic bifurcations in the unfolding of the nilpotent singularity of codimension 4 in R^4
Casas P.S., Drubi F. and Ibánez S. **122**

Different approximations of the parameter for low-order iterative methods with memory
Chicharro F.I., Garrido N., Sarría I. and Orcos L. **130**

Designing new derivative-free memory methods to solve nonlinear scalar problems
Cordero A., Garrido N., Torregrosa J.R. and Triguero P. **135**

Iterative processes with arbitrary order of convergence for approximating generalized inverses
Cordero A., Soto-Quirós P. and Torregrosa J.R. **141**

FCF formulation of Einstein equations: local uniqueness and numerical accuracy and stability
Cordero-Carrión I., Santos-Pérez S. and Cerdá-Durán P. **148**

New Galilean spacetimes to model an expanding universe
De la Fuente D. **155**

Numerical approximation of dispersive shallow flows on spherical coordinates
Escalante C. and Castro M.J. **160**

New contributions to the control of PDEs and their applications
Fernández-Cara E. **167**

Saddle-node bifurcation of canard limit cycles in piecewise linear systems
Fernández-García S., Carmona V. and Teruel A.E. **172**

On the amplitudes of spherical harmonics of gravitational potencial and generalised products of inertia
Floría L. **177**

Turing instability analysis of a singular cross-diffusion problem
Galiano G. and González-Tabernero V. **184**

Weakly nonlinear analysis of a system with nonlocal diffusion
Galiano G. and Velasco J. **192**

What is the humanitarian aid required after tsunami?
González-Vida J.M., Ortega S., Macías J., Castro M.J., Michelini A. and Azzarone A. **197**

On Keller-Segel systems with fractional diffusion
Granero-Belinchón R. **201**

An arbitrary high order ADER Discontinous Galerking (DG) numerical scheme for the multilayer shallow water model with variable density
Guerrero Fernández E., Castro Díaz M.J., Dumbser M. and Morales de Luna T. **208**

Picard-type iterations for solving Fredholm integral equations
Gutiérrez J.M. and Hernández-Verón M.A. **216**

High-order well-balanced methods for systems of balance laws based on collocation RK ODE solvers
Gómez-Bueno I., Castro M.J., Parés C. and Russo G. **220**

An algorithm to create conservative Galerkin projection between meshes
Gómez-Molina P., Sanz-Lorenzo L. and Carpio J. **228**

On iterative schemes for matrix equations
Hernández-Verón M.A. and Romero N. **236**

A predictor-corrector iterative scheme for improving the accessibility of the Steffensen-type methods
Hernández-Verón M.A., Magreñán A.A., Martínez E. and Sukhjit S. **242**

Recent developments in modeling free-surface flows with vertically-resolved velocity profiles using moments
Koellermeier J. **247**

Stability of a one degree of freedom Hamiltonian system in a case of zero quadratic and cubic terms
Lanchares V. and Bardin B. **253**

Minimal complexity of subharmonics in a class of planar periodic predator-prey models
López-Gómez J., Muñoz-Hernández E. and Zanolin F. **258**

On a non-linear system of PDEs with application to tumor identification
Maestre F. and Pedregal P. **265**

Fractional evolution equations in dicrete sequences spaces
Miana P.J. **271**

KPZ equation approximated by a nonlocal equation
Molino A. **277**

Symmetry analysis and conservation laws of a family of non-linear viscoelastic wave equations
Márquez A. and Bruzón M. **284**

Flux-corrected methods for chemotaxis equations
Navarro Izquierdo A.M., Redondo Neble M.V. and Rodríguez Galván J.R. **289**

Ejection-collision orbits in two degrees of freedom problems
Ollé M., Álvarez-Ramírez M., Barrabés E. and Medina M. **295**

Teaching experience in the Differential Equations Semi-Virtual Method course of the Tecnológico de Costa Rica
Oviedo N.G. **300**

Nonlinear analysis in lorentzian geometry: the maximal hypersurface equation in a generalized Robertson-Walker spacetime
Pelegrín J.A.S. **307**

Well-balanced algorithms for relativistic fluids on a Schwarzschild background
Pimentel-García E., Parés C. and LeFloch P.G. **313**

Asymptotic analysis of the behavior of a viscous fluid between two very close mobile surfaces
Rodríguez J.M. and Taboada-Vázquez R. **321**

Convergence rates for Galerkin approximation for magnetohydrodynamic type equations
Rodríguez-Bellido M.A., Rojas-Medar M.A. and Sepúlveda-Cerda A. **325**

Asymptotic aspects of the logistic equation under diffusion
Sabina de Lis J.C. and Segura de León S. **332**

Analysis of turbulence models for flow simulation in the aorta
Santos S., Rojas J.M., Romero P., Lozano M., Conejero J.A. and García-Fernández I. **339**

Overdetermined elliptic problems in onduloid-type domains with general nonlinearities
Wu J. **344**

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
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(pp. 155–159)

New Galilean spacetimes to model an expanding universe

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Abstract

We introduce a new family relevant in the context of a generalized Newton-Cartan Theory: the Galilean Generalized Robertson-Walker spacetimes. We study its geometrical structure and analyse the completeness of its inextensible free falling observers. Additionally, we find some sufficient geometric conditions which guarantee a global splitting of a Galilean spacetime as a Galilean Generalized Robertson-Walker spacetime.

1. Introduction

General Relativity is so far the most accurate and successful theory to describe the spacetime structure and the gravitational phenomena. The evolution of the universe on a large scale was aptly described in the first half of the 20th century by means of the Robertson-Walker cosmological models (or fairly, Friedmann-Lemaître-Robertson-Walker models). These models assume that the matter distribution and the "space relative to the family of observers commovil with the matter" are homogeneous and isotropic. These hypotheses may be weakened in order to describe a universe in a more accurate scale. With this objective, much more recently, new cosmological models have been introduced, as the Generalized Robertson-Walker (GRW) spacetimes [5]. This kind of relativistic spacetimes has been intensively studied from a mathematical perspective (see, for instance, [8], [12, 13], [15], [19, 20].)

However, the geometric formulation of the Newtonian's Gravitation, firstly postulated by E. Cartan [10, 11], after the appearance of the Einstein's General Relativity Theory, is still of interest and significant for several reasons.

On one hand, it formulates the classical Newtonian gravitation as a covariant theory and shows that certain results previously considered as characteristic or singular of the theory of Relativity are shared by the (geometric) gravitational Newton-Cartan Theory. In fact, the Newtonian gravity also arises as a consequence of the curvature of a connection in the spacetime, which does not come from any semi-Riemannian metric. Moreover, in the geometric formulation of Newtonian's Gravity Theory, the spacetime structure is dynamical in the sense that it participates in the unfolding of physics rather than being a fixed backdrop against which it unfolds (see [16] and classical references therein).

On the other hand, it allows to establish from an accurate and intrinsic way the limit relation between the Newtonian theory of Gravitation and General Relativity.

The notion of symmetry is clearly basic in Physics. On a geometrical spacetime model, symmetry is usually based on the assumption of the existence of a one-parameter group of transformations generated by a Killing or, more generally, by a conformal vector field (see, [22]). Another important question is that a geometric approach enables possible generalizations of Newtonian Theory, via the assumption of certain symmetries on Galilean spacetimes (see Section 2), which are the geometrical "arena" for the Newton-Cartan gravitation. So, in [17] the author studies the symmetry imposed on a Galilean spacetime by the cosmological principle, obtaining the Galilean model analogous to the relativistic Robertson-Walker spacetimes.

In this work, we introduce a new family of Galilean geometrical models, which generalize the non-relativistic Robertson-Walker spacetimes, in the same way as GRW spacetimes generalize the Friedmann-Lemaître-Robertson-Walker spacetimes: the Galilean Generalized Robertson-Walker (GGRW) spacetimes (Sect. 3). A GGRW spacetime possesses an infinitesimal symmetry given by the existence of a timelike irrotational conformally Leibnizian (ICL) vector field. Several geometrical properties and physical interpretations for this family of spacetimes are given in Section 3, as the possible existence of singularities or the completeness of its free falling observers. Section 4 is devoted to the study of Galilean spacetimes admitting a timelike irrotational conformally Leibnizian vector field . We show that an ICL Galilean spacetime must be locally a GGRW spacetime. Finally, Section 5 is devoted to face the following kind of splitting problems: under what geometrical assumptions an ICL spacetime globally decomposes as a GGRW spacetime.

2. Set up

Recall that a *Leibnizian* structure on a (non-relativistic) spacetime M is a pair (Ω, g) consisting of a differential 1-form $\Omega \in \Lambda^1(M)$, nowhere null ($\Omega_p \neq 0, \forall p \in M$) and a positive definite metric g on its kernel. Specifically, let us denote by $\mathrm{An}(\Omega) = \{v \in TM\,,\, \Omega(v) = 0\}$ the smooth n-distribution induced on M by Ω. If we denote by $\Gamma(TM)$ the set of smooth vector fields on M, we may construct the subset $\Gamma(\mathrm{An}(\Omega)) = \{V \in \Gamma(TM) \,/\, V_q \in \mathrm{An}(\Omega),\ \forall p \in M\}$. So, the map

$$g : \Gamma(\mathrm{An}(\Omega)) \times \Gamma(\mathrm{An}(\Omega)) \longrightarrow C^\infty(M),\ \ (V, W) \mapsto g(V, W),$$

is smooth, bilinear, symmetric and positive definite. Hence, M is endowed with a sub-Riemannian structure defined on the bundle $\mathrm{An}(\Omega)$, i.e., the annihilator of the degenerate metric $\Omega \otimes \Omega$ (see [6] and [7], for details). The triad, $(M, \Omega, g))$ is called Leibnizian spacetime.

Points of M are usually called *events*. The Euclidean vector space $(\mathrm{An}(\Omega_p)\,, g_p)$ is called the *absolute space* at $p \in M$, and the linear form Ω_p is the *absolute clock* at p. A tangent vector $v \in T_pM$ is named *spacelike* if $\Omega_p(v) = 0$ and, otherwise, *timelike*. Additionally, if $\Omega_p(v) > 0$ (resp. $\Omega_p(v) < 0$), v points out the *future* (resp. the *past*).

An *observer* in a Leibnizian spacetime M is a timelike future unit smooth curve $\gamma : J \longrightarrow M$, i.e., its velocity γ' satisfies that $\Omega(\gamma'(s)) = 1$ for all $s \in J$. The parameter s is called the *proper time* of the observer γ. A vector field $Z \in \Gamma(TM)$ with $\Omega(Z) = 1$ is called a *field of observers*, this is, its integral curves are observers.

When the smooth distribution $\mathrm{An}(\Omega)$ is integrable (equivalently, if the absolute clock Ω satisfies $\Omega \wedge d\Omega = 0$), the Leibnizian spacetime (M, Ω, g) is said to be *locally sincronizable*, and making use of the Frobenius Theorem (see [21]), it may be foliated by a family of spacelike hypersurfaces $\{\mathcal{F}_\lambda\}$. In this case, it is well-known that each $p \in M$ has a neighbourhood where $\Omega = f\,dt$, for certain smooth functions $f > 0$, t, and the hypersurfaces $\{t = \text{constant}\}$ locally coincide with a leaf of the foliation $\mathcal{F}$. Thus, any observer may be synchronized through the "compromise time" t, obtained rescaling its proper time. In the more restrictive case $d\Omega = 0$, then the Leibnizian spacetime (M, Ω, g) is called *proper time locally synchronizable*, and one has, locally, $\Omega = dt$. Now, observers are synchronized directly by its proper time (up to a constant). When Ω is exact, $\Omega = dt$ for some function $t \in C^\infty(M)$, which is called the *absolute time function*. In this case, any observer may be assumed to be parametrized by t. Notice that the notion of (local and local proper time) synchronizability is intrinsic to the Leibnizian structure, applicable for every observer, in contrast to the relativistic setting, where the analogous concepts have meanings only for fields of observers.

According to [7], a field of observers is called *Leibnizian* if the stages Φ_s of its local flows are *Leibnizian diffeomorphisms*, that is, they preserve the absolute clock and space, i.e.,

$$\Phi_s^*\Omega = \Omega, \qquad \text{and} \qquad \Phi_s^* g = g.$$

On the other hand, the inertia principle must be codified through a connection on the spacetime. However, a Leibnizian structure has not a canonical affine connection associated. Then, it is required to introduce a compatible connection with the absolute clock Ω and the space metric g, i.e., a connection ∇ such that

(a) $\nabla\Omega = 0$ (equivalently, $\Omega(\nabla_X Y) = X(\Omega(Y))$ for any $X, Y \in \Gamma(TM)$).

(b) $\nabla g = 0$ (i.e., $Z(g(V, W)) = g(\nabla_Z V, W) + g(\nabla_Z W, V)$ for any $Z \in \Gamma(TM)$ and V, W spacelike vector fields).

Such a connection is named *Galilean*. A *Galilean spacetime* (M, Ω, g, ∇) is a Leibnizian spacetime endowed with a Galilean connection ∇. In addition, ∇ is said *symmetric* if its torsion vanishes identically ($\mathrm{Tor}_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$). From a physical point of view, a symmetric connection is desirable since it is completely determined by its geodesics, i.e., by the free falling observers of M. From now on, we will only consider symmetric Galilean connections on the spacetime.

Given two Galilean spacetimes (M, Ω, g, ∇) and $(M', \Omega', g', \nabla')$, a diffeomorphism $F : M \longrightarrow M'$ is said to be *Galilean* if $F^*\Omega' = \Omega$, $F^*g' = g$ and $F^*\nabla' = \nabla$, i.e., $\nabla'_{dF(X)} dF(Y) = \nabla_X Y$.

For each fixed field of observers Z on a Galilean spacetime (M, Ω, g, ∇), the *gravitational field* induced by ∇ in Z is given by the spacelike vector field $\mathcal{G} = \nabla_Z Z$. The *vorticity* or *Coriolis field* of Z is the 2−form $\omega(Z) = \frac{1}{2}\mathrm{Rot}(Z)$, defined as

$$\omega(Z)(V, W) = \frac{1}{2}\Big(g(\nabla_V Z, W) - g(\nabla_W Z, V)\Big) \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)).$$

The main result of [7, Th.5.27] claims that, for a fixed field of observers Z on a Leibnizian spacetime (M, Ω, g) with $d\Omega = 0$, the set of all symmetric Galilean connections is bijectively mapped onto $\Big(\Gamma(TM), \Lambda^2(\mathrm{An}(\Omega))\Big)$. Each symmetric Galilean connection ∇ is mapped to $\Big(\mathcal{G}(Z), \mathrm{Rot}(Z)\Big)$. Thus, the gravitational field and the vorticity of a field of observers determine a unique symmetric Galilean geometry of the spacetime.

Additionally, a Leibnizian field of observers Z in a Galilean spacetime (M, Ω, g, ∇) is named *Galilean* if it is affine for ∇, that is, $L_Z \nabla = 0$, where L denotes the Lie derivative. Finally, a Galilean spacetime is said *Newtonian* if the (symmetric) connection ∇ restricted to the spacelike vectors is flat, and it admits an irrotational Galilean field of observers. This kind of spacetimes has traditionally represented the classical (non-relativistic) geometric model of gravity.

3. Galilean Generalized Robertson-Walker spacetimes

In this section we introduce a new family of Galilean geometric models, which are the classical version of the relativistic Generalized Robertson-Walker spacetimes defined in [5].

Definition 3.1 Let $I \subseteq \mathbb{R}$ be a real interval, (F, h) a n-dimensional connected Riemannian manifold, and $f \in C^\infty(I)$ a smooth positive function on I. A Galilean spacetime (M, Ω, g, ∇) is called *Galilean Generalized Robertson-Walker spacetime (GGRW)* if $M = I \times F$, $\Omega = d\pi_I$, g is the restriction to the bundle $\mathrm{An}(\Omega)$ of the following (degenerate) metric on M,

$$\overline{g} = (f \circ \pi_I)^2 \, \pi_F^* h, \tag{3.1}$$

where π_I, π_F are the canonical projections onto the open interval I and the fiber F respectively, and ∇ is the only symmetric Galilean connection on M such that

$$\nabla_{\partial_t} \partial_t = 0, \qquad and \qquad \mathrm{Rot}\, \partial_t = 0, \tag{3.2}$$

where $\partial_t = \partial/\partial t$ is the global coordinate vector field associated to $t := \pi_I$.

The vector field ∂_t defines a field of observers in M ($\Omega(\partial_t) = 1$), which we will call *commovil observers*, by the similarity with the relativistic Robertson-Walker spacetimes. Then, the conditions (3.2) in above definition mean that commovil observers are free falling and they do not rotate. Notice that from [7, Th.5.27], the conditions (3.2) determine the (symmetric) Galilean connection on M.

Example Let us consider a GGRW with $I = \mathbb{R}$ and $F = \mathbb{R}^n$ endowed with the usual Euclidean metric. If $f(t) =$ constant, then the Galilean connection coincides with the standard flat connection of the affine space $\mathbb{R}^{n+1}$. In addition, the commovil observers satisfy the necessary conditions to assure the Newtonian character of this spacetime. More physically relevant examples are given in the next section.

3.1. Completeness of free falling observers in a GGRW spacetime

We now proceed to analyze when the inextensible free falling trajectories in a GGRW spacetime are complete. Physically we are looking for geometric assumptions that guarantee that every free falling observer *lives forever*.

First, we have an analogous result to the geodesic normalization lemma in semi-Riemannian manifolds.

Lemma 3.2 *Let γ be a geodesic in a GGRW spacetime. Then, $\Omega(\gamma')$ is constant along the trajectory of γ.*

The relevant cases correspond with $\Omega(\gamma') = 0$ or 1. The first one ($\Omega(\gamma') = 0$) means that γ is spacelike and contained in a leaf $\mathcal{F}_t$ of the foliation of Ω. As ∇ coincides with the Levi-Civita connection of $(\mathcal{F}_t, f(t)^2 h)$, the completeness of this kind of geodesics is equivalent to the geodesic completeness of (F, h). Thus, from now on we will deal with free falling observers (γ geodesic with $\Omega(\gamma') = 1$).

Theorem 3.3 *A GGRW spacetime is geodesically complete if and only if $I = \mathbb{R}$ and the fiber (F, h) is (geodesically) complete.*

4. Irrotational conformally Leibnizian spacetimes

In this section we present a wider family of Galilean spacetimes which locally exhibit the structure of a GGRW spacetime. As a previous step, we introduce the concept of conformally Leibnizian field of observers, generalizing the well-known notion of Leibnizian observer.

Definition 4.1 Let (M, Ω, g) be a Leibnizian spacetime. A vector field X is called *spatially conformally Leibnizian* vector field if

$$L_X \Omega = \mu \, \Omega, \tag{4.1}$$

and the Lie derivative of the absolute space metric satisfies

$$L_X g = 2\lambda \, g, \tag{4.2}$$

for some smooth functions $\lambda, \mu \in C^\infty(M)$. If, additionally, both functions coincide, i.e., $\lambda = \mu$, then X is named *conformally Leibnizian* vector field.

Note that a conformally Leibnizian vector field is Leibnizian if and only if the conformal factor λ is identically zero [7].

Remark 4.2 Condition (4.1) may be also expressed as

$$d\Omega(X,Y) + Y(\Omega(X)) = \mu\,\Omega(Y), \qquad \forall Y \in \Gamma(TM),$$

and means that distribution $\mathrm{An}(\Omega)$ is invariant along the flow of vector field X. So, if this distribution is integrable, the flow of X carries each leaf of the foliation to another one. Analogously, assumption (4.2) is equivalent to

$$X(g(V,W)) = \lambda\, g(V,W) + g([X,V],W) + g([X,W],V), \qquad \forall V,W \in \Gamma(\mathrm{An}(\Omega)).$$

The following result shows that GGRW spacetimes admit a timelike conformally Leibnizian vector field.

Proposition 4.3 *Let $(M = I \times F, \Omega = dt, g, \nabla)$ be a GGRW spacetime with scale factor $f \in C^\infty(I)$. Then, the vector field $K := (f \circ \pi_I)\, \partial_t$ is irrotational and conformally Leibnizian and, consequently, it satisfies the identity*

$$\nabla_X K = (f' \circ \pi_I)\, X, \qquad \forall X \in \Gamma(TM). \tag{4.3}$$

Definition 4.4 Let (M, Ω, g, ∇) be a Galilean spacetime, whose absolute clock is closed ($d\Omega = 0$). If M admits a timelike vector field $K \in \Gamma(TM)$ satisfying

$$\nabla_X K = \rho\, X, \qquad \forall X \in \Gamma(TM), \ \ where\ \rho \in C^\infty(M), \tag{4.4}$$

M is called *Irrotational Conformally Leibnizian spacetime* (ICL).

Remark 4.5 Notice that condition (4.4) directly implies that K is conformally Leibnizian and $\mathrm{Rot}(K)(V,W) = 0$, for all spacelike vector fields V, W.

As a first consequence of Definition 4.4, we obtain that functions $\Omega(K)$ and ρ are constant on each leaf of the foliation induced by Ω.

Lemma 4.6 *Let (M, Ω, g, ∇) be a ICL spacetime with irrotational conformally Leibnizian vector field K and conformal factor ρ. Then*

$$V\big(\Omega(K)\big) = 0 \quad and \quad V(\rho) = 0, \qquad \forall V \in \Gamma(\mathrm{An}(\Omega)).$$

We have just seen that each GGRW is an ICL spacetime. Next theorem ensures that any ICL spacetime is locally a GGRW spacetime.

Theorem 4.7 *Let (M, Ω, g, ∇) be an ICL spacetime. For each $p \in M$, there exist an open neighbourhood of p, $\mathcal{U}$, and a Galilean diffeomorphism $\Psi : N \longrightarrow \mathcal{U}$, where N is a GGRW spacetime.*

5. Global GGRW decompositions

We know that an ICL spacetime is locally a GGRW spacetime. Now, our aim here consists in looking for additional assumptions on the geometry of an ICL spacetime which lead to a global splitting as a GGRW spacetime. This type of question has been yet discussed several times in the relativistic setting (see for instance, [8], [14], [15] and [4]), i.e., under what conditions on the geometry of a relativistic spacetime, this admits a global decomposition as a warped product space or, in particular, as a GRW spacetime.

Theorem 5.1 *A Gailean spacetime $\big(M, \Omega, g, \nabla\big)$, whose 1-form Ω is exact, admits a global decomposition as a GGRW spacetime if and only if it is an ICL spacetime with a timelike irrotational conformally vector field K, such that the flow of the associated field of observers, $Z := \frac{1}{\Omega(K)} K$, is well defined and onto in a domain $I \times \mathcal{F}$ for some interval $I \subseteq \mathbb{R}$ and some leaf of the foliation $\mathcal{F}$ induced by Ω.*

Remark 5.2 (i) Note that the hypothesis on the absolute clock Ω automatically holds when the spacetime is simply connected. (ii) Observe that the assumption on the flow of Z trivially holds when Z is complete.

Taking into account the previous Remark, we can assert

Corollary 5.3 *Let (M, Ω, g, ∇) be an ICL spacetime with timelike irrotational conformally Leibnizian vector field K. If the absolute clock Ω is exact and $\frac{1}{\Omega(K)}K$ is complete, then M globally splits as a GGRW spacetime.*

To end this work, we present a global splitting result when the spacetime is spatially compact, that is, when the leaves of the spacelike foliation are compact.

Theorem 5.4 *Let (M, Ω, g, ∇) be an ICL spacetime with Ω exact. If the leaves of the foliation induced by Ω are compact, then M is a GGRW spacetime.*

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