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Universidad de Oviedo

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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics SĒMA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the SĒMA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier “Pancho” Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: “a mathematician is a device for turning coffee into theorems”. Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Iterative processes with arbitrary order of convergence for approximating generalized inverses

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Abstract

A family of iterative schemes for finding approximate inverses of nonsingular matrices is suggested and established analytically. This class of methods can be used for finding the Moore-Penrose inverse of a rectangular complex matrix. The order of convergence is stated in each case, depending on the first non-zero parameter. For different examples, the accessibility of some schemes, that is, the set of initial estimations leading to convergence, is analyzed in order to select those with wider sets. This wideness is related with the value of the first non-zero value of the parameters defining the method. Finally, some numerical examples are provided to confirm the theoretical results and to show the feasibility and effectiveness of the new methods.

1. Introduction

Computing the matrix inverse of nonsingular matrices of higher size is difficult and is a time consuming task. Generally speaking, in wide variety of topics, one must compute the inverse or particularly the generalized inverses to comprehend and realize significant features of the involved problems.

In the last decade, many iterative schemes of different orders have been designed for approximating the inverse or some generalized inverse (Moore-Penrose inverse, Drazin inverse, etc.) of a complex matrix A . In this paper, we focus our attention on constructing a new class of iterative methods, free of inverse operators and with arbitrary order of convergence, for finding the inverse of a nonsingular complex matrix. We also study the proposed class for computing the Moore-Penrose inverse of complex rectangular matrices. The designed family depends on several real parameters, which by taking particular values provide us numerous known methods constructed by other authors with different procedures.

The most known iterative scheme for computing the inverse A^{-1} of a nonsingular complex matrix A is the Schulz's method whose iterative expression is

$$X_{k+1} = X_k(2I - AX_k), \quad k = 0, 1, \dots \quad (1.1)$$

where I is the identity matrix with the same size of A . Schulz in [8] demonstrated the convergence of sequence $\{X_k\}_{k \geq 0}$, obtained from (1.1), to the inverse A^{-1} is guaranteed if the eigenvalues of matrix $I - AX_0$ are lower than 1. Taking into account that the residuals $E_k = I - AX_k$, $k = 0, 1, \dots$ satisfy $\|E_{k+1}\| \leq \|E_k\|^2$, expression (1.1) has quadratic convergence. In general, in the Schulz-type methods it is common to use as initial approach $X_0 = \alpha A^*$ or $X_0 = \alpha A$, where $0 < \alpha < 2/\rho(A^*A)$, where A^* is the conjugate transpose of A and $\rho(\cdot)$ the spectral radius. In this paper, we use in the case of inverses and also in generalized inverses, the initial estimation $X_0 = \beta A^*/\|A\|^2$. We also study the values of the parameter β that guarantee convergence.

Li et al. in [5] proposed the family of iterative methods

$$X_{k+1} = X_k \left(\nu I - \frac{\nu(\nu-1)}{2} AX_k + \frac{\nu(\nu-1)(\nu-2)}{3!} (AX_k)^2 - \dots + (-1)^{\nu-1} (AX_k)^{\nu-1} \right), \quad \nu = 2, 3, \dots$$

with $X_0 = \alpha A^*$. They proved the convergence of ν -order of $\{X_k\}_{k \geq 0}$ to the inverse of matrix A . This class was used by Chen et al. in [2] and by Li et al. in [19] for approximating the Moore-Penrose inverse.

Soleymani et al. in [18] also constructed a fourth-order iterative scheme for calculating the inverse and the Moore-Penrose inverse, with iterative expression

$$X_{k+1} = \frac{1}{2} X_k (9I - AX_k (16I - AX_k (14I - AX_k (6I - AX_k))))), \quad k = 0, 1, \dots$$

On the other hand, Stanimirović et al. in [16] designed the following scheme of order eleven for computing the generalized outer inverse $A_{T,S}^{(2)}$

$$X_{k+1} = X_k (I + (R_k + R_k^2)(I + (R_k^2 + R_k^4)(I + R_k^4))), \quad k = 0, 1, \dots$$

being $R_k = I - AX_k$, $k = 0, 1, \dots$

Kaur et al. in [4], by using also the hyperpower iterative method, designed the following scheme of order five for obtaining the weighted Moore-Penrose inverse

$$X_{k+1} = X_k(5I - 10AX_k + 10(AX_k)^2 - 5(AX_k)^3 + (AX_k)^4), \quad k = 0, 1, \dots$$

These papers are some of the manuscripts that have been published to approximate the inverse of a nonsingular matrix or some of the generalized inverses of arbitrary matrices. In this paper, we design a parametric family of iterative schemes with arbitrary order of convergence that contains many of the methods constructed up to date. For each fixed value of the order of convergence, we still have a class of iterative methods depending on several parameters.

The rest of this manuscript is organized as follows. Section 2 is devoted to the construction of the proposed class of iterative schemes, proving its convergence to the inverse of a nonsingular complex matrix, with arbitrary order of convergence. In Section 3, it is proven that the same family of iterative methods is able to converge to the Moore-Penrose inverse of a complex matrix of size $m \times n$. Some particular cases of this class are found in Section 4, corresponding to existing methods proposed by different authors. A wide range of numerical test are also found in Section 5, checking the robustness and applicability of the proposed methods on different kinds of matrices. With some conclusions and the references used finishes this manuscript.

2. A class of iterative schemes for matrix inversion

In this section, we present a parametric family of iterative schemes for approximating the inverse of nonsingular matrices and we prove the order of convergence of the different members of the family. First, we define the following polynomial matrix.

Definition 2.1 Let $U \in \mathbb{C}^{m \times m}$ be a complex square matrix and $p > 0$ a positive integer number. We define the polynomial matrix $H_p(U)$ as

$$H_p(U) = \sum_{j=1}^p (-1)^{j-1} C_p^j U^{j-1} = C_p^1 I - C_p^2 U + C_p^3 U^2 + \dots + (-1)^{p-1} C_p^p U^{p-1},$$

where C_p^j is the combinatorial number $C_p^j = \binom{p}{j} = \frac{p!}{j!(p-j)!}$.

The following technical result can be proven by using mathematical induction with respect to parameter p .

Lemma 2.2 Let $p > 0$ be a positive integer and $U \in \mathbb{C}^{m \times m}$. Then $(I - U)^p = I - UH_p(U)$.

Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix and $p > 1$ a positive integer. Let $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ be a set of real parameters such that $\alpha_i \in [0, 1]$, for $i = 1, 2, \dots, p-1$, $\alpha_p \in]0, 1]$ and $\sum_{i=1}^p \alpha_i = 1$.

We assume a sequence of complex matrices $\{X_0, X_1, \dots, X_n, \dots\}$, of size $m \times m$, satisfying following conditions:

- (a) $\|I - AX_0\| = \gamma_0 < 1$,
- (b) $I - AX_{k+1} = \sum_{i=1}^p \alpha_i (I - AX_k)^i$.

We consider the family of methods with iterative expression

$$X_{k+1} = X_k \sum_{i=1}^p \alpha_i H_i(AX_k), \quad k = 0, 1, \dots \quad (2.1)$$

For each positive integer p , $p > 1$, we have a different class of iterative methods, whose order of convergence depends on the value of parameters α_i , $i = 1, 2, \dots, p$.

In the following results, the convergence of these schemes to the inverse of matrix A is proven.

Proposition 2.3 Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix and $p > 1$ a positive integer. Let us consider the sequence of complex matrices constructed as

$$X_{k+1} = X_k \sum_{i=1}^p \alpha_i H_i(AX_k), \quad k = 0, 1, \dots,$$

where $\alpha_i \in [0, 1]$, for $i = 1, 2, \dots, p-1$, $\alpha_p \in]0, 1]$ and $\sum_{i=1}^p \alpha_i = 1$. Then, condition

$$I - AX_{k+1} = \sum_{i=1}^p \alpha_i (I - AX_k)^i,$$

is equivalent to

$$X_{k+1} = X_k \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right). \quad (2.2)$$

By mathematical induction it is also easy to prove the following result.

Proposition 2.4 *Let us consider sequence $\{X_k\}_{k \geq 0}$ obtained from expression (2.1). If $\|I - AX_0\| < 1$, then*

$$\|I - AX_k\| < 1, \quad k = 1, 2, \dots$$

From these previous results, we can establish the following convergence theorem.

Theorem 2.5 *Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix and an initial guess $X_0 \in \mathbb{C}^{m \times m}$. Let $\alpha_1, \dots, \alpha_p$ be nonnegative real numbers such that $\alpha_i \in [0, 1]$, $\alpha_p \neq 0$ and $\sum_{i=1}^p \alpha_i = 1$. If $\|I - AX_0\| < 1$, then sequence $\{X_k\}_{k \geq 0}$, obtained by (2.1), converges to A^{-1} with convergence order q for any $p > 1$, where $q = \min_{i=1,2,\dots,p} \{i \mid \alpha_i \neq 0\}$.*

In the next section, we extend the iterative schemes (2.1) for finding the Moore-Penrose inverse of any complex rectangular matrix.

3. A class of iterative schemes for computing Moore-Penrose inverse

Let A be a $m \times n$ complex matrix. The Moore-Penrose inverse of A (pseudoinverse), denoted by A^\dagger , is the unique $n \times m$ matrix X satisfying

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

This generalized inverse plays an important role in several fields, such as eigenvalue problems and the linear least square problems [3]. It can be obtained, explicitly, from the singular value decomposition of A but, with a high computational cost. Therefore, it is interesting to have efficient iterative methods to approximate this matrix. In this section, we prove how family (2.1) allows us to compute the pseudoinverse with the same order of convergence that in the previous section, where the inverse of a square matrix was calculated. First, we establish the following technical result, that is proven by mathematical induction, although other authors state similar results in the context of outer inverses (see, for example, [17]).

Lemma 3.1 *Let us consider $X_0 = \alpha A^*$, where $\alpha \in \mathbb{R}$, and sequence $\{X_k\}_{k \geq 0}$ generated by family (2.1). For any $k \geq 0$, it is satisfied*

$$(X_k A)^* = X_k A, \quad (AX_k)^* = AX_k, \quad X_k A A^\dagger = X_k, \quad A^\dagger A X_k = X_k. \quad (3.1)$$

Now, some technical results are presented.

Lemma 3.2 ([2]) *Let $A \in \mathbb{C}^{m \times n}$ and $X_0 = \alpha A^*$ be, where $\alpha < \frac{1}{\sigma_1^2}$ and σ_1 is the largest singular value of A . Then $\|(X_0 - A^\dagger)A\| < 1$.*

Lemma 3.3 *Let $A \in \mathbb{C}^{m \times n}$ and $\{X_k\}_{k \geq 0}$ be the sequence generated by (2.1). Let us consider $E_k = X_k - A^\dagger$, $k = 0, 1, \dots$. Then,*

1. $\|X_k - A^\dagger\| \leq \|E_k A\| \|A^\dagger\|$,
2. $(I - A^\dagger A)E_k A = \mathbf{0}$.

Lemma 3.4 Let $A \in \mathbb{C}^{m \times n}$ and $\{X_k\}_{k \geq 0}$ be the sequence generated by (2.1). By using $E_k = X_k - A^\dagger$, defined in the previous lemma, we have

$$E_{k+1}A = \sum_{i=1}^p \alpha_i (-1)^{i-1} (E_k A)^i, \quad k = 0, 1, \dots \quad (3.2)$$

Finally, we can state the following convergence result.

Theorem 3.5 Let $A \in \mathbb{C}^{m \times n}$ and $q = \min_{i=1,2,\dots,p} \{\alpha_i \neq 0\}$. Then, sequence $\{X_k\}_{k \geq 0}$ generated by (2.1) converges to the Moore-Penrose inverse A^\dagger with q th-order provided that $X_0 = \alpha A^*$, where $\alpha < \frac{1}{\sigma_1^2}$ is a constant and σ_1 is the largest singular value of A .

4. Some known members of the proposed class

The family of iterative schemes (2.1) is a generalization of other known methods constructed with different techniques. Now, we describe some of them.

1. For any $p > 1$, if $\alpha_1 = \dots = \alpha_{p-1} = 0$ and $\alpha_p = 1$, then we obtain the method proposed by Li and Li. (see Eq. (2.3) in [5] for inverse case and Eq. (2.1) in [2] for pseudoinverse one). Recall that method proposed by Li and Li generalizes the Newton-Schultz ($p = 2$) and Chebyshev method ($p = 3$).
2. On the other hand, if $\alpha_i = 0$ for $i = 1, 2, \dots, 8$, $\alpha_9 = \alpha_{12} = 1/8$ and $\alpha_{10} = \alpha_{11} = 3/8$, then we get the method proposed by Soleymani and Stanimirovic (see Eq. (12) in [9]).
3. Also, expression (2.1) gives us the method proposed by Toutounian and Soleymani (see Eq. (18) in [18]), if $\alpha_4 = 1/2$ and $\alpha_5 = 1/2$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
4. When the only not null parameter is $\alpha_7 = 1$, then we obtain method proposed by Soleymani (see Eq. (18) in [11]).
5. In a similar way, if the only parameter different from zero is $\alpha_6 = 1$, then the method proposed by Soleymani, Stanimirovic and Zaka (see Eq. (2.4) in [14]) is obtained.
6. When $\alpha_i = 0$ for $i = 1, 2, \dots, 7$, $\alpha_8 = \alpha_{10} = 1/4$ and $\alpha_9 = 2/4$, the resulting scheme is that proposed by Soleymani in Eq. (9) in [12].
7. The method proposed by Soleymani et al in [13], Eq. (10), appears if $\alpha_1 = \dots = \alpha_8 = 0$, $\alpha_9 = 7/9$ and $\alpha_{10} = 2/9$.
8. The scheme proposed by Razavi, Kerayechian, Gachpazan and Shateyi, (see Eq. (16) in [7]) is obtained if we choose $\alpha_1 = \dots = \alpha_9 = 0$, $\alpha_{10} = \alpha_{12} = 1/4$ and $\alpha_{11} = 1/2$ in Equation (2.1).
9. When the first eight parameters are null, $\alpha_9 = 343/729$, $\alpha_{10} = 294/729$, $\alpha_{11} = 84/729$ and $\alpha_{12} = 8/729$, we get the scheme proposed by Al-Fhaid et al in [1], Eq. (5).
10. If $\alpha_7 = 9/16$, $\alpha_8 = 6/16$, $\alpha_9 = 1/16$ and the rest of parameters are zero, then the scheme proposed by Soleymani is found (see Eq. (3.1) in [10]).
11. When $p = 12$ and the only parameters different from zero are $\alpha_9 = \alpha_{12} = 1/8$ and $\alpha_{10} = \alpha_{11} = 3/8$, therefore the method proposed by Liu and Cai. see Eq. (4) in [6] is obtained.
12. If $\alpha_2 = 0$, $\alpha_1 = 1 - \alpha$ and $\alpha_3 = \alpha$, where $\alpha \in (0, 1]$, then we find the method proposed by Srivastava and Gupta in [15]), Eq. (6).

5. Numerical examples

In this section, we check the performance of the proposed schemes, on small and large-scale matrices. Among them, we work with the Hilbert matrix as an example of ill-conditioned matrix. These numerical tests have been made with Matlab R2018b, by using double precision arithmetics. The convergence is checked by means of the stopping criterium of the residual, $\|AX_k - I\| < 10^{-6}$ and a maximum of 200 iterations. In all cases, the initial estimation taken is $X_0 = \beta \frac{A^T}{\|A\|^2}$, being A the matrix whose inverse we are estimating and choosing values of parameter β close to 0.7.

In Tables 1 - 2, elements $\alpha_1 = 0$, $\alpha_1 = 0.6$ and $\alpha_1 = 0.8$ for the class $p = 2$ (all of them with $\alpha_2 = 1 - \alpha_1$) are compared with the members of class $p = 3$ corresponding to $\alpha_1 = 0$ and $\alpha_2 = 0$, $\alpha_2 = 0.6$ and $\alpha_2 = 0.8$, where $\alpha_3 = 1 - \alpha_2$. The comparison is made through the number of iterations needed to converge (it) and the residual $\|AX_k - I\|$, denoted by (res). If the method does not converge (typically giving “NaN”), it is denoted by “nc” in the column of iterations; if the scheme simply needs more than 200 iterations to converge, it is denoted by > 200 .

In Table 1 the numerical results correspond to a Leslie matrix of size 100×100 , and in Table 2 the results generated by a Hilbert matrix of size 5×5 are shown.

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.6$		$\alpha_1 = 0.8$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	18	6.9e-12	55	8.5e-7	113	9.2e-7	11	2.9e-8	14	6.4e-7	16	2.4e-10
1.5	17	4.2e-9	54	7.7e-7	111	8.8e-7	11	4.8e-12	14	2.4e-9	15	1.4e-7
2	53	3.7e-11	53	8.3e-7	107	9.6e-7	33	8.0e-11	14	1.3e-11	15	1.4e-9
2.5	nc	-	52	9.8e-7	108	9.2e-7	nc	-	13	3.9e-7	nc	-
3	nc	-	52	7.5e-7	107	9.2e-7	nc	-	13	3.7e-8	nc	-
3.5	nc	-	nc	-	106	9.5e-7	nc	-	26	1.1e-12	nc	-
4	nc	-	nc	-	106	8.1e-7	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	105	8.7e-7	nc	-	nc	-	nc	-
5	nc	-	nc	-	104	9.6e-7	nc	-	nc	-	14	1.2e-12
5.5	nc	-	nc	-	104	8.5e-7	nc	-	nc	-	nc	-
6	nc	-	nc	-	> 200	-	nc	-	nc	-	nc	-

Tab. 1 Numerical results for a Leslie matrix of size 100×100

In Table 1, we notice that for large-scale (100×100) Leslie matrix, the numerical results obtained by $p = 2$, $\alpha_1 = 0$ and $\alpha_2 = 0.6$ show convergence to the inverse matrix even when parameter β of the initial estimation is not close to zero. However, in these cases the number of iterations is very high. Regarding the lowest number of iterations needed to converge, the best method corresponds to $p = 3$, $\alpha_1 = 0$ and $\alpha_2 = 0.6$ as it holds low number of iterations and high value of β .

Table 2 corresponds to a test on a 5×5 Hilbert matrix. It is clear that the number of iterations is high due to the bad conditioning of the matrix. Nevertheless, the performance is, in general similar to previous cases.

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.2$		$\alpha_1 = 0.4$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	42	3.9e-9	54	5.7e-7	72	4.3e-7	27	2.3e-11	34	9.5e-11	37	1.07e-7
1.5	41	4.9e-7	53	9.3e-7	71	4.8e-7	26	5.1e-8	33	1.5e-7	37	9.8e-11
2	56	5.7e-8	53	4.2e-7	70	6.9e-7	nc	-	33	2.3e-9	36	3.9e-7
2.5	nc	-	nc	-	70	4.5e-7	nc	-	33	4.8e-11	nc	-
3	nc	-	nc	-	nc	-	nc	-	33	1.3e-11	nc	-
3.5	nc	-	nc	-	nc	-	nc	-	32	2.1e-8	nc	-
4	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
5.5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
6	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-

Tab. 2 Numerical results for a Hilbert matrix of size 5×5

Finally, Table includes the results of pseudoinverse calculation for a random matrix of size 300×200 . In this case, Chebyshev’s method performs better than the most of schemes under study, as it need a very low number of iterations to converge to the pseudoinverse, although $p = 2$ can converge even with values of $\beta = 6$ or higher.

6. Conclusions

In this paper, we have developed a parametric family of iterative methods for computing inverse and pseudoinverse of a complex matrix, having arbitrary order of convergence. Moreover, we have shown in Theorems 2.5 and 3.5 that the order of the suggested method in (2.1) depends on the first non-zero parameter. The proposed parametric

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.6$		$\alpha_1 = 0.8$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	19	5.4e-8	56	6.7e-7	109	9.7e-7	13	4.9e-15	16	1.2e-9	18	5.7e-14
1.5	19	1.3e-11	55	6.0e-7	107	9.3e-7	12	4.3e-8	16	4.2e-13	17	3.8e-10
2	18	5.4e-8	54	6.5e-7	106	8.1e-7	12	1.5e-10	15	2.1e-8	17	6.8e-13
2.5	nc	-	53	7.7e-7	104	9.7e-7	nc	-	15	6.1e-10	nc	-
3	nc	-	52	9.7e-7	103	9.7e-7	nc	-	15	2.1e-11	nc	-
3.5	nc	-	52	7.7e-7	103	8.0e-7	nc	-	15	3.5e-8	nc	-
4	nc	-	nc	-	102	8.5e-7	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	101	9.2e-7	nc	-	nc	-	nc	-
5	nc	-	nc	-	101	8.1e-7	nc	-	nc	-	nc	-
5.5	nc	-	nc	-	100	9.0e-7	nc	-	nc	-	nc	-
6	nc	-	nc	-	100	8.1e-7	nc	-	nc	-	nc	-

Tab. 3 Numerical results for the estimation of the pseudoinverse of a random matrix of size 300×200 with $X_0 = \beta \frac{A^T}{\|A\|^2}$

family in (2.1) is a generalization of other methods which are obtained for particular values of the parameters. The numerical experiments show the feasibility and effectiveness of the new methods, for both nonsingular and rectangular matrices with or without full rank and arbitrary size.

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