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Editors:
Rafael Gallego, Mariano Mateos

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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Designing new derivative-free memory methods to solve nonlinear scalar problems

Alicia Cordero[1], **Neus Garrido**[2], **Juan R. Torregrosa**[1], **Paula Triguero**[1]
1. *Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, València, Spain*
2. *Escuela Superior de Ingeniería y Tecnología, Universidad Internacional de La Rioja, Logroño, Spain*

Abstract

In this work, two families of iterative methods without derivatives have been designed using the composition of iterative schemes and the inclusion of weight functions. The convergence analysis of the two-step and the three-step families is presented, showing the necessary conditions that must be satisfied by the weight functions to have order four and six, respectively. From them, two methods with memory have been derived, improving their order of convergence and their efficiency. All the methods are tested and compared with other known methods in the approxitation of the roots of different nonlinear functions. The results show the improvement of the classes after including memory.

1. Introduction

It is becoming a need in many scientific and technological disciplines to solve a nonlinear equation or a system of nonlinear equations. We describe this nonlinear problem for the scalar case as $f(x) = 0$, where $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ and D is an open set. Due to the lack of analytical methods for solving these type of nonlinear problems, the implementation of iterative processes to solve them has become more frequent.
The use of iterative methods for solving nonlinear problems has increased in the recent decades. These iterative processes generate a sequence of points closer and closer to the solution, so that an approximation to the root with the required precision is obtained as a solution to the problem.
There is a wide literature related to iterative schemes for approximating simple roots of nonlinear functions (see [8], [1] and [5] and the references therein). Among them, the most classical iterative algorithm is Newton's method, with iterative expression

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots$$

Given an initial estimation x_0 to the root of $f(x) = 0$, Newton's method converges with quadratic order.

There are several ways to quantify the quality of an iterative method, in particular the speed of convergence and its computational cost. One point methods, such as Newton's scheme, are known for their simplicity and low computational cost, but have slow convergence. For this reason, the number of multipoint methods designed to increase the order of convergence has grown exponentially.

The most common iterative schemes are those that use only the previous iteration to obtain the next approximation. They are called methods without memory. However, Kung and Traub conjectured in [4] that the order of these scalar methods can not be grater than 2^{d-1}, where d is the number of different functional evaluations performed on each iteration of the method. However, iterative methods with memory, that is, methods that use more than one previous iterate, do not have any upper bound on their order of convergence.

On the other hand, many nonlinear functions have a derivative that is difficult to calculate or whose expression is not known. For this reason, Steffensen [7] proposed to approximate the derivative of Newton's method by

$$f'(x_k) \approx f[x_k + f(x_k), x_k] = \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}$$

and then replacing it on its iterative structure, obtaining the well-known Steffensen's method:

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}, \qquad k = 0, 1, 2, \ldots,$$

with quadratic order.
In addition, it is known that the composition of two iterative methods with orders p_1 and p_2 results in a method with order of convergence $p_1 \cdot p_2$, but whose computational cost increases significantly. However, the use of

weight functions in the composition of iterative schemes allows the design of methods that increase the order of convergence without adding a high number of new functional evaluations.

In this paper, we propose the use of the above techniques to design methods with higher order of convergence and adding the minimal computational cost. For this purpose, we have organized the contents as follows. In Section 2 we present a new derivative-free iterative family with two steps and a real parameter. We analyse its convergence and the possibility of increasing the order by using previous iterations. Section 3 is devoted to extend the two-step family to a three-step family holding the same iterative structure. The convergence of this family and an approximation for the parameter to increase the order of convergence leading to a method with memory are also analysed. In Section 4 we test the performance of the methods studied in this work for solving different nonlinear equations. Finally, the conclusions of the study are summarized in section 5.

2. Derivative-free iterative family with two steps

In this section, we will use several techniques to design new iterative schemes that improve the performance of Newton and Steffensen's methods and also have the possibility of increasing the order of convergence. First, we propose a derivative-free family obtained by composing methods and using weight functions. Then, we will extend this family to higher order memory methods without adding new functional evaluations.

The starting family of iterative methods presented in this work is obtained by the composition of Steffensen's method and the addition of a real parameter β and a weight function H. The proposed scheme is as follows:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[w_k, x_k]}, \\ x_{k+1} &= y_k - H(\mu_k)\frac{f(y_k)}{f[y_k, x_k]}, \end{aligned} \qquad k = 0, 1, 2, \ldots, \tag{2.1}$$

where $w_k = x_k + \beta f(x_k)$, $\beta \in \mathbb{R} - \{0\}$, and the weight function variable is defined by $\mu = \frac{f(y)}{f(w)}$. We denote $M4_\beta$ the iterative family (2.1).

Theorem 2.1 shows the conditions that the weight function must satisfy to obtain order four for any value of the parameter.

Theorem 2.1 *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real sufficiently differentiable function in a convex set D and let $\alpha \in D$ be a simple root of $f(x) = 0$. If x_0 is close enough to α and $H(\mu)$ satisfies $H(0) = H'(0) = 1$ and $|H''(0)| < \infty$, then sequence $\{x_k\}$ generated by family $M4_\beta$ converges to α with order of convergence 4 for any value of $\beta \in \mathbb{R}$, $\beta \neq 0$, being its error equation*:

$$e_{k+1} = \frac{1}{2}c_2(1+\beta f'(\alpha))(-2c_3(1+\beta f'(\alpha)) + c_2^2(6+4\beta f'(\alpha) - H_2))e_k^4 + O(e_k^5), \tag{2.2}$$

where $H_2 = H''(0)$, $e_k = x_k - \alpha$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j \geq 2$.

From the error equation (2.2), we can observe that family $M4_\beta$ is fourth-order convergent for any value of β. According to the Kung and Traub conjecture [4], $M4_\beta$ is a family of optimal iterative methods as the number of different functional evaluations is three, i.e, $f(x_k)$, $f(y_k)$ and $f(w_k)$, so the maximum value $4 = 2^{3-1}$ is reached. Moreover, the value of $\beta = -\frac{1}{f'(\alpha)}$ leads to a method of the family with order five. As the solution α is unknown, we can not use this value to fix the parameter and increase the order of convergence. Following the guideliness in [3], we can approximate the derivative of f in the solution as

$$f'(\alpha) \approx f[x_k, x_{k-1}] = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}},$$

so we propose the following approximation of the parameter, which varies on each iteration of the method:

$$\beta_k = -\frac{1}{f[x_k, x_{k-1}]}. \tag{2.3}$$

Let us note that with parameter β_k defined in (2.3) the number of different functional evaluations has not been increased, because $f(x_k)$ and $f(x_{k-1})$ are functional evaluations that were already being performed by the method at iterations k and $k-1$, respectively.

The replacement of (2.3) in the iterative structure (2.1) of $M4_\beta$ gives a method with memory, denoted $MM1$, that belongs to the family and also with higher order of convergence than the original family as Theorem 2.2 states.

Theorem 2.2 *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real sufficiently differentiable function in a convex set D and let $\alpha \in D$ be a simple root of $f(x) = 0$. Let us suppose that $H(\mu)$ satisfies $H(0) = H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. If x_0 is close enough to α, method $MM1$ converges to α with order of convergence*:

$$p = 2 + \sqrt{6} \approx 4.4495.$$

In addition to the approximation with memory considered, we could develop approximations using higher order interpolating polynomials. However, we have studied the resulting iterative family after adding a new step in the iterative scheme of family $M4_\beta$.

3. Derivative-free iterative family with three steps

Following the same iterative structure than family $M4_\beta$, we propose to add a step in order to accelerate the convergence. In this sense, we propose to replicate the last step of the family and add a new weight function G, so we obtain the following three-step family of iterative schemes:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[w_k, x_k]}, \\ z_k &= y_k - H(\mu_k)\frac{f(y_k)}{f[y_k, x_k]}, \qquad k = 0, 1, 2, \dots, \\ x_{k+1} &= z_k - G(\nu_k)\frac{f(z_k)}{f[z_k, y_k]} \end{aligned} \tag{3.1}$$

where $w_k = x_k + \beta f(x_k)$, $\beta \in \mathbb{R} - \{0\}$, and the weight function variables are $\mu = \frac{f(y)}{f(w)}$ and $\nu = \frac{f(z)}{f(y)}$. We denote the resulting three-step family as family $M6_\beta$. The analysis of its order if convergence is shown below.

Theorem 3.1 *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real sufficiently differentiable function in a convex set D and let $\alpha \in D$ be a simple root of $f(x) = 0$. Let us suppose that the weight functions $H(\mu)$ and $G(\nu)$ hold*:

- $H(0) = 1$, $H'(0) = 1$, $|H''(0)| < \infty$.
- $G(0) = 1$, $|G'(0)| < \infty$.

Then, if x_0 is close enough to α, the sequence $\{x_k\}$ generated by family $M6_\beta$ converges to α with order of convergence 6 for any value of $\beta \in \mathbb{R}$, $\beta \neq 0$, The error equation of the family is given by:

$$\begin{aligned} e_{k+1} = &-\frac{c_2}{4}(1 + \beta f'(\alpha))\left(-2c_3(1 + \beta f'(\alpha)) + c_2^2(6 + 4\beta f'(\alpha) - H_2)\right) \\ &\cdot \left(-2c_3(1 + \beta f'(\alpha))G_1 + c_2^2(-2 + 6G_1 + 2\beta f'(\alpha)(-1 + 2G_1) - G_1 H_2)\right) e_k^6 + O(e_k^7), \end{aligned} \tag{3.2}$$

where $G_1 = G'(0)$, $H_2 = H''(0)$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j \geq 2$. In addition, if we set $H_2 = 2$, the error equation (3.2) *turns into*

$$e_{k+1} = c_2(2c_2^2 - c_3)(c_2^2(1 - 2G_1) + c_3 G_1)(1 + \beta f'(\alpha))^3 e_k^6 + O(e_k^7). \tag{3.3}$$

In the same way as family $M4_\beta$, the term $1 + \beta f'(\alpha)$ appears in the error equation (3.3), so we can use the same approximation for the parameter in order to cancel the lower term in the error equation. Then, we replace the parameter $\beta_k = -\frac{1}{f[x_k, x_{k-1}]}$ in (3.1) obtaining a method of family $M6_\beta$. The resulting iterative scheme has been denoted $MM2$ and is a method with memory without additional functional evaluations. The improvement of the order of convergence is described in Theorem 3.2.

Theorem 3.2 *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real sufficiently differentiable function in a convex set D and $\alpha \in D$ a simple root of $f(x) = 0$. Let us suppose that $H(\mu)$ and $G(\eta)$ are real functions satisfying*:

- $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ *and* $|H'''(0)| < \infty$,
- $G(0) = 1$, $|G'(0)| < \infty$.

Then, if x_0 is close enough to α, method $MM2$ converges to α with order of convergence:

$$p = 3 + 2\sqrt{3} \approx 6.4641.$$

To compare different iterative methods from the point of view of their computational cost, Ostrowski [6] introduced the efficiency index $I = p^{1/d}$, where p is the order of the method and d is the number of funcional evaluations. Table 1 summarises the order and functional evaluations of the proposed methods and their efficiency index. We can see that all the methods improve the efficiency of Newtona and Steffensen's method. In turn, methods with memory $MM1$ and $MM2$ improve the efficiency with respect to the original families $M4_\beta$ and $M6_\beta$, respectively.

Method	p	d	I
Newton	2	2	1.4142
Steffensen	2	2	1.4142
$M4_\beta$	4	3	1.5847
$MM1$	4.4495	3	1.6448
$M6_\beta$	6	4	1.5650
$MM2$	6.4641	4	1.5945

Tab. 1 Efficiency indices

4. Numerical results

In this section, we perforn numerical experiments to test the features of the proposed methods. With this aim, they are used to solve different nonlinear problems. We also compare our methods with the classical iterative schemes of Newton and Steffensen.

All the methods require an initial estimation x_0 to the root of the nonlinear function. In addition, to check the numerical performance of families $M4_\beta$ and $M6_\beta$ we have used the weight functions:

$$H(\mu) = 1 + \mu + \mu^2,$$

$$G(\nu) = 1 + \nu + \nu^2,$$

such that they hold the convergence conditions stated in Theorems 2.2 and 3.2. The real parameter β has been set to $\beta = 1$, having Steffensen's method in the first step, and arbitrarily to $\beta = 5$.

The solution of the following nonlinear functions has been approximated:

- $f_1(x) = e^{-x} + 2\sin(x) - x + 3.5$, $\alpha \approx 3.273938$.
- $f_2(x) = \cos(x) - x$, $\alpha \approx 0.73908513$.
- $f_3(x) = (x-1)^3 - 1$, $\alpha = 2$.

In order to compare the theoretical order of convergence of the methods with their practical implementation, we use the approximated computational order of convergence, ACOC, introduced by the authors in [2] and defined by

$$ACOC = \frac{\ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}, \qquad k = 2, 3, \ldots$$

The numerical implementation has been done using Matlab R2018b with variable precision arithmetics of 2000 digits of mantissa. Tables 2, 3 and 4 show the results obtained for f_1, f_2 and f_3, respectively. For each method, we have shown the number of iterations, the difference between the two last iterations, the value of the function in the last iterate and the ACOC. Taking an initial estimation x_0, the iterative process stops when $|x_{k+1} - x_k| < 10^{-100}$ or $|f(x_{k+1})| < 10^{-100}$, with a maximum of 50 iterations.

We can observe in Tables 2 and 3 that the best results are given by the methods with memory $MM1$ and $MM2$. Both methods approximate the solution with high precision and the lowest number of iterations. In addition, methods belonging to families $M4_\beta$ and $M6_\beta$ are also competitive. In all cases the ACOC is near the theoretical order of convergence, being the higher value, as expected, in method $MM2$.

Finally, in Table 4 we can see an example where Steffensen's method and families $M4_\beta$ and $M6_\beta$ for $\beta = 1$ do not work properly. However, for $\beta = 5$ the performance is good and again methods with memory remain the most competitive.

x_0	Method	$iter$	$\lvert x_{k+1}-x_k\rvert$	$\lvert f(x_{k+1})\rvert$	ACOC
2	Newton	8	2.0845e–87	6.5561e–175	2.0000
	Steffensen	7	1.5294e–56	7.1309e–113	2.0000
	$M4_1$	5	2.3692e–66	2.1794e–264	4.0000
	$M4_5$	5	3.0444e–42	3.0444e–42	4.0041
	$MM1$	4	3.7619e–74	3.3998e–330	4.5071
	$M6_1$	4	4.8657e–61	2.0361e–364	6.0079
	$M6_5$	4	2.1005e–44	4.4813e–262	5.9678
	$MM2$	3	1.3534e–31	5.4909e–205	6.2994

Tab. 2 Numerical results for $f_1(x)$

x_0	Method	$iter$	$\lvert x_{k+1}-x_k\rvert$	$\lvert f(x_{k+1})\rvert$	ACOC
1	Newton	7	1.7955e–83	1.1913e–166	2.0000
	Steffensen	7	5.4267e–89	7.3307e–178	2.0000
	$M4_1$	4	2.4716e–74	1.0299e–296	4.0000
	$M4_5$	5	4.926e–67	1.9443e–265	4.0000
	$MM1$	3	5.6456e–30	7.4958e–133	4.0126
	$M6_1$	3	2.389e–41	4.0033e–247	6.0180
	$M6_5$	4	2.1274e–71	2.6129e–424	6.0041
	$MM2$	3	1.6466e–64	2.667e–416	6.0214

Tab. 3 Numerical results for $f_2(x)$

5. Conclusions

Two new derivative-free families of iterative methods have been introduced. The starting point has been an optimal two-step family with a real parameter and order four. After analyzing its order of convergence, the parameter has been approximated using two previous iterations, resulting in a method with memory with higher order of convergence than the original family and without additional functional evaluations. Then, we have extended the initial family to a three-step scheme with order six following a similar iterative estructure and the same real parameter. A new method with memory has been designed using the approximation of the parameter with memory as in the initial family and also improving the order of convergence. In both cases, the schemes not only improve the order but also the efficiency index with respect to the starting families. Finally, it has been verified that the theoretical analysis carried out in this work is consistent with the practical implementation of the methods. For this purpose, the proposed methods have been used to approximate roots of nonlinear test functions, obtaining the best results in the methods with memory.

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x_0	Method	$iter$	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	ACOC
1.5	Newton	10	1.7506e–90	9.1937e–180	2.0000
	Steffensen	nc			
	$M4_1$	40	6.8579e–32	1.7695e–123	3.9979
	$M4_5$	7	2.4187e–60	4.3803e–236	4.0000
	$MM1$	5	8.8702e–78	5.9533e–343	4.4425
	$M6_1$	nc			
	$M6_5$	9	1.1125e–27	2.5886e–158	5.7084
	$MM2$	4	1.3332e–100	1.162e–645	6.5487

Tab. 4 Numerical results for $f_3(x)$

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