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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Variable time-step modal methods to integrate the time-dependent neutron diffusion equation

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Abstract

The time-dependent neutron diffusion equation can describe the power evolution inside a nuclear reactor core. One approach to integrate this time-dependent equation is the modal method. This methodology is based on assuming that the solution can be decomposed as a finite sum of time-dependent amplitudes multiplied by shape functions (obtained by solving a partial eigenvalue problem), which are updated along the transient. In this work, different controls, that adapt the time-step according to the state of the transient, are implemented. Several benchmark problems show the competitiveness of the methodology.

1. Introduction

The evolution of the neutron power inside of reactor core can be described by the time-dependent multigroup neutron diffusion equation [11]. This equation is an approximation of the neutron transport equation that assumes that the neutron current is proportional to the gradient of the scalar neutron flux by means of a diffusion coefficient. For two energy-groups and without considering up-scattering, this model can be expressed as

$$\begin{aligned} \mathcal{V}\frac{d}{dt}\Phi + \mathcal{L}\Phi &= (1-\beta)\mathcal{F}\Phi + \sum_{k=1}^{K}\lambda_k^d \chi C_k, \\ \frac{d}{dt}C_k &= \beta_k \mathcal{F}_1\Phi - \lambda_k^d C_k, \qquad k = 1,\dots,K\,, \end{aligned} \tag{1.1}$$

where the time-dependent operators are

$$\begin{aligned} \mathcal{L} &= \begin{pmatrix} -\vec{\nabla}\cdot(D_1\vec{\nabla}) + \Sigma_{a_1} + \Sigma_{12} & 0 \\ -\Sigma_{12} & -\vec{\nabla}\cdot(D_2\vec{\nabla}) + \Sigma_{a_2} \end{pmatrix}, \\ \mathcal{F} &= \begin{pmatrix} \nu\Sigma_{f_1} & \nu\Sigma_{f_2} \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{F}_1 = \begin{pmatrix} \nu\Sigma_{f_1} & \nu\Sigma_{f_2} \end{pmatrix}, \\ \mathcal{V} &= \begin{pmatrix} 1/v_1 & 0 \\ 0 & 1/v_2 \end{pmatrix}, \qquad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \end{aligned} \tag{1.2}$$

In the previous expressions, D is the diffusion coefficient, Σ_a is the absorption cross-section, Σ_{12} is the scattering cross-section from the first group to the second group, Σ_f is the fission cross-section, ν is the average number of neutrons produced in each fission and v_g is the velocity of the neutrons. Subindex $g(=1,2)$ denotes the energy group. The first and the second group are known as fast and thermal group, respectively. Thus, Φ_1 and Φ_2 are the fast and thermal neutron fluxes. The concentration of delayed neutron precursors is represented by C_k, where subindex k denotes the delayed group k. β_k is the fraction of delayed neutrons that satisfies $\sum_{k=1}^{K}\beta_k = \beta$. λ_k^d is the neutron decay constant. All magnitudes and variables are, in general, time and space-dependent.

A Galerkin finite element method is applied for the spatial discretization of the neutron diffusion equation to obtain a semi-discrete system of ordinary differential equations [13]. Usually, this system of differential equations is stiff due to, among other things, the fast variation of the neutron flux and the presence of both prompt and delayed neutrons that leads to time scales of different orders of magnitude.

Several methodologies of different types have been studied to integrate this semi-discrete equation. First, one can use implicit differential schemes such as the backward differential method or higher order differential schemes [5, 14]. There are also many works that apply a quasi-static method that decomposes the solution as a product of two functions: an amplitude function that only depends on the time and a shape function that depends on space and time but its variation in time is assumed to be slow. These functions are approximated with two

different schemes that are coupled. In this work, to integrate the time-dependent neutron diffusion equation, a modal method is used. This approach assumes that the solution can be described as the sum of several amplitude functions multiplied by shape functions or modes. This expansion has a strong interest to approximate the solution for some types of transient problems, such as the ones defined from out-of-phase oscillations or local perturbations, where more than one shape function is necessary. The shape functions are obtained computing the eigenfunctions associated with the dominant eigenvalues (larger in magnitude) of the λ-modes problem

$$\mathcal{L}\phi_m = \frac{1}{\lambda_m}\mathcal{F}\phi_m\,, \tag{1.3}$$

where $\mathcal{L}$ and $\mathcal{F}$ are the operators defined in Equation (1.2) for a given configuration of the reactor. This generalized eigenvalue problem is obtained forcing the criticality of the system and describes the steady-state, which is the initial condition for the problem (1.1). Other possibilities for the shape functions have been studied, but they are not as efficient numerically as the λ-modes [4].

The shape functions can be fixed along the transient and one can use the modes associated with the static problem but this implies, in some transients, the necessity of using a high number of them in the expansion to obtain accurate results. To reduce the number of modes, Miró et al. proposed an updated modal method where the shape functions are updated at some time-steps [10]. In this last work, the time-step to update the modes was a fix value that had to be selected before beginning the computation. This strategy leads, in some cases, to use a too small time-step to assure the accuracy of the computations, which implies an unnecessary large computational cost. In this work, we propose some adaptive time-step controls that estimate an updating time-step depending on the transient analysed such that the results obtained are accurate enough with reasonable computational demands.

The rest of the paper is organized as follows. Section 2 briefly describes the finite element method used for the spatial discretization. Section 3 exposes the updated modal method. Section 4 presents the adaptive time-step controls analysed. Section 5 contains the numerical results obtained to test the proposed methodology. Finally, Section 6 collects the main conclusions of this work.

2. Spatial discretization. Finite element method

To approximate the solution, the differential system (1.1) is discretized. For the spatial discretization, a continuous Galerkin finite element method (FEM) is applied to obtain the semi-discrete system of differential equations (see details in [14])

$$\begin{aligned} V\frac{\mathrm{d}\tilde{\Phi}}{\mathrm{d}t} + L\tilde{\Phi} &= (1-\beta)F\tilde{\Phi} + \sum_{k=1}^{K}\lambda_k^d XC_k, \\ \frac{\mathrm{d}XC_k}{\mathrm{d}t} &= \beta_k F\tilde{\Phi} - \lambda_k^d XC_k, \qquad k = 1,\dots,K\,, \end{aligned} \tag{2.1}$$

where L, F, V and X are the matrices obtained from the spatial discretization of operators $\mathcal{L}$, $\mathcal{F}$, $\mathcal{V}$ and $\mathcal{X}$, respectively. Vectors $\tilde{\Phi}$ and C_k are the corresponding coefficients of Φ and C_k in terms of the Lagrange polynomials, which are the polynomials used in the finite element method. The FEM has been implemented by using the open source finite elements library Deal.II ([2]). Henceforth, the notation has been simplified by removing the tildes of the discrete operators from the original notation to the vectors Φ and ϕ.

For the λ-modes problem (1.3), the algebraic problem associated with the spatial discretization has the following structure

$$L\phi_m = \frac{1}{\lambda_m}F\phi_m, \tag{2.2}$$

where $\tilde{\phi}_m$ are the algebraic vectors of coefficients associated with the functions ϕ_m.

Associated with the λ-modes problem one can define the adjoint problem [6]

$$L^T\phi_l^\dagger = \frac{1}{\lambda_l}F^T\phi_l^\dagger, \tag{2.3}$$

where L^T and F^T are the matrices obtained from the spatial discretization of the adjoint operators $\mathcal{L}^\dagger$ and $\mathcal{F}^\dagger$. They also correspond to the transpose matrices of L and F. The solutions of the adjoint modes problem $\phi_l^\dagger$, $l = 1,\dots,q$ satisfy the biorthogonality condition

$$\langle\phi_l^\dagger, F\phi_m\rangle = \langle\phi_m^\dagger, F\phi_m\rangle\delta_{l,m}, \quad l,m = 1,\dots,q, \tag{2.4}$$

where $\langle\,,\rangle$ denotes the inner product for vectors and $\delta_{l,m}$ is the Kronecker's delta.

3. Time discretization. Updated modal method

From the finite element discretization, a semi-discrete system of ordinary differential equations is obtained that must be integrated over the time. In this work, the updated modal expansion is used for this purpose. This is a generalization of the traditional modal method that updates the eigenfunctions used in the expansions to avoid using a high number of modes [10].

To apply this method, the time domain is divided into several intervals $[t_i, t_i + \Delta t_i] = [t_i, t_{i+1}]$ and the neutron flux in this interval is decomposed in terms of q dominant λ-modes as

$$\Phi^i(\vec{r}, t) = \sum_{m=1}^{q} n_m^i(t)\phi_m^i(\vec{r}), \quad t \in [t_i, t_{i+1}], \tag{3.1}$$

where $\phi_m^i(\vec{r})$ is the unitary eigenvector associated with the m-th dominant eigenvalue of the λ-modes problem (1.3) associated with the configuration of the reactor in time $t = t_i$

$$L^i\phi_m^i = \frac{1}{\lambda_m^i}F^i\phi_m^i, \tag{3.2}$$

and $n_m^i(t)$ is the amplitude coefficient associated, that is only time-dependent. The matrices L^i and F^i correspond to the matrices L and F at time $t = t_i$.

Along the transient, it is assumed that only the magnitudes included in the operators L and F are time-dependent. The precursor data and the velocities are considered constant. The matrices L and F are expressed as

$$L(t) = L^i + \delta L^i(t), \qquad F(t) = F^i + \delta F^i(t), \quad t \in [t_i, t_{i+1}]. \tag{3.3}$$

The transient is initialized by forcing the criticality of the reactor dividing the fission cross-sections by λ_1^0 and using the steady-state neutron flux as initial condition.

To apply the modal methodology, the expressions (3.1) and (3.3) are substituted in the discretized neutron diffusion equation (2.1) and the resulting expression is then collapsed on the left by the adjoint λ-modes to obtain a system of $q(K + 1)$ ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{N}^i = \mathbf{T}^i\mathbf{N}^i, \tag{3.4}$$

where

$$\mathbf{N}^i = \begin{pmatrix} n_1^i \cdots n_q^i & c_{11}^i \cdots c_{q1}^i & \cdots & c_{1K}^i \cdots c_{qK}^i \end{pmatrix}^T, \tag{3.5}$$

$$\mathbf{T}^i = \left(\begin{array}{c|ccc} \Lambda_i^{-1}((1-\beta)I - [\lambda_i]^{-1} - \Delta L^i + (1-\beta)\Delta F^i) & \Lambda_i^{-1}\lambda_1^d & \cdots & \Lambda_i^{-1}\lambda_K^d \\ \hline \beta_1(I + \Delta F^i) & -\lambda_1^d I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_K(I + \Delta F^i) & 0 & \cdots & -\lambda_K^d I \end{array}\right), \tag{3.6}$$

and

$$\begin{aligned} &\Lambda_{lm}^i = \langle \phi_l^{\dagger,i}, V\phi_m^i\rangle, \qquad \Delta L_{lm}^i = \langle \phi_l^{\dagger,i}, \delta L^i\phi_m^i\rangle, \\ &\Delta F_{lm}^i = \langle \phi_l^{\dagger,i}, \delta F^i\phi_m^i\rangle, \quad c_{lk}^\lambda = \langle \phi_l^{\dagger,i}, XC_k\rangle. \end{aligned} \tag{3.7}$$

The matrix block $[\Lambda]_i$ is a diagonal matrix whose elements are the dominant λ-modes λ_m^i. The initial conditions at $t = 0$ are

$$\begin{aligned} &n_1^0(0) = 1, && n_m^0(0) = 0,\ m = 2, \ldots, q, \\ &c_{1k}^0(0) = \frac{\beta_k}{\lambda_k^d}\langle \phi_1^{\dagger,0}, F_0\phi_1^0\rangle, && c_{mk}^0(0) = 0,\ m = 2, \ldots, q, \quad k = 1, \ldots, K, \end{aligned} \tag{3.8}$$

with ϕ_1^0 and $\phi_1^{\dagger,0}$, the corresponding eigenvector and its adjoint associated with the dominant eigenvalue λ_1^0. That are obtained from the problem in the initial configuration.

The initial conditions at t_i to integrate the system in the interval $[t_i, t_{i+1}]$ must be defined to ensure the continuity of the solution. These initial conditions will be calculated from the solution in the previous interval $[t_{i-1}, t_i]$, the eigenvectors associated with direct modes (ϕ_m^i) and the adjoint modes ($\phi_l^{\dagger,i}$). Therefore, the computation of the solution in the interval $[t_i, t_{i+1}]$ uses the solution of the previous interval $[t_{i-1}, t_i]$.

First, the initial conditions for n_m^i in the interval $[t_i, t_{i+1}]$ are defined. The vector $\Phi(t_i)$ can be computed by using the expansion in the interval $[t_{i-1}, t_i]$ as

$$\Phi^{i-1}(t_i) = \sum_{m=1}^{q} n_m^{i-1}(t_i)\phi_m^{i-1}. \tag{3.9}$$

Assuming the continuity of $\Phi(t)$ on all its domain, that is $\Phi^{i-1}(t_i) = \Phi^i(t_i)$, and collapsing the expression (3.1) at $t = t_i$ by the adjoint modes it is obtained that the amplitude coefficients must be equal to

$$n_m^i(t_i) = \frac{\langle \phi_m^{\dagger,i}, F^i \Phi^{i-1}(t_i) \rangle}{\langle \phi_m^{\dagger,i}, F^i \phi_m^i \rangle}. \tag{3.10}$$

To compute the initial conditions for the concentration of the precursor k at time t_i, $c_{l,k}^i(t_i)$, the coefficients computed in the previous integration for $t \in [t_{i-1}, t_i]$ and the adjoint modes are used. It is supposed that

$$\phi_l^{\dagger,i} = \sum_{m=1}^{q} a_{lm} \phi_m^{\dagger,i-1}. \tag{3.11}$$

Using the biorthogonality relation of the adjoint λ-modes and Equation (3.11) it is obtained that

$$a_{lm} = \frac{\langle \phi_l^{\dagger,i}, F^{i-1} \phi_m^{i-1} \rangle}{\langle \phi_m^{\dagger,i-1}, F^{i-1} \phi_m^{i-1} \rangle}. \tag{3.12}$$

Therefore, the precursors coefficients at time t_i can be computed as

$$c_{l,k}^i(t_i) = \langle \phi_l^{\dagger,i}, XC_k \rangle(t_i) = \sum_{m=1}^{q} a_{lm} \langle \phi_m^{\dagger,i-1}, XC_k \rangle(t_i) = \sum_{m=1}^{q} a_{lm} c_{m,k}^{i-1,\lambda}(t_i). \tag{3.13}$$

Note that the system of ODEs (3.4) is much smaller than the system (2.1) if a moderate number of modes is used in expansion (3.1). In this work, this stiff system is integrated with a backward differentiation formula implemented in the CVODE solver from the SUNDIALS library [1, 7]. This code has implemented an adaptive time step and it is initialized with a time step of 10^{-3}s.

4. Adaptive time-step control

Previous works [10], update the modes with a fix time-step that is selected before beginning the computation. This implies the necessity of selecting a time-step that can lead to results with unpredictable errors. If a small time-step is used to reduce the errors, the computational cost also increases and this small time-step may be not necessary in some stages of the transient. Consequently, it is interesting to have an algorithm to adapt the time-step during the transient. To define it, two fundamental points must be studied: an error estimation and a control to select the time-step based on this error.

4.1. Estimation of local error

The error obtained for the modal expansion essentially comes from the assumption that the neutron flux can be described as a finite linear combination of the spatial modes, because the set of q modes do not form a complete basis of the function space. Therefore, larger spatial variations in the flux will imply larger errors in the modal method. We define several types of errors to estimate these spatial variations.

Modal difference error One way to estimate how the neutron flux will change in the interval $[t_i, t_{i+1}]$ is to compute the modes in the next time t_{i+1} and observe the differences between the modes at t_i and the modes at t_{i+1} as

$$\varepsilon_{md}^i = \max_{m=1,\dots,q} \frac{\|\phi_m^{i-1} - \phi_m^i\|_1}{\|\phi_m^{i-1}\|_1} k_{md}\,. \tag{4.1}$$

Modal residual error Other possibility to estimate this change is computing the residual error of the modes at t_i on the problem corresponding to the time t_{i+1} as

$$\varepsilon_{mr} = \max_{m=1,\dots,q} \frac{\|F^i \phi_m^{i-1} - \lambda_m^{i-1} L^i \phi_m^{i-1}\|_1}{\|\phi_m^{i-1}\|_1} k_{mr}\,.$$

Cross-section perturbation error In nuclear reactor systems the neutron flux shape change will depend on the variation in the cross-sections. Thus, we define an error

$$\varepsilon_{xs} = \sum_c \frac{\|\mathrm{XS}^{i-1}(c) - \mathrm{XS}^i(c)\|_1}{\|\mathrm{XS}^{i-1}(c)\|_1} k_{xs},$$

where c denotes the different cells of the spatial discretization of the reactor and XS is one type of cross-section that depends on the perturbation applied to generate the transient.

In the previous error estimations the constants k_{md}, k_{mr}, k_{xs}, are defined to adjust the accuracy of the approximation and their values will depend on the transient analyzed.

4.2. Time-step control

Two strategies are defined to compute the time-step from the error estimations. Both compute the new time-step Δt_i from the previous one Δt_{i-1} .

Banded control The first control computes this time-step in a fixed way as

$$\Delta t_i = \begin{cases} \Delta 2 t_{i-1}, & \varepsilon < 1.0, \\ \Delta t_{i-1}, & 1.0 < \varepsilon < 2.0, \\ \Delta t_{i-1} \,/\, 2, & 1.0 < \varepsilon, \end{cases} \tag{4.2}$$

where ε is one of the error estimations presented in Section 4.1.

Dynamic control It is based on control algorithms of other differential methods implemented for stiff problems [15]. In particular, the time-step Δt_i is computed as

$$\Delta t_i = \Delta t_{i-1} \min\{2.0, \max\{0.5, \sqrt{1.0/\varepsilon}\}\}, \tag{4.3}$$

where ε is some error defined in Section 4.1.

Finally, a minimum time-step and maximum time-step to avoid using very high or very small time-steps are used. These values are defined as $\Delta t_{\min} = \Delta t_0/2$, $\Delta t_{\max} = 50\Delta t_0$, where Δt_0 is the initial Δt.

5. Numerical results

The performance of the variable time-step updated modal methodology is tested using two type of reactor transients. In the finite element method, Lagrange polynomials of degree 3 are used because usually, this degree gives accurate results for usual reactor calculations [14].

The solution of the partial eigenvalue problems has been computed with a hybrid method by using a residual error of 10^{-7} (see more details in [3]). Moreover, for the implementation, a matrix-free technique is applied where the matrices of the system are not assembled. Matrix-vector products are computed 'on the fly' in a cell-based interface.

To analyzed the results, different relative errors for the neutron power are computed. The neutron power distribution, P, is defined as

$$\mathrm{P}(\vec{r}, t) = \Sigma_{f1}\Phi_1(\vec{r}, t) + \Sigma_{f2}\Phi_2(\vec{r}, t).$$

The Local Error (LE) at time t and Mean Power Error (MPE) in the interval $[t_0, t_N]$ are given by

$$\mathrm{LE}(t) = \frac{\|\mathrm{P}(t) - \mathrm{P}^{\mathrm{ref}}(t)\|_1}{\|\mathrm{P}^{\mathrm{ref}}(t)\|_1} \cdot 100, \qquad \mathrm{MPE} = \frac{1}{(t_N - t_0)} \sum_{n=1}^{N} \mathrm{LE}(t_n)(t_n - t_{n-1}),$$

where $\mathrm{P}^{\mathrm{ref}}(t)$ is the reference power at time t, that is computed from the solution obtained with a backward differential method (BKM) of first order and time-step of 0.001 s. [14].

This methodology has been implemented in C++ based on data structures provided by the library Deal.ii [2], PETSc [1]. It has been incorporated to the open-source neutronic code `FEMFFUSION`. It approximates the neutron diffusion equation and the steady-state $\mathrm{SP_N}$ equations by using a high order finite element method. The full description and the source code of `FEMFFUSION` is available in [12].

The computer used in the computations has been an Intel® Core™ i7-4790 @3.60GHz×8 processor with 32Gb of RAM running on Ubuntu GNU/Linux 18.04 LTS.

5.1. Langenbuch OOP transient

The Langenbuch reactor is a small LWR core with 77 fuel assemblies and two types of fuel [9]. The Langenbuch-OOP transient is defined from two local sinusoidal perturbations that are out-of-phase between them. They are expressed as,

$$\Sigma_{fg}(t) = \Sigma_{fg}(0) + \delta\Sigma_{fg}(t) \qquad g = 1, 2. \tag{5.1}$$

The perturbation 1 ($\mathrm{P_1}$) and the perturbation 2 ($\mathrm{P_1}$), marked in the Figure 1 with dash pattern, are given by

$$\delta\Sigma_{fg}^{P_1}(t) = 5 \cdot 10^{-4} \sin(2\pi t), \qquad \delta\Sigma_{fg}^{P_2}(t) = 5 \cdot 10^{-4} \sin(2\pi t + \pi), \qquad g = 1, 2. \tag{5.2}$$

This transient is followed during 2 s. The number of modes for the modal method has been set to $q = 3$ because the out-of-phase perturbations cannot be described with only one mode.

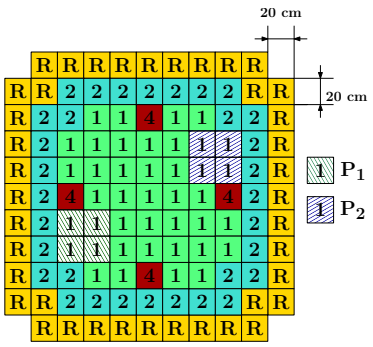
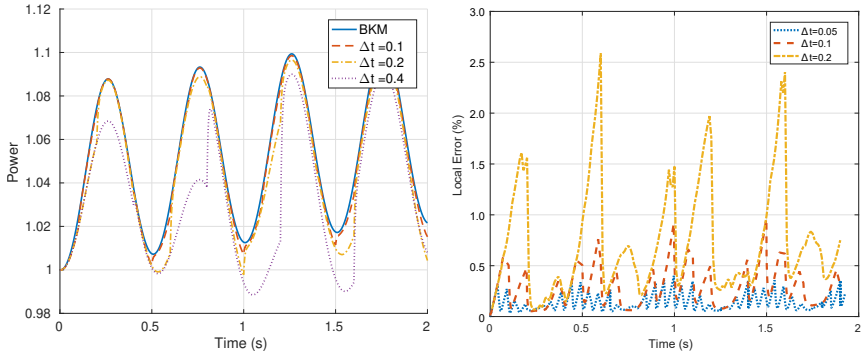


Fig. 1 Radial location of the perturbation areas for the Langenbuch-OPP transient.



(a) Relative power (b) Local Error

Fig. 2 Relative power and local error (%) obtained with the updated modal method with 3 modes for the Langenbuch-OPP transient.

Figure 2(a) displays the evolution of the relative global power computed with the BKM and the updated modal method with several fixed time-steps (Δt). Figure 2(b) displays the local error (LE). Large errors between the BKM and the updated modal method are produced when the perturbations reach their maximum value. However, these differences are reduced for smaller time-steps. Errors are not constant along the transient and the use of a control for the updating time-step is convenient to improve the efficiency of the method.

First, the adaptive control is analysed. The time-step to initialize the time-step control is set to $\Delta t_0 = 0.05$ s. The fission cross-section is used for the *cross-section perturbation error*. Table 1 shows Mean Power Errors and CPU times obtained by setting the different error estimations, control errors and accuracy coefficients k. The *modal difference error* (ε_{md}) is very expensive because it needs to compute the modes at the start of each time interval. The *cross-section perturbation error* (ε_{xs}) gives lower errors than the *modal residual error* (ε_{mr}), but by requiring more time. In the type of controls, the dynamic control gives similar approximations than the banded control, but also using more CPU time.

Tab. 1 Mean Power Errors (MPE) and CPU time obtained with the adaptive time-step modal method for the Langenbuch-OOP transient.

Type of Error	**Banded Control**		**Dynamic Control**	
k	**MPE (%)**	**CPU Time**	**MPE (%)**	**CPU Time**
ε_{md}				
1.0	3.901	3 min	3.748	5 min
2.0	2.902	22 min	1.681	14 min
5.0	1.641	32 min	1.519	20 min
ε_{mr}				
50	6.383	3 min	2.117	3 min
100	1.011	7 min	1.162	8 min
200	1.531	7 min	0.713	18 min
ε_{xs}				
0.5	1.338	4 min	0.809	7 min
1.0	0.647	11 min	0.656	12 min
2.0	0.617	13 min	0.607	13 min

Table 2 compares the mean power errors for the updated modal method with several fixed time-steps Δt and for the adaptive modal method with the *cross-section perturbation residual error*, *dynamic* control time-step and

$k_{xs} = 0.5$. The updated modal method with fix $\Delta t = 0.2$ s and the adaptive modal method uses similar CPU times, but the approximation obtained with the adaptive control is a 20 % more accurate.

Tab. 2 Errors and CPU time obtained to integrate the Langenbuch-OOP transient.

	BKM	**Updated Modal** $\Delta t = 0.2$ s	**Updated Modal** $\Delta t = 0.1$ s	**Updated Modal** $\Delta t = 0.05$ s	**Adaptive Modal**
MPE (%)		0.988	0.558	0.187	0.809
CPU Time (min)	195	7	12	30	7

5.2. AER-DYN-001 transient

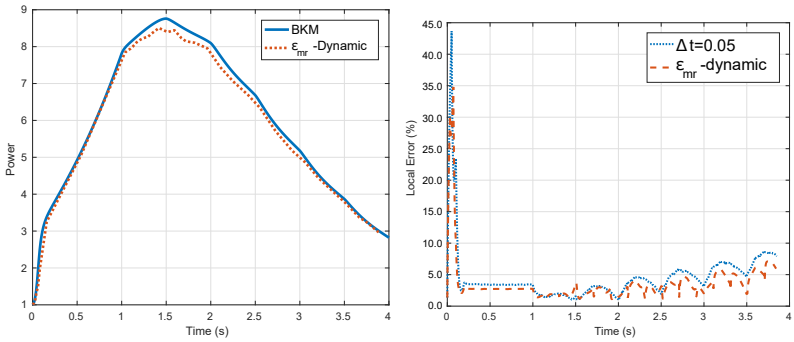
The AER-DYN-001 problem was introduced in [8]. It corresponds to an asymmetric control rod ejection accident without any feedback in a large hexagonal VVER440 reactor. The discretization of the reactor core is composed of 15 156 cells to have a system of 3 361 970 degrees of freedoms for a degree in the FEM equal to 3. Two dominant modes are used for the modal method because one eigenvalue gives non-accurate approximations [8].

Table 3 shows the mean power error (MPE) and the CPU time obtained with the updated modal method with fix time-steps and the adaptive control. Very small time-steps are necessary in the updated modal method to approximate accurately the drop out of the bar at the beginning of the transient, but then these small values are not necessary. Thus, different time-steps are interesting to be used along the transient. The adaptive updated modal method with modal residual error, dynamic control time-step and $k_{mr} = 100$ is applied. This solution has smaller mean power error than the rest of the solutions computed with the fixed updated modal method and this is also computed in less CPU time.

Tab. 3 Comparison of the BKM and the updated modal method for the AER-DYN-001 transient.

Method	**Δ*t*** (s)	**MPE** (%)	**CPU Time** (h)
BKM	0.01	-	89
Updated modal method	0.01	5.90	140
Updated modal method	0.05	4.60	38
Updated modal method	0.10	4.86	23
Adaptive updated modal method	-	3.59	17

Figure 3(a) shows the relative power obtained with the updated modal method with adaptive time step and the BKM. Figure 3(b) displays the local errors of the updated modal method using a fix time step $\Delta t = 0.05$ s and the adaptive update modal method. It observed that the adaptive modal method reduces the local error in the first times, but also reduces the local error beyond $t = 1$ s.



(a) Relative power (b) Local Error

Fig. 3 Relative power, local error (%) and time-step (Δt) obtained with the updated modal method and the adaptive updated modal method with 2 modes for the AER-DYN-001 transient.

6. Conclusions

An updated modal method with a variable time-step is proposed to integrate the neutron diffusion equation, where the updating time-step is selected in function on different types of errors and controls.

Numerical results show that the *modal residual error* and the *cross-section perturbation error* are good estimators to control the time-step update. However, the *modal difference error* has been shown computationally very expensive. In the type of controls, the dynamic control error is more adapted to the local errors, but there are not relevant differences between the dynamic and the banded time step control. Moreover, different coefficients k are defined to adjust the accuracy obtain in the different errors. Values of $k_{md} \approx 2.0$, $k_{mr} \approx 100$ and $k_{xs} \approx 0.5$ are recommended.

Finally, numerical results show that the time-step control for the updated modal method decreases the errors with similar or smaller CPU times than if the updated modal method is applied with fix time-step.

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