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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Fractionary iterative methods for solving nonlinear problems

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Abstract

In recent years, some point-to-point fractional Newton-type methods have been proposed to find roots of nonlinear equations using fractional derivatives. We present several Newton-type methods based on Caputo fractional derivative. For each case, the order of convergence of the proposed methods is stated, and some numerical tests are carried out in order to observe their performance, in practice. Convergence to different roots depending on the order of the derivative is observed and also differences among the methods in terms of the percentage of converging starting points.

1. Introduction

Leibnitz and L'Höpital created the concept of the semi-derivative at 1695, giving birth to fractional calculus. Also Riemann, Liouville and Euler were interested in this idea. From then, fractional calculus has evolved from theoretical aspects to the applications in many real world problems (see [2, 5, 8, 10, 12]): medicine, mechanical engineering, economics, ... In numerical analysis, we are focused in the area of research of iterative methods for solving nonlinear equations $f(x) = 0$. A large amount of these procedures are Newton-like, that is, they involve in their iterative expressions the evaluation of the nonlinear function f and its first derivative f' at each iterate. In this context, we question ourselves how would affect to the convergence order p of these schemes the replacement of integer derivatives by fractional ones. In particular, we introduce the Caputo fractional derivative, and study the convergence of these fractional methods. We would like to answer this and other questions for both point-to-point and multipoint schemes.

For the sake of completeness, we introduce in what follows some concepts about fractional derivatives and the series developments necessary to prove the convergence results.

1.1. Preliminary concepts

Now, we introduce some general concepts such as the Caputo fractional derivative [10, 11] and the fractional Taylor series [4, 9].

The first concept that we define is the Gamma function, as:

$$\Gamma(x) = \int_0^{+\infty} u^{x-1}e^{-u}du,$$

whenever $x > 0$. This function is a generalization of the factorial function to the complex plane, taking into account that $\Gamma(1) = 1$ and $\Gamma(n+1) = n!$, when $n \in \mathbb{N}$. As we will see in the following section, it appears in the iterative expressions of fractional iterative methods, being necessary for reaching the order of convergence of the iterative scheme.

Definition 1.1 (Caputo fractional derivative of order α) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an element of $C^{+\infty}([a, x])$ $(-\infty < a < x < +\infty)$, with $\alpha \geq 0$ and $n = [\alpha] + 1$, being $[\alpha]$ the integer part of α. Then, the Caputo fractional derivative of order α of $f(x)$ is defined as follows:

$$(cD_a^\alpha)f(x) = \begin{cases} \dfrac{1}{\Gamma(n-a)}\displaystyle\int_a^x \dfrac{d^n f(t)}{dt^n}\dfrac{dt}{(x-t)^{\alpha-n+1}}, & \alpha \notin \mathbb{N}, \\ \dfrac{d^{n-1}f(x)}{dx^{n-1}}, & \alpha = n-1 \in \mathbb{N}\cup\{0\}. \end{cases} \tag{1.1}$$

Moreover, to prove the order of convergence of the iterative fractional methods, a generalization of the classical Taylor series expansion of $f(x)$ around the zero of the nonlinear function, $\bar{x}$, is needed. Further on, this development also uses the Caputo fractional derivatives, see [9] (with $\rho = 1$).

Theorem 1.2 (*Taylor series expansion by using Caputo fractional derivatives* [9]) *Let us suppose that* $cD_a^{j\alpha} f(x) \in C([a,b])$*, for* $j = 1, 2, \ldots, n+1$*, where* $\alpha \in (0,1]$*, then we have*

$$f(x) = \sum_{i=0}^{n} cD_a^{i\alpha} f(a) \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} + cD_a^{(n+1)\alpha} f(\xi) \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}, \tag{1.2}$$

with $a \le \xi \le x$*, for all* $x \in (a,b]$ *where* $cD_a^{n\alpha} = cD_a^{\alpha} \cdot cD_a^{\alpha} \cdots cD_a^{\alpha}$ *(n times composition).*

We develop the work in the following order. In Section 2 we state our iterative methods and expose their convergence results. In the next section, numerical tests are performed, paying special attention to the convergence rates and the roots the methods converge to. Finally, expose the conclusions obtained and some open questions.

2. Iterative methods designed by using fractional derivatives

In this section, we introduce high-order one-point and multi-point fractional iterative methods based on the methods of Newton and Traub methods, stating the conditions that must be assued in order to achieve their order of convergence, which depend on the order of the fractional derivative.

Theorem 2.1 ([1]) *Let* $f : D \subset \mathbb{R} \to \mathbb{R}$ *be a continuous function whose fractional derivatives of order* $k\alpha$ *are defined for any positive integer* k *and any* α*,* $0 < \alpha < 1$*, on the interval* D *containing the zero* $\bar{x}$ *of* $f(x)$ *and let the fractional derivatives of Caputo type,* $cD_a^{\alpha} f(x)$*, be continuous and non-singular at* $\bar{x}$*. Also, let us suppose that* x_0 *is an initial approximation close enough to* $\bar{x}$*. Then the order of local convergence of Newton's fractional method*

$$x_{k+1} = x_k - \Gamma(\alpha+1) \frac{f(x_k)}{cD_a^{\alpha} f(x_k)}, \; k = 0, 1, \ldots, \tag{2.1}$$

of Caputo type is at least 2α*, where* $0 < \alpha \le 1$*, with the error equation*

$$e_{k+1} = \frac{\Gamma(2\alpha+1) - (\Gamma(\alpha+1))^2}{(\Gamma(\alpha+1))^3} C_2 e_k^{2\alpha} + O[e_k^{3\alpha}).$$

We denote the iterative method (2.1) as CFN1. However, another kind of fractional iterative method can be designed, fixing the order of convergence to be at least one, as can be seen in the following result.

Theorem 2.2 ([3]) *Let* $f : D \subset \mathbb{R} \to \mathbb{R}$ *be a continuous function with fractional derivatives of order* $k\alpha$ *defined for any positive integer* k *and* $\alpha \in (0,1]$ *defined on the open interval* D *containing the zero* $\bar{x}$ *of* $f(x)$*. Additionally, let us suppose that* $cD_a^{\alpha} f(x)$ *is continuous and not zero at* $\bar{x}$*. Then, the order of convergence of the Caputo type fractional Newton method with iterative scheme*

$$x_{k+1} = x_k - \left(\Gamma(\alpha+1) \frac{f(x_k)}{cD_a^{\alpha} f(x_k)} \right)^{1/\alpha}, \; k = 0, 1, 2, \ldots \tag{2.2}$$

(denoted by CFN2) is at least $\alpha + 1$*, and its error equation is*

$$e_{k+1} = \frac{\Gamma(2\alpha+1) - (\Gamma(\alpha+1))^2}{\alpha(\Gamma(\alpha+1))^2} C_2 e_k^{\alpha+1} + O[e_k^{2\alpha+1}].$$

On the other hand, higher-order iterative schemes can be designed, following this structure. A Traub-type fractional-order method can be defined also by means of Caputo derivatives. In the following result, the convergence conditions and its fractional order of convergence are stated.

Theorem 2.3 ([3]) *Let* $f : D \subset \mathbb{R} \to \mathbb{R}$ *be a the continuous function with fractional derivatives of order* $k\alpha$*, for any positive integer* k *and* $\alpha \in (0,1]$*, in the open interval* D *holding the zero of* $f(x)$*, denoted by* $\bar{x}$*. Let us suppose* $cD_a^{\alpha} f(x)$ *is continuous and not null at* $\bar{x}$*. Additionally, let us consider an initial estimation* x_0*, close enough to* $\bar{x}$*. Therefore, the local convergence order of CFT method with iterative expression*

$$x_{k+1} = y_k - \left(\Gamma(\alpha+1) \frac{f(y_k)}{cD_a^{\alpha} f(x_k)} \right)^{1/\alpha}, \; k = 0, 1, \ldots, \tag{2.3}$$

where y_k *is obtained using* (2.2)*, being* $\alpha^2 + 2\alpha + 1 < 3\alpha + 1$*, is at least* $2\alpha + 1$*. Its error equation is*

$$e_{k+1} = -\frac{\Gamma(2\alpha+1)}{\alpha^2(\Gamma(\alpha+1))^2} \frac{(\Gamma(\alpha+1))^2 - \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} C_2^2 e_k^{2\alpha+1} + O[e_k^{\alpha^2+2\alpha+1}].$$

3. Numerical results

In the following section, the numerical performance of these schemes is tested. We are going to test a nonlinear function in order to make a comparison between the presented methods. It is important to point out that in any case a comparison is being made with the classical methods (when $\alpha = 1$).

To get these results, we have used Matlab R2018b with double precision arithmetics, $|x_{k+1} - x_k| < 10^{-8}$ or $|f(x_{k+1})| < 10^{-8}$ as stopping criteria, and a maximum of 500 iterations. For the calculation of the gamma function, $\Gamma(x)$, we use the program presented in [6], where gamma function is calculated with 15 digits of accuracy along the real axis and 13 elsewhere in $\mathbb{C}$. Moreover, in all the numerical tests, we used $a = 0$.

Our test function is $f(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.21$ with roots $\bar{x}_1 = 0.82366 + 0.24769i$, $\bar{x}_2 = 0.82366 - 0.24769i$, $\bar{x}_3 = -2.62297$, $\bar{x}_4 = -0.584$, $\bar{x}_5 = -0.21705 + 0.99911i$ and $\bar{x}_6 = -0.21705 - 0.99911i$.

We observe that Newton-type methods (Table 1) with Caputo derivative, for the same value of x_0 and the same values of α, converge to the different roots in more iterations than fractional Traub's methods. It also can be observed that Newton- and Traub-type schemes require approximately the same values of α to converge. Also, it has been observed in practice that, for wide ranges of initial guesses, the same x_0 defines a sequence converging to different roots of the nonlinear function depending on the value of α.

	CFN1 method				CFN2 method			
α	$\bar{x}$	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	iter	$\bar{x}$	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	iter
0.6	-	0.29821	28.343	500	-	1.7603e-07	0.0035619	500
0.65	-	0.17488	11.329	500	-	4.1154e-08	6.7515e-04	500
0.7	-	0.058499	2.98929	500	$\bar{x}_4$	9.9926e-09	1.1322e-04	432
0.75	$\bar{x}_4$	9.6537e-09	4.1645e-07	151	$\bar{x}_4$	9.8524e-09	4.6756e-05	230
0.8	$\bar{x}_4$	8.5475e-09	3.0465e-07	50	$\bar{x}_4$	9.6579e-09	1.8943e-05	124
0.85	$\bar{x}_4$	9.468e-09	2.606e-07	28	$\bar{x}_4$	9.9396e-09	7.7541e-06	67
0.9	$\bar{x}_4$	3.9203e-09	7.3851e-08	19	$\bar{x}_4$	9.109e-09	2.6706e-06	37
0.95	$\bar{x}_4$	2.5822e-09	2.4894e-08	13	$\bar{x}_4$	7.3622e-09	6.4461e-07	20
1	$\bar{x}_4$	3.0876e-06	8.8694e-10	6	$\bar{x}_4$	3.0876e-06	8.8694e-10	6

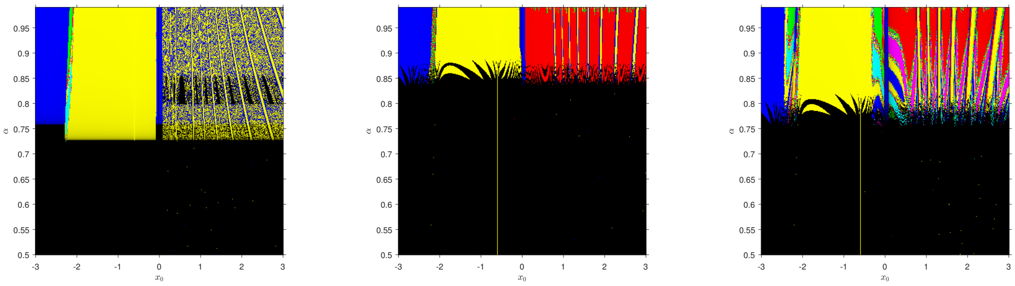
Tab. 1 Fractional Newton results for $f(x)$ with Caputo derivative and initial estimation $x_0 = -1.5$

	CFT method			
α	$\bar{x}$	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	iter
0.6	-	6.2898e-08	0.0012681	500
0.65	-	1.1562e-08	1.8867e-04	500
0.7	$\bar{x}_4$	9.9588e-09	6.9453e-05	268
0.75	$\bar{x}_4$	9.9889e-09	2.7995e-05	138
0.8	$\bar{x}_4$	9.5606e-09	1.0693e-05	73
0.85	$\bar{x}_4$	9.4657e-09	4.0225e-06	39
0.9	$\bar{x}_4$	6.8084e-09	1.0286e-06	22
0.95	$\bar{x}_4$	5.2078e-09	1.8928e-07	12
1	$\bar{x}_4$	2.2023e-10	5.329e-15	5

Tab. 2 Fractional Traub results for $f(x)$ with Caputo derivative and initial estimation $x_0 = -1.5$

Now, we are going to analyze the dependence on the initial estimation of Newton- and Traub-type methods by using convergence planes defined in [7]. In them (see, for example, Figure 1a) the abscissa axis corresponds to the starting guess and the fractional index α appears in the ordinate axis. A mesh of 400×400 points is used. Points that are not painted in black color correspond to those pairs of initial estimations and values of α that converge to one of the roots with a tolerance of 10^{-3}. Different colors mean convergence to different roots. Therefore, when a point is painted in black, this shows that no root is found in a maximum of 500 iterations. Moreover, for all convergence planes, the percentage of convergent pairs (x_0, α) is calculated, in order to compare the performance of the methods.

In Figure 1, we can see that CFN1 and CFT methods have higher percentage of convergence than CFN2. We can also see, that there are intervals for x_0 such that the same fractional iterative method with different values of the order of the fractional derivative can lead us to converge to different roots. It can be useful in order to find all the roots of a function with few computational effort.



(a) CFN1, 49.54% convergence

(b) CFN2, 27.15% convergenc

(c) CFT, 41.5% convergence

Fig. 1 Convergence planes of proposed methods on $f(x)$ with $-3 \leq x_0 \leq 3$

4. Concluding remarks

Fractional Newton- and Traub-type schemes have been designed by using Caputo derivatives. The convergence properties of these procedures imply always (at least) linear convergence, reaching order 2α, $1+\alpha$ and $1+2\alpha$, respectively. Some numerical tests have been done, and the dependence on the initial estimation has been observed.

It can be concluded that Traub-type procedures can improve Newton-type ones, not only because they require fewer iterations, higher or similar percentages of convergence. Moreover, the test made have shown that, for some problems, the methods using fractional derivatives reach different solutions with the same initial estimations.

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