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Editors:
Rafael Gallego, Mariano Mateos

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Foreword

It is with great pleasure that we present the Proceedings of the 26$^{\text{th}}$ Congress of Differential Equations and Applications / 16$^{\text{th}}$ Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier “Pancho” Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: “a mathematician is a device for turning coffee into theorems”. Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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A semi-implicit Lagrange-projection-type finite volume scheme exactly well-balanced for 1D shallow-water system

C. Caballero-Cárdenas[1], **M. J. Castro**[1], **T. Morales de Luna**[2], **M. L. Muñoz-Ruiz**[1]
1. Universidad de Málaga, Spain
2. Universidad de Córdoba, Spain

Abstract

In this work we present a numerical approximation of the shallow water equations based on a Lagrange-projection-type finite volume scheme. For the Lagrangian step we propose two different versions of the scheme, two explicit and other two semi-implicit that ensure first and second order of accuracy. The projection on the Eulerian coordinates will always be done explicitly, preserving the total order of the scheme. Special care is done for ensuring the well-balanced properties of the scheme. Several numerical experiments are included in order to illustrate the good behavior of the proposed schemes.

1. Introduction

Let us consider the shallow water equations (SWE), given by

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\dfrac{h^2}{2}\right) = -gh\partial_x z, \end{cases} \tag{1.1}$$

where $z(x)$ denotes a given smooth topography and $g > 0$ is the gravity constant. The primitive variables are the water depth $h \geq 0$ and its velocity u, which both depend on the space and time variables, respectively, $x \in \mathbb{R}$ and $t \in [0, \infty)$. We assume that the initial water depth $h(x,0) = h_0(x)$ and velocity $u(x,0) = u_0(x)$ are given.

The use of Lagrangian coordinates allows to describe the flow by following the fluid motion. With this in mind, for any given "fluid particle", ξ, we consider the characteristic curves

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = u(x(\xi,t),t), \\ x(\xi, 0) = \xi, \end{cases}$$

and given any function $(x,t) \mapsto \mathbf{U}(x,t)$ in Eulerian coordinates, we denote by

$$\overline{\mathbf{U}}(\xi,t) = \mathbf{U}(x(\xi,t),t)$$

its counterpart in Lagrangian coordinates.

Moreover, we define

$$L(\xi,t) = \frac{\partial x}{\partial \xi}(\xi,t),$$

which implies that $\partial_t L(\xi,t) = \partial_\xi \overline{u}(\xi,t)$.

Since system (1.1) can be written for smooth solutions as

$$\begin{cases} \partial_t h + u\partial_x h + h\partial_x u = 0, \\ \partial_t(hu) + u\partial_x(hu) + hu\partial_x u + \partial_x\left(g\dfrac{h^2}{2}\right) = -gh\partial_x z, \end{cases}$$

after multiplying both equations by $L(\xi,t)$ and setting $p = gh^2/2$ we obtain

$$\begin{cases} \partial_t(L\overline{h}) = 0, \\ \partial_t(L\overline{hu}) + \partial_\xi \overline{p} + g\overline{h}\partial_\xi \overline{z} = 0. \end{cases} \tag{1.2}$$

The Lagrangian-projection scheme can be interpreted as a two-step algorithm consisting in first solving the system in Lagrangian coordinates (1.2), which is known as the Lagrangian step, and then projecting the results in Eulerian coordinates, which is known as the Projection step. See [9] for more details. This strategy allows us to decouple the acoustic and transport phenomena and to design a natural implicit-explicit and large time steps could be considered with a CFL restriction based on the slower transport waves and not on the acoustic ones. We address the reader to [5–8] for further details. In this work we follow the strategy described in [3, 10] for the definition of the LP scheme and [4] to ensure its well-balanced character. Concerning the well-balanced property, we refer the reader to [1] for a different approach, and to [2, 4] and the references therein, for a review on this topic.

2. The Lagrange-projection numerical algorithm

Space and time will be discretized using a space step Δx and a time step Δt in a set of cells $[x_{i-1/2}, x_{i+1/2}]$ and instants $t^n = n\Delta t$, for $\in \mathbb{Z}$, $n \in \mathbb{N}$. We consider Δx and Δt constants for simplicity. We define $x_{i+1/2} = i\Delta x$ and $x_i = (x_{i-1/2} + x_{i+1/2})/2$, the cell interfaces and the cell centers, respectively. We consider for the variable ξ the same space discretization as for x, that is, $\Delta\xi = \Delta x$, $\xi_{i+1/2} = x_{i+1/2}$ and $\xi_i = x_i$ for all $i \in \mathbb{Z}$.

Let $\mathbf{U} = (h, hu)^t$. For a given initial condition $x \mapsto \mathbf{U}^0(x)$, we will consider a discrete initial condition $\mathbf{U}_i^0$, which approximates $\dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}^0(x)dx$, for $i \in \mathbb{Z}$. The proposed algorithm aims at computing an approximation $\mathbf{U}_i^n$ of

$$\mathbf{U}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x, t^n)dx,$$

where $x \mapsto \mathbf{U}(x, t^n)$ is the exact solution of (1.1) at time t^n, $n \in \mathbb{N}$. Given the sequence $\{\mathbf{U}_i^n\}_i$, it is a matter of defining the sequence $\{\mathbf{U}_i^{n+1}\}_i$, $n \in \mathbb{N}$, since $\{\mathbf{U}_i^0\}_i$ is assumed to be known.

Using these notation, the overall Lagrange-projection algorithm can be described as follows: for a given discrete state $\mathbf{U}_i^n = (h, hu)_i^n, i \in \mathbb{Z}$, that describes the system at instant t^n, the computation of the approximation $\mathbf{U}_i^{n+1} = (h, hu)_i^{n+1}$ at the next time level is a two-step process defined by

1. update $\mathbf{U}_i^n$ to $\overline{\mathbf{U}}_i^{n+1}$ by approximating the solution of (1.2);
2. update $\overline{\mathbf{U}}_i^{n+1}$ to $\mathbf{U}_i^{n+1}$ by projecting the solution back to the Eulerian coordinates.

3. The Lagrangian step

Concerning the Lagrangian step, we will approximate the solution of (1.2) in two ways: explicitly and implicitly.

For simplicity, from now on we will focus on the flat topography case, which corresponds to $\partial_x z = 0$. The non-flat case can be carried out by adapting the schemes presented here as proposed in [3, 4, 10]. A detailed description of the well-balanced schemes will be done during the presentation.

Following [3, 8], we will now consider a relaxation approach of the Lagrangian formulation:

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \Pi = \lambda(p - \Pi), \\ \partial_t \Pi + a^2 \partial_m u = 0, \end{cases} \tag{3.1}$$

where $\tau = 1/h$, $\tau\partial_x = \partial_m$, a is a constant satisfying the subcharacteristic condition $a > h\sqrt{gh}$ and $\lambda \to \infty$.

From a numerical point of view, the strategy consists in first solving (3.1) with $\lambda = 0$, that is,

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \Pi = 0, \\ \partial_t \Pi + a^2 \partial_m u = 0, \end{cases} \tag{3.2}$$

and then setting $\Pi = p(\tau)$ before going to the following iteration.

Defining the two new variables $\overrightarrow{w} = \Pi + au$ and $\overleftarrow{w} = \Pi - au$, system (3.2) can be written as

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t \overrightarrow{w} + a\partial_m \overrightarrow{w} = 0, \\ \partial_t \overleftarrow{w} - a\partial_m \overleftarrow{w} = 0. \end{cases} \tag{3.3}$$

In what follows, we present a first order fully explicit and semi-implicit scheme and their extensions to second order. The second order extension is performed using a second order reconstruction operator combined with a suitable second order time integration.

3.1. Explicit Lagrangian schemes

First order explicit Lagrangian scheme

The Lagrangian step for the first order explicit scheme can be written as (see [3] for details)

$$\begin{cases} (L\overline{h})_i^{n+1} = h_i^n, \\ (L\overline{hu})_i^{n+1} = (hu)_i^n - \dfrac{\Delta t}{\Delta x}(\pi^*_{i+1/2} - \pi^*_{i-1/2}), \end{cases}$$

where

$$L_i^{n+1} = 1 + \frac{\Delta t}{\Delta x}(u^*_{i+1/2} - u^*_{i-1/2})$$

and

$$\pi^*_{i+1/2} = \frac{\overrightarrow{w}_i^n + \overleftarrow{w}_{i+1}^n}{2} = \frac{\pi_i^n + \pi_{i+1}^n}{2} - \frac{a}{2}(u_{i+1}^n - u_i^n),$$
$$u^*_{i+1/2} = \frac{\overrightarrow{w}_i^n - \overleftarrow{w}_{i+1}^n}{2a} = \frac{u_i^n + u_{i+1}^n}{2} - \frac{1}{2a}(\pi_{i+1}^n - \pi_i^n).$$

Second order explicit Lagrangian scheme
The second order explicit scheme, after using a MUSCL-Hancock reconstruction, can be described again as

$$\begin{cases} (L\overline{h})_i^{n+1} = h_i^n, \\ (L\overline{hu})_i^{n+1} = (hu)_i^n - \dfrac{\Delta t}{\Delta x}(\pi^*_{i+1/2} - \pi^*_{i-1/2}), \end{cases}$$

with

$$L_i^{n+1} = 1 + \frac{\Delta t}{\Delta x}(u^*_{i+1/2} - u^*_{i-1/2}),$$

where now

$$\pi^*_{i+1/2} = \frac{\overrightarrow{w}_{i+1/2-}^{n+1/2} + \overleftarrow{w}_{i+1/2+}^{n+1/2}}{2},$$
$$u^*_{i+1/2} = \frac{\overrightarrow{w}_{i+1/2-}^{n+1/2} - \overleftarrow{w}_{i+1/2+}^{n+1/2}}{2a},$$

and

$$\overrightarrow{w}_{i+1/2-}^{n+1/2} = \overrightarrow{w}_i^n + \frac{\Delta x}{2}\delta\overrightarrow{w}_i^n - \frac{a\Delta t}{2h_i^n}\delta\overrightarrow{w}_i^n,$$
$$\overleftarrow{w}_{i+1/2+}^{n+1/2} = \overleftarrow{w}_{i+1}^n - \frac{\Delta x}{2}\delta\overleftarrow{w}_{i+1}^n + \frac{a\Delta t}{2h_{i+1}^n}\delta\overleftarrow{w}_{i+1}^n.$$

In the previous expression, $\delta\overrightarrow{w}_i^n$ and $\delta\overleftarrow{w}_{i+1}^n$ are approximations of the space derivative of $\overrightarrow{w}(x_i, t^n)$ and $\overleftarrow{w}(x_{i+1}, t^n)$ respectively, that are computed by means of a limiter that avoids the appearance of spurious oscillations in the presence of discontinuities. In this work we use:

$$\delta\overrightarrow{w}_i^n = \phi_L(d_{i,l}, d_{i,r})d_{i,l} + \phi_R(d_{i,l}, d_{i,r})d_{i,r} \tag{3.4}$$

where

$$d_{i,l} = \frac{\overrightarrow{w}_i^n - \overrightarrow{w}_{i-1}^n}{\Delta x},\; d_{i,r} = \frac{\overrightarrow{w}_{i+1}^n - \overrightarrow{w}_i^n}{\Delta x}.$$

and

$$\phi_L(a,b) = \begin{cases} \dfrac{|b|}{|a|+|b|}, \text{ if } |a|+|b| > 0, \\ 0, \text{ otherwise.} \end{cases} \text{ and } \phi_R(a,b) = \begin{cases} \dfrac{|a|}{|a|+|b|}, \text{ if } |a|+|b| > 0, \\ 0 \text{ otherwise.} \end{cases}$$

3.2. Implicit Lagrangian schemes

First order implicit Lagrangian scheme
In this section we present the first order implicit scheme for the Lagrangian step. Note, that formally, the explicit and the implicit schemes have a similar expression, but now, $\pi^*_{i+1/2}$ and $u^*_{i+1/2}$ have to be evaluated at time $t = t^{n+1}$:

$$\begin{cases} (L\overline{h})_i^{n+1} = h_i^n, \\ (L\overline{hu})_i^{n+1} = (hu)_i^n - \dfrac{\Delta t}{\Delta x}(\pi^*_{i+1/2} - \pi^*_{i-1/2}), \end{cases}$$

with

$$L_i^{n+1} = 1 + \frac{\Delta t}{\Delta x}(u^*_{i+1/2} - u^*_{i-1/2}),$$

where now

$$\pi^*_{i+1/2} = \frac{\overrightarrow{w}_i^{n+1} + \overleftarrow{w}_{i+1}^{n+1}}{2},$$
$$u^*_{i+1/2} = \frac{\overrightarrow{w}_i^{n+1} - \overleftarrow{w}_{i+1}^{n+1}}{2a},$$

and $\overrightarrow{w}_i^{n+1}$, $\overleftarrow{w}_i^{n+1}$ are the solutions of

$$\overrightarrow{w}_i^{n+1} = \overrightarrow{w}_i^{n} - \frac{a\Delta t}{h_i^n \Delta x}(\overrightarrow{w}_i^{n+1} - \overrightarrow{w}_{i-1}^{n+1}), \tag{3.5}$$

$$\overleftarrow{w}_i^{n+1} = \overleftarrow{w}_i^{n} + \frac{a\Delta t}{h_i^n \Delta x}(\overleftarrow{w}_{i+1}^{n+1} - \overleftarrow{w}_{i}^{n+1}), \tag{3.6}$$

Note that due to the special form of (3.5) and (3.6), $\overrightarrow{w}_i^{n+1}$ and $\overleftarrow{w}_i^{n+1}$ can be computed in a very simple way.

Second order implicit Lagrangian scheme

The second order implicit scheme is defined combining the second order Adams-Moulton scheme for the time integration, and a second order reconstruction procedure. The resulting scheme reads as follows

$$\begin{cases} (L\overline{h})_i^{n+1} = h_i^n, \\ (L\overline{hu})_i^{n+1} = (hu)_i^n - \dfrac{\Delta t}{2\Delta x}\left(\pi^{*,n}_{i+1/2} - \pi^{*,n}_{i-1/2} + \pi^{*,n+1}_{i+1/2} - \pi^{*,n+1}_{i-1/2}\right), \end{cases}$$

with

$$L_i^{n+1} = 1 + \frac{\Delta t}{2\Delta x}\left(u^{*,n}_{i+1/2} - u^{*,n}_{i-1/2} + u^{*,n+1}_{i+1/2} - u^{*,n+1}_{i-1/2}\right),$$

where

$$\pi^{*,\#}_{i+1/2} = \frac{\overrightarrow{w}^{\#}_{i+1/2-} + \overleftarrow{w}^{\#}_{i+1/2+}}{2},$$
$$u^{*,\#}_{i+1/2} = \frac{\overrightarrow{w}^{\#}_{i+1/2-} - \overleftarrow{w}^{\#}_{i+1/2+}}{2a},$$

where # stands for n or $n+1$. The space reconstruction is performed as follows

$$\overrightarrow{w}^{\#}_{i+1/2\pm} = \overrightarrow{w}^{\#}_i \mp \frac{\Delta x}{2}\left(\phi_L(d^n_{i,l}, d^n_{i,r})d^{\#}_{i,l} + \phi_R(d^n_{i,l}, d^n_{i,r})d^{\#}_{i,r}\right).$$

We can define $\overleftarrow{w}^{\#}_{i+1/2\pm}$ in a similar way. Note that the limiters are always evaluated at time t^n, therefore, the systems that define $\overrightarrow{w}_i^{n+1}$ and $\overleftarrow{w}_i^{n+1}$, respectively, remain linear and have the following form:

$$\overrightarrow{w}_i^{n+1} = \overrightarrow{w}_i^{n} - \frac{a\Delta t}{2h_i^n\Delta x}\left(\overrightarrow{w}^{n}_{i+1/2-} - \overrightarrow{w}^{n}_{i-1/2-} + \overrightarrow{w}^{n+1}_{i+1/2-} - \overrightarrow{w}^{n+1}_{i-1/2-}\right),$$
$$\overleftarrow{w}_i^{n+1} = \overleftarrow{w}_i^{n} + \frac{a\Delta t}{2h_i^n\Delta x}\left(\overleftarrow{w}^{n}_{i+1/2+} - \overleftarrow{w}^{n}_{i-1/2+} + \overleftarrow{w}^{n+1}_{i+1/2+} - \overleftarrow{w}^{n+1}_{i-1/2+}\right).$$

4. The projection step

Once the system in Lagrangian coordinates is solved, the result has to be projected in Eulerian coordinates. This step will always be done explicitly.

For doing the projection of $L\overline{U}(\xi, t)$ on the Eulerian cells $(x_{i-1/2}, x_{i+1/2})$, we need to compute

$$\mathbf{U}_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x,t)dx.$$

Given $t \geq 0$ we define $\hat{\xi}_{i+1/2}(t)$ such that

$$x(\hat{\xi}_{i+1/2}(t), t) = x_{i+1/2}.$$

Thus, for any time $T \geq 0$, $\hat{\xi}_{i+1/2}(T)$ corresponds to the origin of the characteristic $x(\hat{\xi}_{i+1/2}, t)$ such that at time $t = T$ coincides with $x_{i+1/2}$.

Using this notation, we can write

$$\mathbf{U}_i(t) = \frac{1}{\Delta x}\int_{x(\hat{\xi}_{i-1/2}(t),t)}^{x(\hat{\xi}_{i+1/2}(t),t)} \mathbf{U}(x,t)dx = \frac{1}{\Delta x}\int_{\hat{\xi}_{i-1/2}(t)}^{\hat{\xi}_{i+1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi,$$

and we can split the integral as follows

$$\mathbf{U}_i(t) = \frac{1}{\Delta x}\int_{\hat{\xi}_{i-1/2}(t)}^{\xi_{i-1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi + \frac{1}{\Delta x}\int_{\xi_{i-1/2}(t)}^{\xi_{i+1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi + \frac{1}{\Delta x}\int_{\xi_{i+1/2}(t)}^{\hat{\xi}_{i+1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi.$$

Note that the middle integral in the right-hand-side equals $(L\overline{\mathbf{U}})_i(t)$, which is known from the Lagrangian step. Therefore,

$$\mathbf{U}_i^{n+1} = (L\overline{\mathbf{U}})_i^{n+1} + \frac{1}{\Delta x}\int_{\hat{\xi}_{i-1/2}}^{\xi_{i-1/2}} L(\xi,t^{n+1})\overline{\mathbf{U}}(\xi,t^{n+1})d\xi + \frac{1}{\Delta x}\int_{\xi_{i+1/2}}^{\hat{\xi}_{i+1/2}} L(\xi,t^{n+1})\overline{\mathbf{U}}(\xi,t^{n+1})d\xi. \tag{4.1}$$

It remains to evaluate the other two integrals. We will now present the corresponding first and second order numerical schemes for doing this.

4.1. First order projection scheme

The previous integrals can be approximated in the following way:

$$\frac{1}{\Delta x}\int_{\xi_{i+1/2}}^{\hat{\xi}_{i+1/2}} L(\xi,t^{n+1})\overline{\mathbf{U}}(\xi,t^{n+1})d\xi = \frac{\hat{\xi}_{i+1/2}-\xi_{i+1/2}}{\Delta x}(L\overline{\mathbf{U}})_{i+1/2}^{n+1},$$

where

$$(L\overline{\mathbf{U}})_{i+1/2}^{n+1} = \begin{cases} (L\overline{\mathbf{U}})_i^{n+1} \text{ for } \xi_{i+1/2} > \hat{\xi}_{i+1/2}, \\ (L\overline{\mathbf{U}})_{i+1}^{n+1} \text{ for } \xi_{i+1/2} \leq \hat{\xi}_{i+1/2}. \end{cases}$$

Moreover, since for sufficiently small Δt we can use the approximation

$$\hat{\xi}_{i+1/2} = x_{i+1/2} - \Delta t u_{i+1/2}^*,$$

then from (4.1) we obtain

$$\mathbf{U}_i^{n+1} = (L\overline{\mathbf{U}})_i^{n+1} - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^*(L\overline{\mathbf{U}})_{i+1/2}^{n+1} - u_{i-1/2}^*(L\overline{\mathbf{U}})_{i-1/2}^{n+1}\right).$$

4.2. Second order projection scheme

In order to obtain a second order approximation of the integrals in (4.1), we will consider a linear reconstruction of the cell averages of $(L\overline{\mathbf{U}})_i^{n+1}$ and the velocities $u_{i+1/2}^{*,n+1}$ that are continuously defined at the intercells.

It can be seen that we can write (4.1) again as

$$\mathbf{U}_i^{n+1} = (L\overline{\mathbf{U}})_i^{n+1} - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^*(L\overline{\mathbf{U}})_{i+1/2}^{n+1} - u_{i-1/2}^*(L\overline{\mathbf{U}})_{i-1/2}^{n+1}\right),$$

where

$$(L\overline{\mathbf{U}})_{i+1/2}^{n+1} = \begin{cases} (L\overline{\mathbf{U}})_{i+1/2-}^{n+1} \text{ for } u_{i+1/2}^* > 0, \\ \\ (L\overline{\mathbf{U}})_{i+1/2+}^{n+1} \text{ for } u_{i+1/2}^* \leq 0, \end{cases}$$

and

$$\left(L\overline{\mathbf{U}}\right)_{i+1/2-}^{n+1} = \frac{1}{L_i^{n+1}}\left(\left(L\overline{\mathbf{U}}\right)_i^{n+1} + \frac{1}{2}\left(\delta L\overline{\mathbf{U}}\right)_i^{n+1}\left(\Delta x - \frac{\Delta t}{L_i^{n+1}}u_{i+1/2}^{*,n+1}\right)\right),$$

$$\left(L\overline{\mathbf{U}}\right)_{i+1/2+}^{n+1} = \frac{1}{L_{i+1}^{n+1}}\left(\left(L\overline{\mathbf{U}}\right)_{i+1}^{n+1} + \frac{1}{2}\left(\delta L\overline{\mathbf{U}}\right)_{i+1}^{n+1}\left(-\Delta x - \frac{\Delta t}{L_{i+1}^{n+1}}u_{i+1/2}^{*,n+1}\right)\right).$$

In the previous expressions, $\left(\delta L\overline{\mathbf{U}}\right)_i^{n+1}$ and $\left(\delta L\overline{\mathbf{U}}\right)_{i+1}^{n+1}$ are approximations of the derivatives of $L\overline{\mathbf{U}}$ at time t^{n+1} at x_i and x_{i+1}, respectively, that are computed using (3.4).

$(L\overline{\mathbf{U}})_{i-1/2}^{n+1}$ is defined in a similar way.

Now, the previous numerical schemes could be extended to the general case of non-flat topography. For that we follow [3, 4] to ensure the well-balanced properties of the schemes. More details will be shown during the presentation.

5. Numerical experiments

To be consistent with the numerical scheme presented previously, we only consider here a simple test case with flat bottom topography.

The aim of this test if to check that the numerical schemes that have been proposed achieves the expected order of accuracy. Let us consider a flat topography in the interval $[-5, 5]$, with initial zero velocity ($u = 0$) and an initial water depth with a small perturbation given by

$$h(x, 0) = 1 + 0.1 \exp\left(-x^2\right).$$

Figures 1 and 2 correspond to this initial condition.

Periodic boundary conditions are considered.

Tables 1 and 2 show the errors corresponding to the different methods. Notice that the errors for the explicit and the semi-implicit schemes decrease as the number of cells increase at the expected rate.

	Explicit scheme - Order 1				Explicit scheme - Order 2			
No. of	h		q		h		q	
cells	Error	Order	Error	Order	Error	Order	Error	Order
25	6.09E-2	-	1.60E-1	-	3.69E-2	-	1.18E-1	-
50	3.70E-2	0.72	9.97E-2	0.68	1.21E-2	1.61	3.86E-2	1.61
100	2.07E-2	0.84	5.63E-2	0.82	3.16E-3	1.93	1.03E-2	1.90
200	1.07E-2	0.95	2.95E-2	0.93	7.80E-4	2.02	2.58E-3	2.00
400	5.20E-3	1.05	1.44E-2	1.04	1.89E-4	2.05	6.27E-4	2.04

Tab. 1 Errors in L^1 norm and convergence rates for the explicit LP schemes of order 1 and 2.

	Implicit scheme - Order 1				Implicit scheme - Order 2			
No. of	h		q		h		q	
cells	Error	Order	Error	Order	Error	Order	Error	Order
25	7.99E-2	-	2.03E-1	-	3.80E-2	-	1.19E-1	-
50	5.13E-2	0.64	1.35E-1	0.59	1.27E-2	1.58	3.99E-2	1.57
100	3.05E-2	0.75	8.19E-2	0.73	3.35E-3	1.93	1.07E-2	1.90
200	1.66E-2	0.88	4.52E-2	0.86	8.00E-4	2.07	2.55E-3	2.07
400	8.19E-3	1.02	2.26E-2	1.00	1.85E-4	2.12	5.89E-4	2.12

Tab. 2 Errors in L^1 norm and convergence rates for the semi-implicit LP schemes of order 1 and 2.

We now consider a uniform mesh composed by 400 cells. The solution at time $t = 0.5$ for variables h and u using the order 1 and order 2 explicit schemes is shown in Figures 3 and 4. The same for the semi-implicit schemes can be seen in 5 and 6 with CFL value 0.5 and in Figures 7 and 8 with CFL value 2. Observe that the semi-implicit second order scheme with CFL=2 behaves similar to the explicit scheme with CFL=0.5.

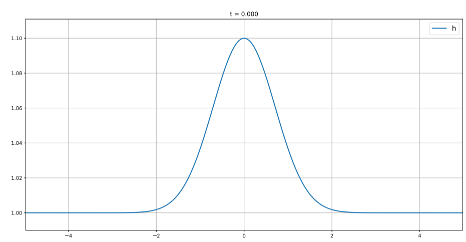

Fig. 1 Initial condition for the variable h.

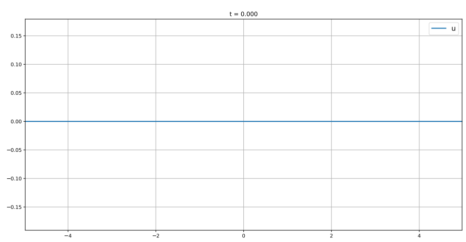

Fig. 2 Initial condition for the variable u.

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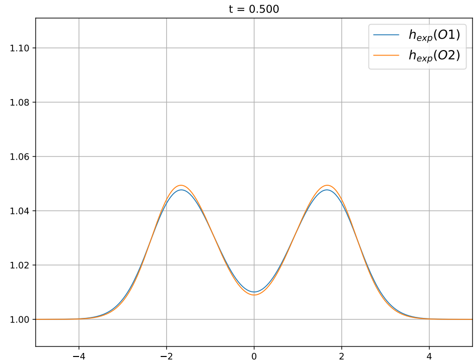

Fig. 3 Solution at time $t = 0.5$ for the variable h using the explicit schemes of order 1 and 2.

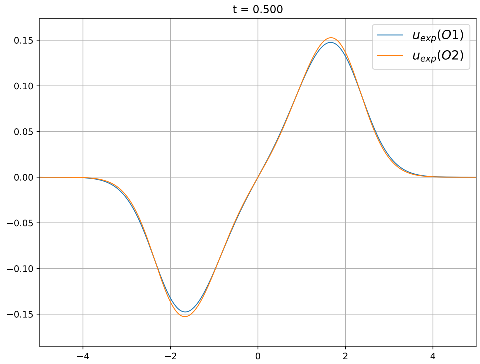

Fig. 4 Solution at time $t = 0.5$ for the variable u using the explicit schemes of order 1 and 2.

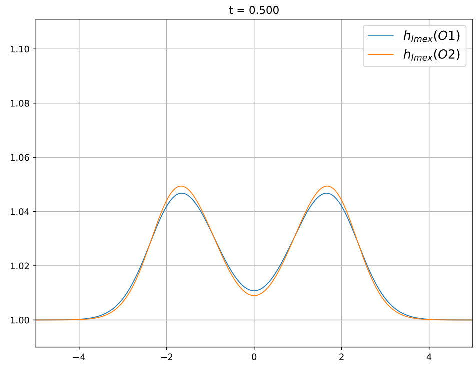

Fig. 5 Solution at time $t = 0.5$ for the variable h using the semi-implicit schemes of order 1 and 2 with CFL 0.5.

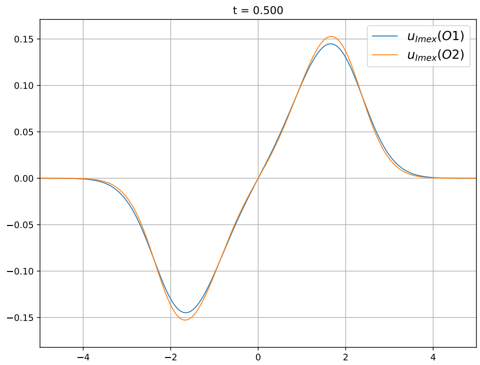

Fig. 6 Solution at time $t = 0.5$ for the variable u using the semi-implicit schemes of order 1 and 2 with CFL 0.5.

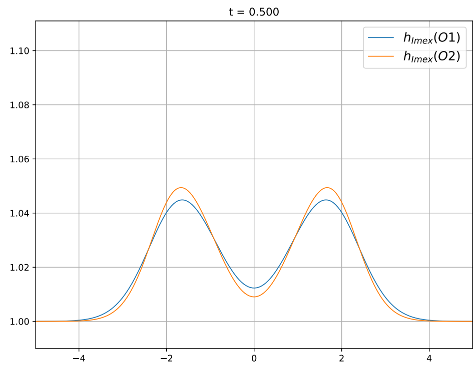

Fig. 7 Solution at time $t = 0.5$ for the variable h using the semi-implicit schemes of order 1 and 2 with CFL 2.

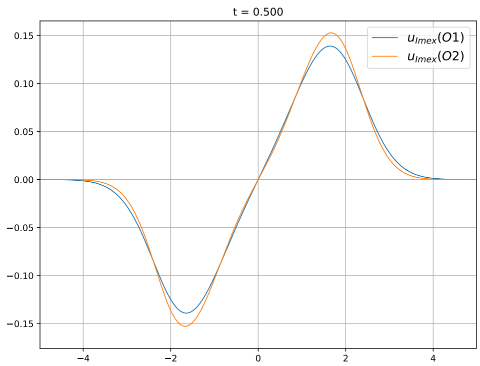

Fig. 8 Solution at time $t = 0.5$ for the variable u using the semi-implicit schemes of order 1 and 2 with CFL 2.

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