Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

CEDYA CMA 2020 GIJÓN

SēMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

Editors:
Rafael Gallego, Mariano Mateos

Universidad de Oviedo
Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
http: www.uniovi.es/publicaciones
servipub@uniovi.es

ISBN: 978-84-18482-21-2

Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

The Local Organizing Committee from the Universidad de Oviedo

Scientific Committee

- Juan Luis Vázquez, Universidad Autónoma de Madrid
- María Paz Calvo, Universidad de Valladolid
- Laura Grigori, INRIA Paris
- José Antonio Langa, Universidad de Sevilla
- Mikel Lezaun, Euskal Herriko Unibersitatea
- Peter Monk, University of Delaware
- Ira Neitzel, Universität Bonn
- JoséÁngel Rodríguez, Universidad de Oviedo
- Fernando de Terán, Universidad Carlos III de Madrid

Sponsors

- Sociedad Española de Matemática Aplicada
- Departamento de Matemáticas de la Universidad de Oviedo
- Escuela Politécnica de Ingeniería de Gijón
- Gijón Convention Bureau
- Ayuntamiento de Gijón

Local Organizing Committee from the Universidad de Oviedo

- Pedro Alonso Velázquez
- Rafael Gallego
- Mariano Mateos
- Omar Menéndez
- Virginia Selgas
- Marisa Serrano
- Jesús Suárez Pérez del Río

Contents

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
Gijón, 14-18 junio 2021
(pp. 73–81)

On volume constraint problems related to the fractional Laplacian

J. C. Bellido[1] **and Alejandro Ortega**[2]
1. *Universidad de Castilla–La Mancha, Spain,* josecarlos.bellido@uclm.es
2. *Universidad Carlos III de Madrid, Spain,* alortega@math.uc3m.es

Abstract

In this work we study volume constraint problems involving the nonlocal operator $(-\Delta)^s_\delta$ depending upon a parameter $\delta > 0$ called horizon. We analyze the associated linear and spectral problems and the behavior of these volume constraint problems when $\delta \to 0^+$ and $\delta \to +\infty$. We prove spectral convergence to the classical Laplacian as $\delta \to 0^+$ under a suitable scaling and spectral convergence to the fractional Laplacian as $\delta \to +\infty$.

1. Introduction

We study volume constraint elliptic problems driven by a nonlocal operator closely related to the well-known fractional Laplace operator. In particular, given an open bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and $\delta > 0$, a parameter called *horizon*, let us define the problem

$$\left\{ \begin{array}{rll} (-\Delta)^s_\delta u = f & \text{in} & \Omega, \\ u = 0 & \text{on} & \partial_\delta \Omega, \end{array} \right. \qquad (P^s_\delta)$$

where,

$$(-\Delta)^s_\delta u(x) = c_{N,s} P.V. \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy,$$

with $c_{N,s} = \frac{2^{2s} s \Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}} \Gamma(1-s)}$ a normalization constant and $\partial_\delta \Omega$ the nonlocal boundary given by

$$\partial_\delta \Omega = \{y \in \mathbb{R}^N \backslash \Omega : |x-y| < \delta \text{ for some } x \in \Omega\}.$$

Nonlocal and fractional elliptic problems have attracted a great attention in the mathematical community in the last decades, coming from fields as nonlocal diffusion [4, 13], statistical mechanics [2] and continuum mechanics, including peridynamics, [17, 20, 25]. Nonlocal variational problems are also important in the characterization of Sobolev spaces [9,18,21]. Interesting surveys on the fractional Laplacian and nonlocal elliptic problems are [19,22].

The operator $(-\Delta)^s_\delta$ is not new, and it has been addressed in different studies in the literature before. In view of the definition of $(-\Delta)^s_\delta$, it is clear that long-range interactions are neglected and only those exerted at distance smaller than $\delta > 0$ are taken into account, i.e., the horizon $\delta > 0$ represents the range of interactions. In this sense, the operator $(-\Delta)^s_\delta$, pertaining to the class of nonlocal elliptic operators, it is clearly inspired by peridynamics, where the elastic energy is computed through a double integral of a pairwise potential function, and it could actually be seen as a *peridynamic fractional Laplacian*. Peridynamics is a nonlocal continuum model for Solid Mechanics proposed by Silling, cf. [25]. The main difference with classical theory relies on the nonlocality, reflected in the fact that points separated by a positive distance exert a force upon each other. Since the use of gradients is avoided, peridynamics is a suitable framework for problems where discontinuities, such as fractures, appear naturally. In [16] a numerical study comparing $(-\Delta)^s_\delta$ with the fractional Laplacian, the spectral fractional Laplacian and the regional Laplacian is performed. In [1], the Fourier multiplier associated to $(-\Delta)^s_\delta$ is computed and, as a consequence, convergence of $(-\Delta)^s_\delta u(x)$ to $(-\Delta)u(x)$, for sufficiently smooth u, is obtained as $\delta \to 0^+$ or $s \to 1^-$. Also, $(-\Delta)^s_\delta$ was studied in [14] in connection with the fractional Laplacian, $(-\Delta)^s = (-\Delta)^s_\infty$, and with the motivation of computing numerical approximations. Notice that taking the limit as $\delta \to +\infty$ one recovers, at least formally, the usual nonlocal elliptic problem driven by the fractional Laplace operator with boundary condition on the complementary of the domain Ω.

In this work the limit properties of $(-\Delta)^s_\delta$, both as $\delta \to 0^+$ and as $\delta \to +\infty$, are addressed. In particular, by means of Γ-convergence techniques, we show, cf. [7], convergence of solutions and spectral stability, i.e., convergence of eigenvalues and eigenfunctions, to the classical Laplacian and to the fractional Laplacian as $\delta \to 0^+$ and as $\delta \to +\infty$ respectively. Therefore, the operator $(-\Delta)^s_\delta$ can be seen as an intermediate operator in between the local Laplacian and the fractional Laplacian.

The results for the case $\delta \to 0^+$ rely on a general Γ-convergence result from [6], while the results for the case $\delta \to +\infty$ are based on Γ-convergence properties of monotone sequences.

Closely related to our work is [3], where spectral stability as $\delta \to 0^+$ for certain nonlocal problems is shown without explicitly appealing to Γ-convergence. The advantage of the Γ-convergence approach is its adaptability to a nonlinear setting. Regarding this nonlinear setting, the spectral convergence of the fractional p-Laplacian to the classical p-Laplacian as $s \to 1^-$ is shown, by means of Γ-convergence techniques, in [11]. We extend the results of this work about spectral behavior to the nonlinear case dealing with the *peridynamic fractional p-Laplacian* in [8], where we obtain analogous results to those of [11] regarding the fractional p-Laplacian.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^N$ be a regular bounded domain and consider the Sobolev space $H^s(\Omega) = \{v \in L^2(\Omega) : \|v\|_{H^s(\Omega)} < \infty\}$, where $\|v\|^2_{H^s(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{H^s(\Omega)}$ being $|\cdot|_{H^s(\Omega)}$ the Gagliardo semi-norm,

$$|v|^2_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dydx.$$

Next, denoting by $\Omega^c = \mathbb{R}^N \backslash \Omega$, let us set the energy space $\mathcal{H}^s_0(\Omega) = \{v \in H^s(\mathbb{R}^N) : v = 0 \text{ on } \Omega^c\}$ endowed with the norm inherited from $H^s(\mathbb{R}^N)$. Let us note that, given $v \in \mathcal{H}^s_0(\Omega)$, although $v = 0$ on Ω^c, the norms $\|v\|_{H^s(\Omega)}$ and $\|v\|_{\mathcal{H}^s_0(\Omega)}$ are not the same. Indeed, denoting by $\mathcal{D} = \left(\mathbb{R}^N \times \mathbb{R}^N\right) \backslash \left(\Omega^c \times \Omega^c\right)$, we have the strict inclusion $\Omega \times \Omega \subsetneq \mathcal{D}$. Then, the norm $\|\cdot\|_{\mathcal{H}^s_0(\Omega)}$ takes into account the interaction between Ω and Ω^c, i.e.,

$$\|v\|^2_{\mathcal{H}^s_0(\Omega)} = \|v\|^2_{H^s(\mathbb{R}^N)} = \|v\|^2_{L^2(\Omega)} + \iint\limits_{\mathcal{D}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dydx.$$

Therefore, the space $\mathcal{H}^s_0(\Omega)$ is the appropriate space to deal with homogeneous elliptic boundary value problems involving the fractional Laplace operator,

$$(-\Delta)^s_\infty u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy.$$

On the other hand, by the fractional Sobolev inequality, cf. [15, Th. 6.5], we can renormize the space $\mathcal{H}^s_0(\Omega)$ and consider it endowed with the norm

$$|||v|||^2_{\mathcal{H}^s_0} = \iint\limits_{\mathcal{D}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dydx.$$

Next, given an horizon $\delta > 0$, let us define the (*nonlocally*) completed domain $\Omega_\delta = \Omega \cup \partial_\delta \Omega$, and the energy space $\mathbb{H}^s(\Omega_\delta) = \{v \in L^2(\Omega_\delta) : \|v\|_{\mathbb{H}^s(\Omega_\delta)} < \infty\}$ where $\|v\|^2_{\mathbb{H}^s(\Omega_\delta)} = \|v\|^2_{L^2(\Omega_\delta)} + |v|^2_{\mathbb{H}^s(\Omega_\delta)}$ with

$$|v|^2_{\mathbb{H}^s(\Omega_\delta)} = \int_{\Omega_\delta} \int_{\Omega_\delta \cap B(x,\delta)} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dydx.$$

Note that, because of [5, Prop. 6.1], the spaces $\mathbb{H}^s(\Omega_\delta)$ and $H^s(\Omega_\delta)$ are isomorphic. In order to deal with the boundary value problem P^s_δ, we define the energy space $\mathbb{H}^{\delta,s}_0(\Omega) = \{v \in \mathbb{H}^s(\Omega_\delta) : v \equiv 0 \text{ on } \partial_\delta\Omega\}$ endowed with the norm inherited from $\mathbb{H}^s(\Omega_\delta)$. Let us notice that, given a function $v \in \mathbb{H}^{\delta,s}_0(\Omega)$, although we have $v = 0$ on $\partial_\delta\Omega = \Omega_\delta \backslash \Omega$, the norms $\|v\|_{\mathbb{H}^s(\Omega)}$ and $\|v\|_{\mathbb{H}^{\delta,s}_0(\Omega)}$ are not the same. Indeed, if $v = 0$ on $\partial_\delta\Omega$, since $\mathbb{H}^s(\Omega) = \{v \in L^2(\Omega) : \|v\|_{\mathbb{H}^s(\Omega)} < \infty\}$ with $\|v\|^2_{\mathbb{H}^s(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{H^s(\Omega)}$ and

$$\|v\|^2_{\mathbb{H}^{\delta,s}_0(\Omega)} = \|v\|^2_{L^2(\Omega)} + \iint\limits_{\mathcal{D}_\delta} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dydx,$$

with $\mathcal{D}_\delta = \Big(\Omega_\delta \times \big(\Omega_\delta \cap B(x,\delta)\big)\Big) \Big\backslash \Big(\partial_\delta\Omega \times \big(\partial_\delta\Omega \cap B(x,\delta)\big)\Big)$, we have the strict inclusion $\big(\Omega \times (\Omega \cap B(x,\delta))\big) \subsetneq \mathcal{D}_\delta$. Hence, the norm $\|\cdot\|_{\mathbb{H}^{\delta,s}_0(\Omega)}$ takes into account the interaction between Ω and $\partial_\delta\Omega$ in the sense that

$$|v|^2_{\mathbb{H}^s(\Omega_\delta)} = |v|^2_{H^s(\Omega)} + \int_{\partial_\delta\Omega} \int_{\Omega \cap B(x,\delta)} \frac{|v(y)|^2}{|x-y|^{N+2s}} dydx + \int_\Omega \int_{\partial_\delta\Omega \cap B(x,\delta)} \frac{|v(x)|^2}{|x-y|^{N+2s}} dydx.$$

Therefore, the space $\mathbb{H}^{\delta,s}_0(\Omega)$ is the appropriate space to deal with homogeneous elliptic boundary value problems involving the operator $(-\Delta)^s_\delta$. Moreover, comparing the norms $\|\cdot\|_{\mathcal{H}^s_0(\Omega)}$ and $\|\cdot\|_{\mathbb{H}^{\delta,s}_0(\Omega)}$ we observe that $\partial_\delta\Omega$

plays the role of Ω^c. Indeed, the sets Ω_δ and $\Omega_\delta \cap B(x,\delta)$ will lead to the complete space $\mathbb{R}^N$ for $\delta \to +\infty$, the set $\Omega \cap B(x,\delta)$ will eventually reach the set Ω for $\delta > 0$ big enough and the sets $\partial_\delta\Omega$ and $\partial_\delta\Omega \cap B(x,\delta)$ will reach Ω^c for $\delta \to +\infty$. In fact, $\mathcal{D}_{\delta_1} \subset \mathcal{D}_{\delta_2}$ for $\delta_1 < \delta_2$ and $\mathcal{D}_\delta \to \mathcal{D}$ as $\delta \to +\infty$.
Due to [5, Prop. 6.1] and [5, Lem. 6.2], we have $|v|_{\mathbb{H}^s(\Omega_\delta)} \le \|v\|_{\mathbb{H}^s(\Omega_\delta)} \le c|v|_{\mathbb{H}^s(\Omega_\delta)}$, for a positive constant $c \in \mathbb{R}$ and, then, we can renormize the space $\mathbb{H}_0^{\delta,s}(\Omega)$ and consider it endowed with the norm

$$|||v|||^2_{\mathbb{H}_0^{\delta,s}} = \int_{\Omega_\delta}\int_{\Omega_\delta\cap B(x,\delta)} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dydx.$$

As a consequence, we have the following, cf. [7, Lem. 2] and [7, Lem. 4] respectively.

Lemma 2.1 *The space $\mathbb{H}_0^{\delta,s}(\Omega)$ is a Hilbert space endowed with norm $|||\cdot|||_{\mathbb{H}_0^{\delta,s}}$ induced by the scalar product*

$$\langle u,v\rangle_{\mathbb{H}_0^{\delta,s}} = \int_{\Omega_\delta}\int_{\Omega_\delta\cap B(x,\delta)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx.$$

Analyze convergence phenomena for $\delta \to +\infty$ will require to study the relation between $\mathcal{H}_0^s(\Omega)$ and $\mathbb{H}_0^{\delta,s}(\Omega)$.

Lemma 2.2 *For any $\delta > 0$, the spaces $\mathbb{H}_0^{\delta,s}(\Omega)$ and $\mathcal{H}_0^s(\Omega)$ are isomorphic. In particular, there exists a constant $C = C(\delta) > 1$ such that $C(\delta) \to 1$ as $\delta \to +\infty$ and*

$$|||\cdot|||^2_{\mathbb{H}_0^{\delta,s}} \le |||\cdot|||^2_{\mathcal{H}_0^s} \le C(\delta)|||\cdot|||^2_{\mathbb{H}_0^{\delta,s}} \quad \textit{for all } \delta > 0.$$

Now we make precise the definition of weak solution of problem P^s_δ.

Definition 2.3 We say that $u \in \mathbb{H}_0^{\delta,s}(\Omega)$ is a weak solution to problem P^s_δ if, for all $v \in \mathbb{H}_0^{\delta,s}(\Omega)$,

$$\frac{c_{N,s}}{2}\langle u,v\rangle_{\mathbb{H}_0^{\delta,s}} = \langle f,v\rangle_{L^2(\Omega)}.$$

3. Main Results

We present now the main results of the work. To that end, let us set $\kappa(N,s) = \frac{4N(1-s)}{\sigma_{N-1}c_{N,s}}$ with σ_{N-1} the surface of the unitary sphere $\mathbb{S}^{N-1}$ and $\partial_\infty\Omega = \mathbb{R}^N\backslash\Omega$ and consider the following problems,

$$RP^s_\delta \equiv \begin{cases} (-\Delta)^s_\delta u = \frac{\delta^{2(1-s)}}{\kappa(N,s)}f & \text{in } \Omega, \\ u = 0 & \text{on } \partial_\delta\Omega, \end{cases} \quad P^1_0 \equiv \begin{cases} (-\Delta)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad P^s_\infty \equiv \begin{cases} (-\Delta)^s_\infty u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial_\infty\Omega, \end{cases}$$

Our main results regarding the linear problems are the following, cf. [7, Th. 2] and [7, Th. 3].

Theorem 3.1 *Let $u^{\delta,s}$ and $u^{0,1}$ be the solutions of RP^s_δ and P^1_0 respectively. Then, up to a subsequence,*

$$u^{\delta,s} \to u^{0,1} \textit{ in } L^2(\Omega) \quad \textit{as } \delta \to 0^+.$$

Theorem 3.2 *Let $u^{\delta,s}$ and $u^{\infty,s}$ be the solutions of P^s_δ and P^s_∞ respectively. Then, up to a subsequence,*

$$u^{\delta,s} \to u^{\infty,s} \textit{ in } L^2(\Omega) \quad \textit{as } \delta \to +\infty.$$

We continue with existence and stability issues for the eigenvalue problem

$$\begin{cases} (-\Delta)^s_\delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial_\delta\Omega. \end{cases} \qquad (EP^s_\delta)$$

Using [5, Prop. 6.1, Lem. 6.2] and following [24, Prop. 9], we prove the following, cf. [7, Prop. 2].

Proposition 3.3 *Let $\delta > 0$, $s \in (0,1)$, $N > 2s$ and $\Omega \subset \mathbb{R}^N$ an open bounded set with Lipschitz boundary. Then, the following hold:*

1. *Problem EP^s_δ has a first positive eigenvalue that can be characterized as*

$$\lambda_1^{\delta,s} = \min_{\substack{u\in\mathbb{H}_0^{\delta,s}(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \frac{c_{N,s}}{2}|||u|||^2_{\mathbb{H}_0^{\delta,s}} = \frac{c_{N,s}}{2}\left|\left|\left|\varphi_1^{\delta,s}\right|\right|\right|^2_{\mathbb{H}_0^{\delta,s}},$$

where $\varphi_1^{\delta,s}\in\mathbb{H}_0^{\delta,s}(\Omega)$, is a nonnegative eigenfunction. In addition, the first eigenvalue $\lambda_1^{\delta,s}$ is simple.

2. *The eigenvalues of EP^s_δ are a countable set $\{\lambda_k^{\delta,s}\}_{k\in\mathbb{N}}$ satisfying*

$$0<\lambda_1^{\delta,s}<\lambda_2^{\delta,s}\le\ldots\le\lambda_k^{\delta,s}\le\ldots \quad \text{and} \quad \lambda_k^{\delta,s}\to+\infty \qquad \text{as } k\to+\infty.$$

Furthermore, for any $k\in\mathbb{N},\ k\ge2$ the eigenvalues can be characterized as

$$\lambda_k^{\delta,s} = \min_{\substack{u\in\mathbb{P}_k^{\delta}\\ \|u\|_{L^2(\Omega)}=1}} \frac{c_{N,s}}{2}|||u|||^2_{\mathbb{H}_0^{\delta,s}} = \frac{c_{N,s}}{2}\left|\left|\left|\varphi_k^{\delta,s}\right|\right|\right|^2_{\mathbb{H}_0^{\delta,s}}.$$

where $\mathbb{P}_k^\delta=\{u\in\mathbb{H}_0^{\delta,s}(\Omega):\langle u,\varphi_j^{\delta,s}\rangle_{\mathbb{H}_0^{\delta,s}}=0,\ j=1,\ldots,k-1\}$ and an eigenfunction $\varphi_k^{\delta,s}\in\mathbb{P}_k^\delta$.

3. *The set of eigenfunctions $\{\varphi_k^{\delta,s}\}_{k\in\mathbb{N}}$ is an orthogonal basis of $\mathbb{H}_0^{\delta,s}(\Omega)$ and an orthonormal basis of $L^2(\Omega)$.*

4. *For any $k\in\mathbb{N}$, the eigenvalue $\lambda_k^{\delta,s}$ has finite multiplicity, $1\le m_k^{\delta,s}<\infty$ for all $k\in\mathbb{N}$.*

Moreover, arguing as in [23, Prop. 4], we also deduce the following, cf. [7, Lem. 5].

Lemma 3.4 *Let $\varphi_k^{\delta,s}\in\mathbb{H}_0^{\delta,s}(\Omega)$ be an eigenfunction of EP^s_δ, then $\varphi_k^{\delta,s}\in L^\infty(\Omega)$ for any $k\in\mathbb{N}$.*

Finally, we present the main results about the behavior of EP^s_δ when $\delta\to0^+$ and $\delta\to+\infty$. To that end, let us consider the eigenvalue problems,

$$EP_0^1\equiv\begin{cases}(-\Delta)\varphi=\lambda\varphi & \text{in} \quad \Omega,\\ \varphi=0 & \text{on} \quad \partial\Omega,\end{cases} \quad \text{and} \quad EP^s_\infty\equiv\begin{cases}(-\Delta)^s_\infty\varphi=\lambda\varphi & \text{in} \quad \Omega,\\ \varphi=0 & \text{on} \quad \mathbb{R}^N\backslash\Omega.\end{cases}$$

It is well known cf. [12] that the problem EP_0^1 has a countable set of eigenvalues that we denote by $\{\lambda_k^{0,1}\}_{k\in\mathbb{N}}$ and such that

$$0<\lambda_1^{0,1}<\lambda_2^{0,1}\le\ldots\le\lambda_k^{0,1}\le\ldots \quad \text{and} \quad \lambda_k^{0,1}\to+\infty \text{ as } k\to+\infty.$$

Denoting by $m_k^{0,1}$ the multiplicity of the eigenvalue $\lambda_k^{0,1}$, we have $1\le m_k^{0,1}<\infty$ for all $k\in\mathbb{N}$. Moreover, there exists a countable set of eigenfunctions $\{\varphi_k^{0,1}\}_{k\in\mathbb{N}}$ that is an orthogonal basis of $H_0^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. The first eigenvalue is simple and $\varphi_1^{0,1}>0$ in Ω.

Concerning the fractional eigenvalue problem, Servadei and Valdinoci proved, cf. [24], that EP^s_∞ has a countable set of eigenvalues that we denote by $\{\lambda_k^{\infty,s}\}_{k\in\mathbb{N}}$ and such that

$$0<\lambda_1^{\infty,s}<\lambda_2^{\infty,s}\le\ldots\le\lambda_k^{\infty,s}\le\ldots, \quad \text{and} \quad \lambda_k^{\infty,s}\to+\infty \text{ as } k\to+\infty.$$

Denoting by $m_k^{\infty,s}$ the multiplicity of the eigenvalue $\lambda_k^{\infty,s}$, we have $1\le m_k^{\infty,s}<\infty$ for all $k\in\mathbb{N}$. Moreover, there exists a countable set of eigenfunctions $\{\varphi_k^{\infty,s}\}_{k\in\mathbb{N}}$ that is an orthogonal basis of

$$\mathbb{H}_0^{\infty,s}(\Omega)=\left\{v\in L^2(\Omega):\iint_{\mathcal{D}}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dydx<\infty,\ v=0 \text{ a.e. on } \mathbb{R}^N\backslash\Omega\right\},$$

and an orthonormal basis of $L^2(\Omega)$. The first eigenvalue is also simple and $\varphi_1^{\infty,s}\ge0$ in Ω.

We relate the eigenvalues and eigenfunctions of EP^s_δ to those of the eigenvalue problems EP_0^1 and EP^s_∞ through the following results, cf. [7, Th. 4] and [7, Th. 5].

Theorem 3.5 *Let $\{(\lambda_k^{\delta,s}, \varphi_k^{\delta,s})\}_{k\in\mathbb{N}}$ be the set of eigenvalues and eigenfunctions of $(-\Delta)^s_\delta$ with homogeneous Dirichlet boundary condition on $\partial_\delta\Omega$ and let $\{(\lambda_k^{0,1}, \varphi_k^{0,1})\}_{k\in\mathbb{N}}$ be the set of eigenvalues of $(-\Delta)$ with homogeneous Dirichlet boundary condition on $\partial\Omega$. Then,*

$$\kappa(N,s)\frac{\lambda_k^{\delta,s}}{\delta^{2(1-s)}} \to \lambda_k^{0,1} \quad \text{as } \delta \to 0^+,$$

for $\kappa(N,s) = \frac{4N(1-s)}{\sigma_{N-1}c_{N,s}}$. Moreover, there exists a subsequence (that we do not relabel) such that, for every $k \in \mathbb{N}$,

$$\varphi_k^{\delta,s} \to \varphi_k^{0,1} \text{ in } L^2(\Omega) \quad \text{as } \delta \to 0^+.$$

As a consequence, $m_k^{\delta,s} \to m_k^{0,1}$ as $\delta \to 0^+$, for any $k \geq 1$.

As Theorems 3.1 and 3.5 show, even though the fractionality parameter s keeps fixed, the local problem driven by $(-\Delta)$ is recovered, under the appropriate rescaling, as $\delta \to 0^+$.

Theorem 3.6 *Let $\{(\lambda_k^{\delta,s}, \varphi_k^{\delta,s})\}_{k\in\mathbb{N}}$ be the set of eigenvalues and eigenfunctions of $(-\Delta)^s_\delta$ with homogeneous Dirichlet boundary condition on $\partial_\delta\Omega$ and let $\{(\lambda_k^{\infty,s}, \varphi_k^{\infty,s})\}_{k\in\mathbb{N}}$ be the set of eigenvalues of $(-\Delta)^s_\infty$ with homogeneous Dirichlet boundary condition on $\mathbb{R}^N\setminus\Omega$. Then,*

$$\lambda_k^{\delta,s} \to \lambda_k^{\infty,s} \quad \text{as } \delta \to +\infty,$$

and there exists a subsequence (that we do not relabeled) such that for every $k \in \mathbb{N}$,

$$\varphi_k^{\delta,s} \to \varphi_k^{\infty,s} \text{ in } L^2(\Omega) \quad \text{as } \delta \to +\infty.$$

As a consequence, $m_k^{\delta,s} \to m_k^{\infty,s}$ as $\delta \to \infty$, for any $k \geq 1$.

3.1. Γ-convergence

This section includes some results about Γ-convergence that play a crucial role in the proof of the above stated results. The limit process in the sense of Γ-convergence, denoted by $\xrightarrow{\Gamma}$, is the right concept of limit for variational problems since, together with equicoercivity or compactness, it implies that minimizers of I_δ converge to minimizers of I as well as their energies. A nice account on Γ-convergence is provided in [10]. We present now a result about Γ-convergence of functionals proved in [6] that is the core of the proof of Theorem 3.1 and Theorem 3.5. Let us consider a functional of the form

$$I(u) = \int_\Omega \int_{\Omega\cap B(x,\delta)} \omega(x-y, u(x)-u(y))dydx,$$

for a *potential function* $\omega(x,y) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}$ verifying that, for some $\beta \in \mathbb{R}$, the following limit exists,

$$\omega^\circ(x,y) = \lim_{t\to 0^+} \frac{1}{t^\beta}\omega(tx,ty).$$

Let $\overline{\omega}^c : \mathbb{R}^N \mapsto \mathbb{R}$ be the *limit density convexification* of $\overline{\omega}$ defined as $\overline{\omega}^c = \sup\{v : v \leq \overline{\omega} \text{ and } v \text{ convex}\}$, where $\overline{\omega} : \mathbb{R}^N \mapsto \mathbb{R}$ is the *limit density* of ω,

$$\overline{\omega}(F) = \int_{\mathbb{S}^{N-1}} \omega^\circ(z, Fz)d\sigma(z).$$

Under the hypotheses stated below, given the sequence of rescaled functionals

$$I_\delta(u) = \frac{N+\beta}{\delta^{N+\beta}} \int_\Omega \int_{\Omega\cap B(x,\delta)} \omega(x-y, u(x)-u(y))dydx,$$

we have,

$$I_\delta(u) \xrightarrow{\Gamma} I_0(u) = \int_\Omega \overline{\omega}^c(\nabla u)dx.$$

In particular, the above Γ-convergence is ensured by the next result, that also provides the compactness of uniformly bounded energy sequences. Let us set $\tilde{\Omega} = \{z = x-y \ : \ x, y \in \Omega\}$ and $\mathcal{A}_\delta = \{v \in L^p(\Omega) : v = 0 \text{ on } \partial_\delta\Omega\}$.

Theorem 3.7 ([6, *Th.* 1]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and $\omega : \tilde{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ satisfying the hypotheses $(H1)$-$(H2)$ below. Then, the following holds*:

a) *Compactness: For each $\delta > 0$, let $u_\delta \in \mathcal{A}_\delta$ such that $\sup_\delta I_\delta(u_\delta) < +\infty$. Then, there exist $u \in W_0^{1,p}(\Omega)$ such that, for a subsequence, $u_\delta \to u$ strong in $L^p(\Omega)$ as $\delta \to 0^+$.*

b) Γ*-liminf inequality*: *For each $\delta > 0$ let $u_\delta \in \mathcal{A}_\delta$ and $u \in W_0^{1,p}(\Omega)$ such that $u_\delta \to u$ strong in $L^p(\Omega)$ as $\delta \to 0^+$. Then, $I_0(u) \le \liminf_{\delta\to 0^+} I_\delta(u_\delta)$.*

c) Γ*-limsup inequality*: *For each $\delta > 0$ and $u \in W_0^{1,p}(\Omega)$ there exist $u_\delta \in \mathcal{A}_\delta$, called recovery sequence, such that $u_\delta \to u$ strong in $L^p(\Omega)$ as $\delta \to 0^+$ and $\limsup_{\delta\to 0^+} I_\delta(u_\delta) \le I_0(u)$.*

For a general potential function $\omega(x, y)$ the hypotheses of Theorem 3.7 are quite involved but, if $\omega(x, y) = f(x)g(y)$ for f a Lebesgue measurable function and g a Borel measurable and convex function, the necessary hypotheses are:

H1) There exists constants $c_0, c_1 > 0$ and $h \in L^1(\mathbb{S}^{N-1})$ with $h \ge 0$ such that, for some $1 < p < +\infty$ and $0 \le \alpha < N + p$,

$$c_0 \frac{|y|^p}{|x|^\alpha} \le f(x)g(y) \le c_1 h\left(\frac{x}{|x|}\right) \frac{|y|^p}{|x|^\alpha} \quad \text{for } x \in \widetilde{\Omega},\ y \in \mathbb{R}.$$

H2) The functions $f^\circ(x) = \lim_{t\to 0^+} t^\alpha f(tx)$ and $g^\circ(y) = \lim_{t\to 0^+} \frac{1}{t^p} g(ty)$, are continuous and, for each compact $K \subset \mathbb{R}$,

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{S}^{N-1}} |t^\alpha f(tx) - f^\circ(x)| = 0 \quad \text{and} \quad \lim_{t\to 0^+} \sup_{K\subset\mathbb{R}} |\frac{1}{t^p} g(ty) - g^\circ(y)| = 0.$$

The following is a straightforward consequence, cf. [10], of the Γ-convergence and the compactness provided by Theorem 3.7. Notice that under previous hypothesis existence of minimizers for I_δ is guaranteed, cf. [5].

Corollary 3.8 *In the conditions of Theorem* 3.7, *let $u_\delta \in \mathbb{H}_0^{\delta,s}(\Omega)$ be a minimizer of I_δ, for any $\delta > 0$. Then, there exists $u_0 \in H_0^1(\Omega)$ a minimizer of I_0 such that, up to a subsequence,*

$$u_\delta \to u_0 \text{ strong in } L^2(\Omega) \quad \text{as } \delta \to 0^+ \quad \text{and} \quad I_\delta(u_\delta) \to I_0(u_0) \quad \text{as } \delta \to 0^+.$$

3.2. Taking the horizon $\delta \to 0^+$

One of the main steps to prove Theorem 3.1 and Theorem 3.5 is the following result, cf. [7, Lem. 6], concerning the Γ-convergence of the energy functional defining the eigenvalues. Among other things, it shows that, up to the appropriate scaling, all the functionals $I_{\delta,s}$ will Γ-converge to the same Γ-limit independently of s.

Lemma 3.9 *Let us consider the scaled functional*

$$I_{\delta,s}(u) = \frac{2(1-s)}{\delta^{2(1-s)}} \int_{\Omega_\delta} \int_{\Omega_\delta \cap B(x,\delta)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx,$$

defined on $\mathbb{H}_0^{\delta,s}(\Omega)$. Then, the Γ-limit of $I_{\delta,s}(u)$ as $\delta \to 0^+$ is given by

$$I_0(u) = \frac{\sigma_{N-1}}{N} \int_\Omega |\nabla u(x)|^2 dx.$$

The proof of Theorem 3.1 follows from Lemma 3.9 and Corollary 3.8. To prove Theorem 3.5, we use Lemma 3.9 and Corollary 3.8, from where we get the convergence of the first eigenvalue under the appropriate scaling, namely,

$$\kappa(N, s) \frac{\lambda_1^{\delta,s}}{\delta^{2(1-s)}} \to \lambda_1^{0,1} \quad \text{as } \delta \to 0^+.$$

Moreover, we also get $\varphi_1^{\delta,s} \to \varphi_1^{0,1}$ strong in $L^2(\Omega)$ as $\delta \to 0^+$. Next, we construct a recovery sequence by projecting appropriately on the second eigenspace $\mathbb{P}_2^\delta = \{u \in \mathbb{H}_0^{\delta,s}(\Omega) : \langle u, \varphi_1^{\delta,s} \rangle_{\mathbb{H}_0^{\delta,s}} = 0\}$. Thanks to the strong $L^2(\Omega)$-convergence of the first eigenfunction the convergence of the second eigenvalue and the second eigenfunction follows. To conclude we argue inductively.

In order to clarify why the scaling in Γ-convergence result is natural in our context, it is of interest to deduce from the classical *localization* result of Bourgain, Brezis and Mironescu the upper bound

$$\lim_{\delta\to 0^+} \frac{\kappa(N,s)}{\delta^{2(1-s)}}\lambda_1^{\delta,s} \le \lambda_1^{0,1}.$$

Let $\{\rho_n(x)\}_{n\in\mathbb{N}}$ be a sequence of radial mollifiers, i.e.,

$$\rho_n(x)=\rho_n(|x|),\ \rho_n(x)\ge 0 \text{ and } \int \rho_n(x)dx=1 \quad \text{and satisfying} \quad \lim_{n\to\infty}\int_\varepsilon^\infty \rho_n(r)r^{N-1}=0\ \forall\varepsilon>0.$$

Theorem 3.10 ([9, *Th.* 2]) *Assume* $u\in L^p(\Omega)$, $1<p<\infty$. *Then, for a constant* $C=C(N,p)>0$, *we have*

$$\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(x-y)dydx = C\int_\Omega |\nabla u|^p dx.$$

with the convention that $\int_\Omega |\nabla u|^p dx=\infty$ *if* $u\notin W^{1,p}(\Omega)$.

Since $H_0^1(\Omega)\subset \mathbb{H}_0^{\delta,s}(\Omega)$ for all $\delta>0$, we have

$$\lambda_1^{\delta,s} = \min_{\substack{u\in\mathbb{H}_0^{\delta,s}(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \frac{c_{N,s}}{2}\int_{\Omega_\delta}\int_{\Omega_\delta\cap B(x,\delta)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx \le \frac{c_{N,s}}{2}\int_{\Omega_\delta}\int_{\Omega_\delta\cap B(x,\delta)} \frac{|\psi_1^\delta(x)-\psi_1^\delta(y)|^2}{|x-y|^{N+2s}}dydx,$$

being ψ_1^δ the first eigenfunction $\varphi_1^{0,1}$ of the Laplace operator ($L^2(\Omega)$-normalized) extended by zero on $\partial_\delta\Omega$. In order to apply Theorem 3.10, let us rewrite the above inequality as

$$\lambda_1^{\delta,s} \le \int_{\Omega_\delta}\int_{\Omega_\delta} \frac{|\psi_1^\delta(x)-\psi_1^\delta(y)|^2}{|x-y|^2}\rho_\delta(|x-y|)dydx,$$

with $\rho_\delta(z)=\dfrac{c_{N,s}}{2}\dfrac{\chi_{B(0,\delta)}(|z|)}{|z|^{N+2(s-1)}}$ and χ_A the characteristic function of the set A. Since

$$\int \rho_\delta(z)dz = \frac{\sigma_{N-1}c_{N,s}}{4(1-s)}\delta^{2(1-s)},$$

the sequence of radial mollifiers $\overline{\rho}_\delta(z) = \dfrac{4(1-s)}{\sigma_{N-1}}\dfrac{1}{\delta^{2(1-s)}}\dfrac{\chi_{B(0,\delta)}(|z|)}{|z|^{N+2(s-1)}}$ satisfy the hypotheses of Theorem 3.10. Then, because of Theorem 3.10, we conclude

$$\lim_{\delta\to 0^+}\frac{4(1-s)}{\sigma_{N-1}c_{N,s}}\frac{\lambda_1^{\delta,s}}{\delta^{2(1-s)}} \le \lim_{\delta\to 0^+}\int_{\Omega_\delta}\int_{\Omega_\delta}\frac{|\psi_1^\delta(x)-\psi_1^\delta(y)|^2}{|x-y|^2}\overline{\rho}_\delta(|x-y|)dydx = C\int_\Omega |\nabla\varphi_1^{0,1}|^2dx,$$

since $\psi_1=0$ on $\partial_{\delta_0}\Omega$ and $\psi_1=\varphi_1^{0,1}$ in Ω. Since for $p=2$ the constant $C=C(N,p)$ appearing in Theorem 3.10 takes the value $C(N,2)=\frac{1}{N}$, taking in mind that $\|\varphi_1^{0,1}\|_{L^2(\Omega)}=1$, the desired bound follows.

3.3. Taking the horizon $\delta\to+\infty$

Because of the definition of the operator $(-\Delta)_\delta^s$, as a restriction of the fractional Laplacian, it is plausible that if we take $\delta\to+\infty$ one recovers the definition of the standard fractional Laplacian, namely,

$$\lim_{\delta\to+\infty}(-\Delta)_\delta^s u(x) = c_{N,s}P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy.$$

A result in this line was given in [14, Th. 3.1], where it is showed the explicit convergence rate

$$\|u^{\delta,s}-u^{\infty,s}\|_{\mathbb{H}_0^{\delta,s}} \le \frac{c}{(\delta-I)^{2s}}\|u^{\infty,s}\|_{L^2(\Omega)},$$

being $u^{\delta,s}$ and $u^{\infty,s}$ the solutions of P_δ^s and P_∞^s respectively, $c>0$ is a constant independent of δ and $I=I(\Omega)$ a constant depending on the diameter of Ω. This is an important result from the point of view of the numerical approximation of problems involving the fractional Laplacian but its proof strongly relies on the linearity of the problem P^s_δ. Instead, the proof of Theorem 3.2 and Theorem 3.6 are based on a general result about Γ-convergence that works for both the linear and nonlinear setting. We exploit this advantage to address the p -fractional Laplacian case, cf. [8], and extend the results of this work to the nonlinear setting.

The following Γ-convergence result, cf. [7, Lem. 7], is in the core of the proofs of Theorems 3.2 and Theorem 3.6. This result is analogous to Lemma 3.9 in relation to the proofs of Theorem 3.1 and Theorem 3.5.

Lemma 3.11 *Let us consider the functional*

$$\mathcal{E}_{\delta,s}(u) = \int_{\Omega_\delta}\int_{\Omega_\delta\cap B(x,\delta)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx.$$

defined on $\mathbb{H}_0^{\delta,s}(\Omega)$. *Then, the* Γ*-limit of* $\mathcal{E}_{\delta,s}(u)$ *is given by*

$$\mathcal{E}_{\infty,s}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx \quad \text{as } \delta \to +\infty.$$

The above Lemma is an easy consequence of the monotonicity in $\delta > 0$ of the sequence of functionals $\mathcal{E}_{\delta,s}(u)$ and Γ-convergence properties. Indeed, since the sequence of functionals $\mathcal{E}_{\delta,s}(u)$ with $\delta \to +\infty$ is a monotone increasing sequence and functionals $\mathcal{E}_{\delta,s}$ are lower semicontinuous, cf. [5], because of [10, Remark 1.40], we conclude $\mathcal{E}_{\delta,s}(u) \xrightarrow{\Gamma} \mathcal{E}_{\infty,s}(u)$ as $\delta \to +\infty$.

The proof of Theorem 3.2 follows from Lemma 3.11 combined with the monotonicity in $\delta > 0$ together with Lemma 2.2, the compact embedding of $\mathcal{H}_0^s(\Omega)$ into $L^2(\Omega)$, cf. [15, Cor. 7.2], and the fact that Γ-convergence implies the convergence of the minimizers.

The proof of Theorem 3.6 follows by combining Lemma 3.11 and Lemma 2.2 with similar arguments to those used in Theorem 3.5.

Acknowledgements

This work was carried out while the second author was a postdoctoral researcher at Universidad de Castilla–La Mancha in Ciudad Real, funded by project MTM2017-83740-P of the *Agencia Estatal de Investigación, Ministerio de Ciencia e Innovación* (Spain).

References

[1] B. Alali and N. Albin, *Fourier multipliers for nonlocal Laplace operators*, Applicable Analysis, **0** (2019), pp. 1–21.

[2] G. Alberti and G. Bellettini, *A non-local anisotropic model for phase transitions: asymptotic behaviour of rescaled energies*, European J. Appl. Math., **9**, pp. 261–284, 1998.

[3] F. Andrés and J. Muñoz, *A type of nonlocal elliptic problem: existence and approximation through a Galerkin-Fourier method*, SIAM J. Math. Anal., **47**, pp. 498–525, 2015.

[4] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. J. Toledo-Melero, *Nonlocal diffusion problems*, vol. **165** of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

[5] J. C. Bellido and C. Mora-Corral, *Existence for nonlocal variational problems in peridynamics*, SIAM J. Math. Anal., **46**, pp. 890–916, 2014.

[6] J. C. Bellido, C. Mora-Corral and P. Pedregal, *Hyperelasticity as a* Γ*-limit of peridynamics when the horizon goes to zero*, Calc. Var. Partial Differential Equations, **54**, pp. 1643–1670, 2015.

[7] J.C. Bellido and A. Ortega, *A restricted nonlocal operator bridging together the Laplacian and the fractional Laplacian*, Calc. Var. **60**, 71, 2021.

[8] J.C. Bellido and A. Ortega, *Spectral Stability for the Peridynamic Fractional p-Laplacian*, Appl. Math. Optim., 2021.

[9] J. Bourgain, H. Brezis and P. Mironescu, *Another look at Sobolev spaces*, in Optimal control and partial differential equations, IOS, Amsterdam, pp. 439–455, 2001.

[10] A. Braides, Γ*-convergence for beginners*, vol. **22** of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2002.

[11] L. Brasco, E. Parini and M. Squassina, *Stability of variational eigenvalues for the fractional* p*-Laplacian*, Discrete Contin. Dyn. Syst., **36**, pp. 1813–1845, 2016.

[12] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[13] C. Bucur and E. Valdinoci, *Nonlocal diffusion and applications*, vol. **20** of Lecture Notes of the Unione Matematica Italiana, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[14] M. D'Elia and M. Gunzburger, *The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator*, Comput. Math. Appl., **66**, pp. 1245–1260, 2013.

[15] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136**, pp. 521–573, 2012.

[16] S. Duo, H. Wang and Y. Zhang, *A comparative study on nonlocal diffusion operators related to the fractional Laplacian*, Discrete Contin. Dyn. Syst. Ser. B, **24**, pp. 231–256, 2019.

[17] A. Evgrafov and J. C. Bellido, *From non-local Eringen's model to fractional elasticity*, Math. Mech. Solids, **24**, pp. 1935–1953, 2019.

[18] G. Leoni and D. Spector, *Characterization of Sobolev and BV spaces*, J. Funct. Anal., **261**, pp. 2926–2958, 2011.

[19] A. Lischke, G. Pang, M. Gulian and et al., *What is the fractional Laplacian? A comparative review with new results*, J. Comput. Phys., **404**, 109009, 62 pp., 2020.

[20] T. Mengesha and Q. Du, *Characterization of function spaces of vector fields and an application in nonlinear peridynamics*, Nonlinear Anal., **140**, pp. 82–111, 2016.

[21] A. C. Ponce, *A new approach to Sobolev spaces and connections to Γ-convergence*, Calc. Var. Partial Differential Equations, **19**, pp. 229–255, 2004.

[22] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: a survey*, Publ. Mat., **60**, pp. 3–26, 2016.

[23] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Communications on Pure & Applied Analysis, **12**, pp. 2445–2464, 2013.

[24] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., **33**, pp. 2105–2137, 2013.

[25] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids, **48**, pp. 175–209, 2000.