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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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A numerical method to solve Maxwell's equations in 3D singular geometry

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Abstract

We propose a new method to solve the 3D Maxwell equations in axisymmetric singular domains, containing reentrant corner or edges. By doing a Fourier analysis, one arrives to a sequence of singular problems set in 2D domains, and 3D solutions are computed by solving 2D problems, depending on a Fourier mode k. For each k, the solution is decomposed into a regular and a singular part. The regular part is computed with a finite element method. The singular part belongs to a finite-dimensional subspace and is computed by an appropriate numerical approach, *only* for the modes $k = 0, \pm 1, 2$. The total the solution is then reconstructed, based on a non stationary variational formulation. Numerical examples will be shown.

1. Introduction

This article is part of the efforts made in the framework of non-smooth problems, i.e. problems set in non convex curvilinear polyhedra: such domains containing reentrant edges, they generate singularities in Maxwell's equations solutions. From a more intuitive point of view, the term singularities means that such geometrical features can generate, in their vicinity, very strong electromagnetic fields, that have to be carefully handled and are often difficult to compute. Moreover, as shown in [2], the impossibility of correctly handling these singularities may have drastic consequences on the phenomenon one wants to model.

In this context, many methods have been proposed to compute the solution to the Maxwell equations. We can mention the edge finite element method, introduced by Nédélec [9], that has demonstrated efficiency for the static and eigenvalue problems. More recently, discontinuous Galerkin method has been introduced [8] and have been extensively studied since then. In [5], Brenner et al. have also proposed an adaptive finite element method that works in dimension two.

Nevertheless, it is interesting for some applications to have a *continuous approximation* of the solutions, that can capture both the curl and the divergence of the electromagnetic fields, for instance when coupling the Maxwell equations in other equations, like the Vlasov one, see [3]. But the latter works only in convex (curvilinear) polyhedra.

In this paper, we consider three-dimensional axisymmetric domains with non axisymmetric data. Due to the axisymmetric assumption, the singular computational domain can be reduced to a subset of $\mathbb{R}^2$. However, the data being *arbitrary*, i.e. not necessarily axisymmetric, the electromagnetic field and other vector quantities still belong to $\mathbb{R}^3$. Hence, we take advantage that the domain is transformed into a two-dimensional one, and based on a Fourier analysis in the third dimension, one arrives to a sequence of singular problems set in a 2D singular domain. We then derive a variational formulation from which we propose a finite element method to solve the problem and numerically compute the solution.

2. Setting of the problem

We consider an axisymmetric bounded and simply connected Lipschitz domain Ω in $\mathbb{R}^3$, with a boundary Γ, $\mathbf{n}$ being the unit outward normal to Γ. We denote by c and ε_0 the speed of light and the dielectric permittivity respectively.

Hence, the evolution of a time-dependent electromagnetic field $\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t)$ propagating in vacuum is governed by Maxwell's equations[1]:

$$\frac{\partial \mathbf{E}}{\partial t} - c^2 \mathbf{curl}\ \mathbf{B} = -\frac{1}{\varepsilon_0}\mathbf{J}, \tag{2.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\ \mathbf{E} = 0, \tag{2.2}$$

$$\text{div}\ \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{2.3}$$

$$\text{div}\ \mathbf{B} = 0\,, \tag{2.4}$$

where $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ are the charge and current densities, that depend on the space variable $\mathbf{x}$ and on the time variable t. These equations are supplemented with perfect conductor boundary conditions, and homogeneous initial conditions at initial time $t = 0$.

We assume now that the domain Ω is axisymmetric, limited by the surface of revolution Γ, and we denote by ω and γ_b their intersections with a meridian half-plane. The boundary $\partial\omega := \gamma$ corresponds to $\gamma_a \cup \gamma_b$, where either $\gamma_a = \emptyset$ when γ_b is a closed contour (i.e. Ω does not contain the axis), or γ_a is the segment of the axis lying between the extremities of γ_b, see Fig.1. The natural coordinates for this domain are the cylindrical coordinates (r, θ, z), with the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. A meridian half-plane is defined by the equation θ =constant, and (r, z) are Cartesian coordinates in this half-plane.

However, even if we assumed symmetry of revolution for the domain Ω, we *do not assumed* such a symmetry for the data. Consequently, the problem can not be reduced to a two-dimensional one by assuming that derivative with respect to the azimuthal variable θ vanishes, i.e. $\partial/\partial\theta = 0$, as made for example in [1]: we have to continue to deal with a three-dimensional problem.

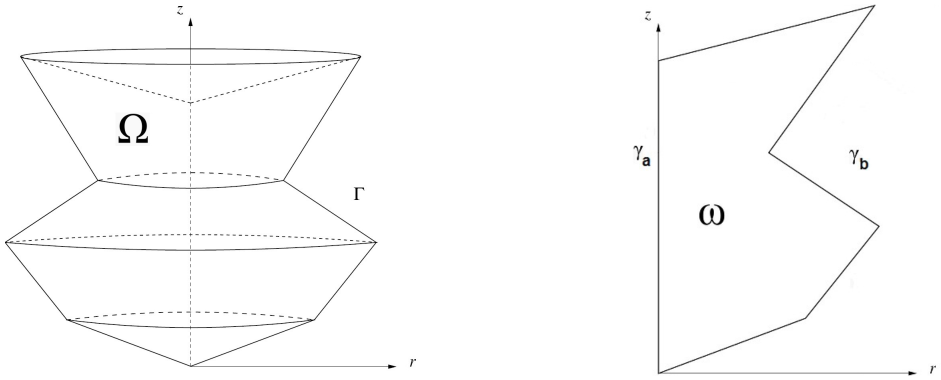

Fig. 1 Example of 3D domain Ω, and its corresponding 2D intersection with meridian half-plane ω.

Following [3], it is more efficient if one wishes to use nodal finite element methods, for instance for charge particle simulations as in the context of Vlasov-Maxwell computations, to eliminate the magnetic field $\mathbf{B}$ (respectively the electric field $\mathbf{E}$) from Eqs. (2.1-2.4). Hence, Maxwell's equations reduce to two second-order wave equations for each field separately:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2 \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = -\frac{1}{\varepsilon_0}\frac{\partial \mathbf{J}}{\partial t},$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + c^2 \mathbf{curl}\,\mathbf{curl}\,\mathbf{B} = \frac{1}{\varepsilon_0}\mathbf{curl}\,\mathbf{J}\,,$$

the constraints equations, namely divergence and boundary conditions, still holding.

3. Two-dimensional space reduction

As the data we consider are not axisymmetric, one can not perform $\partial/\partial\theta = 0$. However, one can use the cylindrical symmetry of the domain Ω to characterize the quantities defined on it, through their Fourier series in θ, the coefficients of which being functions defined on ω.

[1]In the text, names of function spaces of scalar fields usually begin by an italic letter, whereas they begin by a bold letter for spaces of vector fields.

Hence, we will consider, for a given vector field $\mathbf{w}(r,\theta,z)$

$$\mathbf{w}(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \mathbf{w}^k(r,z)\, e^{ik\theta},$$

and the following weighted Lebesgue space

$$L^2_r(\omega) := \left\{ w \text{ measurable on } \omega : \iint_\omega |w(r,z)|^2\, r dr dz < \infty \right\},$$

that will be the space of Fourier coefficients (at all modes) of functions[2] in $\mathbf{L}^2(\Omega)$.

At the same time, let us also define the space of relevant Fourier coefficients for the electromagnetic fields. It is easy to check that, for $\mathbf{w} \in \mathbf{H}(\mathrm{div}\,;\Omega)$, resp. $\mathbf{H}(\mathbf{curl}\,;\Omega)$, one has

$$\mathrm{div}\,\mathbf{w} = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \mathrm{div}_k \mathbf{w}^k e^{ik\theta} \text{ resp. } \mathbf{curl}\,\mathbf{w} = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \mathbf{curl}_k \mathbf{w}^k e^{ik\theta},$$

where the operators for each mode k are defined as

$$\mathrm{div}_k \mathbf{w} := \frac{1}{r}\frac{\partial(rw_r)}{\partial r} + \frac{ik}{r} w_\theta + \frac{\partial w_z}{\partial z}; \qquad (\mathbf{curl}_k \mathbf{w})_r := \frac{ik}{r} w_z - \frac{\partial w_\theta}{\partial z};$$

$$(\mathbf{curl}_k \mathbf{w})_\theta := \frac{\partial w_r}{\partial z} - \frac{\partial w_z}{\partial r}; \qquad (\mathbf{curl}_k \mathbf{w})_z := \frac{1}{r}\left(\frac{\partial(rw_\theta)}{\partial r} - ikw_r\right).$$

The regularity of $\mathbf{w}$ only depends on the regularity of its Fourier components $\mathbf{w}^k$, for $k \in \mathbb{Z}$. Let us now introduce the spaces for the **curl** and div operators

$$\mathbf{H}_0(\mathbf{curl}\,;\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}\,;\Omega) : \mathbf{v}\times\mathbf{n}|_\Gamma = 0\} \text{ and } \mathbf{H}_0(\mathrm{div}\,;\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{div}\,;\Omega) : \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0\}.$$

Hence, electric and magnetic field naturally belongs to the spaces

$$\mathbf{X}(\Omega) = \mathbf{H}_0(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,\mathbf{v};\Omega) \text{ and } \mathbf{Y}(\Omega) = \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}_0(\mathrm{div}\,\mathbf{v};\Omega)$$

As a consequence, a function $\mathbf{v}$ belongs to $\mathbf{X}(\Omega)$ if and only if, for all $k \in \mathbb{Z}$, its Fourier coefficients $\mathbf{v}^k$ belong to the space $\mathbf{X}_{(k)}(\omega)$ defined by

$$\mathbf{X}_{(k)}(\omega) = \{\mathbf{v}^k \in \mathbf{L}^2_r(\omega),\ \mathbf{curl}_k \mathbf{v}^k \in \mathbf{L}^2_r(\omega), \mathrm{div}_k \mathbf{v}^k \in \mathbf{L}^2_r(\omega), \mathbf{v}^k \times \mathbf{n}_{|\gamma_b} = 0\}.$$

In a similar way, one introduces the space $\mathbf{Y}_{(k)}(\omega)$ for the Fourier coefficients of elements of $\mathbf{Y}(\Omega)$, namely

$$\mathbf{Y}_{(k)}(\omega) = \{\mathbf{v}^k \in \mathbf{L}^2_r(\omega),\ \mathbf{curl}_k \mathbf{v}^k \in \mathbf{L}^2_r(\omega), \mathrm{div}_k \mathbf{v}^k \in \mathbf{L}^2_r(\omega), \mathbf{v}^k \cdot \mathbf{n}_{|\gamma_b} = 0\}$$

A useful property concerning these spaces (see [6]) is that $\mathbf{X}_{(k)}(\omega)$ and $\mathbf{Y}_{(k)}(\omega)$ are independent of k, for $|k| \geq 2$. This allows us to compute the singular subspaces only for the modes $|k| \leq 2$, while the modes ± 2 will be use to compute all the higher modes $|k| > 2$.

Applying the dimension reduction, and using the linearity of the Maxwell equations together with the orthogonality of the different Fourier modes, we can reduce the 3D equations to a series of 2D formulations solved by the Fourier coefficients $(\mathbf{E}_k, \mathbf{B}_k)$, for each mode k, where the operators $\mathbf{curl}_k$ and div_k are involved. Let us introduce the operator $a_k(\cdot,\cdot)$ defined by

$$a_k(\mathbf{u},\mathbf{v}) = (\mathbf{curl}_k \mathbf{u}, \mathbf{curl}_k \mathbf{v}) + (\mathrm{div}_k \mathbf{u}, \mathrm{div}_k \mathbf{v}). \tag{3.1}$$

We get that each mode $\mathbf{E}^k$ is solution to the following variational formulation:
find $\boldsymbol{E}^k(t) \in \boldsymbol{X}_{(k)}(\omega)$ such that, for all $\boldsymbol{F} \in \boldsymbol{X}_{(k)}(\omega)$:

$$\frac{d^2}{dt^2}\left(\mathbf{E}^k(t), \mathbf{F}\right) + c^2 a_k\left(\mathbf{E}^k(t), \mathbf{F}\right) = -\frac{1}{\varepsilon_0}\left(\partial_t \mathbf{J}^k, \mathbf{F}\right) + \frac{1}{\varepsilon_0}\left(\rho^k, \mathrm{div}_k \mathbf{F}\right), \tag{3.2}$$

[2] In the text, we shall also use the standard spaces and norms

where ρ^k and $\mathbf{J}^k$ denote the Fourier coefficients of the charge and current density ρ and $\mathbf{J}$ respectively, that depend (in space) only on (r, z).

In the same way one gets that the Fourier coefficients $\mathbf{B}_k(t)$ verify the variational formulation, for each mode k: *find $\boldsymbol{B}^k(t) \in \boldsymbol{Y}_{(k)}(\omega)$ such that, for all $\boldsymbol{C} \in \boldsymbol{Y}_{(k)}(\omega)$:*

$$\frac{d^2}{dt^2}\left(\mathbf{B}^k(t), \mathbf{C}\right) + c^2\, a_k\left(\mathbf{B}^k(t), \mathbf{C}\right) = \frac{1}{\varepsilon_0}\left(\mathbf{curl}_{\,k}\mathbf{J}^k, \mathbf{C}\right). \tag{3.3}$$

4. Decomposition in regular and singular parts

Due to the geometrical reduction, the geometrical singularities remain in the two-dimensional domain ω (see Figure 1). We briefly recall here some useful results helpful to understand the construction of the numerical method (see [2], [6] for details). As a first step, we introduce, for each Fourier mode k, the regular subspaces $\mathbf{X}^R_{(k)}$ and $\mathbf{Y}^R_{(k)}$, defined by:

$$\mathbf{X}^R_{(k)} := \mathbf{X}_{(k)} \cap \mathbf{H}^1_r(\omega)\,, \text{ and } \mathbf{Y}^R_{(k)} := \mathbf{Y}_{(k)} \cap \mathbf{H}^1_r(\omega)\,.$$

These subspaces are regular, in the sense that they coincide to the spaces of solutions in the case of a regular domain. Using now that $\mathbf{X}^R_{(k)}$ and $\mathbf{Y}^R_{(k)}$ are closed subspaces of $\mathbf{X}_{(k)}$ and $\mathbf{Y}_{(k)}$ respectively, we deduce the following decomposition

$$\mathbf{X}_{(k)} = \mathbf{X}^R_{(k)} \oplus \mathbf{X}^S_{(k)} \qquad \text{and} \qquad \mathbf{Y}_{(k)} = \mathbf{Y}^R_{(k)} \oplus \mathbf{Y}^S_{(k)}\,,$$

where $\mathbf{X}^S_{(k)}$ and $\mathbf{Y}^S_{(k)}$ are singular subspaces, equal to $\{0\}$ if the domain Ω (or equivalently ω) is regular.

The second step is to characterize these singular spaces, that have been proved to be finite dimensional. We have

Theorem 4.1 *The singular spaces $X^S_{(k)}$ and $Y^S_{(k)}$ are of finite dimension, namely*

- *For $k = 0$*

$$\dim Y^S_{(k)} := N_B = \textit{ number of reentrant edges,}$$
$$\dim X^S_{(k)} := N_E = N_B + \textit{ number of conical points with vertex angle } > \frac{\pi}{\beta},\ (\beta \simeq 1.3731)$$

- *For $k \neq 0$*

$$\dim Y^S_{(k)} := N_B = \dim X^S_{(k)} := N_E = \textit{ number of reentrant edges.}$$

From these properties, one can decompose, for each mode k, the electromagnetic field $(\mathbf{E}^k, \mathbf{B}^k)$ into a regular and a singular part, namely

$$(\mathbf{E}^k(t), \mathbf{B}^k(t)) = (\mathbf{E}^k_R(t), \mathbf{B}^k_R(t)) + (\mathbf{E}^k_S(t), \mathbf{B}^k_S(t))\,. \tag{4.1}$$

Moreover, since the singular spaces are of finite dimension, one can introduce their respective basis $(\mathbf{x}^k_{S,j})_{j=1,N_E}$ and $(\mathbf{y}^k_{S,j})_{j=1,N_B}$ for a given Fourier mode k. Using now that these basis are time independent, one can express the singular parts $\mathbf{E}^k_S(t)$ and $\mathbf{B}^k_S(t)$ as

$$\mathbf{E}^k_S(t) = \sum_{j=1}^{N_E} \mu^k_{E,j}(t)\mathbf{x}^k_{S,j} \qquad \text{and} \qquad \mathbf{B}^k_S(t) = \sum_{j=1}^{N_B} \mu^k_{B,j}(t)\mathbf{y}^k_{S,j}\,,$$

where $\mu^k_{E,j}(t)$ and $\mu^k_{B,j}(t)$ are smooth functions in time (at least continuous). As a consequence, the decomposition (4.1) of the electromagnetic, that will be useful for the numerical method, can be finally expressed, for each k,

$$\mathbf{E}^k(t) = \mathbf{E}^k_R(t) \oplus \sum_{j=1}^{j=N_E} \mu^k_{E,j}(t)\mathbf{x}^k_{S,j}, \qquad \mathbf{B}^k(t) = \mathbf{B}^k_R(t) \oplus \sum_{j=1}^{j=N_B} \mu^k_{B,j}(t)\mathbf{y}^k_{S,j}\,. \tag{4.2}$$

From a numerical point of view, as explained above, it is sufficient to compute them only for $k = -1, 0, 1, 2$. As these basis are not time-dependent, the computations will be carried out only once as an initialization procedure. This has been previously presented in [4] where details can be found.

5. Solving the time-dependent problem

In this section, we present the case of the magnetic field formulation. The electric field formulation is similar and can be derived in the same way. Therefore, we consider the variational formulation (3.3), in which we substitute the decomposition of the magnetic field (4.2) in regular and singular parts. Using that the singular basis $\mathbf{y}^k_{S,j}$ are time-independent, and denoting by $''$ the second derivative in time, we get

$$\frac{d^2}{dt^2}\left(\mathbf{B}^k_R(t),\mathbf{C}\right)+\sum_{j=1}^{N_B}(\mu^k_{B,j})''\left(\mathbf{y}^k_{S,j},\mathbf{C}\right)+c^2\,a_k\left(\mathbf{B}^k_R(t),\mathbf{C}\right)+c^2\sum_{j=1}^{N_B}\mu^k_{B,j}(t)\,a_k\left(\mathbf{y}^k_{S,j},\mathbf{C}\right)$$
$$=\frac{1}{\varepsilon_0}\left(\mathbf{curl}\,_k\mathbf{J}^k,\mathbf{C}\right),\qquad \forall\mathbf{C}\in\mathbf{Y}^R_{(k)}(\omega)\,. \tag{5.1}$$

In addition, we add to the space of test functions $\mathbf{Y}_R(\omega)$ the fonctions $(\mathbf{y}^k_{S,j})_{j=1,N_B}$. This yields the N_B additional equations

$$\frac{d^2}{dt^2}\left(\mathbf{B}^k_R(t),\mathbf{y}^k_{S,i}\right)+\sum_{j=1}^{N_B}(\mu^k_{B,j})''\left(\mathbf{y}^k_{S,j},\mathbf{y}^k_{S,i}\right)+c^2\,a_k\left(\mathbf{B}^k_R(t),\mathbf{y}^k_{S,i}\right)+c^2\sum_{j=1}^{N_B}\mu^k_{B,j}(t)\,a_k\left(\mathbf{y}^k_{S,j},\mathbf{y}^k_{S,i}\right)$$
$$=\frac{1}{\varepsilon_0}\left(\mathbf{curl}\,_k\mathbf{J}^k,\mathbf{y}^k_{S,i}\right),\quad \forall\mathbf{y}^k_{S,i}\in\mathbf{Y}^S_{(k)}(\omega),\,1\le i\le N_B\,.$$

Moreover, using the orthogonality for each k of $\mathbf{Y}^R_{(k)}$ and $\mathbf{Y}^S_{(k)}$ with respect to the equivalent scalar product $a_k(\cdot,\cdot)$ defined by (3.1), we can eliminate the corresponding terms in the formulations above. This variational formulation is finally expresses as
Find $(\boldsymbol{B}^k_R,\boldsymbol{\mu}^k_B)\in\boldsymbol{Y}^R_{(k)}\times\mathbb{R}^{N_B}$ *such that*

$$\begin{cases}\left(\dfrac{\partial^2\mathbf{B}^k_R(t)}{\partial t^2},\mathbf{C}\right)+\displaystyle\sum_{j=1}^{N_B}(\mu^k_{B,j})''\left(\mathbf{y}^k_{S,j},\mathbf{C}\right)+c^2\,a_k\left(\mathbf{B}^k_R(t),\mathbf{C}\right)\\ \qquad\qquad=\dfrac{1}{\varepsilon_0}\left(\mathbf{curl}\,_k\mathbf{J}^k,\mathbf{C}\right),\quad\forall\mathbf{C}\in\mathbf{Y}^R_{(k)}(\omega)\,,\\ \left(\dfrac{\partial^2\mathbf{B}^k_R(t)}{\partial t^2},\mathbf{y}^k_{S,i}\right)+\displaystyle\sum_{j=1}^{N_B}(\mu^k_{B,j})''\left(\mathbf{y}^k_{S,j},\mathbf{y}^k_{S,i}\right)+c^2\sum_{j=1}^{N_B}\mu^k_{B,j}(t)\,a_k\left(\mathbf{y}^k_{S,j},\mathbf{y}^k_{S,i}\right)\\ \qquad\qquad=\dfrac{1}{\varepsilon_0}\left(\mathbf{curl}\,_k\mathbf{J}^k,\mathbf{y}^k_{S,i}\right),\quad\forall\mathbf{y}^k_{S,i}\in\mathbf{Y}^S_{(k)}(\omega).\end{cases} \tag{5.2}$$

From a computational point of view, it is worth to rewrite the bilinear form $a_k\,(\cdot,\cdot)$ involved above, depending on the values of k. Performing a simple integration by parts shows that

$$\begin{aligned}a_k\,(\mathbf{u},\mathbf{v})\;&=\;a_0\,(\mathbf{u}_m,\mathbf{v}_m)+k^2\left(\frac{\mathbf{u}_m}{r},\frac{\mathbf{v}_m}{r}\right)+(\mathbf{curl}\,u_\theta,\mathbf{curl}\,v_\theta)+k^2\left(\frac{u_\theta}{r},\frac{v_\theta}{r}\right)\\&+\;\imath kB\,(\mathbf{u},\mathbf{v})\;+\imath kC\,(\mathbf{u},\mathbf{v})\;,\end{aligned}$$

where $a_0(\cdot,\cdot)$ denotes the operator $a_k(\cdot,\cdot)$ for $k=0$ (namely in the "full" axisymmetric case), $\mathbf{u}_m:=(u_r,u_z)$ and the vector **curl** of a scalar field w is defined by

$$\mathbf{curl}\,w:=-\partial_z w\mathbf{e}_r+r^{-1}\partial_r\,(rw)\,\mathbf{e}_z\,.$$

In addition, the two bilinear forms $B\,(\mathbf{u},\mathbf{v})$ and $C\,(\mathbf{u},\mathbf{v})$ are defined by

$$B\,(\mathbf{u},\mathbf{v}):=\int_{\gamma_b}(\mathbf{u}_m\cdot\mathbf{n})\,\bar{v}_\theta-\mathbf{u}_\theta\,(\bar{\mathbf{v}}_m\cdot\mathbf{n})\,d\gamma\,,$$

and

$$C\,(\mathbf{u},\mathbf{v}):=\int\int_\omega 2\,(u_\theta\bar{v}_r-u_r\bar{v}_\theta)\,\frac{d\omega}{r}\,.$$

Note that the term $B\,(\mathbf{u},\mathbf{v})$ is vanishes as soon $\mathbf{u}\cdot\mathbf{n}=\mathbf{v}\cdot\mathbf{n}=0$, that is the case for the magnetic field, due to the perfect conductor boundary condition. The same is true if $\mathbf{u}\times\mathbf{n}=\mathbf{v}\times\mathbf{n}=0$, that is the case for the electric field. In addition, the term $C\,(\mathbf{u},\mathbf{v})$ is not singular despite the presence of $1/r$ in the integral. Indeed, only on the boundary γ_a one may have $r=0$, but $u_\theta=v_\theta=0$ (that is in practice B^k_θ or E^k_θ for the electric case) due the symmetry condition on the axis γ_a.

Starting from this variational formulation, we are now ready to derive a finite element approximation. Let $\mathbf{Y}^{R,h}_{(k)} \subset \mathbf{Y}^{R}_{(k)}$ be the space of discretized test functions of dimension N_h. We actually used the P_2 finite element, and denote by $\mathcal{T}_h$ the mesh of ω made of triangles K_h. Then, the approximation space for the vector fields is made of functions which are component-wise P_2-conforming on the triangulation.

Let now $\mathbf{B}^{k,h}(t) = \mathbf{B}^{k,h}_R(t) + \sum_{j=1}^{N_B} \mu^{k,h}_{B,j}(t)\,\mathbf{y}^{k,h}_{S,j}$ be the discrete solution. After discretization in space, the semi-discretized variational formulation is written (with the addition of the index ^{h}) in the same way as (5.2). It can be expressed equivalently as a linear system:

$$\frac{d^2}{dt^2}\mathbb{M}_{rr}B^k_R \;+\; \mathbb{M}^k_{rs}\mu^{k\,\prime\prime}_B \;+\; c^2\mathbb{K}^k_{rr}B^k_R \;=\; \frac{1}{\varepsilon_0}\mathbb{R}^k_{rr}J^k\;, \tag{5.3}$$

$$\frac{d^2}{dt^2}\mathbb{M}^k_{sr}B^k_R \;+\; \mathbb{M}^k_{ss}\mu^{k\,\prime\prime}_B \;+\; c^2\mathbb{K}^k_{ss}\mu^{k\,\prime\prime}_B \;=\; \frac{1}{\varepsilon_0}\mathbb{R}^k_{sr}J^k\;, \tag{5.4}$$

where $\mathbb{M}_{rr}$ denotes the mass matrix that does not depend to the Fourier mode k, $\mathbb{M}^k_{rs}$ is a (N_h, N_B) rectangular matrix coming from the integral over ω of the product of the N_B singular functions $\mathbf{y}^{k,h}_{S,j}$ by the basis functions of $\mathbf{Y}^{R,h}_{(k)}$, $\mathbb{M}^k_{sr}$ being its transpose. Similarly, the matrix $\mathbb{K}^k_{rr}$ is associated to the term $a_k\left(\mathbf{B}^k_R(t), \mathbf{C}\right)$ $\mathbb{R}^k_{rr}$ coming from the source term with $\mathbf{curl}_{\,k}\mathbf{J}^k$, and μ^k_B standing for the vector of $\mathbb{R}^{N_B}$ of entries $(\mu^k_{B,j})$. Finally, $\mathbb{M}^k_{ss}$ and $\mathbb{K}^k_{ss}$ are the "singular" mass and rigidity matrices of dimension (N_B, N_B), associated to the term $\left(\mathbf{y}^k_{S,j}, \mathbf{y}^k_{S,i}\right)$ and $a_k\left(\mathbf{y}^k_{S,j}, \mathbf{y}^k_{S,i}\right)$ respectively. For these singular matrices, the computation must be carried out precisely in the neighborhood of the singularities by using a quadrature formula of high order.

We then perform a time discretization involving a second-order explicit (leap-frog) scheme. Here the notation X^n (resp. X^{n+1}) stands for a variable X at time $t^n = n\Delta t$ (resp. $t^{n+1} = (n+1)\Delta t$), where Δt is the time-step. F^n, G^n, H^n is the set of quantities known at time t^n for each equation of the scheme (5.3)-(5.4), which can be rewritten as

$$\mathbb{M}_{rr}B^{k,n+1}_R + \mathbb{M}^k_{rs}\mu^{k,n+1}_B = F^{k,n}\;, \tag{5.5}$$

$$\mathbb{M}^k_{sr}B^{k,n+1}_R + \mathbb{M}^k_{ss}\mu^{k,n+1}_B = G^{k,n}\;. \tag{5.6}$$

To solve this linear system, a convenient way is to decouple $\mu^{k,n+1}_B$ and the unknown $B^{k,n+1}_R$as proposed in [2] for a two-dimensional Cartesian Maxwell system of equations, in the case of $N_B = 1$. The method developed here is more general, since it is also adapted to a domain with $N_B \geq 1$. For this purpose, we simply substitute (5.5)$-\mathbb{M}^k_{rs}(\mathbb{M}^k_{ss})^{-1}$(5.6) to obtain a system where $\mu^{k,n+1}_B$ does no appear anymore. It remains now to invert this system to compute $B^{k,n+1}_R$, and then, at the corresponding time, the value $\mu^{k,n+1}_B$ by solving (5.6).

Compared to the system one would obtained in a regular domain, the additional effort is essentially the computation of the matrix $(\mathbb{M}^k_{ss})^{-1}$. $\mathbb{M}^k_{ss}$ being a symmetric definite positive matrix (by construction) of dimension (N_B, N_B), i.e. a few units (and often $N_B = 1$), $(\mathbb{M}^k_{ss})^{-1}$ is very easy to compute once and for all, for any mode k, $|k| \leq 2$.

6. Numerical results

We present here numerical results to illustrate the method. For the sake of simplicity, we restrict ourselves to a domain with only one singular point. Hence, we will consider a 3-D top hat domain Ω with a reentrant circular edge, that corresponds, for a given θ, to an L-shaped 2-D domain ω with a reentrant corner. We introduce an unstructured mesh of ω made up of triangles, with no particular refinement near the reentrant corner. The variational formulations are approximated by a finite element method with FreeFem++ package [7]. The singular basis being computed as described in [4], we focus here on the computation of the time-dependent solutions. As in the previous section, we will concentrate on the magnetic case.

In addition, we assume that a perfectly conducting boundary condition is imposed on ω, and we want to numerically compute $\mathbf{B}^k(t) = \mathbf{B}^k_R(t) + \mu^k_B(t)\mathbf{y}^k_S$, assuming that singular basis $\mathbf{y}^k_S$ was already computed. More precisely, we are interested in computing the magnetic field $\mathbf{B}^k(t)$ created by a current loop, with initial conditions set to zero, and a current defined by $\mathbf{J}(t) = 10\sin(\lambda t)\mathbf{e}_\theta$, with a frequency $\lambda/2\pi = 2.5$GHz. The support of this current is a little disc centered around the middle of the domain. This current generates a wave that propagates circularly around the current source. Physically, as long as the wave has not reached the reentrant corner, the field is smooth.

Let t_I be the impact time of the wave on the reentrant corner. Then, if one writes $\mathbf{B}^k(t) = \mathbf{B}_R^k(t) + \mu_B^k(t)\mathbf{y}_S^k$, $\mu_B^k(t) = 0$ for all t lower than t_I, and $\mathbf{B}^k(t)$ and $\mathbf{B}_R^k(t)$ coincide. On the other hand, for $t > t_I$, $\mu_B^k(t) \neq 0$ (and so $\mu_B^k(t)\mathbf{y}_S^k$ is) and the total field differs from its regular part.

This behavior is illustrated, for $k = 1$, on Figures 2 and 3. Similar results are obtained for other values of k.

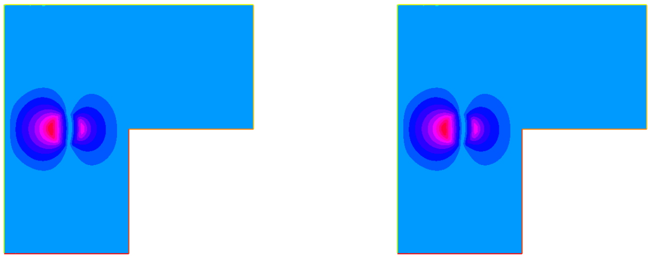

Fig. 2 $\mathbf{B}^1(t_1)$ and $\mathbf{B}_R^1(t_1)$, for $t_1 < t_I$ (case $k = 1$), z-component.

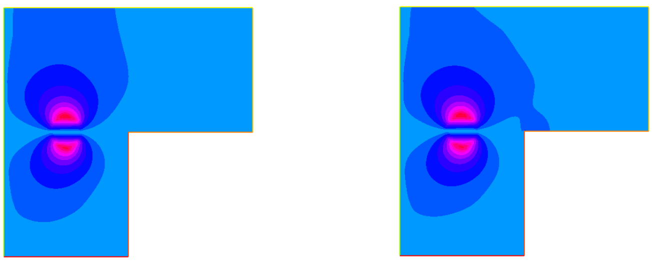

Fig. 3 $\mathbf{B}^1(t_2)$ and $\mathbf{B}_R^1(t_2)$, for $t_2 > t_I$ (case $k = 1$), r-component, in 2D and 3D view.

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