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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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An optimized sixth-order explicit RKN method to solve oscillating systems

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Abstract

Optimization of the sixth-order explicit Runge-Kutta-Nyström method with six stages derived by El-Mikkawy and Rahmo using the phase-fitted and amplification-fitted techniques with constant step-size is developed in this paper. The new method integrates exactly the common test: $y'' = -w^2 y$. The local truncation error of the new method is computed, showing that the order of convergence is maintained. The stability analysis is addressed, showing that the developed method has a periodicity interval. The numerical experiments demonstrate the high performance of the proposed scheme compared to other existing RKN codes with six stages and same order.

1. Introduction

This paper aim to effectively solve the special second-order initial-value problem of the form

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0, \tag{1.1}$$

assuming that their solutions are oscillatory, where $y \in \Re^d$ and $f : \Re \times \Re^d \to \Re^d$ are sufficiently differentiable. In recent and past years, the search of new numerical algorithms to effectively solve (1.1) has brought the attention of many researchers due to the great role the problem played in so many areas of applied sciences. To solve (1.1) directly, the class of Runge-Kutta-Nyström (RKN) methods has been largely used. Regarding the effective use of these methods, some RKN methods of sixth-order with six stages have been developed in [6], [7], and [1]. A lot of adapted RKN methods have been developed, which are of less algebraic-order than the constructed method in this paper. To mention a few, we cite those in [2, 3, 9–11, 13]. Recently, Demba et al. [4, 5] derived two new explicit RKN methods trigonometrically adapted for solving the kind of problems in (1.1).
This work aims at the development of a new phase- and amplification-fitted sixth order explicit RKN method with six stages based on the sixth order method of the RKN6(4)6ER pair given in [1] for solving the problem in (1.1). The derived method solves exactly the test equation $y'' = -w^2 y$. The numerical experiments reveal the effectiveness of the developed method compared to standard RKN codes of sixth order with six stages.

The remaining part of this paper is organized in this way: the basic theory of explicit RKN methods, the definitions of phase-lag and amplification error, and the definitions regarding the stability analysis are addressed in Section 2. Section 3 is devoted to the construction of the new code, to determine the order and error analysis, and to bring some details about the periodicity interval of the derived code. Some numerical examples are presented in Section 4, showing the good performance of the proposed scheme. Comments on the obtained results are given in Section 5, and finally, Section 6 gives a conclusion.

2. Fundamental Concepts

2.1. Explicit Runge-Kutta-Nyström Methods

An explicit RKN method with r stages is generally expressed by the formulas:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{l=1}^{r} b_l f(x_n + c_l h, Y_l), \tag{2.1}$$

$$y'_{n+1} = y'_n + h \sum_{l=1}^{r} d_l f(x_n + c_l h, Y_l), \tag{2.2}$$

$$Y_l = y_n + c_l hy'_n + h^2 \sum_{j=1}^{l-1} a_{lj} f(x_n + c_j h, Y_j), \quad l = 1, \ldots, r, \tag{2.3}$$

where as usual, y_{n+1} and y'_{n+1} denote approximations for $y(x_{n+1})$ and $y'(x_{n+1})$, respectively, and the grid points on the integration interval $[x_0, x_N]$ are given by $x_j = x_0 + jh,\ j = 0, 1, \ldots, N$, with h a fixed step-size.
The above method may be formulated compactly using the Butcher array in the form

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \end{array}$$

being $A = (a_{ij})_{r\times r}$ a matrix of coefficients, $c = (c_1, c_2, \ldots, c_r)^T$ is the vector of stages, and $b = (b_1, b_2, \ldots, b_r)^T$, $d = (d_1, d_2, \ldots, d_r)^T$ are two vectors containing the remaining coefficients of the method. For short, this can be denoted as (c, A, b, d).

Definition 1 ([8]) *An explicit Runge-Kutta-Nyström method as given in equations* (2)–(4) *is said to have algebraic order k if at any grid point x_{n+1} it holds*

$$\begin{cases} y_{n+1} - y(x_n + h) = O(h^{k+1}), \\ y'_{n+1} - y'(x_n + h) = O(h^{k+1}). \end{cases} \tag{2.4}$$

2.2. Analysis of Phase-lag, amplification error and stability

Applying the RKN method in (2)–(4) to the test equation $y'' = -w^2 y$, the phase-lag, amplification error and the linear stability analysis are derived. In particular, letting $\tilde{h} = -(wh)^2$, the approximate solution provided by (2)–(4) verifies the recurrence equation:

$$L_{n+1} = E(\tilde{h})L_n,$$

where

$$L_{n+1} = \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix},\ L_n = \begin{bmatrix} y_n \\ hy'_n \end{bmatrix},\ E(\tilde{h}) = \begin{bmatrix} 1 + \tilde{h}b^T N^{-1} e & wh(1 + \tilde{h}b^T N^{-1} c) \\ -whd^T N^{-1} e & 1 + \tilde{h}d^T N^{-1} c \end{bmatrix},$$

$N = I - \tilde{h}A$, with I the identity matrix of dimension six, $A = \left(a_{ij}\right)_{6\times 6}$, b, c, d are the corresponding matrix and vectors collecting the coefficients, and $e = [1, 1, 1, 1, 1, 1]^T$.

For enough small values of $\mu = wh$, it can be assumed that the matrix $E(\tilde{h})$ possesses conjugate complex eigenvalues [15]. Under this assumption, an oscillatory numerical solution should be provided by the method. The oscillatory character depends on the eigenvalues of the stability matrix $E(\tilde{h})$. The characteristic equation of this matrix can be expressed as:

$$\lambda^2 - \lambda Tr(E(\tilde{h})) + Det(E(\tilde{h})) = 0. \tag{2.4}$$

Theorem 2 ([1]) *If we apply to the common test equation $y'' = -w^2 y$ the Runge-Kutta-Nyström scheme in* (2)–(4), *we get the formula for calculating directly the phase-lag* (*or dispersion error*) $\Psi(\mu)$ *given by*:

$$\Psi(\mu) = \mu - cos^{-1}\left(\frac{Tr(E(\tilde{h}))}{2\sqrt{Det(E(\tilde{h}))}}\right). \tag{2.4}$$

If $\Psi(\mu) = O(\mu^{l+1})$, then it is said that the method has a phase-lag of order l. For an explicit RKN method, $Tr(E(\tilde{h})$ and $Det(E(\tilde{h})$ are polynomials in μ (in case of an implicit RKN method these would be rational functions).

Definition 3 *An explicit Runge-Kutta-Nyström method as given in equations* (2)–(4) *is said to be phase-fitted, if the phase-lag is zero.*

Definition 4 ([1]) *For the Runge-Kutta-Nyström method given in equations* (2)–(4)*, the value* $\beta(\mu) = 1 - \sqrt{Det(E(\tilde{h})}$ *is known as the amplification error* (*or dissipative error*)*. If* $\beta(\mu) = O(\mu^{s+1})$*, then it is said that the method has an amplification error of order s.*

Definition 5 *An explicit Runge-Kutta-Nyström method as given in equations* (2)–(4) *is said to be amplification-fitted if the amplification-error is zero.*

Definition 6 *An interval* $(-\tilde{h}_b, 0)$*,* $\tilde{h}_b \in \Re$ *is named as the primary interval of periodicity of the method in* (2)–(4)*, if* $\tilde{h}$ *is the highest value such that for all* $\tilde{h}_b \in (-\tilde{h}_b, 0)$*,* $|\lambda_{1,2}| = 1$ *and* $\lambda_1 \neq \lambda_2$*. Where* $\lambda_{1,2}$ *are the solutions of the equation in* (2.2).

3. Development of the new scheme

In this section, we will obtain a sixth order explicit phase- and amplification-fitted RKN scheme based on the higher-order method in the RKN6(4)6ER embedded pair derived by El-Mikkawy and Rahmo in [7], which we named as RKN6-6ER. The coefficients of the sixth order RKN method in [7] are shown in Table 1 with the correct value of a_{54} as given in [1].

Tab. 1 The RKN6-6ER Method in [7]

0						
$\frac{1}{77}$	$\frac{1}{11858}$					
$\frac{1}{3}$	$-\frac{7189}{17118}$	$\frac{4070}{8559}$				
$\frac{2}{3}$	$\frac{4007}{2403}$	$-\frac{589655}{355644}$	$\frac{25217}{118548}$			
$\frac{13}{15}$	$-\frac{4477057}{843750}$	$\frac{13331783894}{2357015625}$	$-\frac{281996}{5203125}$	$\frac{563992}{7078125}$		
1	$\frac{17265}{2002}$	$-\frac{1886451746}{212088107}$	$\frac{22401}{31339}$	$\frac{2964}{127897}$	$\frac{178125}{5428423}$	
	$-\frac{341}{780}$	$\frac{386683451}{661053840}$	$\frac{2853}{11840}$	$\frac{267}{3020}$	$\frac{9375}{410176}$	0
	$-\frac{341}{780}$	$\frac{29774625727}{50240091840}$	$\frac{8559}{23680}$	$\frac{801}{3020}$	$\frac{140625}{820352}$	$\frac{847}{18240}$

In order to get the new adapted scheme, we equate to zero the phase-lag $\Psi(\mu)$ and the amplification error $\beta(\mu)$, and we get the system:

$$\begin{cases} \Psi(\mu) = 0 \\ \beta(\mu) = 0. \end{cases} \tag{3.0}$$

We solve this system considering the coefficients in Table 1 except two of them which are taking as unknowns. Specifically, we take b_5 and d_5 as unknowns. We obtain the following values:

$$
\begin{aligned}
b_5 \quad = \quad & \frac{2503125}{410176M}\Bigg(-1258632233707368303463680000000 + 524994684043706387148025080000\,\mu^2 \\
& +38027832783293925493906168800\,\mu^4 - 4230511004002098685547254 5000\,\mu^6 + 638949635090375307952 5017100\,\mu^8 \\
& -396360945814751886526623990\,\mu^{10} + 123936749198262707148 85995\,\mu^{12} - 163757382111950819488686\,\mu^{14} \\
& +443880244626070278520\,\mu^{16} - 355613551745891361931008 0000\,\mu^2 \cos(\mu) \\
& +79692959576555263252169856 00\,\mu^6 \cos(\mu) - 125718020321097360886329600\,\mu^8 \cos(\mu) \\
& -742693155585909485806937 08800\,\mu^4 \cos(\mu) + 1258632233707368303463680000000\,\cos(\mu) \Bigg),
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
d_5 \quad = \quad & \frac{625}{820352M}\Bigg(4882682690886773063720\,\mu^{20} - 17664354381917313486 92196\,\mu^{18} + 142671286498878012015349560\,\mu^{16} \\
& -1012622614389216610961601 5370\,\mu^{14} + 4754939043963115273766323 26825\,\mu^{12} \\
& +546881973050840782778527104 00\,\mu^{10} \cos(\mu) - 1039168019912554487955565 2445650\,\mu^{10} \\
& -10381900296589462467492329664000\,v^8 \cos(v) + 113086760758089573241298829586500\,\mu^8 \\
& +25369020404906010573240339840 0000\,\mu^6 \cos(\mu) - 38172183245988102106377647719 5000\,\mu^6 \\
& -17758939055466816939883595736 60000\,\mu^4 - 6305505579734821871351779238 40000\,\mu^4 \cos(\mu) \\
& +3199753041551405164628715874 5000000\,\mu^2 - 67210961279973467404960512 0000000\,\mu^2 \cos(v) \\
& +75612331439970150830580576000000000\,\cos(\mu) - 75612331439970150830580576000000000 \Bigg),
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
M \quad = \quad & \mu^2\Bigg(-288033107434255930802343750000000 + 480055179057093218003906250000 00\,\mu^2 \\
& +2409864727821008473951031 25000\,\mu^4 - 211575854747321234593653037500\,\mu^6 \\
& +27693379469414224574322792750\,\mu^8 - 1543565245575968927989765335\,\mu^{10} \\
& +55158851048499641449369350\,\mu^{12} - 861578557170344748268248\,\mu^{14} + 2441341345443386531860\,\mu^{16} \Bigg).
\end{aligned}
$$

The corresponding Taylor series expansions in powers of μ are given by

$$
\begin{aligned}
b_5 &= \frac{9375}{410176} - \frac{261461}{93847723200}\mu^6 + \frac{20361401}{369525410100000}\mu^8 - \frac{177044709462626977}{866977960060782108000000 0}\mu^{10} \\
&+ \frac{113475585753433129 22557}{887568686612225683065000000000}\mu^{12} - \frac{1014777911601836484322385 39}{136685577738282755192010000000000000}\mu^{14} + \cdots \quad , \\
d_5 &= \frac{140625}{820352} - \frac{1}{213290280}\mu^6 - \frac{618923}{739050820200}\mu^8 - \frac{1251344791}{93120403345200000}\mu^{10} \\
&- \frac{190297638076116325219}{7396405721768547358875000000}\mu^{12} + \frac{35276945432092739240316 79}{99407692900569276503280000000000 0}\mu^{14} + \cdots \quad .
\end{aligned}
\tag{3.3}
$$

As expected, when $\mu \to 0$, the newly obtained coefficients b_5, d_5 become the coefficients of the counterpart scheme in the original method. The new adapted RKN scheme will be named as PFAFRKN6-6ER.

3.1. Order of Convergence

This section is devoted to present the local truncation error of the proposed method and to get the order of convergence. This is accomplished by using the usual tool of Taylor expansions. The local truncation errors (LTE) at the point x_{n+1} of the solution and the first derivative are given respectively by:

$$
\begin{aligned}
LTE &= y_{n+1} - y(x_n + h), \\
LTE_{der} &= y'_{n+1} - y'(x_n + h).
\end{aligned}
\tag{3.3}
$$

Proposition 7 *The corresponding LTEs of the formulas to provide the solution and the derivative with the new RKN method are, respectively*:

$$
\begin{aligned}
LTE &= \frac{h^7}{213290280}(f_y)^2(f_x + f_y y') + O(h^8), \\
LTE_{der} &= \frac{h^7}{5040}(f_{xxxxxx} + 15(y')^4 f_{yyyy} y'' + 60(y')^3 f_{xyyyy} y'' + 60y' f_{xxxyy} y'' + 90(y')^2 f_{xxyyy} y'' + 21 f_y f_{yxx} y'' \\
&+60y'' f_{xyy} f_x + 15y'' f_{yy} f_{xx} + 18(y'')^2 f_{yy} f_y + 90y' f_{xyyy}(y'')^2 + 45(y')^2 f_{yyyy}(y'')^2 + 33(y')^2 (f_{yy})^2 y'' \\
&+48y' f_{xy} f_{yxx} + 10 f_y f_{xy} f_x + 12(f_y)^2 y' f_{xy} + 60y' f_{xxyy} f_x + 60(y')^2 f_{xyyy} f_x + 20(y')^3 f_{yyyy} f_x \\
&+24 f_y y' f_{xxxy} + 30y' f_{xyy} f_{xx} + 15(y')^2 f_{yyy} f_{xx} + 6y' f_{yy} f_{xxx} + 78(y')^2 f_{xyy} f_{xy} + 66(y')^2 f_y f_{xxyy} \\
&+33(y')^2 f_{yy} f_{yxx} + 64(y')^3 f_y f_{xyyy} + 36(y')^3 f_{yyy} f_{xy} + 48(y')^3 f_{yy} f_{xyy} + 21(f_y)^2 (y')^2 f_{yy} \\
&+21(y')^4 f_y f_{yyyy} + 21(y')^4 f_{yyy} f_{yy} + 15(y'')^3 f_{yyy} + 45(y'')^2 f_{xxyy} + 15y'' f_{xxxxy} + 18y''(f_{xy})^2 \\
&+(f_y)^3 y'' + (y')^6 f_{yyyyyy} + 6(y')^5 f_{xyyyyy} + (f_y)^2 f_{xx} + 6 f_{xxx} f_{xy} + f_y f_{xxxx} + 20 f_x f_{xxxy} \\
&+6y' f_{xxxxxy} + 15 f_{yxx} f_{xx} + 15(y')^4 f_{xxyyyy} + 15(y')^2 f_{xxxxyy} + 20(y')^3 f_{xxxyyy} + 10 f_{yy}(f_x)^2 \\
&+81(y')^2 f_{yyy} f_y y'' + 60y' f_{yyy} f_x y'' + 102y' f_y f_{xyy} y'' + 66y' f_{yy} f_{xy} y'' + 30y' f_{yy} f_y f_x) + O(h^8),
\end{aligned}
\tag{3.4}
$$

*from which we can infer that the PFAFRKN*6-6*ER method has order six.*

3.2. Periodicity interval of the new method

Using the Maple package, from the definition in (6), the following result can be readily obtained.

Proposition 8 *The newly derived method, PFAFRKN*6-6*ER, has* $(-39.11, 0)$ *as the primary interval of periodicity.*

4. Some Numerical Examples

To assess the performance of the new scheme, we have considered the following RKN codes of the same order and stages to get fair comparisons:

- PFAFRKN6-6ER: The constructed adapted RKN code developed here,
- RKN6-6ER: An explicit sixth-order six stage RKN method presented in [7],
- RKN6-6ER-PFAF: An optimized explicit sixth-order six stage RKN method derived by Anastassi and Kosti in [1],
- RKN6-6FM: An explicit sixth-order six stage RKN method developed by Dormand et al. in [6].

We will consider different oscillatory problems appeared in the literature to test the performance of the above methods:

Problem. (Non-linear System in [14])

$$\begin{aligned} y_1'' + w^2 y_1 &= \frac{2y_1y_2 - \sin(2wx)}{(y_1{}^2 + y_2{}^2)^{\frac{3}{2}}}, \quad y_1(0) = 1, y_1'(0) = 0, \\ y_2'' + w^2 y_2 &= \frac{y_1{}^2 - y_2{}^2 - \sin(2wx)}{(y_1{}^2 + y_2{}^2)^{\frac{3}{2}}}, \quad y_2(0) = 0, y_2'(0) = w, \quad x \in [0, 4000], \end{aligned}$$

with exact solution given by

$$y_1(x) = \cos(wx), \quad y_2(x) = \sin(wx).$$

To use the adapted methods we have taken the parameter value $w = 5$.

To tests the accuracy of the considered methods we have taken the integration interval $[x_0, x_N] = [0, T]$, with step-length, $h = \frac{x_N - x_0}{N} = \frac{T}{N}$, where the end point takes different values, $T = 100, 1000, 4000$.

The numerical data is given in Tables 2, considering different step-sizes h. The tables contain the maximum absolute errors

$$Max\ Abs\ Err = \max_{n=1,2,\ldots,N} ||y(x_n) - y_n||.$$

Tab. 2 Numerical data corresponding to Problem

h	Methods	T = 100	T = 1000	T = 4000
0.05	PFAFRKN6-6ER	3.802533(-10)	2.155096(-9)	9.277232(-9)
	RKN6-6ER	4.282131(-8)	1.632977(-7)	1.632977(-7)
	RKN6-6ER-PFAF	1.899990(-8)	7.334019(-8)	7.667858(-8)
	RKN6-6FM	1.953175(-7)	7.542634(-7)	7.542634(-7)
0.075	PFAFRKN6-6ER	9.475666(-9)	3.697725(-8)	3.697725(-8)
	RKN6-6ER	7.249395(-7)	2.799656(-6)	2.804762(-6)
	RKN6-6ER-PFAF	1.214422(-5)	4.709903(-5)	4.713322(-5)
	RKN6-6FM	2.236286(-6)	8.631087(-6)	8.634346(-6)
0.1	PFAFRKN6-6ER	9.349917(-8)	3.600327(-7)	3.600327(-7)
	RKN6-6ER	5.491464(-6)	2.106615(-5)	2.106615(-5)
	RKN6-6ER-PFAF	1.223785(-3)	4.727541(-3)	4.727541(-3)
	RKN6-6FM	1.261951(-5)	4.880468(-5)	4.881228(-5)
0.125	PFAFRKN6-6ER	5.305980(-7)	2.048570(-6)	2.048570(-6)
	RKN6-6ER	2.598789(-5)	1.008581(-4)	1.008581(-4)
	RKN6-6ER-PFAF	4.319413(-2)	2.315899(-1)	2.324977(-1)
	RKN6-6FM	4.804257(-5)	1.875787(-4)	1.876650(-4)

In order to show even more the efficiency of the developed PFAFRKN6-6ER code, we present the efficiency curve for the considered problem for $T = 100$. In Figure 1 the logarithm of the maximum absolute global error versus the logarithm of the total number of function evaluations have been plotted. It can be observed the good behavior of the new code.

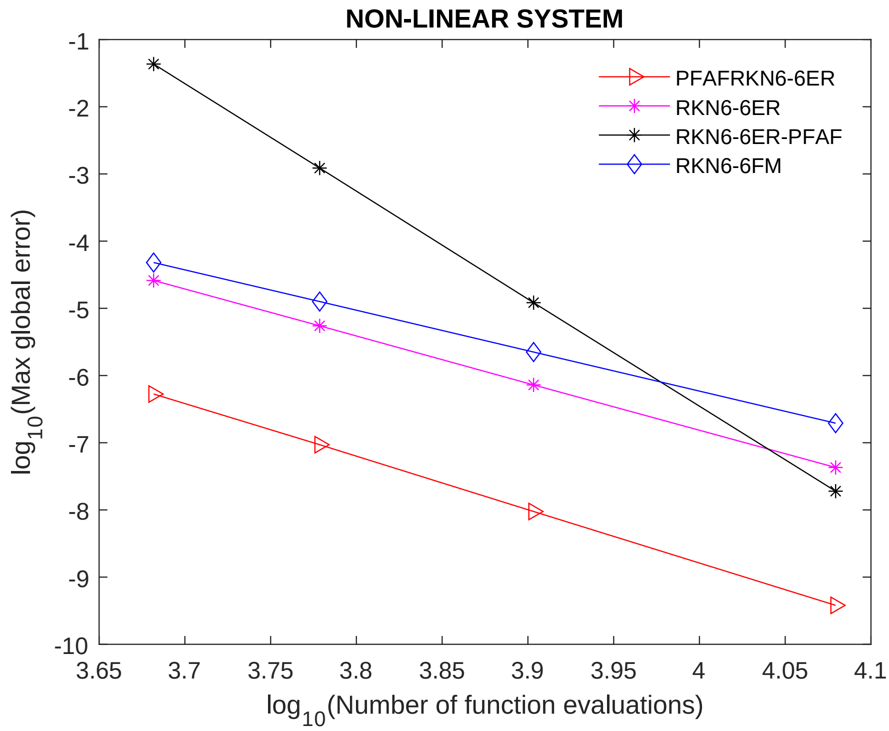

Fig. 1 Efficiency curves corresponding to the Problem

To further demonstrate the efficiency of the constructed PFAFRKN6-6ER code, we present in Figures 2 the logarithms of the maximum absolute global errors versus the CPU time used. It can be observed the good behavior of the new code.

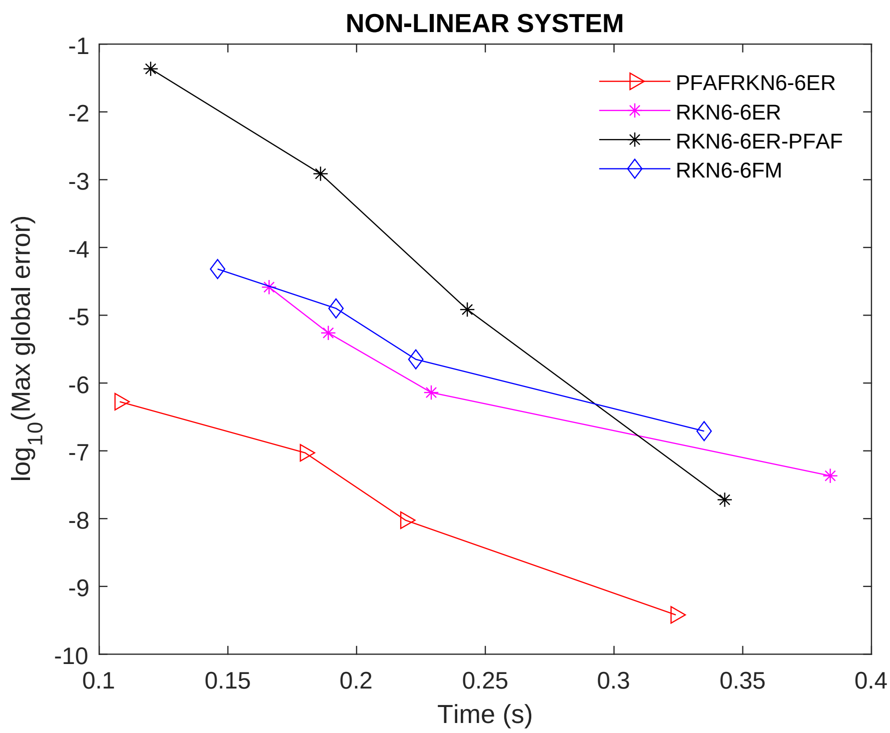

Fig. 2 Efficiency curves corresponding to Problem

5. Discussion

The new PFAFRKN6-6ER code gives minimum error norm, minimum number of function evaluation per steps, and minimum computational cost (Time(s)). Table 2 and Figures 1 – 2 put an evidence that PFAFRKN6-6ER is a very efficient scheme. Therefore, we can say that PFAFRKN6-6ER is more appropriate for solving the type of problem in (1.1) than the other existing RKN methods of order 6 with six stages in the literature.

6. Conclusion

In this study, we have used the methodology for constructing the phase-fitted and amplification-fitted methods to develop an efficient new explicit phase- and amplification-fitted RKN code based on the RKN6-6ER method due to El-Mikkawy and Rahmo [7]. The new developed method has two variable coefficients depending on the parameter $\mu = wh$, which is usually known as the parameter frequency [12, 16]. We computed the local truncation error of the new method, confirming that the order of convergence of the underlying code is maintained. In addition, the periodicity interval of the new code has been obtained. The obtained numerical results clearly show that PFAFRKN6-6ER is more accurate and efficient than other sixth-order six-stage RKN codes in the literature.

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