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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics SēMA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the SēMA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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On numerical approximations to diffuse-interface tumor growth models

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Abstract

This work is devoted to developing new numerical schemes for a tumor-nutrient PDE model. It is based on phase field equations for the tumor variable and a diffusive equation for the nutrient one, coupled by reaction terms and cross-diffusion terms. The model conserves the sum of tumor+nutrient and has a dissipative energy law.

We introduce two different time-discrete schemes: one is based on an Eyre-type decomposition of the energy and the other is an energy quadratization scheme. Both (continuous) Finite Elements and Discontinuous Galerkin are used for space discretization.

The schemes are compared analytically and computationally.

1. Introduction

In [3] this tumor-nutrient PDE model is proposed:

$$\partial_t u = \nabla \cdot (M_u \nabla \mu_u) + \delta P_0 u_+ (\mu_n - \mu_u) \qquad \text{in } \Omega \times (0,T], \qquad (1.1a)$$

$$\mu_u = f'(u) - \varepsilon^2 \Delta u - \chi_0 n \qquad \text{in } \Omega \times (0,T], \qquad (1.1b)$$

$$\partial_t n = \nabla \cdot (M_n \nabla \mu_n) - \delta P_0 u_+ (\mu_n - \mu_u) \qquad \text{in } \Omega \times (0,T], \qquad (1.1c)$$

$$\mu_n = \frac{1}{\delta} n - \chi_0 u \qquad \text{in } \Omega \times (0,T], \qquad (1.1d)$$

$$\nabla u \cdot \mathbf{n} = M_u \nabla \mu_u \cdot \mathbf{n} = M_n \nabla n \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times (0,T], \qquad (1.1e)$$

$$u(x,0) = u_0, \quad n(x,0) = n_0 \qquad \text{in } \Omega, \qquad (1.1f)$$

where u is the tumor cell volume fraction and n is the nutrient density. Here M_u, M_n are nonnegative mobility functions, $\varepsilon, \delta, \chi_0 > 0$ and $f(u) = \Gamma u^2 (1-u)^2$, $\Gamma > 0$. Then, $f(u)$ is a polynomial double-well potential with stable values 0 and 1.

This model is conservative in the sense that $\partial_t \left(\int_\Omega u + n \right) = 0$ and the following energy is dissipated:

$$E(u,n) := \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 + \int_\Omega f(u) - \chi_0 \int_\Omega un + \frac{1}{2\delta} \int_\Omega n^2. \qquad (1.2)$$

Specifically, the following energy law holds:

$$\frac{d}{dt} E = - \int_\Omega M_u |\nabla \mu_u|^2 - \int_\Omega M_n |\nabla \mu_n|^2 - \delta P_0 \int_\Omega u_+ (\mu_n - \mu_u)^2.$$

Well-posedness of this model is studied in [1, 2] for the case $\chi_0 = 0$ while, to the best knowledge of the authors, it is still an open problem for the cross-difussion case ($\chi_0 \neq 0$).

Also in [3], an Eyre-type (convex-splitting) time-discrete numerical scheme is studied which conserves the mass and has non-increasing discrete energy, giving several numerical examples using (continuous) Finite Elements (FE) as space discretization.

In this work, we introduce a slight modification of the nonlinear time semidiscretization shown in [3] which will lead to a first order linear scheme. Besides, we propose another semidiscrete-time scheme based on Invariant Energy Quadratization (IEQ). Furthermore, we extend the previous FE scheme to a Discontinuous Galerkin space discretization (DG). Finally, we compare the results both theoretically and computationally.

2. Discrete schemes

Troughout this section we show the different aforementioned time-semidiscrete and space-semidiscrete schemes. In particular, we focus on the theoretical properties of the Eyre-FE and IEQ-DG fully-discrete schemes. The proofs of the results will appear together with a numerical comparison of the other different time-space combinations in a forthcoming paper.

2.1. Fully-discrete Eyre-FE scheme

The Eyre-type scheme consist of splitting the energy functional (1.2) into two terms

$$E(u, n) = E_i(u, n) + E_e(u, n),$$

where $E_i(u, n)$ is a convex term (that we will treat implicitly) and $E_e(u, n)$ is a non-convex term (which we will treat explictly) so as to avoid the nonlineaity of the model (1.1a)–(1.1f).

In [3] the following splitting is considered

$$E_i(u, n) := \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 + \frac{3\Gamma}{2} \int_\Omega u^2 - \chi_0 \int_\Omega un + \left(\frac{\alpha}{2} + \frac{1}{2\delta}\right) \int_\Omega n^2,$$

$$E_e(u, n) := \Gamma \int_\Omega \left(u^4 - 2u^3 - \frac{1}{2}u^2\right) - \frac{\alpha}{2} \int_\Omega n^2,$$

where $\alpha \in \mathbb{R}$ is a stabilization parameter which must satisfy that $\alpha > \frac{\chi_0^2}{3\Gamma}$ in order to make $E_i(\cdot, \cdot)$ convex with respect to u and n independently.

Regarding the previous splitting, we propose the following linear scheme using a FE space-semidiscretization associated with a triangulation family $\mathcal{T}_h$ of Ω: find, for each $m \in \mathbb{N} \cup \{0\}$, $u^{m+1}, n^{m+1}, \mu_u^{m+1} \in \mathbb{P}_k^{cont}(\mathcal{T}_h)$ such that for every $\overline{\mu}_u, \overline{u}, \overline{\mu}_n \in \mathbb{P}_k^{cont}(\mathcal{T}_h)$,

$$\left(\delta_t u^{m+1}, \overline{\mu}_u\right)_{L^2(\Omega)} = -\left(M_u(u^m)\nabla \mu_u^{m+1}, \nabla \overline{\mu}_u\right)_{(L^2(\Omega))^d} + \delta P_0\left(u_+^m(\mu_n^{m+1} - \mu_u^{m+1}), \overline{\mu}_u\right)_{L^2(\Omega)}, \tag{2.1a}$$

$$\left(\mu_u^{m+1}, \overline{u}\right)_{L^2(\Omega)} = \varepsilon^2\left(\nabla u^{m+1}, \nabla \overline{u}\right)_{(L^2(\Omega))^d} + 3\Gamma\left(u^{m+1}, \overline{u}\right)_{L^2(\Omega)}$$

$$+ \Gamma\left(4(u^m)^3 - 6(u^m)^2 - u^m, \overline{u}\right)_{L^2(\Omega)} - \chi_0\left(n^{m+1}, \overline{u}\right)_{L^2(\Omega)}, \tag{2.1b}$$

$$\left(\delta_t n^{m+1}, \overline{\mu}_n\right)_{L^2(\Omega)} = -\left(M_n(n^m)\nabla \mu_n^{m+1}, \nabla \overline{\mu}_n\right)_{(L^2(\Omega))^d} - \delta P_0\left(u_+^m(\mu_n^{n+1} - \mu_u^{m+1}), \overline{\mu}_n\right)_{L^2(\Omega)}, \tag{2.1c}$$

where we denote $\delta_t u^{m+1} = \frac{u^{m+1}-u^m}{\Delta t}$ and

$$\mu_n^{m+1} = \frac{1}{\delta}n^{m+1} + \alpha\left(n^{m+1} - n^m\right) - \chi_0 u^{m+1}. \tag{2.2}$$

Remark 2.1 The previous time-semidiscrete scheme (2.1) differs from the one proposed in [3] in the way we treat u_+ in the reaction terms. Whereas in [3] this term is approximated using a Crank-Nicolson scheme, we just treat this term explicitly in order to avoid the nonlinearity.

Despite using a lower-order time-discretization scheme than in [3] for one of the terms in (2.1a)–(2.1c), we can also afford first-order consistency in time. In particular, using the ideas in [3] we get the following result.

Theorem 2.2 *The scheme* (2.1a)–(2.1c) *has the following properties*:

1. *If there is a smooth enough solution u, μ_u, n of* (1.1a)–(1.1f)*, then the scheme has first-order consistency in time.*

2. *The mass is conserved in the following sense*: $\int_\Omega(u^{m+1} + n^{m+1}) = \int_\Omega(u^m + n^m)$ *for $m \geq 0$.*

3. *If $u^i \in \left[\frac{1}{2} - \frac{1}{\sqrt{3}}, \frac{1}{2} + \frac{1}{\sqrt{3}}\right]$ for $i \in \{m, m+1\}$, $m \geq 0$, then $E(u^{m+1}, n^{m+1}) \leq E(u^m, n^m)$.*

Remark 2.3 It is granted by the previous result that the energy of the solution of the scheme (2.1a)–(2.1c) decreases as long as the solution u belongs to the interval $\left[\frac{1}{2} - \frac{1}{\sqrt{3}}, \frac{1}{2} + \frac{1}{\sqrt{3}}\right] \approx [-0.08, 1.08]$, where $E_e(\cdot, \cdot)$ is concave. Nevertheless, the model (1.1a)–(1.1f) has no maximum principle so the solution may not be bounded by this range. Hence, it is not guaranteed that the energy always decreases with this scheme (2.1a)–(2.1c).

2.2. Fully-discrete IEQ-DG scheme

The following space discretization of the variational formulation of the previous model (1.1a)–(1.1f) using Symmetric Interior Penalty (SIP) Discontinuous Galerkin is introduced: find $u(\cdot, t), n(\cdot, t), \mu_u(\cdot, t) \in \mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$ for a.e. $t \in (0, T)$ such that

$$\int_\Omega \partial_t u \overline{\mu}_u = -a_h^{\text{sip}}(M_u(u); \mu_u, \overline{\mu}_u) + \delta P_0 \int_\Omega u_+(\mu_n - \mu_u)\overline{\mu}_u, \tag{2.3a}$$

$$\int_\Omega \mu_u \overline{u} = a_h^{\text{sip}}(\varepsilon^2; u, \overline{u}) + \int_\Omega f'(u)\overline{u} - \chi_0 \int_\Omega n\overline{u}, \tag{2.3b}$$

$$\int_\Omega \partial_t n \overline{\mu}_n = -a_h^{\text{sip}}(M_n(n); \mu_n, \overline{\mu}_n) - \delta P_0 \int_\Omega u_+(\mu_n - \mu_u)\overline{\mu}_n, \tag{2.3c}$$

$$\mu_n = \frac{1}{\delta} n - \chi_0 u, \tag{2.3d}$$

for every $\overline{\mu}_u, \overline{u}, \overline{\mu}_n \in \mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$, with $a_h^{\text{sip}}(\cdot; \cdot, \cdot)$ the SIP-blinear form defined for $v, w \in \mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$ as in [4], where this kind of IEQ-DG schemes are studied for the Cahn-Hilliard equations:

$$a_h^{\text{sip}}(M(a); v, w) := \sum_{K \in \mathcal{T}_h} \int_K M(a) \nabla_h v \cdot \nabla_h w - \sum_{e \in \mathcal{E}_h^i} \int_e M(\{a\}) (\{\nabla_h v \cdot \mathbf{n}_e\} [[w]] + \{\nabla_h w \cdot \mathbf{n}_e\} [[v]])$$

$$+ \sigma \sum_{e \in \mathcal{E}_h^i} \int_e \frac{1}{h_e} M(\{a\}) [[v]] [[w]].$$

Regarding the IEQ time-semidiscretization we have that, taking $B > 0$ and defining the artifical variable $U = \sqrt{f(u) + B}$ and $H(u) = \frac{f'(u)}{2\sqrt{f(u)+B}}$, then $f'(u) = 2H(u)U(u)$ and

$$\partial_t U = H(u)\partial_t u. \tag{2.4}$$

Now, we will approximate (2.4) in the $(m + 1)$-th time iteration in two steps: first, we use the projection operator $\Pi_h \colon L^2(\Omega) \longrightarrow \mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$ to calculate $U_h^m = \Pi_h U^m$ and then we use the semidiscrete scheme

$$\frac{U^{m+1} - U_h^m}{\Delta t} = H(u^m)\frac{u^{m+1} - u^m}{\Delta t},$$

where $U^0 = \sqrt{f(u^0) + B}$.

Therefore, we propose the next fully discrete IEQ-DG scheme: find, for each $m \in \mathbb{N} \cup \{0\}$, $u^{m+1}, n^{m+1}, \mu_u^{m+1} \in \mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$ such that for every $\overline{\mu}_u, \overline{u}, \overline{\mu}_n \in \mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$,

$$\left(\delta_t u^{m+1}, \overline{\mu}_u\right) + a_h^{\text{sip}}(M_u(u^m); \mu_u^{m+1}, \overline{\mu}_u) = \delta P_0 \left(u_+^m(\mu_n^{m+1} - \mu_u^{m+1}), \overline{\mu}_u\right), \tag{2.5a}$$

$$\left(\mu_u^{m+1}, \overline{u}\right) = a_h^{\text{sip}}(\varepsilon^2; u^{m+1}, \overline{u}) + \left(2H(u^m)U^{m+1}, \overline{u}\right) - \chi_0(n^m, \overline{u}), \tag{2.5b}$$

$$\left(\delta_t n^{m+1}, \overline{\mu}_n\right) + a_h^{\text{sip}}(M_n(n^m); \mu_n^{m+1}, \overline{\mu}_n) = -\delta P_0 \left(u_+^m(\mu_n^{m+1} - \mu_u^{m+1}), \overline{\mu}_n\right), \tag{2.5c}$$

where

$$\mu_n^{m+1} = \frac{1}{\delta} n^{m+1} - \chi_0 u^{m+1}, \tag{2.6a}$$

$$U^{m+1} = U_h^m + H(u^m)(u^{m+1} - u^m). \tag{2.6b}$$

In practice, in order to solve the previous system of equations (2.5a)–(2.5c) minimising the computational costs, in each time step we do the following:

1. We introduce the expressions (2.6a)–(2.6b) in (2.5a)–(2.5c) and we solve the system of equations.

2. The approximation U_h^{m+1} is obtained by projecting (2.6b) into $\mathbb{P}_k^{\text{disc}}(\mathcal{T}_h)$.

Theorem 2.4 *The scheme* (2.5a)–(2.5c) *has the following properties*:

1. *The mass is conserved*: $\int_\Omega (u^{m+1} + n^{m+1}) = \int_\Omega (u^m + n^m)$.

2. *The following energy law holds*:

$$\delta_t E_h(u^{m+1}, U_h^{m+1}, n^{m+1})$$
$$+ a_h^{sip}(M_u(u^m); \mu_u^{m+1}, \mu_u^{m+1}) + a_h^{sip}(M_n(n^m); \mu_n^{m+1}, \mu_n^{m+1}) + \delta P_0 \int_\Omega u_+^m (\mu_n^{m+1} - \mu_u^{m+1})^2 \le 0$$

for the modified energy

$$E_h(u, U_h, n) = a_h^{sip}\left(\frac{\varepsilon^2}{2}; u, u\right) + \int_\Omega U_h^2 - \chi_0 \int_\Omega un + \frac{1}{2\delta} \int_\Omega n^2.$$

Remark 2.5 Observe that the EQ-DG scheme (2.5a)–(2.5c) is unconditionally energy-stable for a modified energy obtained by using the artificial variable U. Nonetheless, if the approximation is good enough (for small Δt and h) we have that $E_h(u^m, n^m) + B|\Omega| \approx E_h(u^m, U_h^m, n^m)$ where now

$$E_h(u, n) = a_h^{sip}\left(\frac{\varepsilon^2}{2}; u, u\right) + \int_\Omega f(u) - \chi_0 \int_\Omega un + \frac{1}{2\delta} \int_\Omega n^2,$$

is the natural discrete energy of the model in $\mathbb{P}_k^{disc}(T_h)$.

2.3. Numerical experiments

In this section we show several numerical experiments with the purpose of comparing both the Eyre-FE (2.1a)–(2.1a) and the IEQ-DG (2.5a)–(2.5c) schemes and studying some properties of the tumor model (1.1a)–(1.1f) by reproducing some results of the papers [3, 8].

Example 1. Circular tumor growth
We show the results we got with the two schemes for a initial small tumor (radius 0.1) located in the center of the domain $\Omega = [-1, 1]^2$, considering, at the beginning, the extracellular water to be completely nutrient-rich, $n_0 = 1$, in Ω.

We take the parameters $\varepsilon = 0.005$, $\delta = 0.01$, $P_0 = 0.1$, $\chi_0 = 0.05$, $\Gamma = 0.045$ y $\alpha = \frac{\chi_0^2}{3\Gamma} + 0.1$ as in [3] so as to reproduce its results. Likewise, we will take the mobility functions $M_u(u) = 200u^2$ and $M_n = \delta$. In the case of the IEQ-DG scheme we use $\sigma = 15$ and $B = 1$.

A time step and a mesh size $h \approx 0.04$ are used together with polynomials of order $k = 2$.
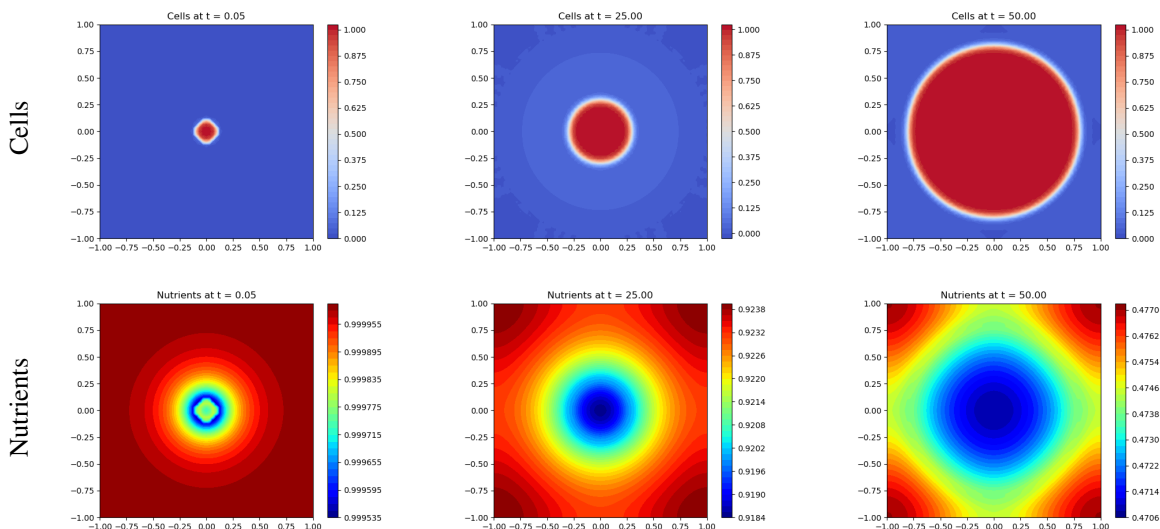
Fig. 1 Example 1. Cells and nutrients at different time steps ($\Delta t = 5 \cdot 10^{-2}$) with the **Eyre-FE** scheme (2.1).

Comparing Fig. 1 and Fig. 2, despite the qualitative behaviour of the schemes seem to be similar, the scheme IEQ-DG is much more unstable in time and a significantly smaller time step is needed to control the spurious oscillations over time. The time step used is $\Delta t = 0.05$ for Eyre-FE and $\Delta t = 0.002$ for IEQ-DG. This issue makes it very difficult to reach the final time $T = 50$ with the IEQ-DG scheme as it was easily done using the scheme
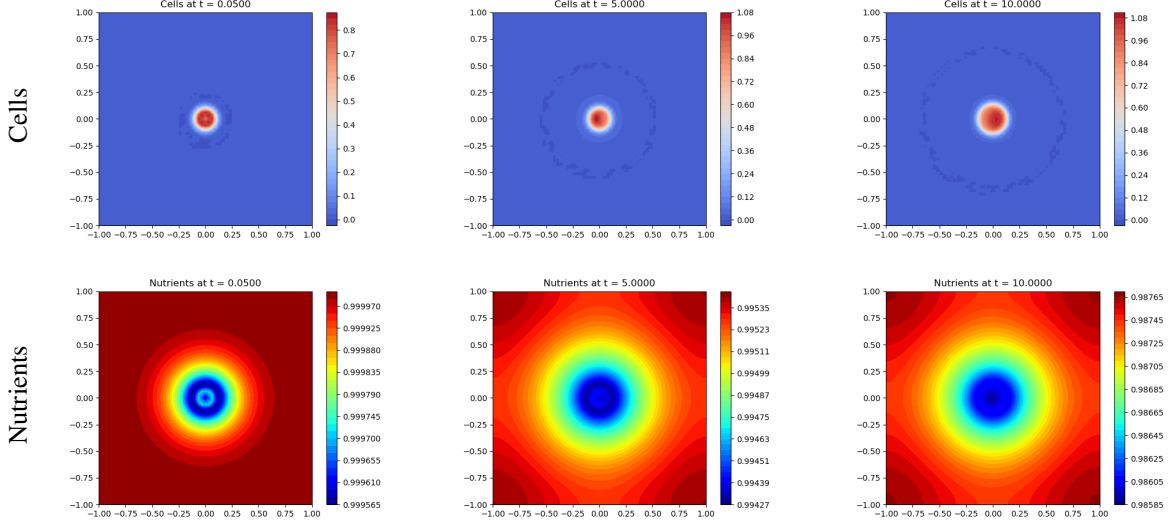
Fig. 2 Example 1. Cells and nutrients at different time steps ($\Delta t = 2 \cdot 10^{-3}$) with the **IEQ-DG** scheme (2.5).
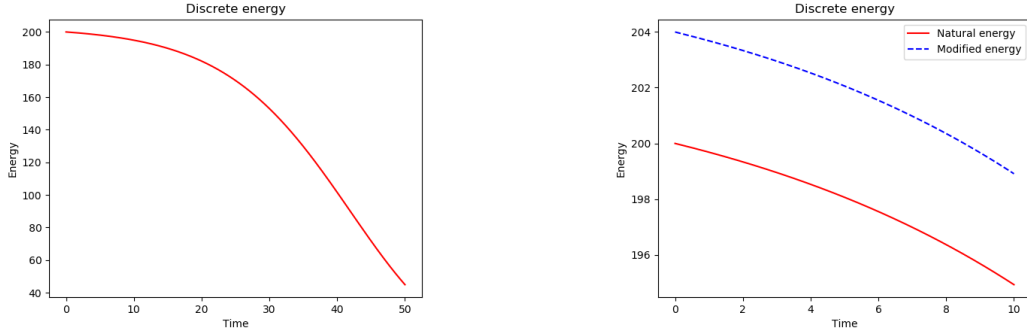

Fig. 3 Example 1. Discrete energy with the **Eyre-FE** scheme (2.1) on the left and with the **IEQ-DG** scheme (2.5) on the right.

Eyre-FE. The unstability of these well-known and widely studied IEQ time-semidiscretization technique has been spotted in several papers such as [5–7].

In both cases the mass is conserved and the energy decreases (Fig. 3). Notice that on the right of Fig. 3 we can observe that both the natural and the modified energies differ only in the constant $B|\Omega| = 4$.

Example 2. Elliptic tumor with nutrient source growth

Now, we slightly modify the model as it was done in [3] in order to increase the interaction between the tumor and the nutrients. To this aim, we guess that the diffusion of the nutrients is much faster than the growth of the tumor cells so we consider the elliptic version of the n-equation, taking $\partial_t n = 0$. Moreover, we change $\nabla n \cdot \mathbf{n} = 0$ to $n = 1$ on $\partial\Omega$.

$$\partial_t u = M_u \Delta \mu_u + \delta P_0 u_+ (\mu_n - \mu_u) \qquad \text{in } \Omega \times (0, T), \qquad (2.7a)$$

$$\mu_u = F'(u) - \varepsilon^2 \Delta u - \chi_0 n \qquad \text{in } \Omega \times (0, T), \qquad (2.7b)$$

$$0 = M_n \Delta \mu_n - \delta P_0 u_+ (\mu_n - \mu_u) \qquad \text{in } \Omega \times (0, T), \qquad (2.7c)$$

$$\mu_n = \frac{1}{\delta} n - \chi_0 u \qquad \text{in } \Omega \times (0, T), \qquad (2.7d)$$

$$\nabla u \cdot \mathbf{n} = \nabla \mu_u \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times (0, T), \qquad (2.7e)$$

$$n = 1 \qquad \text{on } \partial\Omega \times (0, T), \qquad (2.7f)$$

$$u(x, 0) = u_0, \quad n(x, 0) = n_0 \qquad \text{in } \Omega, \qquad (2.7g)$$
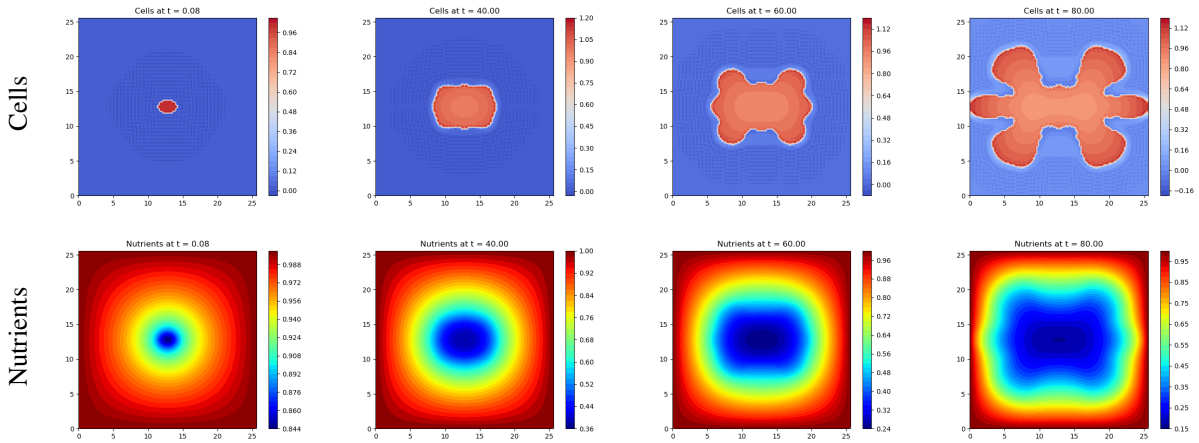
where $u_0, n_0 \in L^2(\Omega)$.

Fig. 4 Example 2. Cells and nutrients of the variant (2.7) with the **Eyre-FE** scheme (2.1).
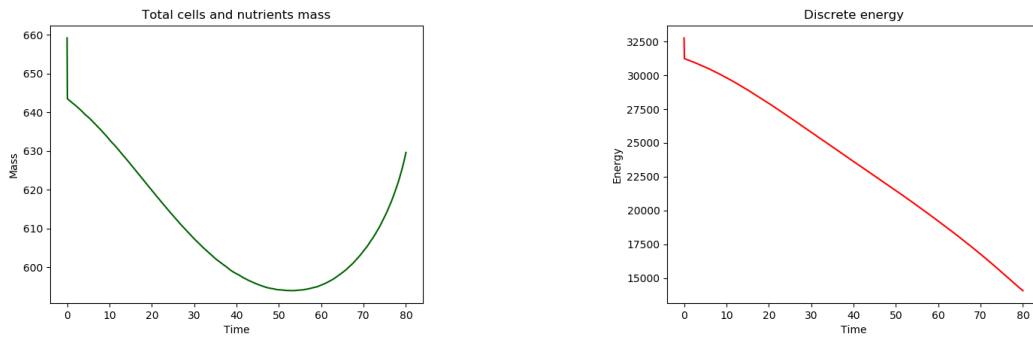

Fig. 5 Example 2. Mass and discrete energy of the variant (2.7) with the **Eyre-FE** scheme (2.1).

Consider the domain $\Omega = [0, 25.6]^2$. We take $u_0 = 1$ in the region $\left\{(x, y) \in \mathbb{R}^2 : \frac{(x-12.8)^2}{1.7} + \frac{(y-12.8)^2}{0.9} \leq 1\right\}$ and $n_0 = 1$ in Ω.

We keep the same parameters than in the example 1 as it is done in [3]. We take $\Delta t = 8 \cdot 10^{-2}$ and $h \approx 0.36$ with polynomials of order $k = 1$.

The Eyre-FE scheme (2.1a)–(2.1c) is used to obtain the solution that is shown in Fig. 4. It is remarkable to underline the evolution of shape of the tumor over time, forming buds towards the higher levels of nutrients as in [3].

Nontheless, the solution escapes from the range $[0, 1]$ which is due to the lack of maximum principle of the model (1.1) and, consequently, of the Eyre-FE scheme (2.1a)–(2.1c). Moreover, some oscillations are produced which may be due to the bad approximations of the cross-diffusion terms as $\chi_0 \neq 0$.

In this case, the energy and mass functions that we obtain with the modification of the model (2.7a)–(2.7g) are shown in the Fig. 5. Now, the energy may not be dissipative in general and neither is tumor + nutrient conserved.

Example 3. Aggregation of circular tumors
We show a simulation of the aggregation process of three circular tumors as it was done in [8]. To this aim, we consider the domain $\Omega = [-1, 1]^2$ and we guess that there are three tumors in the regions $\overline{B}((0.2, 0.2), 0.01)$, $\overline{B}((0.3, -0.5), 0.01)$ and $\overline{B}((-0.15, -0.15), 0.03)$ with the maximum concentration of nutrients in the extracellular water $n_0 = 1$.

This time we use the IEQ-DG scheme (2.5a)–(2.5c). We take the parameters $\varepsilon = 0.02$, $\delta = 0.01$, $P_0 = 100$, $\chi_0 = 0$, $\Gamma = 0.045$ and $B = 1$. We consider the constant mobility functions $M_u = 1$ and $M_n = \delta$. Moreover we use polynomials of order $k = 1$ with a penalization parameter $\sigma = 4$ and we take $\Delta t = 10^{-4}$ and $h \approx 0.02$.
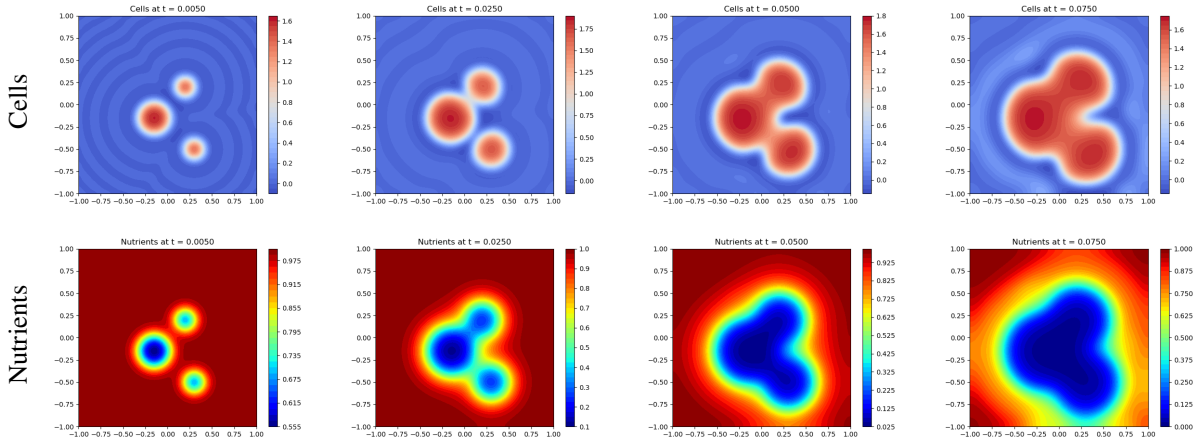
Fig. 6 Example 3. Cells and nutrients at different time steps with the **IEQ-DG** scheme (2.5).

Looking at the Fig. 6, the solution is not bounded in the interval $[0, 1]$ and some remarkable oscillations are produced as there is no maximum principle and we are not properly approximating the cross diffusion terms.

The mass is conserved and both the natural and the modified energies have the same decreasing behaviour (Fig. 7) differing only in the constant $B|\Omega| = 4$.
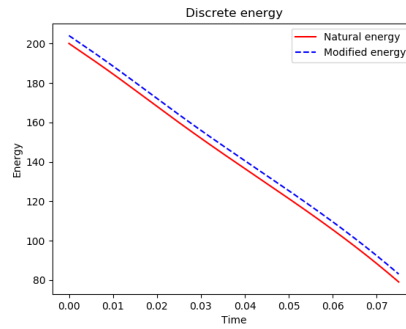

Fig. 7 Example 3. Discrete energy with the **IEQ-DG** scheme (2.5).

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