On an order-based multivariate median

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Abstract

Although the order-based definition of the univariate median is ubiquitous in statistics, the same order-based definition is typically abandoned when extending the univariate median to higher dimensions. In this paper, an example of order-based multivariate median based on the use of a linear extension of the product order is brought to the attention. Symmetry and internality properties are fulfilled by such order-based multivariate median, however, it turns out that this function fails to fulfil in general other appealing properties such as equivariance to different geometrical transformations, monotonicity properties and continuity. Interestingly, translation and scale equivariance and some monotonicity properties can be guaranteed if the linear extension of the product order is carefully chosen. Finally, it is proved that, unlike in the univariate setting, the finite-sample breakdown point of the order-based multivariate median is 1/n, thus implying that it is not robust in the presence of outliers.

Keywords: Multivariate median; Order; Linear extension; Robustness.

1. Introduction

When dealing with univariate data, the median is commonly described as the point that lies in the middle after an increasing reordering of the points. However, when one moves to a higher dimension $m \geq 2$, there is very little mention of this order-based definition of the median (see, e.g., the definition of R-ordering by Barnett [1]) and other characterizations are used instead [21]. For instance, the spatial median [23, 24] minimizes the sum of Euclidean distances; Tukey's halfspace median [22] is the point that maximizes the halfspace depth; Oja's simplex median [17] is the point that minimizes the sum of volumes of all simplices formed by $m+1$ subsets of data points; the convex hull peeling median [8] is the point that remains after recursively removing the convex hull; and Liu's simplicial depth median [14] lies inside the most simplices formed by $m+1$ subsets of data points.

A possible explanation for the absence of an order-based multivariate median could follow from claims encouraging us to abandon order-based thinking for multivariate data, as those by Kendall [13] "order properties [...] exist only in one dimension" and by Bell and Haller [2] "there is no unique 'natural' concept of rank [for bivariate data]". However, as Barnett warned us back in 1976 [1], "This is not to say that the idea of order or rank is entirely absent from the multivariate scene."

For instance, consider five individuals with the following heights and weights:

Height (in cm)	169	176	178	183	190
Weight (in kg)	65	75	79	82	83

The *size* of an individual can be defined as a bivariate notion formed by both the height and the weight of the individual. It is clear that the five individuals above can be ordered increasingly according to their size:

$$(169,65)^T \leq_2 (176,75)^T \leq_2 (178,79)^T \leq_2 (183,82)^T \leq_2 (190,83)^T \,,$$

where $\leq_2$ denotes the classic product order on $\mathbb{R}^2$ defined by $(a,b)^T \leq_2 (c,d)^T$ if $a \leq c$ and $b \leq d$. One could then think of a straightforward order-based extension of the median to couples that, here, results in the couple $(178,79)^T$.

Unfortunately, there are some situations in which such order-based multivariate median is not explicitly defined. For instance, consider that the five individuals have the following heights and weights:

Height (in cm)	169	176	178	183	190
Weight (in kg)	83	82	79	75	65

All couples above are incomparable with respect to the product order since the individuals are arranged in increasing height but decreasing weight. The order-based multivariate median is then undefined. It is nonetheless evident that a simple choice of linear extension of the product order turns the order-based multivariate median well-defined. For instance, one could order the couples lexicographically [9], firstly by increasing height and, in case two individuals have the same height, by increasing weight. In particular, the (first) lexicographic order $\preceq_{\text{Lex}}$ results in the order below:

$$(169, 83)^T \preceq_{\text{Lex}} (176, 82)^T \preceq_{\text{Lex}} (178, 79)^T \preceq_{\text{Lex}} (183, 75)^T \preceq_{\text{Lex}} (190, 65)^T \,.$$

The resulting order-based multivariate median is $(178, 79)^T$.

As intuitive as this sounds, I am not aware[1] of any such order-based extension of the median that explicitly uses a linear extension of the product order prior to De Miguel et al. [5]. Although their proposal actually aims at extending OWA operators [27, 28] to the multivariate framework, it is straightforward to obtain a definition of order-based multivariate median from their work. In the upcoming section, their proposal is briefly summarized and their terminology is adapted restricting to the context of this paper.

2. The order-based multivariate median

2.1. The linear extension of the product order based on m linearly independent weighted arithmetic means

Consider n points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$, with $m \geq 2$. All $\mathbf{x}_i$ are treated as column vectors and $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is treated as a matrix with m rows and n columns. The j-th component of the point $\mathbf{x}_i$ is denoted by $\mathbf{x}_i(j)$. The product order $\leq_m$ on $\mathbb{R}^m$ is defined as $\mathbf{x}_{i_1} \leq_m \mathbf{x}_{i_2}$ if $\mathbf{x}_{i_1}(j) \leq \mathbf{x}_{i_2}(j)$ for any $j \in \{1, \ldots, m\}$. Obviously, the product order $\leq_m$ is not a linear order on $\mathbb{R}^m$.

A possible way to refine the product order $\leq_m$ is based on the use of a monotone increasing function $f : \mathbb{R}^m \to \mathbb{R}$. More specifically, an extension $\precsim_f$ of the product order $\leq_m$ based on a monotone increasing function $f : \mathbb{R}^m \to \mathbb{R}$ can be defined as $\mathbf{x}_{i_1} \precsim_f \mathbf{x}_{i_2}$ if $\mathbf{x}_{i_1} = \mathbf{x}_{i_2}$ or $f(\mathbf{x}_{i_1}) < f(\mathbf{x}_{i_2})$. In order to assure that the resulting extension is a linear order, several such extensions based on more than one monotone increasing function need to be sequentially considered.

It is quite typical to consider these functions to be weighted arithmetic means, i.e., functions of the form $M_j : \mathbb{R}^m \to \mathbb{R}$ defined as $M_j(\mathbf{x}_i) = \sum_{\ell=1}^m \alpha_{j\ell}\, \mathbf{x}_i(\ell)$ with all $\alpha_{j\ell} \in [0, 1]$ and $\sum_{\ell=1}^m \alpha_{j\ell} = 1$. The vector $(\alpha_{j1}, \ldots, \alpha_{jm})$

[1] Admittedly, one may find examples for which multivariate data is reduced into a one-dimensional space by means of a linear combination of the components, being the median ultimately computed within this one-dimensional space, see, e.g., [3].

is referred to as the vector of weights associated with M_j. The subindex j is used because, as stated in the definition below, a linear extension $\preceq$ of $\leq_m$ arises when m linearly independent weighted arithmetic means $M_1, \dots, M_m : \mathbb{R}^m \to \mathbb{R}$ are considered. It is recalled that m weighted arithmetic means $M_1, \dots, M_m : \mathbb{R}^m \to \mathbb{R}$ are called linearly independent if their vectors of weights are all linearly independent or, equivalently, if the matrix

$$A = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \dots & \alpha_{mm} \end{pmatrix}$$

is regular (invertible).

Definition 1. [5] Consider $m \geq 2$ linearly independent weighted arithmetic means $M_1, \dots, M_m : \mathbb{R}^m \to \mathbb{R}$. The linear extension $\preceq_{\mathbf{M}}$ of $\leq_m$ based on $\mathbf{M} = (M_1, \dots, M_m)$ is defined as $\mathbf{x}_{i_1} \preceq_{\mathbf{M}} \mathbf{x}_{i_2}$ if $\mathbf{x}_{i_1} = \mathbf{x}_{i_2}$ or there exists $k \in \{1, \dots, m\}$ such that

$$\begin{aligned} & M_j(\mathbf{x}_{i_1}) = M_j(\mathbf{x}_{i_2}) \,, \text{ for any } j \in \{1, \dots, k-1\} \,, \\ & M_k(\mathbf{x}_{i_1}) < M_k(\mathbf{x}_{i_2}) \,. \end{aligned}$$

Typical examples of linear extensions of $\mathbb{R}^m$ defined by means of weighted arithmetic means are the lexicographic orders [9], where the considered arithmetic means are the projections, i.e., $M_j(\mathbf{x}_i) = \mathbf{x}_i(\sigma(j))$ with σ a permutation of $\{1, \dots, n\}$. Other common example is Xu and Yager's linear order on $\mathbb{R}^2$ [26][2], where $M_1(\mathbf{x}_i) = \frac{1}{2}\mathbf{x}_i(1) + \frac{1}{2}\mathbf{x}_i(2)$ and $M_2(\mathbf{x}_i) = \mathbf{x}_i(2)$.

2.2. *The order-based multivariate median based on a linear extension of the product order*

The order-based multivariate median can be simply defined by making use of a linear extension of the product order. For an odd number of points, the order-based multivariate median is the point that lies in the middle after increasingly re-ordering all the points according to the considered linear extension of the product order. For an even number of points, the order-based

[2]Note that an equivalent definition of Xu and Yager's linear order on $\mathbb{R}^2$ is admittedly more common. More precisely, M_2 is alternatively defined as $M_2(\mathbf{x}_i) = \mathbf{x}_i(2) - \mathbf{x}_i(1)$. This equivalent definition is here abandoned in order to guarantee M_2 to be a weighted arithmetic mean.

multivariate median may be defined as any convex combination of the two points that lie in the middle after increasingly re-ordering all the points according to the considered linear extension of the product order. Throughout this paper, the centroid (i.e., the componentwise arithmetic mean) of these two points is considered.

Definition 2. Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$. The order-based multivariate median based on $\preceq$ is defined as the function $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$ defined if n is an odd number by

$$F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_{(\frac{n+1}{2})}\,,$$

and if n is an even number by

$$F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{2}\mathbf{x}_{(\frac{n}{2})} + \frac{1}{2}\mathbf{x}_{(\frac{n}{2}+1)}\,,$$

where $\mathbf{x}_{(i)}$ denotes the i-th greatest point among $\mathbf{x}_1, \ldots, \mathbf{x}_n$ according to $\preceq$.

By definition, the order-based multivariate median differs from the centroid if and only if $n > 2$. Obviously, this does not mean that one cannot find lists of n elements with $n > 2$ for which both the centroid and the order-based multivariate median coincide (e.g., lists for which all the points coincide).

Example 1. Consider the points $\mathbf{x}_1 = (1,3)^T$, $\mathbf{x}_2 = (3,1)^T$, $\mathbf{x}_3 = (5,4)^T$, $\mathbf{x}_4 = (6,8)^T$ and $\mathbf{x}_5 = (9,7)^T$, and the following list $\mathbf{M} = (M_1, M_2)$ of weighted arithmetic means, where

$$\begin{aligned} M_1(\mathbf{x}_i) &= \frac{1}{2}\mathbf{x}_i(1) + \frac{1}{2}\mathbf{x}_i(2)\,, \\ M_2(\mathbf{x}_i) &= \mathbf{x}_i(2)\,. \end{aligned}$$

Note that the first weighted arithmetic mean serves us to establish an order between all the points except $(1,3)^T$ and $(3,1)^T$ since it holds that

$$M_1\left(\begin{pmatrix}1\\3\end{pmatrix}\right) = M_1\left(\begin{pmatrix}3\\1\end{pmatrix}\right) < M_1\left(\begin{pmatrix}5\\4\end{pmatrix}\right) < M_1\left(\begin{pmatrix}6\\8\end{pmatrix}\right) < M_1\left(\begin{pmatrix}9\\7\end{pmatrix}\right).$$

Finally, since $M_2\left((3,1)^T\right) = 1 < 3 = M_2\left((1,3)^T\right)$, it follows that:

$$\begin{pmatrix}3\\1\end{pmatrix} \preceq_{\mathbf{M}} \begin{pmatrix}1\\3\end{pmatrix} \preceq_{\mathbf{M}} \begin{pmatrix}5\\4\end{pmatrix} \preceq_{\mathbf{M}} \begin{pmatrix}6\\8\end{pmatrix} \preceq_{\mathbf{M}} \begin{pmatrix}9\\7\end{pmatrix}.$$

The order-based multivariate median of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ and $\mathbf{x}_5$ based on $\preceq_{\mathbf{M}}$ is $\mathbf{x}_3 = (5,4)^T$. An illustration of this procedure is given in Figure 1.

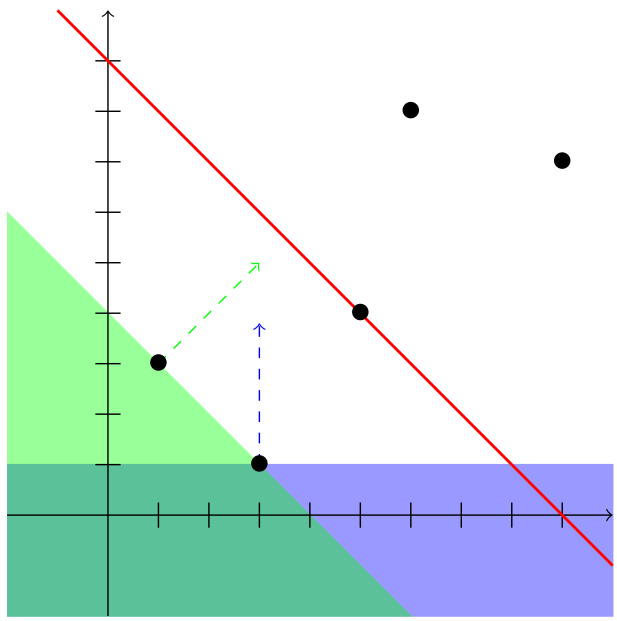

Figure 1: Graphical representation of the process of finding the order-based multivariate median of the points $\mathbf{x}_1 = (1,3)^T$, $\mathbf{x}_2 = (3,1)^T$, $\mathbf{x}_3 = (5,4)^T$, $\mathbf{x}_4 = (6,8)^T$ and $\mathbf{x}_5 = (9,7)^T$ based on $M_1(\mathbf{x}_i) = \frac{1}{2}\mathbf{x}_i(1) + \frac{1}{2}\mathbf{x}_i(2)$ and $M_2(\mathbf{x}_i) = \mathbf{x}_i(2)$. The green area represents the points of $\mathbb{R}^2$ that lead to values of M_1 smaller than those given by $\mathbf{x}_1$ and $\mathbf{x}_2$. The blue area represents the points of $\mathbb{R}^2$ that lead to values of M_2 smaller than that given by $\mathbf{x}_2$. The green and blue dashed arrows respectively represent the direction in which M_1 and M_2 increase. The red line represents the points in the same level set of M_1 as $\mathbf{x}_3$.

3. Properties of the order-based multivariate median

In this section, some basic properties of the order-based multivariate median are studied. Special attention is devoted to the relation of the order-based multivariate median with the componentwise median and the fulfillment of several properties such as symmetry, internality properties, equivariance to different geometrical transformations, monotonicity properties and continuity.

3.1. Relation with the componentwise median

The easiest multivariate extension of the univariate median, referred to as the componentwise median and denoted by F_{CWM}, considers the median in each of the components independently. For example, given $\mathbf{x}_1 = (0,1)^T$, $\mathbf{x}_2 = (1,0)^T$ and $\mathbf{x}_3 = (2,2)^T$, the componentwise median is $F_{\text{CWM}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (1,1)^T$.

It is immediate to see that the order-based multivariate median does not need to coincide with the componentwise median. For instance, it suffices to consider Xu and Yager's linear order on $\mathbb{R}^2$ resulting in $(0,1)^T$ as the order-based multivariate median for the very same points $\mathbf{x}_1 = (0,1)^T$, $\mathbf{x}_2 = (1,0)^T$ and $\mathbf{x}_3 = (2,2)^T$.

Even though the componentwise median and the order-based multivariate median might not coincide, both of them are assured to coincide in case the points are already linearly ordered with respect to $\leq_m$. Note that this result is independent of the considered linear extension of $\leq_m$.

Proposition 1. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$. If the points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly ordered with respect to $\leq_m$, then it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = F_{CWM}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ for any linear extension $\preceq$ of $\leq_m$.*

Proof. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be indexed such that $\mathbf{x}_1 \leq_m \ldots \leq_m \mathbf{x}_n$. For any linear extension $\preceq$ of $\leq_m$, it follows that $\mathbf{x}_1 \preceq \ldots \preceq \mathbf{x}_n$. Moreover, by definition of product order, it follows that $\mathbf{x}_1(i) \leq \ldots \leq \mathbf{x}_n(i)$ for any $i \in \{1, \ldots, m\}$. Thus, it holds that the order-based multivariate median coincides with the componentwise median independently of the choice of linear extension of the product order. $\square$

3.2. Symmetry

One of the most basic properties of a function is that of symmetry, which assures the result to be the same after the input points have been subjected to any permutation.

Definition 3. Consider $n, m \in \mathbb{N}$. A function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is called symmetric if

$$F(\mathbf{x}_1, \dots, \mathbf{x}_n) = F(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(n)}),$$

for any $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any permutation σ of $\{1, \dots, n\}$.

Obviously, as the points to be aggregated are re-ordered according to the considered linear extension, the order-based multivariate median is symmetric.

Proposition 2. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$. The order-based multivariate median $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is symmetric.*

Proof. The result follows from the fact that the order $\preceq$ on $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ does not change after reindexing the points. $\square$

3.3. Internality properties

We recall that the convex hull of $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^m$ is defined as

$$\mathrm{CH}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \left\{ \mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \in \mathbb{R}^m \mid \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\},$$

and the bounding box of $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^m$ is defined as

$$\mathrm{BB}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \left[\min_{i=1}^{n} \mathbf{x}_i(1), \max_{i=1}^{n} \mathbf{x}_i(1) \right] \times \cdots \times \left[\min_{i=1}^{n} \mathbf{x}_i(m), \max_{i=1}^{n} \mathbf{x}_i(m) \right].$$

There exist many types of internality properties for multivariate medians (see, e.g., [10]). Here, the four most classical types of internality are discussed: internality within the points, convex-hull internality (CH-internality, for short), bounding-box internality (BB-internality, for short) and idempotence.

Definition 4. Consider $n, m \in \mathbb{N}$. A function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is called:

(i) *internal within the points* if

$$F(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \{\mathbf{x}_1, \dots, \mathbf{x}_n\},$$

for any $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$;

(ii) *CH-internal* if

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathrm{CH}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$;

(iii) *BB-internal* if

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathrm{BB}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$;

(iv) *idempotent* if

$$F(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{x}\,,$$

for any $\mathbf{x} \in \mathbb{R}^m$.

Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathrm{CH}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \subseteq \mathrm{BB}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, it holds that internality within the points implies CH-internality, which at the same time implies BB-internality. Finally, all three among internality within the points, CH-internality and BB-internality imply idempotence.

Independently of the choice of linear extension of the product order, the order-based multivariate median is CH-internal, BB-internal and idempotent. In case n is an odd number, it additionally is internal within the points.

Proposition 3. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$. The order-based multivariate median $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$:*

(i) is internal within the points if n is and odd number, but is not necessarily internal within the points if n is an even number;

(ii) is CH-internal;

(iii) is BB-internal;

(iv) is idempotent.

Proof. (i) The result follows from the fact that $\mathbf{x}_{(\frac{n+1}{2})} \in \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, whereas this is not necessarily the case for $\frac{1}{2}\mathbf{x}_{(\frac{n}{2})} + \frac{1}{2}\mathbf{x}_{(\frac{n}{2}+1)}$.

(ii) If n is an odd number, then the result follows from (i) and the fact that internality within the points implies CH-internality. If n is an even number, the result follows from the CH-internality of the centroid of two points.

(iii) The result follows from (ii) and the fact that CH-internality implies BB-internality.

(iv) The result follows from (iii) and the fact that BB-internality implies idempotence. $\square$

3.4. Equivariance to different geometrical transformations

The field of multivariate statistics is typically interested in the equivariance to different geometrical transformations. More specifically, the interest lies in whether the order in which the function and the geometrical transformation are applied can be interchanged. Typically studied geometrical transformations within this context are affine transformations, in general, and some specific families of affine transformations such as orthogonal transformations, scalings and translations, in particular.

Definition 5. Consider $n, m \in \mathbb{N}$. A function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is called:

(i) *affine equivariant* if

$$F(\mathbf{A}\mathbf{x}_1 + \mathbf{t}, \ldots, \mathbf{A}\mathbf{x}_n + \mathbf{t}) = \mathbf{A}\, F(\mathbf{x}_1, \ldots, \mathbf{x}_n) + \mathbf{t}\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, any regular $\mathbf{A} \in \mathbb{R}^{m\times m}$ and any $\mathbf{t} \in \mathbb{R}^m$;

(ii) *orthogonal equivariant* if

$$F(\mathbf{O}\mathbf{x}_1, \ldots, \mathbf{O}\mathbf{x}_n) = \mathbf{O}\, F(\mathbf{x}_1, \ldots, \mathbf{x}_n)\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any orthogonal matrix $\mathbf{O} \in \mathbb{R}^{m\times m}$, i.e., a square matrix such that $\mathbf{O}^T = \mathbf{O}^{-1}$;

(iii) *scale equivariant* if

$$F(s\mathbf{x}_1, \ldots, s\mathbf{x}_n) = s\, F(\mathbf{x}_1, \ldots, \mathbf{x}_n)\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any $s > 0$;

(iv) *translation equivariant* if

$$F(\mathbf{x}_1 + \mathbf{t}, \ldots, \mathbf{x}_n + \mathbf{t}) = F(\mathbf{x}_1, \ldots, \mathbf{x}_n) + \mathbf{t}\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any $\mathbf{t} \in \mathbb{R}^m$.

Obviously, affine equivariance implies all other three properties.

The order-based multivariate median is not equivariant to any of the above geometrical transformations in case $n > 2$. However, it turns out that, if the considered linear extension is based on m linearly independent weighted arithmetic means, then the properties of scale equivariance and translation equivariance are fulfilled. The same result does not hold for the properties of affine equivariance and orthogonal equivariance.

Proposition 4. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$. The order-based multivariate median $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$:*

(i) is affine equivariant if and only if $n \leq 2$;

(ii) is orthogonal equivariant if and only if $n \leq 2$;

(iii) is scale equivariant if $n \leq 2$ but is not necessarily scale equivariant if $n > 2$. For any $n \in \mathbb{N}$, $F_{\preceq}$ is assured to be scale equivariant if $\preceq = \preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means;

(iv) is translation equivariant if $n \leq 2$ but is not necessarily translation equivariant if $n > 2$. For any $n \in \mathbb{N}$, $F_{\preceq}$ is assured to be translation equivariant if $\preceq = \preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means.

Proof. The order-based multivariate median coincides with the centroid if and only if $n \leq 2$, and the centroid is known to satisfy all properties above. It only remains to check the case in which $n > 2$.

(i) We first prove the result for $n = 3$. Consider $\mathbf{x} = (1, 0, \ldots, 0)^T$, $\mathbf{y} = (0, 1, 0, \ldots, 0)^T$ and $\mathbf{z} = (0, \ldots, 0)^T$. Since $\preceq$ is a linear extension of $\leq_m$, it holds that $\mathbf{z} \preceq \mathbf{x}$ and $\mathbf{z} \preceq \mathbf{y}$. We distinguish two cases:

- If $\mathbf{x} \preceq \mathbf{y}$, then $F_{\preceq}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}$. Consider $\mathbf{t} = (0, \ldots, 0)^T$ and the rotation matrix A for rotating the first two components 45° counterclockwisely:

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & \ldots & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

  Since $\preceq$ is a linear extension of $\leq_m$, it holds that $A\mathbf{y} \preceq A\mathbf{x}$ and $A\mathbf{z} \preceq A\mathbf{x}$. Therefore, it holds that $F_{\preceq}(A\mathbf{x}, A\mathbf{y}, A\mathbf{z}) \neq A\mathbf{x}$.

- If $\mathbf{y} \preceq \mathbf{x}$, then $F_{\preceq}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{y}$. Consider $\mathbf{t} = (0, \ldots, 0)^T$ and the rotation matrix A' for rotating the first two components 45° clockwisely:

$$A' = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \dots & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Since $\preceq$ is a linear extension of $\leq_m$, it holds that $A'\mathbf{x} \preceq A'\mathbf{y}$ and $A'\mathbf{z} \preceq A'\mathbf{y}$. Therefore, it holds that $F_{\preceq}(A'\mathbf{x}, A'\mathbf{y}, A'\mathbf{z}) \neq A'\mathbf{y}$.

If $n > 3$ is odd, it suffices to additionally consider the points $(-1, 0, \dots, 0)^T$ and $(2, 0, \dots, 0)^T$, both of them with cardinality $\frac{n-3}{2}$ for the case in which $\mathbf{x} \preceq \mathbf{y}$; and the points $(0, -1, 0, \dots, 0)^T$ and $(0, 2, 0, \dots, 0)^T$, both of them with cardinality $\frac{n-3}{2}$ for the case in which $\mathbf{y} \preceq \mathbf{x}$. If $n \geq 4$ is even, it suffices to consider the points for $n-1$ and considering an additional appearance of $\mathbf{x}$ for the case in which $\mathbf{x} \preceq \mathbf{y}$ and an additional appearance of $\mathbf{y}$ for the case in which $\mathbf{y} \preceq \mathbf{x}$.

(ii) The result follows straightforwardly from (i) bearing in mind that the matrices $\mathbf{A}$ and $\mathbf{A}'$ above are orthogonal.

(iii) Consider $m = 2$, $M_1(\mathbf{x}_i) = \mathbf{x}_i(1)$ and $M_2(\mathbf{x}_i) = \mathbf{x}_i(2)$. Let $\preceq_{\mathbf{M}_1}$ and $\preceq_{\mathbf{M}_2}$ be the linear extensions of $\leq_2$ respectively associated with $\mathbf{M}_1 = (M_1, M_2)$ and $\mathbf{M}_1 = (M_2, M_1)$. Let $\preceq_*$ be the linear extension of $\leq_2$ defined as $\mathbf{x}_i \preceq_* \mathbf{x}_j$ if:

(a) $\mathbf{x}_i(1) \leq 3$ and $\mathbf{x}_i \preceq_{\mathbf{M}_1} \mathbf{x}_j$;

(b) $\mathbf{x}_i(1) > 3$, $\mathbf{x}_j(1) > 3$ and $\mathbf{x}_i \preceq_{\mathbf{M}_2} \mathbf{x}_j$.

Consider $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $s = 5$. It follows that $\mathbf{x}_1 \preceq_* \mathbf{x}_2 \preceq_* \mathbf{x}_3$ but $s\mathbf{x}_1 \preceq_* s\mathbf{x}_3 \preceq_* s\mathbf{x}_2$. Thus, it holds that

$$F_{\preceq}(s\mathbf{x}_1, s\mathbf{x}_2, s\mathbf{x}_3) = F_{\preceq}\left(\begin{pmatrix} 0 \\ 10 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 10 \\ 0 \end{pmatrix},$$

and

$$s\, F_{\preceq}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 5\, F_{\preceq}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Therefore, $F_{\preceq}$ is not scale equivariant.

We now prove that scale equivariance is guaranteed if $\preceq=\preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means.

Firstly, it should be noted that scaling does not affect the order $\preceq_{\mathbf{M}}$. Consider $\mathbf{x}_{i_1}$ and $\mathbf{x}_{i_2}$ such that $\mathbf{x}_{i_1} \preceq_{\mathbf{M}} \mathbf{x}_{i_2}$. We distinguish two cases. (a) If $\mathbf{x}_{i_1} = \mathbf{x}_{i_2}$, then it is immediate to see that $s\,\mathbf{x}_{i_1} \preceq_{\mathbf{M}} s\,\mathbf{x}_{i_2}$ for any $s > 0$. (b) If there exists $k \in \{1, \ldots, n\}$ such that

$$\begin{aligned} &M_j(\mathbf{x}_{i_1}) = M_j(\mathbf{x}_{i_2})\,, \text{ for any } j \in \{1, \ldots, k-1\}\,,\\ &M_k(\mathbf{x}_{i_1}) < M_k(\mathbf{x}_{i_2})\,, \end{aligned}$$

then it follows that

$$\begin{aligned} &M_j(s\,\mathbf{x}_{i_1}) = s\,M_j(\mathbf{x}_{i_1}) = s\,M_j(\mathbf{x}_{i_2}) = M_j(s\,\mathbf{x}_{i_2})\,, \text{ for any } j \in \{1, \ldots, k-1\}\,,\\ &M_k(s\,\mathbf{x}_{i_1}) = s\,M_k(\mathbf{x}_{i_1}) < s\,M_k(\mathbf{x}_{i_2}) = M_k(s\,\mathbf{x}_{i_2})\,, \end{aligned}$$

for any $s > 0$. Thus, it holds that $s\,\mathbf{x}_{i_1} \preceq_{\mathbf{M}} s\,\mathbf{x}_{i_2}$.

The scale equivariance of the order-based multivariate median then follows immediately if n is an odd number, and from the fact that the centroid of two points is scale equivariant if n is an even number.

(iv) Consider the counterexample in (iii) with $\mathbf{t} = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$. It follows that $\mathbf{x}_1 \preceq_* \mathbf{x}_2 \preceq_* \mathbf{x}_3$ but $\mathbf{x}_1 + \mathbf{t} \preceq_* \mathbf{x}_3 + \mathbf{t} \preceq_* \mathbf{x}_2 + \mathbf{t}$. Thus, it holds that

$$F_{\preceq}(\mathbf{x}_1 + \mathbf{t}, \mathbf{x}_2 + \mathbf{t}, \mathbf{x}_3 + \mathbf{t}) = F_{\preceq}\left(\begin{pmatrix} 2.5 \\ 4.5 \end{pmatrix}, \begin{pmatrix} 3.5 \\ 3.5 \end{pmatrix}, \begin{pmatrix} 4.5 \\ 2.5 \end{pmatrix}\right) = \begin{pmatrix} 4.5 \\ 2.5 \end{pmatrix}\,,$$

and

$$F_{\preceq}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + \mathbf{t} = F_{\preceq}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 3.5 \end{pmatrix}\,.$$

Therefore, $F_{\preceq}$ is not translation equivariant.

We now prove that translation equivariance is guaranteed if $\preceq=\preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means.

Firstly, it should be noted that a translation does not affect the order $\preceq_{\mathbf{M}}$. Consider $\mathbf{x}_{i_1}$ and $\mathbf{x}_{i_2}$ such that $\mathbf{x}_{i_1} \preceq_{\mathbf{M}} \mathbf{x}_{i_2}$. We distinguish two cases. (a) If $\mathbf{x}_{i_1} = \mathbf{x}_{i_2}$, then it is immediate to see that $\mathbf{x}_{i_1} + \mathbf{t} \preceq_{\mathbf{M}} \mathbf{x}_{i_2} + \mathbf{t}$ for any $\mathbf{t} \in \mathbb{R}^m$. (b) If there exists $k \in \{1, \ldots, n\}$ such that

$$\begin{aligned} &M_j(\mathbf{x}_{i_1}) = M_j(\mathbf{x}_{i_2})\,, \text{ for any } j \in \{1, \ldots, k-1\}\,,\\ &M_k(\mathbf{x}_{i_1}) < M_k(\mathbf{x}_{i_2})\,, \end{aligned}$$

then it follows that

$$\begin{aligned}
&M_j(\mathbf{x}_{i_1}+\mathbf{t})) = M_j(\mathbf{x}_{i_1})+\mathbf{t}) = M_j(\mathbf{x}_{i_2})+\mathbf{t}) = M_j(\mathbf{x}_{i_2}+\mathbf{t}))\,, \text{ for any } j \in \{1,\ldots,k-1\}\,,\\
&M_k(\mathbf{x}_{i_1}+\mathbf{t})) = M_k(\mathbf{x}_{i_1})+\mathbf{t}) < M_k(\mathbf{x}_{i_2})+\mathbf{t}) = M_k(\mathbf{x}_{i_2}+\mathbf{t}))\,,
\end{aligned}$$

for any $\mathbf{t} \in \mathbb{R}^m$. Thus, it holds that $\mathbf{x}_{i_1}+\mathbf{t} \preceq_{\mathbf{M}} \mathbf{x}_{i_2}+\mathbf{t}$.

The translation equivariance of the order-based multivariate median then follows immediately if n is an odd number, and from the fact that the centroid of two points is translation equivariant if n is an even number. □

3.5. Monotonicity properties

Monotonicity is a largely venerated property in the field of aggregation theory that assures that an increase in the inputs does not result in a decrease in the output. Since the set $(\preceq, \mathbb{R}^m)$ is totally ordered given a linear extension $\preceq$ of the product order $\leq_m$, it is evident that the order-based multivariate median based on $\preceq$ is monotone (increasing) with respect to $\preceq$ itself. However, monotonicity of multivariate functions is typically understood quite differently. A taxonomy of different monotonicity properties that have been studied in the context of multivariate data[3] can be found in [19]. In the following, some of these monotonicity properties are presented.

Definition 6. Consider $n, m \in \mathbb{N}$. A function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is called:

(i) *orthomonotone* if, for any $(\mathbf{x}_1,\ldots,\mathbf{x}_n), (\mathbf{y}_1,\ldots,\mathbf{y}_n) \in (\mathbb{R}^m)^n$ and any orthogonal matrix $\mathbf{O} \in \mathbb{R}^{m\times m}$ such that $\mathbf{O}\,\mathbf{x}_i \leq_m \mathbf{O}\,\mathbf{y}_i$ for any $i \in \{1,\ldots,n\}$, it holds that

$$\mathbf{O}\,F(\mathbf{x}_1,\ldots,\mathbf{x}_n) \leq_m \mathbf{O}\,F(\mathbf{y}_1,\ldots,\mathbf{y}_n)\,.$$

(ii) *ultramonotone* if, for any $(\mathbf{x}_1,\ldots,\mathbf{x}_n) \in (\mathbb{R}^m)^n$, any $t_1,\ldots,t_n \geq 0$ and any $\mathbf{u},\mathbf{u}_1,\ldots,\mathbf{u}_n \in \mathbb{R}^m$ such that $\mathbf{u}\cdot\mathbf{u}_i \geq 0$ for any $i \in \{1,\ldots,n\}$, it holds that

$$\Big(F\left(\mathbf{x}_1+t_1\mathbf{u}_1,\ldots,\mathbf{x}_n+t_n\mathbf{u}_n\right) - F\left(\mathbf{x}_1,\ldots,\mathbf{x}_n\right)\Big)\cdot\mathbf{u} \geq 0\,.$$

[3] Other examples of monotonicity properties for multivariate data can be found in [20] when considering the specific space $(\mathbb{R}^m, \leq_m)$.

(iii) *componentwisely monotone* if, for any fixed $j \in \{1, \ldots, m\}$ and any $(\mathbf{x}_1, \ldots, \mathbf{x}_n), (\mathbf{y}_1, \ldots, \mathbf{y}_n) \in (\mathbb{R}^m)^n$ satisfying that $\mathbf{x}_i(j) \leq \mathbf{y}_i(j)$ for any $i \in \{1, \ldots, n\}$, it holds that

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_n)(j) \leq F(\mathbf{y}_1, \ldots, \mathbf{y}_n)(j).$$

(iv) *SP-monotone* if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, any $i \in \{1, \ldots, n\}$, any $t \geq 0$ and any $\mathbf{u} \in \mathbb{R}^m \backslash \{\mathbf{0}\}$, there exists $k \geq 0$ such that

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_i + t\mathbf{u}, \ldots, \mathbf{x}_n) - F(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_n) = k\,\mathbf{u}.$$

(v) *SC-monotone* if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, any $i \in \{1, \ldots, n\}$, any $t \geq 0$ and any $\mathbf{u} \in \mathbb{R}^m \backslash \{\mathbf{0}\}$, it holds that

$$\Big(F(\mathbf{x}_1, \ldots, \mathbf{x}_i + t\mathbf{u}, \ldots, \mathbf{x}_n) - F(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_n)\Big) \cdot \mathbf{u} \geq 0.$$

(vi) *MP-monotone* if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, any $t \geq 0$ and any $\mathbf{u} \in \mathbb{R}^m \backslash \{\mathbf{0}\}$, there exists $k \geq 0$ such that:

$$F(\mathbf{x}_1 + t\mathbf{u}, \ldots, \mathbf{x}_n + t\mathbf{u}) - F(\mathbf{x}_1, \ldots, \mathbf{x}_n) = k\,\mathbf{u}.$$

(vii) *MC-monotone* if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, any $t \geq 0$ and any $\mathbf{u} \in \mathbb{R}^m \backslash \{\mathbf{0}\}$, it holds that

$$\Big(F(\mathbf{x}_1 + t\mathbf{u}, \ldots, \mathbf{x}_n + t\mathbf{u}) - F(\mathbf{x}_1, \ldots, \mathbf{x}_n)\Big) \cdot \mathbf{u} \geq 0.$$

(viii) $\leq_m$-*monotone* if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n), (\mathbf{y}_1, \ldots, \mathbf{y}_n) \in (\mathbb{R}^m)^n$ such that $\mathbf{x}_i \leq_m \mathbf{y}_i$ for any $i \in \{1, \ldots, n\}$, it holds that

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_m F(\mathbf{y}_1, \ldots, \mathbf{y}_n).$$

Note that, unlike the other properties above, MP-monotonicity and MC-monotonicity do not restrict to classical monotonicity in the univariate setting, instead, both properties restrict to the property of weak monotonicity [25]. Interestingly, any among monotonicity and translation equivariance implies weak motonicity. These two implications are also satisfied in the multivariate setting, where any among SP-monotonicity and translation equivariance implies MP-monotonicity and any among SC-monotonicity and translation equivariance implies MC-monotonicity.

In general, the order-based multivariate median is not monotone in any of the senses above in case $n > 2$. However, as translation equivariance is guaranteed if the considered linear extension is based on m linearly independent weighted arithmetic means, then it turns out that the order-based multivariate median is MP-monotone and MC-monotone in such a case.

Proposition 5. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$. The order-based multivariate median $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$:*

(i) is orthomonotone if and only if $n \leq 2$;

(ii) is ultramonotone if and only if $n \leq 2$;

(iii) is componentwisely monotone if and only if $n \leq 2$;

(iv) is SP-monotone if and only if $n \leq 2$;

(v) is SC-monotone if and only if $n \leq 2$;

(vi) is MP-monotone if $n \leq 2$ but is not necessarily MP-monotone if $n > 2$. For any $n \in \mathbb{N}$, $F_{\preceq}$ is assured to be MP-monotone if $\preceq = \preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means;

(vii) is MC-monotone if $n \leq 2$ but is not necessarily MC-monotone if $n > 2$. For any $n \in \mathbb{N}$, $F_{\preceq}$ is assured to be MC-monotone if $\preceq = \preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means;

(viii) is $\leq_m$-monotone if and only if $n \leq 2$;

Proof. The order-based multivariate median coincides with the centroid if and only if $n \leq 2$, and the centroid is known to satisfy all properties above. It only remains to check the case in which $n > 2$.

(i) Theorem 1 in [11] states that the only symmetric, idempotent and orthomonotone function is the centroid. The result follows from the fact that the order-based multivariate median is symmetric, idempotent and differs from the centroid (when $n > 2$).

(ii) Theorems 28 and 41 in [19] imply that the only symmetric, idempotent and ultramonotone function is the centroid. The result follows from the fact that the order-based multivariate median is symmetric, idempotent and differs from the centroid (when $n > 2$).

(iii) Consider any $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}(1) < \mathbf{y}(1)$ and $\mathbf{y}(2) < \mathbf{x}(2)$. We distinguish two cases:

- If $\mathbf{x} \preceq \mathbf{y}$, then there exists $\mathbf{z}$ such that $\mathbf{x}(2) < \mathbf{z}(2)$ and $\mathbf{x} \preceq \mathbf{y} \preceq \mathbf{z}$. Consider $\mathbf{x}_i = \mathbf{x}$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{y}$ for any $i > \frac{n}{2} + 1$. Independently of whether n is odd or even, it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}$. However, if one substitutes $\mathbf{x}_i$ by $\mathbf{z}$ (with $i = \frac{n+1}{2}$ if n is odd and both $i = \frac{n}{2}$ and $i = \frac{n}{2}+1$ if n is even), it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \mathbf{y}$. The result then follows from the fact that $\mathbf{x}(2) < \mathbf{z}(2)$ but $\mathbf{x}(2) > \mathbf{y}(2)$.

- If $\mathbf{y} \preceq \mathbf{x}$, then there exists $\mathbf{z}$ such that $\mathbf{y}(1) < \mathbf{z}(1)$ and $\mathbf{y} \preceq \mathbf{x} \preceq \mathbf{z}$. Consider $\mathbf{x}_i = \mathbf{y}$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{x}$ for any $i > \frac{n}{2} + 1$. Independently of whether n is odd or even, it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{y}$. However, if one substitutes $\mathbf{x}_i$ by $\mathbf{z}$ (with $i = \frac{n+1}{2}$ if n is odd and both $i = \frac{n}{2}$ and $i = \frac{n}{2}+1$ if n is even), it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \mathbf{x}$. The result then follows from the fact that $\mathbf{y}(1) < \mathbf{z}(1)$ but $\mathbf{y}(1) > \mathbf{x}(1)$.

(iv) Theorem 28 in [19] implies that the only symmetric, idempotent and SP-monotone function is the centroid. The result follows from the fact that the order-based multivariate median is symmetric, idempotent and differs from the centroid (when $n > 2$).

(v) Consider any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}(j) > 0$ and $\mathbf{v}(j) > 0$ for any $j \in \{1, \ldots, m\}$, $\mathbf{x}(1) < \mathbf{y}(1)$, $\mathbf{y}(2) < \mathbf{x}(2)$, $(\mathbf{y} - \mathbf{x}) \cdot \mathbf{u} < 0$ and $(\mathbf{x} - \mathbf{y}) \cdot \mathbf{v} < 0$. Note that such values exist (e.g., $\mathbf{x} = (0, \ldots, 0)^T$, $\mathbf{y} = (1, -1, \ldots, -1)^T$, $\mathbf{u} = (0.5, 1, \ldots, 1)^T$ and $\mathbf{v} = (m + 1, 1, \ldots, 1)^T$). We distinguish two cases:

- If $\mathbf{x} \preceq \mathbf{y}$, then there exists $t > 0$ such that $\mathbf{z} = \mathbf{x} + t\mathbf{u}$ satisfies that $\mathbf{x} \preceq \mathbf{y} \preceq \mathbf{z}$. Consider $\mathbf{x}_i = \mathbf{x}$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{y}$ for any $i > \frac{n}{2} + 1$. Independently of whether n is odd or even, it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}$. If n is odd and $\mathbf{x}_{\frac{n+1}{2}}$ is substituted by $\mathbf{z}$, it follows that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \mathbf{y}$. If n is even and $\mathbf{x}_{\frac{n}{2}+1}$ is substituted by $\mathbf{z}$, it follows that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \frac{\mathbf{x}+\mathbf{y}}{2}$. In both cases, the result then follows from the fact that $(\mathbf{y} - \mathbf{x}) \cdot \mathbf{u} < 0$.

- If $\mathbf{y} \preceq \mathbf{x}$, then there exists $t > 0$ such that $\mathbf{z} = \mathbf{y} + t\mathbf{v}$ satisfies that $\mathbf{y} \preceq \mathbf{x} \preceq \mathbf{z}$. Consider $\mathbf{x}_i = \mathbf{y}$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{x}$ for any

$i > \frac{n}{2} + 1$. Independently of whether n is odd or even, it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{y}$. If n is odd and $\mathbf{x}_{\frac{n+1}{2}}$ is substituted by $\mathbf{z}$, it follows that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \mathbf{x}$. If n is even and $\mathbf{x}_{\frac{n}{2}+1}$ is substituted by $\mathbf{z}$, it follows that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \frac{\mathbf{x}+\mathbf{y}}{2}$. In both cases, the result then follows from the fact that $(\mathbf{x} - \mathbf{y}) \cdot \mathbf{v} < 0$.

(vi) Consider the linear extension $\preceq_*$ defined in Proposition 4(iii). Consider $\mathbf{x}_1 = (4,4)^T$, $\mathbf{x}_2 = (7,7)^T$ and $\mathbf{x}_3 = (7,1)^T$. It holds that $\mathbf{x}_3 \preceq_* \mathbf{x}_1 \preceq_* \mathbf{x}_2$, and, therefore, $F_{\preceq_*}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbf{x}_1$. Consider $t = 2$ and $\mathbf{u} = (-1, 0)^T$. It holds that $\mathbf{x}_1 + t\mathbf{u} \preceq_* \mathbf{x}_3 + t\mathbf{u} \preceq_* \mathbf{x}_2 + t\mathbf{u}$, and, therefore, $F_{\preceq_*}(\mathbf{x}_1 + t\mathbf{u}, \mathbf{x}_2 + t\mathbf{u}, \mathbf{x}_3 + t\mathbf{u}) = \mathbf{x}_3 + t\mathbf{u}$. Therefore, it holds that

$$\mathbf{x}_3 + t\mathbf{u} - \mathbf{x}_1 = \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \neq k\mathbf{u}\,,$$

for any $k > 0$. Thus, $F_{\preceq_*}$ is not MP-monotone.

The MP-monotonicity in case $\preceq = \preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means, follows straightforwardly from Proposition 4(iv) and the fact that translation equivariance implies MP-monotonicity (see Proposition 18 in [19]).

(vii) Consider the linear extension $\preceq_*$ defined in Proposition 4(iii). Consider $\mathbf{x}_1 = (4,4)^T$, $\mathbf{x}_2 = (7,7)^T$ and $\mathbf{x}_3 = (7,1)^T$. It holds that $\mathbf{x}_3 \preceq_* \mathbf{x}_1 \preceq_* \mathbf{x}_2$, and, therefore, $F_{\preceq_*}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbf{x}_1$. Consider $t = 2$ and $\mathbf{u} = (-1, 0)^T$. It holds that $\mathbf{x}_1 + t\mathbf{u} \preceq_* \mathbf{x}_3 + t\mathbf{u} \preceq_* \mathbf{x}_2 + t\mathbf{u}$, and, therefore, $F_{\preceq_*}(\mathbf{x}_1 + t\mathbf{u}, \mathbf{x}_2 + t\mathbf{u}, \mathbf{x}_3 + t\mathbf{u}) = \mathbf{x}_3 + t\mathbf{u}$. Therefore, it holds that

$$(\mathbf{x}_3 + t\mathbf{u} - \mathbf{x}_1) \cdot \mathbf{u} = \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 < 0\,.$$

Thus, $F_{\preceq_*}$ is not MC-monotone.

The MC-monotonicity in case $\preceq = \preceq_{\mathbf{M}}$, with $\mathbf{M} = (M_1, \ldots, M_m)$ being m linearly independent weighted arithmetic means, follows straightforwardly from Proposition 4(iv) and the fact that translation equivariance implies MC-monotonicity (see Theorem 14 and Proposition 18 in [19]).

(viii) Consider any $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}(1) < \mathbf{y}(1)$ and $\mathbf{y}(2) < \mathbf{x}(2)$. We distinguish two cases:

- If $\mathbf{x} \preceq \mathbf{y}$, then there exists $\mathbf{z}$ such that $\mathbf{x} \leq_m \mathbf{z}$ and $\mathbf{x} \preceq \mathbf{y} \preceq \mathbf{z}$. Consider $\mathbf{x}_i = \mathbf{x}$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{y}$ for any $i > \frac{n}{2} + 1$. Independently of whether n is odd or even, it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}$. However, if one substitutes $\mathbf{x}_i$ by $\mathbf{z}$ (with $i = \frac{n+1}{2}$ if n is odd and both $i = \frac{n}{2}$ and $i = \frac{n}{2} + 1$ if n is even), it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \mathbf{y}$. The result then follows from the fact that $\mathbf{x} \leq_m \mathbf{z}$ but $\mathbf{x} \not\leq_m \mathbf{y}$.

- If $\mathbf{y} \preceq \mathbf{x}$, then there exists $\mathbf{z}$ such that $\mathbf{y} \leq_m \mathbf{z}$ and $\mathbf{y} \preceq \mathbf{x} \preceq \mathbf{z}$. Consider $\mathbf{x}_i = \mathbf{y}$ for any $i \leq \frac{n}{2}+1$ and $\mathbf{x}_i = \mathbf{x}$ for any $i > \frac{n}{2}+1$. Independently of whether n is odd or even, it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{y}$. However, if one substitutes $\mathbf{x}_i$ by $\mathbf{z}$ (with $i = \frac{n+1}{2}$ if n is odd and both $i = \frac{n}{2}$ and $i = \frac{n}{2} + 1$ if n is even), it holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{z}, \ldots, \mathbf{x}_n) = \mathbf{x}$. The result then follows from the fact that $\mathbf{y} \leq_m \mathbf{z}$ but $\mathbf{y} \not\leq_m \mathbf{x}$.

$\square$

3.6. Continuity

The property of continuity assures that small changes in the input do not cause large changes in the output of a function. Since the set $(\preceq, \mathbb{R}^m)$ is totally ordered given a linear extension $\preceq$ of the product order $\leq_m$, it is evident that the order-based multivariate median based on $\preceq$ is continuous with respect to the order topology induced by $\preceq$ itself. However, as it was the case with the monotonicity properties, continuity of multivariate functions is typically understood quite differently.

Definition 7. Consider $n, m \in \mathbb{N}$. A function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is called (pointwisely) continuous if

$$\lim_{\substack{\mathbf{y}_i \to \mathbf{x}_i \\ i \in \{1,\ldots,n\}}} F(\mathbf{y}_1, \ldots, \mathbf{y}_n) = F(\mathbf{x}_1, \ldots, \mathbf{x}_n)\,,$$

for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$.

It turns out that the oversimplification of an m-dimensional space into a unidimensional space makes the order-based multivariate median, which is known to be continuous in the univariate setting, to no longer be a continuous function in case $n > 2$.

Proposition 6. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$. The order-based multivariate median $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is continuous if and only if $n \leq 2$.*

Proof. The order-based multivariate median coincides with the centroid if and only if $n \leq 2$, and the centroid is known to be continuous. It only remains to check the case in which $n > 2$.

Consider $\mathbf{x} \preceq \mathbf{y}$, $\mathbf{u} = (u_1, \dots, u_m)$ with $u_j > 0$ for any j and such that $(\mathbf{y} - \mathbf{x}) \neq k\mathbf{u}$ for any $k \in \mathbb{R}$. We define $t = \inf\{a > 0 \mid \mathbf{y} \preceq \mathbf{x} + a\mathbf{u}\}$. Note that this t is assured to exist since $\preceq$ is a linear extension of $\leq_m$ and $u_j > 0$ for any j. We define $\mathbf{x}' = \mathbf{x} + t\mathbf{u}$. By definition, it holds that $(\mathbf{y} - \mathbf{x}') \neq k\mathbf{u}$ and $\mathbf{x}' - \varepsilon\mathbf{u} \preceq \mathbf{y} \preceq \mathbf{x}' + \varepsilon\mathbf{u}$ for any $\varepsilon > 0$. We distinguish two cases:

- If $\mathbf{x}' \preceq \mathbf{y}$, then it holds that $\mathbf{x}' \preceq \mathbf{y} \preceq \mathbf{x}' + \varepsilon\mathbf{u}$ for any $\varepsilon > 0$. Consider $\mathbf{x}_i = \mathbf{x}'$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{y}$ for any $i > \frac{n}{2} + 1$. It follows that $F_{\preceq}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{x}'$. In case n is odd, if one considers $\mathbf{y}_{\frac{n+1}{2}} = \mathbf{x}_{\frac{n+1}{2}} + \varepsilon\mathbf{u}$ and $\mathbf{y}_i = \mathbf{x}_i$ for any other i, then $F_{\preceq}(\mathbf{y}_1, \dots, \mathbf{y}_n) = \mathbf{y}$ and, therefore, $F_{\preceq}$ is not continuous. In case n is even, if one considers $\mathbf{y}_{\frac{n}{2}} = \mathbf{x}_{\frac{n}{2}} + \varepsilon\mathbf{u}$, $\mathbf{y}_{\frac{n}{2}+1} = \mathbf{x}_{\frac{n}{2}+1} + \varepsilon\mathbf{u}$ and $\mathbf{y}_i = \mathbf{x}_i$ for any other i, then $F_{\preceq}(\mathbf{y}_1, \dots, \mathbf{y}_n) = \mathbf{y}$ and, therefore, $F_{\preceq}$ is not continuous.

- If $\mathbf{y} \preceq \mathbf{x}'$, then it holds that $\mathbf{x}' - \varepsilon\mathbf{u} \preceq \mathbf{y} \preceq \mathbf{x}'$ for any $\varepsilon > 0$. Consider $\mathbf{x}_i = \mathbf{x}'$ for any $i \leq \frac{n}{2} + 1$ and $\mathbf{x}_i = \mathbf{y}$ for any $i > \frac{n}{2} + 1$. It follows that $F_{\preceq}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{x}'$. In case n is odd, if one considers $\mathbf{y}_{\frac{n+1}{2}} = \mathbf{x}_{\frac{n+1}{2}} - \varepsilon\mathbf{u}$ and $\mathbf{y}_i = \mathbf{x}_i$ for any other i, then $F_{\preceq}(\mathbf{y}_1, \dots, \mathbf{y}_n) = \mathbf{y}$ and, therefore, $F_{\preceq}$ is not continuous. In case n is even, if one considers $\mathbf{y}_{\frac{n}{2}} = \mathbf{x}_{\frac{n}{2}} - \varepsilon\mathbf{u}$, $\mathbf{y}_{\frac{n}{2}+1} = \mathbf{x}_{\frac{n}{2}+1} - \varepsilon\mathbf{u}$ and $\mathbf{y}_i = \mathbf{x}_i$ for any other i, then $F_{\preceq}(\mathbf{y}_1, \dots, \mathbf{y}_n) = \mathbf{y}$ and, therefore, $F_{\preceq}$ is not continuous.

□

4. The finite-sample breakdown point of the order-based multivariate median

The finite-sample breakdown point [7, 12] is a popular measure of robustness that indicates the smallest proportion of contaminated observations that may cause a function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ to take arbitrarily large values.

Definition 8. Consider $n, m \in \mathbb{N}$. Consider $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$. The set $\mathcal{Z}_q(\mathbf{X})$ of q-corrupted (by replacement) lists given $\mathbf{X}$ is defined as the set of all lists $\mathbf{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_n) \in (\mathbb{R}^m)^n$ such that

$$|\{i \in \{1, \dots, n\} \mid \mathbf{x}_i \neq \mathbf{z}_i\}| \leq q\,.$$

The maximum bias caused by q-corruption at $\mathbf{X}$ for a function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is defined as

$$b(F, \mathbf{X}, q) = \sup_{\mathbf{Z} \in \mathcal{Z}_q(\mathbf{X})} ||F(\mathbf{X}) - F(\mathbf{Z})|| \,.$$

The finite-sample breakdown point of a function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ at $\mathbf{X}$ is defined as

$$\epsilon(F, \mathbf{X}) = \inf_{\substack{q \in \{1,\ldots,n\} \\ b(F,\mathbf{X},q) = +\infty}} \frac{q}{n} \,.$$

The finite-sample breakdown point of a function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is defined as

$$\epsilon(F) = \inf_{\mathbf{X} \in (\mathbb{R}^m)^n} \epsilon(F, \mathbf{X}) \,.$$

The finite-sample breakdown point of the univariate median is 0.5, which is the maximum value that can be attained by a translation equivariant function $F : (\mathbb{R}^m)^n \to \mathbb{R}^m$. Some multivariate generalizations of the median, such as the componentwise median and the spatial median, inherit this maximum finite-sample breakdown point value (see [15]), whereas some others, such as Tukey's halfspace median, are still very robust with a finite-sample breakdown point of at least $\frac{1}{m+1}$ [6]. However, there are some other generalizations such as Oja's simplex median (see [16]) or Liu's simplicial depth median (see [4]) that are not very robust. Unfortunately, it can be seen that the order-based multivariate median is of the latter type and its finite-sample breakdown point converges to zero as n tends to infinity.

Proposition 7. *Consider $n, m \in \mathbb{N}$ with $m \geq 2$ and a linear extension $\preceq$ of $\leq_m$ that linearly extends the extension $\precsim_f$ of $\leq_m$ based on an idempotent and monotone increasing function $f : \mathbb{R}^m \to \mathbb{R}$. The finite-sample breakdown point of the order-based multivariate median $F_{\preceq} : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is $\frac{1}{n}$.*

Proof. Since f is monotone increasing there exists at least one unbounded level set of f (see, e.g., Proposition 4.3.2 in [18]). Denote by c the value associated with such level set. Consider $\mathbf{x}_1, \ldots, \mathbf{x}_n$ such that $f(\mathbf{x}_i) < c$ if $i < \frac{n}{2}$; $f(\mathbf{x}_i) = c$ if $i = \frac{n+1}{2}$ in case n is odd and $i \in \{\frac{n}{2}, \frac{n}{2} + 1\}$ in case n is even; and $f(\mathbf{x}_i) > c$ if $i > \frac{n}{2} + 1$. Note that the idempotence of f assures that such points exist. It obviously holds that $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_{\frac{n+1}{2}}$ in case n is odd and $F_{\preceq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{\mathbf{x}_{\frac{n}{2}} + \mathbf{x}_{\frac{n}{2}+1}}{2}$ in case n is even. This result will still hold even if the corresponding point is changed by any other point in its level set. Since the level set is unbounded, this implies that $b(F_{\preceq}, (\mathbf{x}_1, \ldots, \mathbf{x}_n), 1) = +\infty$, and, therefore, $\epsilon(F_{\preceq}, \mathbf{X}) = \epsilon(F_{\preceq}) = \frac{1}{n}$.

□

In particular, the result above implies that, whenever a linear extension of the product order based on m linearly independent weighted arithmetic means is considered, the finite-sample breakdown point of the order-based multivariate median is $\frac{1}{n}$. Note that the result still holds even in case the first weighted arithmetic mean is substituted by a more robust function such as the median. As natural as it is, the order-based multivariate median behaves in a way that inherently depends on the level sets of the considered function. This behaviour is particularly undesirable in the presence of outliers and results in the same finite-sample breakdown point value as the centroid.

5. Concluding remarks

The order-based multivariate median has been proved to be symmetric, CH-internal, BB-internal and idempotent (also being internal within the points for an odd number of points). Properties such as affine equivariance, orthogonal equivariance, most monotonicity properties and continuity are assured to fail, regardless of the choice of linear extension of the product order. However, scale equivariance and translation equivariance (and thus MP-monotonicity and MC-monotonicity), which are not fulfilled in general, are assured to be fulfilled if the linear extension is based on m linearly independent weighted arithmetic means. Interestingly, the finite-sample breakdown point of this order-based multivariate median is $\frac{1}{n}$. This contrasts with the univariate case in which the median attains the maximum possible value of the finite-sample breakdown point ($\frac{1}{2}$) that could be attained by functions that are translation equivariant.

Even though the properties fulfilled by the the order-based multivariate median are not very appealing, it is admittedly still natural to consider such function in some settings. For instance, the in-shoes height of an individual depends on both the actual height of the individual and the height of the shoes that the individual is wearing. In this setting, it might be natural to compare individuals in terms of their in-shoes height at first, and further refine this order by considering the actual height as a tie-breaker for individuals with the same in-shoes height. Note that this would be the result of considering the linear extension of the product order by Xu and Yager ($\mathbf{M} = (M_1, M_2)$, with $M_1(\mathbf{x}_i) = \frac{1}{2}\mathbf{x}_i(1) + \frac{1}{2}\mathbf{x}_i(2)$ and $M_2(\mathbf{x}_i) = \mathbf{x}_i(2)$).

As an illustrative example, consider five individuals with the following actual heights and shoe heights:

Shoe height (in cm)	17	7	3	8	9
Actual height (in cm)	169	176	178	183	190

The five couples above can be ordered increasingly in terms of their in-shoes height:

$$(3, 178)^T \preceq_{\mathbf{M}} (7, 176)^T \preceq_{\mathbf{M}} (17, 169)^T \preceq_{\mathbf{M}} (8, 183)^T \preceq_{\mathbf{M}} (9, 190)^T .$$

The resulting order-based multivariate median is $(17, 169)^T$, which intuitively represents the median in-shoes height of the five individuals. The fact that the shortest individual is the one with the median in-shoes height does not actually matter. Admittedly, this individual could be substituted by a kid on stilts (represented by $(70, 118)^T$) and it will still be considered to be the individual with the median in-shoes height. This is because the in-shoes height is a notion univariate in nature, even though it is here represented as a bivariate notion. Obviously, when jointly discussing the shoe height and the actual height, this kid on stilts should never be considered to be a good representative of the group. Most of the literature on multivariate location points towards this direction and, probably, this is the reason why there is little to no mention of the order-based multivariate median.

As a final comment, it is highlighted that most of the results here presented for the the order-based multivariate median are easily translated into the original setting of OWA operators by De Miguel et al. [5]. Obviously, the result concerning the property of internality within the points would only be fulfilled by order statistics (i.e., OWA operators with an associated vector of weights that only has one non-null element). Special attention should be devoted to the centroid, which is the only order-based extension of an OWA operator in the sense of De Miguel et al. [5] that does not depend on the chosen linear extension of the product order, and satisfies all properties discussed in this paper (but internality within the points).

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