Coupling of virtual element and boundary element methods for the solution of acoustic scattering problems*

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Abstract

This paper extends the applicability of the combined use of the virtual element method (VEM) and the boundary element method (BEM), recently introduced to solve the coupling of linear elliptic equations in divergence form with the Laplace equation, to the case of acoustic scattering problems in 2D and 3D. As a model we consider a bounded obstacle with piecewise constant refractive index, and a time harmonic incident wave, so that the scattered field, and hence the total wave as well, satisfies the homogeneous Helmholtz equation in the unbounded exterior region. The resulting coupled problem is complemented with suitable transmission conditions and the Sommerfeld radiation condition at infinity. The usual primal formulation and the corresponding VEM approach are then employed in the obstacle, which is combined, by means of either the Costabel & Han approach or a modification of it, with the boundary integral equation method in the exterior domain, thus yielding two possible VEM/BEM schemes. The first one of them, which is valid only in 2D, considers the main variable and its normal derivative as unknowns, whereas the second one, valid in 2D and 3D, adds the trace of the original unknown. In both procedures, the above mentioned boundary unknowns are non-virtual, and hence they are approximated by usual finite element subspaces. In addition, the discrete setting certainly requires virtual element subspaces for the main unknowns, and suitable projection and interpolation operators that are employed to define the corresponding discrete bilinear forms. The well-posedness of the continuous and discrete formulations are established, and the key aspects of the associated analyses include the fact that the boundary integral operators of the Helmholtz equation are compact perturbations of those for the Laplacian, the use of the Fredholm alternative, and the introduction of Galerkin projection-type operators. Finally, Cea-type estimates and consequent rates of convergence for the solutions are also derived.

1 Introduction

In the recent paper [8] we introduced and analyzed, up to our knowledge for the first time, the combined use of VEM and BEM for numerically solving transmission problems in 2D and 3D. An elliptic equation in divergence form holding in a bounded region, coupled with the Laplace equation in the corresponding unbounded exterior domain, in addition to transmission conditions on the interface and a suitable radiation condition at infinity, were considered as the respective model. In turn, the Costabel & Han approach and a suitable modification of it that was motivated by the 3D case, but

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not restricted to that dimension, were employed there to combine the primal VEM approach in the interior domain with the boundary integral equation method in the exterior one. In this way, besides the original variable of the model, its normal derivative in 2D, and both its normal derivative and its trace in the 3D case, were introduced as non-virtual unknowns. A priori error estimates and optimal rates of convergence for the solution as well as for a fully calculable projection of the virtual component of it were provided in [8]. Additionally, several numerical examples in 2D illustrating the performance of the VEM/BEM schemes, were also reported there. To some extent, one could argue that, irrespective of the particular transmission problem studied in [8], the main contribution of this work is, perhaps, having settled some fundamentals that would help to apply later on the coupling of VEM and BEM to any other model of interest that has been previously solved by the coupling of BEM with the classical finite element methods (or other Galerkin-type procedure). In this regard, we stress that the advantages of using VEM, which are certainly transferred to its combination with BEM, include the simplicity of the respective coding, and the fact that the elements of the meshes can be chosen as nonoverlapping nonconvex regions of very general shape. For very detailed bibliographic discussions on VEM and BEM, separately, we refer the interested reader to [8, Section 1] and the references there in indicated.

In virtue of the above comments, and aiming to provide further results of interest regarding the coupling of VEM and BEM, we now address its applicability to the numerical solution of acoustic scattering problems in 2D and 3D. The rest of this work is organized as follows. In Section 2 we describe the model problem and establish a corresponding uniqueness result. Then, the boundary integral equation method for the Helmholtz equation, and the coupling procedures to be employed, namely the Costabel & Han one and a suitable modification of it, are introduced in Section 3. Next, in Section 4 we use the Fredholm alternative to prove the well-posedness of the continuous formulations arising from both coupling methods. The 2D discrete VEM/BEM schemes for each one of the coupling procedures from Section 4 are introduced and analyzed in Section 5. More precisely, this section is split into four subsections dealing with some preliminary definitions and results on VEM, the explicit definitions of the discrete schemes, the solvability analysis of each one of them, and the respective a priori error estimates and consequent rates of convergence. In particular, the use of Galerkin projection-type operators and compactness arguments play a key role in the derivation of the associated discrete inf-sup conditions. Finally, in Section 6 we follow basically the same structure of Section 5 to introduce and analyze our discrete VEM/BEM scheme in 3D, which uses the aforementioned modified Costabel & Han coupling method.

We end this section with some notations to be employed throughout the rest of the paper. In particular, given a real number $r \geq 0$ and a polyhedron $\mathcal{O} \subseteq \mathbb{R}^d$, $(d = 2, 3)$, we denote by $\|\cdot\|_{r,\mathcal{O}}$ and $|\cdot|_{r,\mathcal{O}}$, respectively, the norm and seminorm of the usual Sobolev space $\mathrm{H}^r(\mathcal{O})$ (cf. [12]). Also, we use the convention $\mathrm{L}^2(\mathcal{O}) := \mathrm{H}^0(\mathcal{O})$, and for all $t \in (0,1]$ we let $\mathrm{H}^{-t}(\partial\mathcal{O})$ be the dual of $\mathrm{H}^t(\partial\mathcal{O})$ with respect to the pivot space $\mathrm{L}^2(\partial\mathcal{O})$. In addition, we set $\mathcal{P}_{-1} = \{0\}$, and for a nonnegative integer m, $\mathcal{P}_m$ is the space of polynomials of degree $\leq m$. Then, given a domain $D \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, $\mathcal{P}_m(D)$ stands for the restriction of $\mathcal{P}_m$ to D.

2 The model problem

Given $d \in \{2,3\}$, let $\theta : \mathbb{R}^d \to \mathbb{C}$ be a complex-valued function satisfying $\mathrm{Re}\big(\theta(\boldsymbol{x})\big) > 0$ and $\mathrm{Im}\big(\theta(\boldsymbol{x})\big) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$, and such that $1 - \theta(\boldsymbol{x})$ has a compact support in $\mathbb{R}^d$. In addition, let $\kappa > 0$ be given together with a function w satisfying the Helmholtz equation $\Delta w + \kappa^2 w = 0$ in $\mathbb{R}^d$. Then we seek $u : \mathbb{R}^d \to \mathbb{C}$ satisfying

$$\begin{aligned} \Delta u + \kappa^2 \theta(\boldsymbol{x})\, u &= 0 && \text{in} \quad \mathbb{R}^d, \\ u &= w + u^s && \text{in} \quad \mathbb{R}^d, \end{aligned} \tag{2.1}$$

where u^s satisfies a homogeneous Helmholtz equation and the outgoing Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r} - \imath\kappa u^s = o(r^{\frac{1-d}{2}}), \tag{2.2}$$

when $r := |\boldsymbol{x}| \to \infty$ uniformly for all directions $\boldsymbol{x}/|\boldsymbol{x}|$. The system (2.1)-(2.2) governs the propagation of time harmonic acoustic waves of small amplitude in an inhomogeneous and (possibly) absorbing medium. The wave motion is caused by a time harmonic incident field w of amplitude κ. A common choice for w in the $2D$ case is the plane wave $w(\boldsymbol{x}) := \exp(\imath\kappa\boldsymbol{\nu}\cdot\boldsymbol{x})$ where $\boldsymbol{\nu}$ is a fixed unit vector. The solution u of our problem is determined by the scattered field u^s that satisfies the Sommerfeld radiation condition (2.2). We refer to [4, 11] for more information about the physical background of the problem. For the sake of simplicity, we assume here that the obstacle is constituted of diverse materials, each one of them having constant refractive index θ. In this way, we assume that there exists a set of Lipschitz polygons $\{\Omega_i : i = 1,\ldots,I\}$ such that $\mathrm{supp}(1-\theta) = \cup_{i=1}^{I}\overline{\Omega}_i$ and $\theta|_{\Omega_i} \in \mathbb{C}$, $\forall i = 1,\ldots,I$. Concerning the solvability of (2.1)-(2.2), we first have the following result.

Theorem 2.1. *A function $u \in \mathrm{H}^1_{loc}(\mathbb{R}^d)$ satisfying* (2.1)-(2.2) *with $w = 0$ should vanish identically everywhere.*

Proof. We restrict ourselves to $d = 2$, the case $d = 3$ being similar. Let $a > 0$ be such that the support of $1 - \theta(\boldsymbol{x})$ is contained in the disk $B(0,a)$ of radius a centered at the origin. A straightforward application of Green's theorem in $B(0,a)$ gives

$$\int_{|\boldsymbol{x}|=a} u\frac{\partial \bar{u}}{\partial \boldsymbol{n}}\,\mathrm{d}\sigma = \int_{|\boldsymbol{x}|<a}\left(|\nabla u|^2 - \kappa^2\bar{\theta}(\boldsymbol{x})|u|^2\right)\mathrm{d}\boldsymbol{x}\,,$$

from which, taking into account that $\mathrm{Im}[\theta(\boldsymbol{x})] \geq 0$, we deduce that

$$\mathrm{Im}\left(\int_{|\boldsymbol{x}|=a} u\frac{\partial \bar{u}}{\partial \boldsymbol{n}}\,\mathrm{d}\sigma\right) = \kappa^2\int_{|\boldsymbol{x}|<a}\mathrm{Im}[\theta(\boldsymbol{x})]|u|^2\,\mathrm{d}\boldsymbol{x} \geq 0\,.$$

Hence, Rellich's theorem (cf. [4, Theorem 2.12]) ensures that $u(\boldsymbol{x}) = 0$ in $B^e(0,a) := \mathbb{R}^2 \setminus \bar{B}(0,a)$. In turn, we deduce from our hypothesis on θ that we can consider a set of Lipschitz and convex subdomains $\{\Omega_p : p = 1,\ldots,P\}$ satisfying $\cup_{p=1}^{P}\bar{\Omega}_p = \bar{B}(0,a)$ and $\theta_p := \theta|_{\Omega_p} \in \mathbb{C}$ for all $1 \leq p \leq P$. We pick a subdomain Ω_p such that $\partial\Omega_p \cap \partial B(0,a)$ is a segment of positive measure. Let $B(\boldsymbol{x}_0,b)$ be a disk centered at a point $\boldsymbol{x}_0 \in \partial\Omega_p \cap \partial B(0,a)$ with a radius b such that $B(\boldsymbol{x}_0,b) \subseteq \bar{\Omega}_p \cup \bar{B}^e(0,a)$. Since u vanishes in $B^e(0,a)$, we have that

$$\Delta u + \kappa^2\theta_p u = 0 \quad \text{in } \Omega_p \cup B^e(0,a). \tag{2.3}$$

It follows from (2.3) and a classical regularity result for the Laplace operator that $u \in \mathrm{H}^2(B(\boldsymbol{x}_0,b))$. Moreover, by virtue of the unique continuation principle (cf. [14, Lemma 4.15]), the fact that u satisfies (2.3) and vanishes identically in a disk contained in $B(\boldsymbol{x}_0,b) \cap B^e(0,a)$ imply that it should be identically zero in $B(\boldsymbol{x}_0,b)$. We notice now that, as Ω_p is convex, we also have that $u \in \mathrm{H}^2(\Omega_p)$ and we can apply again the unique continuation result as above to prove that u vanishes identically in Ω_p. The same strategy shows that, if two subdomains Ω_p and Ω_q are such that $\bar{\Omega}_p \cap \bar{\Omega}_q$ is a Lipschitz curve with a non–empty interior, then if $u = 0$ in Ω_p implies that u is also identically equal to zero in Ω_q. This proves that u vanishes everywhere in $\mathbb{R}^2$. □

3 The coupling procedures

In this section we describe the continuous version of the two coupling procedures that we plan to utilize for the combination of VEM and BEM. In this regard, we remark in advance that the first discrete scheme to be proposed will work only in 2D, whereas the second one will be valid for both 2D and 3D.

3.1 The boundary integral equation method for Helmholtz

We first discuss the basic aspects of the boundary integral equation method for the Helmholtz equation. To this end, we now introduce a polygonal/polyhedral boundary Γ containing in its interior the support of $1-\theta$. Then Γ separates $\mathbb{R}^d$ into a bounded polygonal/polyhedral domain Ω and the unbounded region Ω^e exterior to Γ. We denote by $\boldsymbol{n}$ the unit normal vector to Γ that is directed towards Ω^e. The scattered field u^s satisfies a homogeneous Helmholtz equation in Ω^e and the radiation condition (2.2). Then, denoting by $H_0^{(1)}$ the Hankel function of order 0 and first type, it can be proved that u^s admits the integral representation

$$u^s(\boldsymbol{x}) = \int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_y} u^s(\boldsymbol{y})\,\mathrm{d}\sigma_y - \int_\Gamma E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\frac{\partial u^s(\boldsymbol{y})}{\partial \boldsymbol{n}}\,\mathrm{d}\sigma_y \qquad \forall \boldsymbol{x}\in\Omega^e\,, \tag{3.1}$$

where

$$E_\kappa(r) := \begin{cases} \dfrac{\imath}{4} H_0^{(1)}(\kappa r) & \text{if } d=2 \\ \dfrac{e^{\imath\kappa r}}{4\pi r} & \text{if } d=3 \end{cases}$$

is the radial outgoing fundamental solution of the Helmholtz equation with wave number κ. We denote by γ and $\gamma_{\boldsymbol{n}}$ the trace and normal trace operators, respectively, on Γ, acting either from Ω or from Ω^e. Then, applying these operators to both sides of (3.1), denoting $\lambda := \gamma_{\boldsymbol{n}}(\nabla u^s) = \dfrac{\partial u^s}{\partial \boldsymbol{n}}$, and taking into account the well-known jump properties of the boundary integral operators, we obtain

$$0 = (\frac{\mathrm{id}}{2} - K_\kappa)\gamma u^s + V_\kappa\lambda\,, \tag{3.2}$$

$$\lambda = -W_\kappa\gamma u^s + (\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}})\lambda\,, \tag{3.3}$$

where id is a generic identity operator, and V_κ, K_κ, $K_\kappa^{\mathtt{t}}$, and W_κ are the boundary integral operators representing the single, double, adjoint of the double, and hypersingular layer potentials, respectively. The latter are formally defined at almost every point $\boldsymbol{x}\in\Gamma$ by

$$\begin{aligned} V_\kappa\lambda(\boldsymbol{x}) &:= \int_\Gamma E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}, \qquad K_\kappa\varphi(\boldsymbol{x}) := \int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\varphi(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}, \\ K_\kappa^{\mathtt{t}}\lambda(\boldsymbol{x}) &:= \int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}, \quad W_\kappa\varphi(\boldsymbol{x}) := -\frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\varphi(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} \end{aligned} \tag{3.4}$$

for suitable functions λ and φ. More precisely, the main mapping properties of these operators are collected in the following lemma.

Lemma 3.1. *The operators*

$$\begin{aligned} V_\kappa:\ H^{-1/2+\sigma}(\Gamma) &\longrightarrow H^{1/2+\sigma}(\Gamma), \qquad K_\kappa:\ H^{1/2+\sigma}(\Gamma) \longrightarrow H^{1/2+\sigma}(\Gamma) \\ K_\kappa^{t}:\ H^{-1/2+\sigma}(\Gamma) &\longrightarrow H^{-1/2+\sigma}(\Gamma), \qquad W_\kappa:\ H^{1/2+\sigma}(\Gamma) \longrightarrow H^{-1/2+\sigma}(\Gamma), \end{aligned}$$

are continuous for all $\sigma\in[-1/2,\ 1/2]$.

Proof. See [15]. □

In turn, the boundary integral operators V_0, K_0, $K_0^{\mathtt{t}}$ and W_0 defined as in (3.4) but in terms of the fundamental solution $E_0(|\boldsymbol{x}-\boldsymbol{y}|)$ of the Laplacian, which is defined as

$$E_0(|\boldsymbol{x}-\boldsymbol{y}|) := \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{if } d=2 \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{if } d=3\,, \end{cases}$$

have the same continuity properties given in Lemma 3.1. Furthermore, in order to recall additional results concerning the particular operators V_0 and W_0, and for additional use throughout the rest of the paper, we now let $\langle \cdot, \cdot \rangle$ be both the inner product in $\mathrm{L}^2(\Gamma)$ and the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ with respect to the pivot space $\mathrm{L}^2(\Gamma)$, and introduce the subspaces

$$\mathrm{H}_0^{1/2}(\Gamma) := \{\varphi \in \mathrm{H}^{1/2}(\Gamma) : \langle 1, \varphi \rangle = 0\}$$

and

$$\mathrm{H}_0^{-1/2}(\Gamma) := \{\mu \in \mathrm{H}^{-1/2}(\Gamma) : \langle \mu, 1 \rangle = 0\}.$$

Then, we can state the following lemma.

Lemma 3.2. *There exist positive constants α_V, C_V, and α_W such that*

$$\langle \bar{\mu}, V_0\mu \rangle \geq \alpha_V \|\mu\|^2_{-1/2,\Gamma} \quad \begin{cases} \forall\, \mu \in \mathrm{H}_0^{-1/2}(\Gamma), & \text{if } d = 2, \\ \forall\, \mu \in \mathrm{H}^{-1/2}(\Gamma), & \text{if } d = 3, \end{cases} \tag{3.5}$$

$$\langle \bar{\mu}, V_0\mu \rangle_\Gamma + \left|\langle \bar{\mu}, 1 \rangle\right|^2 \geq C_V \|\mu\|^2_{-1/2,\Gamma} \quad \forall \mu \in \mathrm{H}^{-1/2}(\Gamma) \quad \text{if } d = 2, \tag{3.6}$$

and

$$\langle W_0\varphi, \bar{\varphi} \rangle \geq \alpha_W \|\varphi\|^2_{1/2,\Gamma} \quad \forall\, \varphi \in \mathrm{H}_0^{1/2}(\Gamma). \tag{3.7}$$

Proof. See [12, 15]. □

Finally, we stress that all the integral operators associated to the Helmholtz equation may be regarded as compact perturbations of the corresponding Laplacian-based operators (see [15]). In fact, we have the following result.

Lemma 3.3. *The operators*

$$V_\kappa - V_0 : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \qquad K_\kappa - K_0 : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$$
$$K^{\mathtt{t}}_\kappa - K^{\mathtt{t}}_0 : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \qquad W_\kappa - W_0 : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),$$

are compact.

3.2 The Costabel & Han coupling

Our first strategy taking advantage of the boundary integral equations (3.2)-(3.3) to reformulate problem (2.1)-(2.2) in the bounded domain Ω is due to Costabel and Han (cf. [5] and [10]). It reads as follows: Find $u : \Omega \to \mathbb{C}$ and $\lambda : \Gamma \to \mathbb{C}$ such that

$$\Delta u + \kappa^2\theta(\boldsymbol{x})\, u = 0 \quad \text{in } \Omega, \tag{3.8}$$
$$\gamma u = \gamma u^s + \gamma w \quad \text{on } \Gamma, \tag{3.9}$$
$$\frac{\partial u}{\partial \boldsymbol{n}} = \lambda + \frac{\partial w}{\partial \boldsymbol{n}} \quad \text{on } \Gamma, \tag{3.10}$$
$$0 = (\frac{\mathrm{id}}{2} - K_\kappa)\gamma u^s + V_\kappa \lambda \tag{3.11}$$
$$\lambda = -W_\kappa \gamma u^s + (\frac{\mathrm{id}}{2} - K^{\mathtt{t}}_\kappa)\lambda. \tag{3.12}$$

Once the Cauchy data γu^s and λ are known, the solution is computed in the exterior domain Ω^e by using the integral representation formula (3.1). It is straightforward to see, according to the analyses provided in [5] and [10], that the variational formulation of problem (3.8)-(3.12) consists in finding $(u,\lambda)\in\mathbb{X}:=\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$ such that

$$\mathbb{A}_\kappa\big((u,\lambda),(v,\mu)\big)=\mathbb{F}(v,\mu):=\langle\frac{\partial w}{\partial\boldsymbol{n}}+W_\kappa\gamma w,\gamma v\rangle+\langle\mu,(\frac{\mathrm{id}}{2}-K_\kappa)\gamma w\rangle\qquad\forall(v,\mu)\in\mathbb{X}\,,\tag{3.13}$$

where

$$\begin{aligned}\mathbb{A}_\kappa\big((z,\xi),(v,\mu)\big)&:=a_\kappa(z,v)+\big\langle W_\kappa\gamma z,\gamma v\big\rangle+\big\langle\mu,V_\kappa\xi\big\rangle\\&+\big\langle\mu,(\frac{\mathrm{id}}{2}-K_\kappa)\gamma z\big\rangle-\big\langle\xi,(\frac{\mathrm{id}}{2}-K_\kappa)\gamma v\big\rangle\end{aligned}\tag{3.14}$$

for all (z,ξ), $(v,\mu)\in\mathbb{X}$, with

$$a_\kappa(z,v):=\int_\Omega\nabla z\cdot\nabla v-\kappa^2\int_\Omega\theta zv\,.\tag{3.15}$$

3.3 The modified Costabel & Han coupling

We now appeal to the modified Costabel & Han approach introduced for the first time in [8, Section 4.2], which consists in considering not only the normal derivative λ but also the trace $\psi:=\gamma u^s$ as boundary unknowns in the formulation. This means that, instead of (3.1), the scattered field is computed as

$$u^s(\boldsymbol{x})=\int_\Gamma\frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial\boldsymbol{n}_y}\psi(\boldsymbol{y})\,\mathrm{d}\sigma_y-\int_\Gamma E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\frac{\partial u^s(\boldsymbol{y})}{\partial\boldsymbol{n}}\,\mathrm{d}\sigma_y\qquad\forall\boldsymbol{x}\in\Omega^e\,,\tag{3.16}$$

whence the corresponding identities (3.2) and (3.3) become

$$0=(\frac{\mathrm{id}}{2}-K_\kappa)\psi+V_\kappa\lambda\,,\tag{3.17}$$

$$\lambda=-W_\kappa\psi+(\frac{\mathrm{id}}{2}-K_\kappa^{\mathtt{t}})\lambda\,.\tag{3.18}$$

In this way, the reformulation of problem (2.1)-(2.2) in the bounded domain Ω now reads as follows: Find $u:\,\Omega\to\mathbb{C}$, $\psi:\,\Gamma\to\mathbb{C}$ and $\lambda:\,\Gamma\to\mathbb{C}$ such that

$$\Delta u+\kappa^2\theta(\boldsymbol{x})\,u=0\quad\text{in }\Omega,\tag{3.19}$$

$$\gamma u=\gamma u^s+\gamma w\quad\text{on }\Gamma,\tag{3.20}$$

$$\psi=\gamma u^s\quad\text{on }\Gamma,\tag{3.21}$$

$$\frac{\partial u}{\partial\boldsymbol{n}}=\lambda+\frac{\partial w}{\partial\boldsymbol{n}}\quad\text{on }\Gamma,\tag{3.22}$$

$$0=(\frac{\mathrm{id}}{2}-K_\kappa)\psi+V_\kappa\lambda\tag{3.23}$$

$$\lambda=-W_\kappa\psi+(\frac{\mathrm{id}}{2}-K_\kappa^{\mathtt{t}})\lambda\,.\tag{3.24}$$

Then, proceeding analogously to [8, Section 4.2], that is integrating by parts (3.19), adding and subtracting the expression $\langle\lambda,\varphi\rangle$ with arbitrary $\varphi\in\mathrm{H}_0^{1/2}(\Gamma)$, imposing weakly the relation $\psi=\gamma u^s$ in $\mathrm{H}^{1/2}(\Gamma)$, and then suitably incorporating (3.22), (3.23), and (3.24) into the resulting terms, we arrive at the variational formulation: Find $(u,\psi,\lambda)\in\widetilde{\mathbb{X}}:=\mathrm{H}^1(\Omega)\times\mathrm{H}_0^{1/2}(\Gamma)\times\mathrm{H}^{-1/2}(\Gamma)$ such that

$$\widetilde{\mathbb{A}}_\kappa\big((u,\psi,\lambda),(v,\varphi,\mu)\big)\;=\;\widetilde{\mathbb{F}}(v,\varphi,\mu):=\langle\frac{\partial w}{\partial\boldsymbol{n}},\gamma v\rangle\;+\;\langle\mu,\gamma w\rangle\qquad\forall\,(v,\varphi,\mu)\in\widetilde{\mathbb{X}}\,,\tag{3.25}$$

where

$$\widetilde{\mathbb{A}}_\kappa\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;:=\; \mathbf{A}_\kappa\big((z,\phi,\xi),(v,\varphi,\mu)\big) \,+\, \big\langle W_\kappa \phi, \varphi\big\rangle \\ +\, \big\langle \mu, V_\kappa \xi\big\rangle \,+\, \big\langle \mu, (\frac{\mathrm{id}}{2} - K_\kappa)\phi\big\rangle \,-\, \big\langle \xi, (\frac{\mathrm{id}}{2} - K_\kappa)\varphi\big\rangle \tag{3.26}$$

for all (z,ϕ,ξ), $(v,\varphi,\mu) \in \widetilde{\mathbb{X}}$, with

$$\mathbf{A}_\kappa\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;:=\; a_\kappa(z,v) \,-\, \big\langle \xi, \gamma v - \varphi\big\rangle \,+\, \big\langle \mu, \gamma z - \phi\big\rangle\,, \tag{3.27}$$

and a_κ given by (3.15).

4 Solvability analysis of the continuous formulations

We now address the solvability analysis of (3.13) and (3.25). For this purpose, we introduce the bilinear forms

$$\mathbb{A}_0\big((z,\xi),(v,\mu)\big) := a_0(z,v) + \left(\int_\Gamma z\right)\left(\int_\Gamma v\right) + \big\langle W_0 \gamma z, \gamma v\big\rangle + \big\langle \mu, V_0 \xi\big\rangle \\ +\, \big\langle \xi, 1\big\rangle\big\langle \mu, 1\big\rangle + \big\langle \mu, (\frac{\mathrm{id}}{2} - K_0)\gamma z\big\rangle - \big\langle \xi, (\frac{\mathrm{id}}{2} - K_0)\gamma v\big\rangle \tag{4.1}$$

for all (z,ξ), $(v,\mu) \in \mathbb{X}$, and

$$\widetilde{\mathbb{A}}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;=\; \mathbf{A}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) + \left(\int_\Gamma z\right)\left(\int_\Gamma v\right) \,+\, \big\langle W_0 \phi, \varphi\big\rangle \\ +\, \big\langle \mu, V_0 \xi\big\rangle + \big\langle \xi, 1\big\rangle\big\langle \mu, 1\big\rangle \,+\, \big\langle \mu, (\frac{\mathrm{id}}{2} - K_0)\phi\big\rangle \,-\, \big\langle \xi, (\frac{\mathrm{id}}{2} - K_0)\varphi\big\rangle \tag{4.2}$$

for all (z,ϕ,ξ), $(v,\varphi,\mu) \in \widetilde{\mathbb{X}}$, where

$$a_0(z,v) := \int_\Omega \nabla z \cdot \nabla v\,, \tag{4.3}$$

and

$$\mathbf{A}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;:=\; a_0(z,v) \,-\, \big\langle \xi, \gamma v - \varphi\big\rangle \,+\, \big\langle \mu, \gamma z - \phi\big\rangle\,. \tag{4.4}$$

It follows from Lemma 3.1 that there exist positive constants $\|\mathbb{A}_\kappa\|$, $\|\mathbb{A}_0\|$, $\|\widetilde{\mathbb{A}}_\kappa\|$ and $\|\widetilde{\mathbb{A}}_0\| > 0$ such that for each $* \in \{\kappa, 0\}$ there hold

$$\big|\mathbb{A}_*\big((z,\xi),(v,\mu)\big)\big| \;\le\; \|\mathbb{A}_*\|\;\|(z,\xi)\|\;\|(v,\mu)\| \qquad \forall\,(z,\xi),\,(v,\mu) \in \mathbb{X} \tag{4.5}$$

and

$$\Big|\widetilde{\mathbb{A}}_*\big((z,\phi,\xi),(v,\varphi,\mu)\big)\Big| \;\le\; \|\widetilde{\mathbb{A}}_*\|\;\|(z,\phi,\xi)\|\;\|(v,\varphi,\mu)\| \qquad \forall\,(z,\phi,\xi),\,(v,\varphi,\mu) \in \widetilde{\mathbb{X}}\,. \tag{4.6}$$

Hereafter, the product spaces $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ are endowed with its Hilbertian norms

$$\|(v,\mu)\|^2 := \|v\|^2_{1,\Omega} + \|\mu\|^2_{-1/2,\Gamma} \qquad \forall\,(v,\mu) \in \mathbb{X}\,,$$

and

$$\|(v,\varphi,\mu)\|^2 := \|v\|^2_{1,\Omega} + \|\varphi\|^2_{1/2,\Gamma} + \|\mu\|^2_{-1/2,\Gamma} \qquad \forall\;(v,\varphi,\mu) \in \widetilde{\mathbb{X}}\,,$$

respectively. Then, by virtue of Lemma 3.2, there exist α_0, $\widetilde{\alpha}_0 > 0$ such that

$$\mathrm{Re}\Big(\mathbb{A}_0\big((v,\mu),(\bar{v},\bar{\mu})\big)\Big) \;\ge\; \alpha_0\;\|(v,\mu)\|^2 \qquad \forall (v,\mu) \in \mathbb{X}\,, \tag{4.7}$$

and

$$\mathrm{Re}\Big(\widetilde{\mathbb{A}}_0\big((v,\varphi,\mu),(\bar v,\bar\varphi,\bar\mu)\big)\Big) \,\geq\, \widetilde{\alpha}_0\,\|(v,\varphi,\mu)\|^2 \qquad \forall (v,\varphi,\mu)\in\widetilde{\mathbb{X}}\,. \tag{4.8}$$

Regarding the ellipticity of $\widetilde{\mathbb{A}}_0$ given by the foregoing equation, we remark here that, because of the inequalities (3.5) and (3.6), the expression $\big\langle \xi,1\big\rangle\big\langle \mu,1\big\rangle$ is needed in the definition of $\widetilde{\mathbb{A}}_0$ (cf. (4.2)) only for the 2D analysis, and hence it will omitted for the 3D one.

Next, we let $\mathbb{X}'$ and $\widetilde{\mathbb{X}}'$ be the duals of $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ pivotal to $\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Gamma)$ and $\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Gamma)\times\mathrm{L}^2(\Gamma)$, respectively, which yields $\mathbb{X}\subset\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Gamma)\subset\mathbb{X}'$ and $\widetilde{\mathbb{X}}\subset\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Gamma)\times\mathrm{L}^2(\Gamma)\subset\widetilde{\mathbb{X}}'$ with dense inclusions. Thus, we denote by $[\,\cdot,\cdot\,]$ the corresponding duality pairings, and let $\mathcal{A}_\kappa:\ \mathbb{X}\to\mathbb{X}'$, $\mathcal{A}_0:\ \mathbb{X}\to\mathbb{X}'$, $\widetilde{\mathcal{A}}_\kappa:\ \widetilde{\mathbb{X}}\to\widetilde{\mathbb{X}}'$, and $\widetilde{\mathcal{A}}_0:\ \widetilde{\mathbb{X}}\to\widetilde{\mathbb{X}}'$ be the linear operators induced by $\mathbb{A}_\kappa$, $\mathbb{A}_0$, $\widetilde{\mathbb{A}}_\kappa$, and $\widetilde{\mathbb{A}}_0$, respectively, that is, for each $*\in\{\kappa,0\}$

$$\big[\mathcal{A}_*(z,\xi),(v,\mu)\big] \,:=\, \mathbb{A}_*\big((z,\xi),(v,\mu)\big)$$

for all (z,ξ), $(v,\mu)\,\in\,\mathbb{X}$, and

$$\big[\widetilde{\mathcal{A}}_*(z,\phi,\xi),(v,\varphi,\mu)\big] \,:=\, \widetilde{\mathbb{A}}_*\big((z,\phi,\xi),(v,\varphi,\mu)\big)$$

for all (z,ϕ,ξ), $(v,\varphi,\mu)\,\in\,\widetilde{\mathbb{X}}$. It is clear from (4.5) and (4.6) that $\mathcal{A}_\kappa$, $\mathcal{A}_0$, $\widetilde{\mathcal{A}}_\kappa$, and $\widetilde{\mathcal{A}}_0$ are all bounded. In addition, (4.7) and (4.8) guarantee that $\mathcal{A}_0$ and $\widetilde{\mathcal{A}}_0$ are isomorphisms. Furthermore, we easily deduce from Lemma 3.3 and the compactness of the canonical injection from $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$, that $\mathcal{A}_\kappa-\mathcal{A}_0\,:\ \mathbb{X}\mapsto\mathbb{X}'$ and $\widetilde{\mathcal{A}}_\kappa-\widetilde{\mathcal{A}}_0\,:\ \widetilde{\mathbb{X}}\mapsto\widetilde{\mathbb{X}}'$ are compact, whence $\mathcal{A}_\kappa$ and $\widetilde{\mathcal{A}}_\kappa$ are Fredholm operators of index zero.

We are now in position to establish the conditions under which problems (3.13) and (3.25) are uniquely solvable.

Theorem 4.1. *Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ. Then, problems* (3.13) *and* (3.25) *are well posed.*

Proof. The proof is adapted from [13, Theorem 3.2]. According to our previous analysis, the Fredholm alternative is applicable and therefore the proof reduces to show uniqueness of solution for (3.13) and (3.25). In what follows we restrict ourselves to (3.13), the proof for (3.25) being analogous. To this end, given a solution $(u_0,\lambda_0)\in\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$ of (3.13) with $w=0$, we introduce the function

$$\widetilde{u}(\boldsymbol{x}) := \begin{cases} u_0(\boldsymbol{x}) & \forall\,\boldsymbol{x}\in\Omega,\\ q(\boldsymbol{x}) := \displaystyle\int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_y} u^s(\boldsymbol{y})\,\mathrm{d}\sigma_y - \int_\Gamma E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\lambda_0\,\mathrm{d}\sigma_y & \forall\,\boldsymbol{x}\in\Omega_e\,. \end{cases}$$

It is easy to verify that u_0 solves the equation

$$\Delta u_0 + \kappa^2\theta(\boldsymbol{x})\,u_0 = 0 \qquad \text{in }\Omega\,, \tag{4.9}$$

and that q is a radiating solution of the Helmholtz equation with wave number κ, that is

$$\Delta q + \kappa^2 q = 0 \qquad \text{in }\Omega_e, \tag{4.10}$$

$$\frac{\partial q}{\partial r} - \imath\kappa q = o(r^{\frac{1-d}{2}}) \qquad r := |\boldsymbol{x}|\to\infty. \tag{4.11}$$

Furthermore, using the jump relations of the acoustic potential layers we obtain the identities

$$\gamma q = (\frac{\mathrm{id}}{2} + K_\kappa)\gamma u_0 - V_\kappa\lambda_0 \quad \text{on} \quad \Gamma\,, \tag{4.12}$$

$$\lambda_0 = -W_\kappa\gamma u_0 + (\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}})\lambda_0 \quad \text{on} \quad \Gamma\,, \tag{4.13}$$

from which, comparing in particular (3.2) and (4.12), we deduce that

$$\gamma q = \gamma u_0. \tag{4.14}$$

In turn, subtracting equations (3.3) and (4.13) yields

$$(\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}}) \left(\frac{\partial u_0}{\partial \boldsymbol{n}} - \lambda_0 \right) = 0\,, \tag{4.15}$$

and using that, under our hypothesis on k, operator $\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}}$ is injective (cf. [4]), we deduce from (4.15) the identity

$$\frac{\partial q}{\partial \boldsymbol{n}} = \frac{\partial u_0}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma\,. \tag{4.16}$$

Finally, equations (4.9), (4.10), (4.14) and (4.16) show that $\widetilde{u} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a solution of (2.1)–(2.2) with $w = 0$, and therefore Theorem 2.1 ensures that such a function $\widetilde{u}$ should vanish identically in $\mathbb{R}^d$, which ends the proof. $\square$

As a consequence of Theorem 4.1, and certainly assuming its hypothesis, we conclude that the operators $\mathcal{A}_\kappa : \mathbb{X} \to \mathbb{X}'$ and $\widetilde{\mathcal{A}}_\kappa : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}'$ are bijective.

5 The discrete VEM/BEM schemes in 2D

In this section we introduce and analyze the two-dimensional discrete VEM - BEM schemes for each one of the coupling procedures described in Sections 3.2 and 3.3. Later on in Section 6, we provide the main distinctive aspects of the application of the second scheme to the 3D case.

5.1 Preliminaries

Given a polygonal domain $\Omega \subseteq \mathbb{R}^2$, we let $\{\mathcal{T}_h\}_h$ be a family of partitions of $\overline{\Omega}$ constituted of connected polygons $E \in \mathcal{T}_h$ of diameter $h_E \le h$, and assume that the meshes $\{\mathcal{T}_h\}_h$ are aligned with each Ω_i, $i = 1, \ldots, I$. The boundary ∂E of each $E \in \mathcal{T}_h$ is subdivided into straight segments e, which are called edges, and we denote by $\mathcal{E}_h$ the set of those contained in Γ, that is $\mathcal{E}_h := \Big\{ \text{edges } e \text{ of } \mathcal{T}_h : \quad e \subseteq \Gamma \Big\}$. In addition, we assume that there exists a constant $\rho \in (0,1)$ with which the family $\{\mathcal{T}_h\}_h$ satisfies the following conditions:

(A1) each E of $\{\mathcal{T}_h\}_h$ is star-shaped with respect to a disk D_E of radius ρh_E,

(A2) for each E of $\{\mathcal{T}_h\}_h$ and for all edges $e \subseteq \partial E$ it holds $|e| \ge \rho h_E$.

Then, given an integer $k \ge 1$, we introduce for each E of $\{\mathcal{T}_h\}_h$ the projection operator $\Pi_k^{\nabla,E}$: $\mathrm{H}^1(E) \to \mathcal{P}_k(T)$, which, given $v \in \mathrm{H}^1(E)$, is uniquely characterized by (see [2])

$$\int_E \nabla(\Pi_k^{\nabla,E} v) \cdot \nabla p + \left(\int_{\partial E} \Pi_k^{\nabla,E} v \right) \left(\int_{\partial E} p \right) = \int_E \nabla v \cdot \nabla p + \left(\int_{\partial E} v \right) \left(\int_{\partial E} p \right), \tag{5.1}$$

for all $p \in \mathcal{P}_k(T)$. Moreover, for each integer $k \ge 0$ we let Π_k^E be the $\mathrm{L}^2(E)$–orthogonal projection onto $\mathcal{P}_k(E)$, and following [1] (see also [8]) we introduce, for $k \ge 1$, the local virtual element space

$$X_h^k(E) := \Big\{ v \in \mathrm{H}^1(E) : \; v|_e \in \mathcal{P}_k(e), \forall e \subseteq \partial E, \, \Delta v \in \mathcal{P}_k(E), \quad \Pi_k^E v - \Pi_k^{\nabla,E} v \in \mathcal{P}_{k-2}(E) \Big\}. \tag{5.2}$$

It can be shown (see [1]) that the degrees of freedom of $X_h^k(E)$ consist of:

i) the values at the vertices of E, and additionally for $k \geq 2$,

ii) the moments of order $\leq k-2$ on the edges of E, and

iii) the moments of order $\leq k-2$ on E.

We are then allowed to construct the global virtual element space by

$$X_h^k := \Big\{ v \in \mathrm{H}^1(\Omega) : \quad v|_E \in X_h^k(E) \quad \forall\, E \in \mathcal{T}_h \Big\} .$$

On the other hand, for any integer $k \geq 0$, we denote by $\mathcal{P}_k(\mathcal{T}_h)$ the space of piecewise polynomials of degree $\leq k$ with respect to $\mathcal{T}_h$, and let $\Pi_k^{\mathcal{T}}$ be the global $\mathrm{L}^2(\Omega)$-orthogonal projection onto $\mathcal{P}_k(\mathcal{T}_h)$, which is assembled cellwise, i.e.

$$(\Pi_k^{\mathcal{T}} v)|_E := \Pi_k^E(v|_E) \qquad \forall\, E \in \mathcal{T}_h\,, \qquad \forall\, v \in \mathrm{L}^2(\Omega)\,. \tag{5.3}$$

It is important to notice that for $k \geq 1$ there holds $\mathcal{P}_k(E) \subseteq X_h^k(E)$, and that the projectors $\Pi_k^{\nabla,E} v$ and $\Pi_k^E v$ are computable for all $v \in X_h^k(E)$.

Hereafter, given any positive functions A_h and B_h of the mesh parameter h, the notation $A_h \lesssim B_h$ means that $A_h \leq C B_h$ with $C > 0$ independent of h, whereas $A_h \simeq B_h$ means that $A_h \lesssim B_h$ and $B_h \lesssim A_h$. Then, under the conditions on $\mathcal{T}_h$, and given in what follows an integer $k \geq 1$, the technique of averaged Taylor polynomials introduced in [6] permits to prove the following error estimates

$$\|v - \Pi_k^E v\|_{0,E} + h_E |v - \Pi_k^E v|_{1,E} \lesssim h_E^{\ell+1} |v|_{\ell+1,E} \quad \forall\, \ell \in \big\{0, 1, ..., k\big\}, \quad \forall\, v \in \mathrm{H}^{\ell+1}(E)\,, \tag{5.4}$$

$$\|v - \Pi_k^{\nabla,E} v\|_{0,E} + h_E \|v - \Pi_k^{\nabla,E} v\|_{1,E} \lesssim h_E^{\ell+1} |v|_{\ell+1,E} \quad \forall\, \ell \in \big\{1, 2, ..., k\big\}, \quad \forall\, v \in \mathrm{H}^{\ell+1}(E)\,. \tag{5.5}$$

In turn, the local interpolation operator $I_k^E : \mathrm{H}^2(E) \to X_h^k(E)$, which is uniquely defined for each $v \in \mathrm{H}^2(E)$ by imposing that $v - I_k^E v$ has vanishing degrees of freedom, satisfies (cf. [3, Lemma 2.23])

$$\|v - I_k^E v\|_{0,E} + h_E \left|v - I_k^E v\right|_{1,E} \lesssim h_E^{\ell+1} |v|_{\ell+1,E} \quad \forall \ell \in \big\{1, 2, ..., k\big\}, \quad \forall\, v \in \mathrm{H}^{\ell+1}(E)\,. \tag{5.6}$$

The corresponding global interpolation operator $I_k^{\mathcal{T}} : \mathcal{C}(\overline{\Omega}) \to X_h^k$ is defined locally as

$$(I_k^{\mathcal{T}} v)|_E \;=\; I_k^E(v|_E) \qquad \forall\, E \in \mathcal{T}_h\,, \qquad \forall\, v \in \mathcal{C}(\overline{\Omega})\,. \tag{5.7}$$

On the other hand, in order to approximate the unknowns $\lambda \in \mathrm{H}^{-1/2}(\Gamma)$ and $\psi \in \mathrm{H}_0^{1/2}(\Gamma)$, we introduce the non-virtual (but explicit) subspaces

$$\Lambda_h^{k-1} := \Big\{ \mu \in \mathrm{L}^2(\Gamma) : \quad \mu|_e \in P_{k-1}(e), \quad \forall e \in \mathcal{E}_h \Big\}\,, \tag{5.8}$$

and

$$\Psi_h^k \;:=\; \Big\{ \varphi_h \in \mathcal{C}^0(\Gamma) : \quad \varphi_h|_e \in \mathcal{P}_k(e) \quad \forall\, e \in \mathcal{E}_h \Big\} \cap \mathrm{H}_0^{1/2}(\Gamma)\,. \tag{5.9}$$

Then, we let $\Pi_{k-1}^{\mathcal{E}} : \mathrm{L}^2(\Gamma) \to \Lambda_h^{k-1}$ and $\mathcal{L}_k^{\mathcal{E}} : \mathcal{C}^0(\Gamma) \to \Psi_h^k$ be the $\mathrm{L}^2(\Gamma)$-orthogonal projection and the corresponding global Lagrange interpolation operator of order k, respectively. In addition, we let $\big\{\Gamma_1, ..., \Gamma_J\big\}$ be the set of segments constituting Γ, and for any $t \geq 0$ we consider the broken Sobolev space $\mathrm{H}_b^t(\Gamma) := \prod_{j=1}^J \mathrm{H}^t(\Gamma_j)$ endowed with the graph norm

$$\|\varphi\|_{t,b,\Gamma}^2 := \sum_{j=1}^{J} \|\varphi\|_{t,\Gamma_j}^2 \quad \forall\, \varphi \in \mathrm{H}_b^t(\Gamma)\,.$$

Next, we recall from [15] the approximation properties of the operators $\Pi_{k-1}^{\mathcal{E}}$ and $\mathcal{L}_k^{\mathcal{E}}$.

Lemma 5.1. *Assume that* $\mu \in \mathrm{H}^{-1/2}(\Gamma) \cap \mathrm{H}_b^r(\Gamma)$ *for some* $r \geq 0$*. Then,*

$$\|\mu - \Pi_{k-1}^{\mathcal{E}} \mu\|_{-t,\Gamma} \lesssim h^{\min\{r,k\}+t} \, \|\mu\|_{r,b,\Gamma} \qquad \forall\, t \in \{0, 1/2\} \,.$$

Proof. See [15, Theorem 4.3.20]. □

Lemma 5.2. *Assume that* $\varphi \in \mathrm{H}^{1/2}(\Gamma) \cap \mathrm{H}_b^{r+1/2}(\Gamma)$ *for some* $r > 1/2$*. Then*

$$\|\varphi - \mathcal{L}_k^{\mathcal{E}} \varphi\|_{t,\Gamma} \lesssim h^{\min\{r+1/2,k+1\}-t} \, \|\varphi\|_{r+1/2,b,\Gamma} \qquad \forall\, t \in \{0, 1/2\} \,.$$

Proof. See [15, Proposition 4.1.50]. □

5.2 The VEM/BEM schemes

For all $E \in \mathcal{T}_h$, we let S_h^E be the symmetric bilinear form defined on $\mathrm{H}^1(E) \times \mathrm{H}^1(E)$ by

$$S_h^E(z, v) := h_E^{-1} \sum_{e \subseteq \partial E} \int_e \pi_k^e z \, \pi_k^e v \qquad \forall\, z,\, v \in \mathrm{H}^1(E)\,, \tag{5.10}$$

where π_k^e is the $\mathrm{L}^2(e)$-projection onto $\mathcal{P}_k(e)$. It is shown in [3, Lemma 3.2] that

$$S_h^E(v, \bar{v}) \simeq a_0^E(v, \bar{v}) \quad \forall v \in X_h^k(E) \quad \text{such that } \Pi_k^{\nabla,E} v = 0, \tag{5.11}$$

where a_0^E is the local version of a_0, that is

$$a_0^E(z, v) := \int_E \nabla z \cdot \nabla v \qquad \forall\, z,\, v \in \mathrm{H}^1(E)\,. \tag{5.12}$$

It is important to notice that S_h^E is computable on $X_h^k(E) \times X_h^k(E)$, and that, by symmetry, there holds

$$S_h^E(z, v) \,\leq\, S_h^E(z, z)^{1/2} \, S_h^E(v, v)^{1/2} \,\lesssim\, a_0^E(z, z)^{1/2} \, a_0^E(v, v)^{1/2}, \tag{5.13}$$

for all $z,\, v \in X_h^k(E)$ satisfying $\Pi_k^{\nabla,E} z = \Pi_k^{\nabla,E} v = 0$. Alternative options for S_h^E restricted to $X_h^k(E) \times X_h^k(E)$ are available in the literature, the simplest one being the inner product of the vectors containing suitably scaled values of the degrees of freedom of the given discrete functions. For the theoretical purposes of the present paper there is no particular reason for using one or other as long as they satisfy the stability condition (5.11). In the forthcoming work [7] we plan to compare different choices of S_h^E from the point of view of their computational implementations and corresponding numerical results. Next, for each $E \in \mathcal{T}_h$ we introduce

$$a_{0,h}^E(z, v) \,:=\, a_0^E(\Pi_k^{\nabla,E} z, \Pi_k^{\nabla,E} v) + S_h^E(z - \Pi_k^{\nabla,E} z, v - \Pi_k^{\nabla,E} v) \quad \forall\, z,\, v \in X_h^k(E)\,, \tag{5.14}$$

and

$$a_{\kappa,h}^E(z, v) := a_{0,h}^E(z, v) - \kappa^2 \theta_E \int_E (\Pi_{k-1}^E z)(\Pi_{k-1}^E v) \quad \forall\, z,\, v \in X_h^k(E)\,, \tag{5.15}$$

where $\theta_E = \theta|_E \in \mathbb{C}$. We also let $a_{0,h}$ and $a_{\kappa,h}$ be the corresponding global extensions of $a_{0,h}^E$ and $a_{\kappa,h}^E$, respectively, that is

$$a_{0,h}(z, v) := \sum_{E \in \mathcal{T}_h} a_{0,h}^E(z, v) \tag{5.16}$$

and

$$a_{\kappa,h}(z, v) := \sum_{E \in \mathcal{T}_h} a_{\kappa,h}^E(z, v) \quad \forall\, z,\, v \in X_h^k. \tag{5.17}$$

Alternatively, we could have proceeded as in [8, eq. (3.9), Section 3.2] and define, instead of (5.14),

$$a_{0,h}^{E}(z,v) := \int_{E} \mathbf{\Pi}_{k-1}^{E}\nabla z \cdot \mathbf{\Pi}_{k-1}^{E}\nabla v + S_{h}^{E}(z - \Pi_{k}^{\nabla,E} z, v - \Pi_{k}^{\nabla,E} v) \quad \forall\, z,\, v \in X_{h}^{k}(E)\,, \tag{5.18}$$

where $\mathbf{\Pi}_{k-1}^{E} : [\mathrm{L}^2(E)]^2 \to [\mathcal{P}_{k-1}(E)]^2$ is the vectorial version of Π_{k-1}^{E}. In this regard, we stress in advance that the results to be provided throughout the rest of the paper, for which we use (5.14), would remain exactly the same if this discrete bilinear form is replaced by (5.18). In turn, since the degrees of freedom associated with the local space $X_h^k(E)$ (cf. (5.2)) also allow the explicit computation of its $\mathrm{L}^2(E)$-orthogonal projection onto the space of polynomials of degree $\leq k$, we stress here that $a_{\kappa,h}^{E}$ could be defined as well by using Π_k^E instead of Π_{k-1}^E in its second term. However, this modification would not improve neither affect the stability nor the rates of convergence of the resulting discrete scheme, as we explain later on after the derivation of the key inequality (5.50), in which the approximation property of Π_{k-1}^E (cf. (5.4)) is employed.

Then, denoting $\mathbb{X}_h^k := X_h^k \times \Lambda_h^{k-1}$, the discrete version of problem (3.13) reduces to: Find $(u_h, \lambda_h) \in \mathbb{X}_h^k$ such that

$$\mathbb{A}_{\kappa,h}\big((u_h,\lambda_h),(v_h,\mu_h)\big) = \mathbb{F}(v_h,\mu_h) \qquad \forall (v_h,\mu_h) \in \mathbb{X}_h^k\,, \tag{5.19}$$

where

$$\begin{aligned}\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),(v_h,\mu_h)\big) &:= a_{\kappa,h}(z_h,v_h) + \big\langle W_\kappa \gamma z_h, \gamma v_h\big\rangle + \big\langle \mu_h, V_\kappa \xi_h\big\rangle \\ &+ \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma z_h\big\rangle - \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma v_h\big\rangle\end{aligned} \tag{5.20}$$

for all (z_h,ξ_h), $(v_h,\mu_h) \in \mathbb{X}_h^k$. In turn, denoting $\widetilde{\mathbb{X}}_h^k := X_h^k \times \Psi_k^h \times \Lambda_h^{k-1}$, the discrete version of problem (3.25) reduces to: Find $(\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h) \in \widetilde{\mathbb{X}}_h^k$ such that

$$\widetilde{\mathbb{A}}_{\kappa,h}\big((\widetilde{u}_h,\widetilde{\psi}_h,\widetilde{\lambda}_h),(v_h,\varphi_h,\mu_h)\big) \;=\; \widetilde{\mathbb{F}}(v_h,\varphi_h,\mu_h) \qquad \forall\, (v_h,\varphi_h,\mu_h) \in \widetilde{\mathbb{X}}_h^k\,, \tag{5.21}$$

where

$$\begin{aligned}\widetilde{\mathbb{A}}_{\kappa,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \;&:=\; \mathbf{A}_{\kappa,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \;+\; \big\langle W_\kappa \phi_h, \varphi_h\big\rangle \\ &+\; \big\langle \mu_h, V_\kappa \xi_h\big\rangle \;+\; \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K_\kappa)\phi_h\big\rangle \;-\; \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K_\kappa)\varphi_h\big\rangle\end{aligned} \tag{5.22}$$

for all (z_h,ϕ_h,ξ_h), $(v_h,\varphi_h,\mu_h) \in \widetilde{\mathbb{X}}_h^k$, with

$$\mathbf{A}_{\kappa,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \;:=\; a_{\kappa,h}(z_h,v_h) \;-\; \big\langle \xi_h, \gamma v_h - \varphi_h\big\rangle \;+\; \big\langle \mu_h, \gamma z_h - \phi_h\big\rangle\,. \tag{5.23}$$

5.3 Solvability analysis

In order to analyze the solvability of (5.19) and (5.21), we now introduce the perturbations of the bilinear forms $\mathbb{A}_{\kappa,h}$ and $\widetilde{\mathbb{A}}_{\kappa,h}$ given, respectively, by

$$\begin{aligned}\mathbb{A}_{0,h}\big((z_h,\xi_h),(v_h,\mu_h)\big) &:= a_{0,h}(z_h,v_h) + \Big\{\int_\Gamma z_h\Big\}\Big\{\int_\Gamma v_h\Big\} + \big\langle W_0 \gamma z_h, \gamma v_h\big\rangle \\ &+ \big\langle \mu_h, V_0 \xi_h\big\rangle + \big\langle \xi_h, 1\big\rangle\big\langle \mu_h, 1\big\rangle + \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K_0)\gamma z_h\big\rangle - \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K_0)\gamma v_h\big\rangle\end{aligned} \tag{5.24}$$

for all $(z_h,\xi_h),(v_h,\mu_h) \in \mathbb{X}_h^k$, and

$$\begin{aligned}\widetilde{\mathbb{A}}_{0,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \;&:=\; \mathbf{A}_{0,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \;+\; \big\langle W_0 \phi_h, \varphi_h\big\rangle \\ &+ \big\langle \mu_h, V_0 \xi_h\big\rangle + \big\langle \xi_h, 1\big\rangle\big\langle \mu_h, 1\big\rangle + \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K_0)\phi_h\big\rangle - \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K_0)\varphi_h\big\rangle\end{aligned} \tag{5.25}$$

for all (z_h, ϕ_h, ξ_h), $(v_h, \varphi_h, \mu_h) \in \widetilde{\mathbb{X}}_h^k$, with

$$\begin{aligned} \mathbf{A}_{0,h}\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) \;:=\; & a_{0,h}(z_h, v_h) + \left\{\int_\Gamma z_h\right\}\left\{\int_\Gamma v_h\right\} \\ & -\big\langle \xi_h, \gamma v_h - \varphi_h\big\rangle \,+\, \big\langle \mu_h, \gamma z_h - \phi_h\big\rangle\,. \end{aligned} \tag{5.26}$$

The boundedness and ellipticity properties of the above bilinear forms are provided by the following two lemmas.

Lemma 5.3. *There exist positive constants M_κ, M_0, $\widetilde{M}_\kappa$ and $\widetilde{M}_0$, independent of h, such that for each $* \in \{\kappa, 0\}$ there hold*

$$|\mathbb{A}_{*,h}\big((z_h, \xi_h), (v_h, \mu_h)\big)| \;\le\; M_* \,\|(z_h, \xi_h)\|\, \|(v_h, \mu_h)\|$$

for all (z_h, ξ_h), $(v_h, \mu_h) \,\in\, \mathbb{X}_h^k$, and

$$|\widetilde{\mathbb{A}}_{*,h}\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big)| \;\le\; \widetilde{M}_* \,\|(z_h, \phi_h, \xi_h)\|\, \|(v_h, \varphi_h, \mu_h)\|$$

for all (z_h, ϕ_h, ξ_h), $(v_h, \varphi_h, \mu_h) \,\in\, \widetilde{\mathbb{X}}_h^k$,

Proof. Starting from the corresponding definitions (cf. (5.20), (5.24), (5.22) and (5.25)), it suffices to employ the mapping properties of the boundary integral operators (cf. Lemma 3.1), and then notice from (5.12), (5.13) and [2], that for each $E \in \mathcal{T}_h$ there holds

$$S_h^E(z_h - \Pi_k^{\nabla,E} z_h, v_h - \Pi_k^{\nabla,E} v_h) \;\lesssim\; |z_h - \Pi_k^{\nabla,E} z_h|_{1,E}\, |v_h - \Pi_k^{\nabla,E} v_h|_{1,E} \;\lesssim\; |z_h|_{1,E}\, |v_h|_{1,E} \tag{5.27}$$

for all z_h, $v_h \in X_h^k(E)$. We omit further details. $\square$

Lemma 5.4. *There exist positive constants β_0 and $\widetilde{\beta}_0$, independent of h, such that*

$$Re\Big\{\mathbb{A}_{0,h}\big((v_h, \mu_h), (\bar{v}_h, \bar{\mu}_h)\big)\Big\} \;\ge\; \beta_0 \,\|(v_h, \mu_h)\|^2 \qquad \forall\, (v_h, \mu_h) \,\in\, \mathbb{X}_h^k\,, \tag{5.28}$$

and

$$Re\Big\{\widetilde{\mathbb{A}}_{0,h}\big((v_h, \varphi_h, \mu_h), (\bar{v}_h, \bar{\varphi}_h, \bar{\mu}_h)\big)\Big\} \;\ge\; \widetilde{\beta}_0 \,\|(v_h, \varphi_h, \mu_h)\|^2 \qquad \forall\, (v_h, \varphi_h, \mu_h) \,\in\, \widetilde{\mathbb{X}}_h^k\,. \tag{5.29}$$

Proof. According now to the definitions (5.24) and (5.25), and proceeding as in the deduction of (4.7) and (4.8), we first apply the positivity properties of the boundary integral operators (cf. Lemma 3.2). Next, noticing from (5.14), (5.12) and (5.11) that for each $E \in \mathcal{T}_h$ there holds

$$\begin{aligned} a_{0,h}^E(v, \bar{v}) \;&=\; |\Pi_k^{\nabla,E} v|_{1,E}^2 + S_h^E(v - \Pi_k^{\nabla,E} v, \bar{v} - \Pi_k^{\nabla,E} \bar{v}) \\ &\gtrsim\; |\Pi_k^{\nabla,E} v|_{1,E}^2 \,+\, |v - \Pi_k^{\nabla,E} v|_{1,E}^2 \;\gtrsim\; |v|_{1,E}^2 \qquad \forall\, v \in X_h^k(E)\,, \end{aligned} \tag{5.30}$$

we arrive at the required inequalities and conclude the proof. $\square$

At this point we remark that, if (5.18) is considered instead of (5.14), then (5.30) would follow exactly as explained in [8, proof of Lemma 3.4]. Also, we stress that, thanks to (3.6), the term $\langle \xi_h, 1\rangle\, \langle \mu_h, 1\rangle$ will not required for the ellipticity (and hence not for the definition) of $\widetilde{\mathbb{A}}_{0,h}$ in the 3D case to be analyzed later on in Section 6.

Then, bearing in mind Lemmas 5.3 and 5.4, and the boundedness estimates (4.5) and (4.6), we can apply the Lax-Milgram lemma to introduce the Galerkin projection-type operators $\mathcal{R}_h : \mathbb{X} \to \mathbb{X}_h^k$ and $\widetilde{\mathcal{R}}_h : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}_h^k$, which, given $(z, \xi) \in \mathbb{X}$ and $(z, \phi, \xi) \in \widetilde{\mathbb{X}}$, are uniquely characterized by

$$\mathbb{A}_{0,h}\big(\mathcal{R}_h(z,\xi),(v_h,\mu_h)\big) \;=\; \mathbb{A}_0\big((z,\xi),(v_h,\mu_h)\big) \qquad \forall\,(v_h,\mu_h) \,\in\, \mathbb{X}_h^k\,, \tag{5.31}$$

and

$$\widetilde{\mathbb{A}}_{0,h}\big(\widetilde{\mathcal{R}}_h(z,\phi,\xi),(v_h,\varphi_h,\mu_h)\big) \;=\; \widetilde{\mathbb{A}}_0\big((z,\phi,\xi),(v_h,\varphi_h,\mu_h)\big) \qquad \forall\,(v_h,\varphi_h,\mu_h) \,\in\, \widetilde{\mathbb{X}}_h^k\,, \tag{5.32}$$

respectively. Moreover, it readily follows from the aforementioned classical lemma that $\mathcal{R}_h : \mathbb{X} \to \mathbb{X}_h^k$ and $\widetilde{\mathcal{R}}_h : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}_h^k$ are uniformly bounded in h with $\|\mathcal{R}_h\| \le \|\mathbb{A}_0\|/\beta_0$ and $\|\widetilde{\mathcal{R}}_h\| \le \|\widetilde{\mathbb{A}}_0\|/\widetilde{\beta}_0$.

The approximation properties of $\mathcal{R}_h$ and $\widetilde{\mathcal{R}}_h$ are established next. As usual, given a finite dimensional subspace X_h of a normed space X, we set for each $x \in X$, $\operatorname{dist}(x, X_h) := \inf\limits_{x_h \in X_h} \|x - x_h\|$.

Theorem 5.1. *There exist positive constants C and $\widetilde{C}$, independent of h, such that*

$$\|\mathcal{R}_h(z,\xi) - (z,\xi)\| \;\le\; C\left\{ \operatorname{dist}\big((z,\xi),\mathbb{X}_h^k\big) \;+\; \Big(\sum_{E\in\mathcal{T}_h} |z - \Pi_k^{\nabla,E} z|_{1,E}^2 \Big)^{1/2} \right\} \tag{5.33}$$

for all $(z, \xi) \in \mathbb{X}$, and

$$\|\widetilde{\mathcal{R}}_h(z,\phi,\xi) - (z,\phi,\xi)\| \;\le\; \widetilde{C}\left\{ \operatorname{dist}\big((z,\phi,\xi),\widetilde{\mathbb{X}}_h^k\big) \;+\; \Big(\sum_{E\in\mathcal{T}_h} |z - \Pi_k^{\nabla,E} z|_{1,E}^2 \Big)^{1/2} \right\} \tag{5.34}$$

for all $(z, \phi, \xi) \in \widetilde{\mathbb{X}}$.

Proof. Given $(z, \xi) \in \mathbb{X}$ and $(z_h, \xi_h) \in \mathbb{X}_h^k$, we first observe by triangle inequality that

$$\|\mathcal{R}_h(z,\xi) - (z,\xi)\| \;\le\; \|(v_h,\mu_h)\| \;+\; \|(z,\xi) - (z_h,\xi_h)\|\,, \tag{5.35}$$

with $(v_h, \mu_h) \;:=\; \mathcal{R}_h(z, \xi) - (z_h, \xi_h) \in \mathbb{X}_h^k$, so that in what follows we focus on estimating $\|(v_h, \mu_h)\|$. In fact, applying the ellipticity property (5.28), the identity (5.31), the boundedness of $\mathbb{A}_0$ (cf. (4.5)), and the fact that the difference between $\mathbb{A}_0$ and $\mathbb{A}_{0,h}$ (cf. (4.1), (5.24)) reduces to $a_0 - a_{0,h}$, we obtain

$$\begin{aligned}
\beta_0\,\|(v_h,\mu_h)\|^2 \;&\le\; \operatorname{Re}\Big\{ \mathbb{A}_{0,h}\big((v_h,\mu_h),(\bar v_h,\bar\mu_h)\big) \Big\} \\
&=\; \operatorname{Re}\Big\{ \mathbb{A}_0\big((z,\xi),(\bar v_h,\bar\mu_h)\big) - \mathbb{A}_{0,h}\big((z_h,\xi_h),(\bar v_h,\bar\mu_h)\big) \Big\} \\
&\le \big|\mathbb{A}_0\big((z,\xi)-(z_h,\xi_h),(\bar v_h,\bar\mu_h)\big)\big| + \big|\mathbb{A}_0\big((z_h,\xi_h),(\bar v_h,\bar\mu_h)\big) - \mathbb{A}_{0,h}\big((z_h,\xi_h),(\bar v_h,\bar\mu_h)\big)\big| \\
&\le\; \|\mathbb{A}_0\|\,\|(z,\xi)-(z_h,\xi_h)\|\,\|(v_h,\mu_h)\| \;+\; \sum_{E\in\mathcal{T}_h} \big|a_0^E(z_h,\bar v_h) - a_{0,h}^E(z_h,\bar v_h)\big|\,.
\end{aligned} \tag{5.36}$$

Then, subtracting and adding $\Pi_k^{\nabla,E} z$ in the first component of the expression $a_{0,h}^E(z_h, \bar v_h)$, using that $a_{0,h}^E(\Pi_k^{\nabla,E} z, v_h) \;=\; a_0^E(\Pi_k^{\nabla,E} z, \Pi_k^{\nabla,E} v_h) \;=\; a_0^E(\Pi_k^{\nabla,E} z, v_h)$ (which follows from (5.14) and after taking $(v, p) \;=\; (v_h, 1)$ and $(v, p) \;=\; (v_h, \Pi_k^{\nabla,E} z)$ in (5.1)), and employing the triangle inequality and the boundedness of a_0^E and $a_{0,h}^E$, the latter being consequence of (5.13), we find that

$$\begin{aligned}
\big|a_0^E(z_h,\bar v_h) - a_{0,h}^E(z_h,\bar v_h)\big| \;&\le\; \big|a_0^E(z_h - \Pi_k^{\nabla,E} z,\bar v_h)\big| + \big|a_{0,h}^E(z_h - \Pi_k^{\nabla,E} z,\bar v_h)\big| \\
&\lesssim\; |z_h - \Pi_k^{\nabla,E} z|_{1,E}\,|v_h|_{1,E} \lesssim \Big\{ |z_h - z|_{1,E} \;+\; |z - \Pi_k^{\nabla,E} z|_{1,E} \Big\}\,|v_h|_{1,E}\,.
\end{aligned}$$

In this way, summing up over $E \in \mathcal{T}_h$, it follows that

$$
\begin{aligned}
\sum_{E\in\mathcal{T}_h} \left| a_0^E(z_h, \bar{v}_h) - a_{0,h}^E(z_h, \bar{v}_h) \right| &\lesssim \left\{ |z_h - z|_{1,\Omega} + \Big(\sum_{E\in\mathcal{T}_h} |z - \Pi_k^{\nabla,E} z|_{1,E}^2 \Big)^{1/2} \right\} |v_h|_{1,\Omega} \\
&\lesssim \left\{ \|(z,\xi) - (z_h,\xi_h)\| + \Big(\sum_{E\in\mathcal{T}_h} |z - \Pi_k^{\nabla,E} z|_{1,E}^2 \Big)^{1/2} \right\} \|(v_h,\mu_h)\| \,,
\end{aligned}
\tag{5.37}
$$

which, combined with (5.36), yields

$$
\|(v_h,\mu_h)\| \lesssim \|(z,\xi) - (z_h,\xi_h)\| + \Big(\sum_{E\in\mathcal{T}_h} |z - \Pi_k^{\nabla,E} z|_{1,E}^2 \Big)^{1/2} .
$$

Hence, replacing the foregoing inequality back into (5.35) and taking infimum with respect to $(z_h, \xi_h) \in \mathbb{X}_h^k$, we arrive at (5.33). In turn, the derivation of (5.34) follows similarly to the previous analysis by noting now that, given $(z, \phi, \xi) \in \widetilde{\mathbb{X}}$ and $(z_h, \phi_h, \xi_h) \in \widetilde{\mathbb{X}}_h^k$, there holds

$$
\|\widetilde{\mathcal{R}}_h(z,\phi,\xi) - (z,\phi,\xi)\| \le \|(v_h,\varphi_h,\mu_h)\| + \|(z,\phi,\xi) - (z_h,\phi_h,\xi_h)\| \,,
$$

with $(v_h, \varphi_h, \mu_h) := \widetilde{\mathcal{R}}_h(z,\phi,\xi) - (z_h,\phi_h,\xi_h) \in \widetilde{\mathbb{X}}_h^k$. Hence, the rest of the proof reduces in this case to apply the ellipticity property (5.29), the identity (5.32), the boundedness of $\widetilde{\mathbb{A}}_0$ (cf. (4.6)), and the fact that, according to (4.2) and (5.25), we obtain

$$
\begin{aligned}
&\widetilde{\mathbb{A}}_0\big((z_h,\phi_h,\xi_h),(\bar{v}_h,\bar{\varphi}_h,\bar{\mu}_h)\big) - \widetilde{\mathbb{A}}_{0,h}\big((z_h,\phi_h,\xi_h),(\bar{v}_h,\bar{\varphi}_h,\bar{\mu}_h)\big) \\
&\qquad = \sum_{E\in\mathcal{T}_h} \left\{ a_0^E(z_h,\bar{v}_h) - a_{0,h}^E(z_h,\bar{v}_h) \right\},
\end{aligned}
\tag{5.38}
$$

which is again estimated by (5.37). We omit further details. □

As a consequence of Theorem 5.1, and employing classical density arguments together with the aproximation properties provided by (5.5), (5.6), and Lemmas 5.1 and 5.2, and using the uniform boundedness of $\mathcal{R}_h$ and $\widetilde{\mathcal{R}}_h$, we deduce that

$$
\lim_{h\to 0} \|\mathcal{R}_h(z,\xi) - (z,\xi)\| = 0 \qquad \forall\, (z,\xi) \in \mathbb{X} \tag{5.39}
$$

and

$$
\lim_{h\to 0} \|\widetilde{\mathcal{R}}_h(z,\phi,\xi) - (z,\phi,\xi)\| = 0 \qquad \forall\, (z,\phi,\xi) \in \widetilde{\mathbb{X}} . \tag{5.40}
$$

In other words, $\mathcal{R}_h$ and $\widetilde{\mathcal{R}}_h$ converge pointwise to the identity operators in $\mathbb{X}$ and $\widetilde{\mathbb{X}}$, respectively.

The well-posedness of the VEM/BEM schemes (5.19) and (5.21), that is their unique solvabilities and associated stability estimates, will follow from the discrete inf-sup conditions for $\mathbb{A}_{\kappa,h}$ and $\widetilde{\mathbb{A}}_{\kappa,h}$, respectively, which are established next. For later use, we now let $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ and $\langle \cdot, \cdot \rangle_{\widetilde{\mathbb{X}}}$ be the inner products of $\mathbb{X}$ and $\widetilde{\mathbb{X}}$, respectively.

Theorem 5.2. *Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ. Then, there exist positive constants h_0, α_κ, and $\widetilde{\alpha}_\kappa$, independent of h, such that for each $h \le h_0$ there hold*

$$
\sup_{\substack{(z_h,\xi_h)\in\mathbb{X}_h^k \\ (z_h,\xi_h)\neq\mathbf{0}}} \frac{\left|\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),(v_h,\mu_h)\big)\right|}{\|(z_h,\xi_h)\|} \ge \alpha_\kappa \|(v_h,\mu_h)\| \qquad \forall\, (v_h,\mu_h) \in \mathbb{X}_h^k \,, \tag{5.41}
$$

and

$$\sup_{\substack{(z_h,\phi_h,\xi_h)\in\widetilde{\mathbb{X}}_h^k\\(z_h,\phi_h,\xi_h)\neq\mathbf{0}}} \frac{\left|\widetilde{\mathbb{A}}_{\kappa,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big)\right|}{\|(z_h,\phi_h,\xi_h)\|} \,\geq\, \widetilde{\alpha}_\kappa\,\|(v_h,\varphi_h,\mu_h)\| \qquad \forall\,(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbb{X}}_h^k\,. \tag{5.42}$$

Proof. We begin with the introduction of some useful tools. In fact, thanks to the bijectivity of $\mathcal{A}_\kappa : \mathbb{X}\to\mathbb{X}'$, we first deduce the existence of a bounded operator $\Theta:\mathbb{X}\to\mathbb{X}$ such that, given $(z,\xi)\in\mathbb{X}$, $\Theta(z,\xi)\in\mathbb{X}$ is uniquely characterized by the identity

$$\mathbb{A}_\kappa\big(\Theta(z,\xi),(v,\mu)\big) \,=\, \langle(z,\xi),(v,\mu)\rangle_{\mathbb{X}} \qquad \forall\,(v,\mu)\in\mathbb{X}\,.$$

It follows, in particular, that

$$\mathbb{A}_\kappa\big(\Theta(z,\xi),(z,\xi)\big) \,=\, \|(z,\xi)\|^2 \qquad \forall\,(z,\xi)\in\mathbb{X}\,. \tag{5.43}$$

Analogously, the bijectivity of $\widetilde{\mathcal{A}}_\kappa : \widetilde{\mathbb{X}}\to\widetilde{\mathbb{X}}'$ implies the existence of a bounded operator $\widetilde{\Theta}:\widetilde{\mathbb{X}}\to\widetilde{\mathbb{X}}$ for which there holds

$$\widetilde{\mathbb{A}}_\kappa\big(\widetilde{\Theta}(z,\phi,\xi),(z,\phi,\xi)\big) \,=\, \|(z,\phi,\xi)\|^2 \qquad \forall\,(z,\phi,\xi)\in\widetilde{\mathbb{X}}\,. \tag{5.44}$$

In addition, we define the compact operators $\mathcal{C}:=\mathcal{A}_\kappa-\mathcal{A}_0:\mathbb{X}\to\mathbb{X}'$ and $\widetilde{\mathcal{C}}:=\widetilde{\mathcal{A}}_\kappa-\widetilde{\mathcal{A}}_0:\widetilde{\mathbb{X}}\to\widetilde{\mathbb{X}}'$. Then, in order to proceed with the proof of (5.41), we consider $(v_h,\mu_h)\in\mathbb{X}_h^k$, set $(z_h^+,\xi_h^+):=\mathcal{R}_h\Theta(v_h,\mu_h)\in\mathbb{X}_h^k$, and observe that certainly

$$\sup_{\substack{(z_h,\xi_h)\in\mathbb{X}_h^k\\(z_h,\xi_h)\neq\mathbf{0}}} \frac{\left|\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),(v_h,\mu_h)\big)\right|}{\|(z_h,\xi_h)\|} \,\geq\, \frac{\left|\mathbb{A}_{\kappa,h}\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big)\right|}{\|(z_h^+,\xi_h^+)\|}\,. \tag{5.45}$$

In turn, adding and subtracting the bilinear forms $\mathbb{A}_\kappa$, $\mathbb{A}_0$, and $\mathbb{A}_{0,h}$, so that

$$\mathbb{A}_{\kappa,h} \,=\, \mathbb{A}_{0,h} + (\mathbb{A}_\kappa-\mathbb{A}_0) + (\mathbb{A}_0-\mathbb{A}_{0,h}) + (\mathbb{A}_{\kappa,h}-\mathbb{A}_\kappa)\,, \tag{5.46}$$

and noticing from the definitions of $\mathbb{A}_\kappa$, $\mathbb{A}_0$, $\mathbb{A}_{\kappa,h}$, and $\mathbb{A}_{0,h}$ (cf. (3.14), (4.1), (5.20), and (5.24)), that

$$\big(\mathbb{A}_0-\mathbb{A}_{0,h}\big)\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \,=\, \int_\Omega \nabla z_h^+\cdot\nabla v_h \,-\, a_{0,h}(z_h^+,v_h) \tag{5.47}$$

and

$$\begin{aligned}\big(\mathbb{A}_{\kappa,h}-\mathbb{A}_\kappa\big)\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \,&=\, a_{\kappa,h}(z_h^+,v_h) \,-\, a_\kappa(z_h^+,v_h)\\ &=\, a_{0,h}(z_h^+,v_h) - \int_\Omega \nabla z_h^+\cdot\nabla v_h \,+\, \kappa^2\sum_{E\in\mathcal{T}_h}\theta_E\int_E\Big\{z_h^+v_h - \big(\Pi_{k-1}^E z_h^+\big)\big(\Pi_{k-1}^E v_h\big)\Big\}\,,\end{aligned} \tag{5.48}$$

we readily arrive at

$$\begin{aligned}\mathbb{A}_{\kappa,h}\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \,&=\, \mathbb{A}_{0,h}\big(\mathcal{R}_h\Theta(v_h,\mu_h),(v_h,\mu_h)\big) \,+\, \big[\mathcal{C}\mathcal{R}_h\Theta(v_h,\mu_h),(v_h,\mu_h)\big]\\ &\quad + \kappa^2\sum_{E\in\mathcal{T}_h}\theta_E\int_E\Big\{z_h^+v_h - \big(\Pi_{k-1}^E z_h^+\big)\big(\Pi_{k-1}^E v_h\big)\Big\}\,.\end{aligned} \tag{5.49}$$

In what follows, I stands for a generic identity operator and θ_M denotes the maximum value of $|\theta_E|$, $E\in\mathcal{T}_h$. Hence, starting from (5.49), and employing the characterization of $\mathcal{R}_h$ (cf. (5.31)), the orthogonality condition satisfied by Π^E_{k-1}, the identity (5.43), the approximation properties of Π^E_{k-1} (cf. (5.4)), and the fact that $\mathcal{R}_h$ is uniformly bounded, we find that

$$
\begin{aligned}
\mathbb{A}_{\kappa,h}\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \;&=\; \mathbb{A}_0\big(\Theta(v_h,\mu_h),(v_h,\mu_h)\big) \;+\; \big[\mathcal{C}\mathcal{R}_h\Theta(v_h,\mu_h),(v_h,\mu_h)\big] \\
&+\, \kappa^2 \sum_{E\in\mathcal{T}_h} \theta_E \int_E \Big\{ z_h^+ v_h - \big(\Pi^E_{k-1} z_h^+\big)\,\big(\Pi^E_{k-1} v_h\big)\Big\} \\
&=\; \mathbb{A}_\kappa\big(\Theta(v_h,\mu_h),(v_h,\mu_h)\big) \;+\; \big[\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\Theta(v_h,\mu_h),(v_h,\mu_h)\big] \\
&+\, \kappa^2 \sum_{E\in\mathcal{T}_h} \theta_E \int_E \Big\{ z_h^+ - \Pi^E_{k-1} z_h^+\Big\}\Big\{ v_h - \Pi^E_{k-1} v_h\Big\} \\
&\ge\; \Big\{1 - \|\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\|\|\Theta\|\Big\}\,\|(v_h,\mu_h)\|^2 \\
&-\, \kappa^2\theta_M \sum_{E\in\mathcal{T}_h} \big\|z_h^+ - \Pi^E_{k-1} z_h^+\big\|_{0,E}\,\big\|v_h - \Pi^E_{k-1} v_h\big\|_{0,E} \\
&\ge\; \Big\{1 - \|\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\|\|\Theta\|\Big\}\,\|(v_h,\mu_h)\|^2 - C h^2 \|(z_h^+,\xi_h^+)\|\,\|(v_h,\mu_h)\| \\
&\ge\; \Big\{1 - \|\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\|\|\Theta\| - C h^2\Big\}\,\|(z_h^+,\xi_h^+)\|\,\|(v_h,\mu_h)\|\,,
\end{aligned}
\tag{5.50}
$$

where C is a positive constant depending on κ and θ_M, but independent of h, and the last inequality uses that $\|(v_h,\mu_h)\| \,\gtrsim\, \|(z_h^+,\xi_h^+)\|$. Finally, the compactness of $\mathcal{C}$ and the pointwise convergence of $\mathcal{R}_h - \mathrm{I}$ to zero (cf. (5.39)) guarantee that $\lim\limits_{h\to 0} \|\mathcal{C}(\mathcal{R}_h - \mathrm{I})\| \,=\, 0$, which, together with the foregoing estimate and (5.45), yield (5.41) for a sufficiently small h_0. We remark here that using Π^E_k instead of Π^E_{k-1} in the original definition of $a^E_{\kappa,h}$ (cf. (5.15)), would not yield any change in the inequality (5.50), and hence neither in the resulting discrete inf-sup condition (5.41). In fact, the local $\mathrm{H}^1(E)$-regularity of z_h^+ and v_h only allows to apply the approximation property (5.4) for $\ell = 0$, so that, irrespective of using Π^E_k or Π^E_{k-1}, the power of h in the last part of (5.50) remains as 2. On the other hand, the discrete inf-sup condition (5.42) is proved similarly to the previous analysis. In fact, given $(v_h,\varphi_h,\mu_h) \in \widetilde{\mathbb{X}}^k_h$, we now set $(z_h^+,\phi_h^+,\xi_h^+) \,:=\, \widetilde{\mathcal{R}}_h\widetilde{\Theta}(v_h,\varphi_h,\mu_h) \in \widetilde{\mathbb{X}}^k_h$, and observe first that $\big(\widetilde{\mathbb{A}}_0 - \widetilde{\mathbb{A}}_{0,h}\big)\big((z_h^+,\phi_h^+,\xi_h^+),(v_h,\varphi_h,\mu_h)\big)$ and $\big(\widetilde{\mathbb{A}}_{\kappa,h} - \widetilde{\mathbb{A}}_\kappa\big)\big((z_h^+,\phi_h^+,\xi_h^+),(v_h,\varphi_h,\mu_h)\big)$ are given exactly by the right hand sides of (5.47) and (5.48), respectively. In this way, the analogue of (5.49) keeps the same term at the end, and the reasoning follows almost verbatim to the steps in (5.50), but now using the characterization of $\widetilde{\mathcal{R}}_h$ (cf. (5.32)), the orthogonality condition satisfied by Π^E_{k-1}, the identity (5.44), the approximation properties of Π^E_{k-1} (cf. (5.4)), the uniform boundedness of $\widetilde{\mathcal{R}}_h$, and the fact that $\|(v_h,\varphi_h,\mu_h)\| \,\gtrsim\, \|(z_h^+,\phi_h^+,\xi_h^+)\|$. The compactness of $\widetilde{\mathcal{C}}$ and the pointwise convergence of $\widetilde{\mathcal{R}}_h - \mathrm{I}$ to zero (cf. (5.40)) complete the proof of (5.42). □

Under the same assumptions of Theorem 5.2, and as a straightforward consequence of (5.41) and (5.42) we deduce that, given $\mathbb{F} \in \mathbb{X}'$, $\widetilde{\mathbb{F}} \in \widetilde{\mathbb{X}}'$, and $h \le h_0$, the VEM/BEM schemes (5.19) and (5.21) have unique solutions $(u_h,\lambda_h) \in \mathbb{X}^k_h$ and $(\widetilde{u}_h,\widetilde{\psi}_h,\widetilde{\lambda}_h) \in \widetilde{\mathbb{X}}^k_h$, respectively. Thus, we can also define the discrete analogues of Θ and $\widetilde{\Theta}$ (though with respect to the second component of the bilinear forms involved), namely the operators $\Theta_h : \mathbb{X}^k_h \to \mathbb{X}^k_h$ and $\widetilde{\Theta}_h : \widetilde{\mathbb{X}}^k_h \to \widetilde{\mathbb{X}}^k_h$, which, given $(v_h,\mu_h) \in \mathbb{X}^k_h$ and $(v_h,\varphi_h,\mu_h) \in \widetilde{\mathbb{X}}^k_h$, are uniquely characterized by the equations

$$
\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),\Theta_h(v_h,\mu_h)\big) \;=\; \langle (z_h,\xi_h),(v_h,\mu_h)\rangle_{\mathbb{X}} \qquad \forall\,(z_h,\xi_h) \in \mathbb{X}^k_h
$$

and

$$
\widetilde{\mathbb{A}}_{\kappa,h}\big((z_h,\phi_h,\xi_h),\widetilde{\Theta}_h(v_h,\varphi_h,\mu_h)\big) \;=\; \langle (z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\rangle_{\widetilde{\mathbb{X}}} \qquad \forall\,(z_h,\phi_h,\xi_h) \in \widetilde{\mathbb{X}}^k_h\,,
$$

which yield, in particular,

$$\mathbb{A}_{\kappa,h}\big((v_h,\mu_h),\Theta_h(v_h,\mu_h)\big) \,=\, \|(v_h,\mu_h)\|^2 \qquad \forall\,(v_h,\mu_h)\in \mathbb{X}_h^k \tag{5.51}$$

and

$$\widetilde{\mathbb{A}}_{\kappa,h}\big((v_h,\varphi_h,\mu_h),\widetilde{\Theta}_h(v_h,\varphi_h,\mu_h)\big) \,=\, \|(v_h,\varphi_h,\mu_h)\|^2 \qquad \forall\,(v_h,\varphi_h,\mu_h)\in \widetilde{\mathbb{X}}_h^k\,. \tag{5.52}$$

In addition, it follows from the above characterizations of Θ_h and $\widetilde{\Theta}_h$ and the discrete inf-sup conditions (5.41) and (5.42), that

$$\|\Theta_h\| \,\le\, \frac{1}{\alpha_\kappa} \qquad \text{and} \qquad \|\widetilde{\Theta}_h\| \,\le\, \frac{1}{\widetilde{\alpha}_\kappa}\,. \tag{5.53}$$

5.4 Error analysis

We now aim to provide a priori error bounds and associated rates of convergence for the solutions of the VEM/BEM schemes (5.19) and (5.21). We begin with the following Cea-type estimates, which make use of $\Pi_{k-1}^{\mathcal{T}}$ (cf. (5.3)), the global $\mathrm{L}^2(\Omega)$-orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{T}_h)$.

Theorem 5.3. *Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ, and let $h_0>0$ be the constant whose existence is guaranteed by Theorem 5.2. Then, there exist constants C, $\widetilde{C}>0$, independent of h, such that for each $h\le h_0$ there hold*

$$\begin{aligned}&\|(u,\lambda)-(u_h,\lambda_h)\|\\ &\quad\le\, C\left\{\mathrm{dist}\big((u,\lambda),\mathbb{X}_h^k\big)+\Big(\sum_{E\in\mathcal{T}_h}\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E}^2\Big)^{1/2}+\big\|u-\Pi_{k-1}^{\mathcal{T}}u\big\|_{0,\Omega}\right\},\end{aligned} \tag{5.54}$$

and

$$\begin{aligned}&\|(u,\psi,\lambda)-(\widetilde{u}_h,\widetilde{\psi}_h,\widetilde{\lambda}_h)\|\\ &\quad\le\, \widetilde{C}\left\{\mathrm{dist}\big((u,\psi,\lambda),\widetilde{\mathbb{X}}_h^k\big)+\Big(\sum_{E\in\mathcal{T}_h}\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E}^2\Big)^{1/2}+\big\|u-\Pi_{k-1}^{\mathcal{T}}u\big\|_{0,\Omega}\right\}.\end{aligned} \tag{5.55}$$

Proof. We begin by observing, thanks to the triangle inequality, that

$$\|(u,\lambda)-(u_h,\lambda_h)\| \,\le\, \|(u,\lambda)-(v_h,\mu_h)\| \,+\, \|(z_h,\xi_h)\| \qquad \forall\,(v_h,\mu_h)\in\mathbb{X}_h^k\,, \tag{5.56}$$

where $(z_h,\xi_h) \,:=\, (u_h,\lambda_h)-(v_h,\mu_h)$. Then, setting $(z_h^+,\xi_h^+) \,:=\, \Theta_h(z_h,\xi_h)\in\mathbb{X}_h^k$, employing the identity (5.51) and the fact that $\mathbb{A}_\kappa\big((u,\lambda),\cdot\big)$ and $\mathbb{A}_{\kappa,h}\big((u_h,\lambda_h),\cdot\big)$ coincide on $\mathbb{X}_h^k$ (which follows from (3.13) and (5.19)), adding and subtracting (v_h,μ_h) in the first component of $\mathbb{A}_\kappa$, using the uniform boundedness of Θ_h (cf. (5.53)) and the identity provided by the first row of (5.48), and then adding and subtracting u in the first component of a_κ, we obtain

$$\begin{aligned}\|(z_h,\xi_h)\|^2 \,&=\, \mathbb{A}_{\kappa,h}\big((u_h,\lambda_h),\Theta_h(z_h,\xi_h)\big) \,-\, \mathbb{A}_{\kappa,h}\big((v_h,\mu_h),\Theta_h(z_h,\xi_h)\big)\\ &=\, \mathbb{A}_\kappa\big((u,\lambda)-(v_h,\mu_h),\Theta_h(z_h,\xi_h)\big)+\big(\mathbb{A}_\kappa-\mathbb{A}_{\kappa,h}\big)\big((v_h,\mu_h),(z_h^+,\xi_h^+)\big)\\ &\le\, \|\mathbb{A}_\kappa\|\,\alpha_\kappa^{-1}\,\|(u,\lambda)-(v_h,\mu_h)\|\,\|(z_h,\xi_h)\| \,+\, \big|a_\kappa(v_h,z_h^+)-a_{\kappa,h}(v_h,z_h^+)\big|\\ &\le\, \Big(\|\mathbb{A}_\kappa\|+\|a_\kappa\|\Big)\,\alpha_\kappa^{-1}\,\|(u,\lambda)-(v_h,\mu_h)\|\,\|(z_h,\xi_h)\| \,+\, \big|a_\kappa(u,z_h^+)-a_{\kappa,h}(v_h,z_h^+)\big|\,.\end{aligned} \tag{5.57}$$

In this way, we now focus on estimating the last term on the right hand side of the foregoing equation. Indeed, according to the definitions of a_κ and $a_{\kappa,h}$ (cf. (3.15) and (5.15) - (5.17)), we first obtain

$$\begin{aligned}\big|a_\kappa(u,z_h^+) - a_{\kappa,h}(v_h,z_h^+)\big| \,&\le\, \sum_{E\in\mathcal{T}_h}\big|a_\kappa^E(u,z_h^+) - a_{\kappa,h}^E(v_h,z_h^+)\big| \\ &\le \sum_{E\in\mathcal{T}_h}\big|a_0^E(u,z_h^+) - a_{0,h}^E(v_h,z_h^+)\big| + \kappa^2\sum_{E\in\mathcal{T}_h}|\theta_E|\,\bigg|\int_E\Big\{u z_h^+ - \big(\Pi_{k-1}^E v_h\big)\big(\Pi_{k-1}^E z_h^+\big)\Big\}\bigg|\,.\end{aligned}\tag{5.58}$$

Next, adding and subtracting $\Pi_k^{\nabla,E}u$ in the first component of $a_0^E(u,z_h^+)$, recalling that there holds $a_0^E(\Pi_k^{\nabla,E}u,z_h^+) = a_{0,h}^E(\Pi_k^{\nabla,E}u,z_h^+)$ (cf. proof of Theorem 5.1), and thanks to the uniform boundedness of $a_{0,h}^E$, we find that

$$\begin{aligned}\big|a_0^E(u,z_h^+) - a_{0,h}^E(v_h,z_h^+)\big| \,&=\, \big|a_0^E(u-\Pi_k^{\nabla,E}u,z_h^+) \,+\, a_0^E(\Pi_k^{\nabla,E}u,z_h^+) - a_{0,h}^E(v_h,z_h^+)\big| \\ &=\, \big|a_0^E(u-\Pi_k^{\nabla,E}u,z_h^+) \,+\, a_{0,h}^E(\Pi_k^{\nabla,E}u - v_h,z_h^+)\big| \\ &\lesssim\, \Big\{\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E} \,+\, \big\|\Pi_k^{\nabla,E}u - v_h\big\|_{1,E}\Big\}\,\|z_h^+\|_{1,E} \\ &\lesssim\, \Big\{\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E} \,+\, \|u - v_h\|_{1,E}\Big\}\,\|z_h^+\|_{1,E}\,.\end{aligned}\tag{5.59}$$

In turn, the orthogonality condition satisfied by Π_{k-1}^E and the triangle inequality yield

$$\begin{aligned}\bigg|\int_E\Big\{u z_h^+ - \big(\Pi_{k-1}^E v_h\big)\big(\Pi_{k-1}^E z_h^+\big)\Big\}\bigg| \,&=\, \bigg|\int_E\Big\{u - \big(\Pi_{k-1}^E v_h\big)\Big\}\,z_h^+\bigg| \\ &\le\, \Big\{\big\|u - v_h\big\|_{0,E} \,+\, \big\|u - \Pi_{k-1}^E u\big\|_{0,E}\Big\}\,\|z_h^+\|_{0,E}\,.\end{aligned}\tag{5.60}$$

Hence, plugging (5.59) and (5.60) in (5.58), and applying the Cauchy-Schwarz inequality, we deduce the existence of a positive constant C_1, depending on κ and θ_M, but independent of h, such that

$$\begin{aligned}&\big|a_\kappa(u,z_h^+) - a_{\kappa,h}(v_h,z_h^+)\big| \\ &\quad\le\, C_1\left\{\Big(\sum_{E\in\mathcal{T}_h}\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E}^2\Big)^{1/2} \,+\, \big\|u-v_h\big\|_{1,\Omega} \,+\, \big\|u-\Pi_{k-1}^{\mathcal{T}}u\big\|_{0,\Omega}\right\}\,\|z_h^+\|_{1,\Omega}\,.\end{aligned}\tag{5.61}$$

Thus, replacing (5.61) back into (5.57), and bounding $\|z_h^+\|_{1,\Omega}$ by $\alpha_\kappa^{-1}\,\|(z_h,\xi_h)\|$, we conclude that

$$\|(z_h,\xi_h)\| \,\le\, C_2\left\{\big\|(u,\lambda)-(v_h,\mu_h)\big\| + \Big(\sum_{E\in\mathcal{T}_h}\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E}^2\Big)^{1/2} + \big\|u-\Pi_{k-1}^{\mathcal{T}}u\big\|_{0,\Omega}\right\}\,,\tag{5.62}$$

where C_2 is a positive constant depending on $\|\mathbb{A}_\kappa\|$, $\|a_\kappa\|$, α_κ, and C_1, but independent of h. Finally, combining (5.56) and (5.62), and then taking infimum over $(v_h,\mu_h)\in\mathbb{X}_h^k$, we arrive at (5.54). On the other hand, the proof of (5.55) follows almost verbatim. In fact, once stated the analogue of (5.56) with an arbitrary $(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbb{X}}_h^k$ and $(z_h,\phi_h,\xi_h) := (\widetilde{u}_h,\widetilde{\psi}_h,\widetilde{\lambda}_h) - (v_h,\varphi_h,\mu_h)$, we set $(z_h^+,\phi_h^+,\xi_h^+) := \widetilde{\Theta}_h(z_h,\phi_h,v_h)\in\widetilde{\mathbb{X}}_h^k$ and proceed analogously to the derivation of (5.57). In this way, observing now from (3.25) and (5.21) that $\widetilde{\mathbb{A}}_\kappa\big((u,\psi,\lambda),\cdot\big)$ and $\widetilde{\mathbb{A}}_{\kappa,h}\big((\widetilde{u}_h,\widetilde{\psi}_h,\widetilde{\lambda}_h),\cdot\big)$ coincide on $\widetilde{\mathbb{X}}_h^k$, and noting from the definitions of $\widetilde{\mathbb{A}}_\kappa$ and $\widetilde{\mathbb{A}}_{\kappa,h}$ (cf. (3.26) - (5.22)) that $\big(\widetilde{\mathbb{A}}_\kappa - \widetilde{\mathbb{A}}_{\kappa,h}\big)\big((v_h,\varphi_h,\mu_h),(z_h^+,\phi_h^+,\xi_h^+)\big)$ also reduces to $a_\kappa(v_h,z_h^+) - a_{\kappa,h}(v_h,z_h^+)$, we realize that it suffices to employ again the upper bound provided by (5.61) to get

$$\begin{aligned}&\|(z_h,\phi_h,\xi_h)\| \\ &\quad\le\, C_3\left\{\big\|(u,\psi,\lambda)-(v_h,\varphi_h,\mu_h)\big\| + \Big(\sum_{E\in\mathcal{T}_h}\big\|u-\Pi_k^{\nabla,E}u\big\|_{1,E}^2\Big)^{1/2} + \big\|u-\Pi_{k-1}^{\mathcal{T}}u\big\|_{0,\Omega}\right\}\,,\end{aligned}$$

where C_3 is a positive constant, independent of h, having basically the same dependences of C_2. The foregoing inequality and the aforementioned analogue of (5.56) imply (5.55) and end the proof. $\square$

We are now in position to establish the announced rates of convergence. To this end, we recall from (5.7) that $I_k^{\mathcal{T}}$ denotes the global virtual element interpolation operator. Then, we have the following result.

Theorem 5.4. *Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ, and that both u and the datum w belong to $\mathrm{H}^1(\Omega) \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\Omega_i)$. In addition, let $h_0 > 0$ be the constant whose existence is guaranteed by Theorem 5.2. Then, there exist constants C_0, $\widetilde{C}_0 > 0$, independent of h, such that for each $h \le h_0$ there hold*

$$\|(u, \lambda) - (u_h, \lambda_h)\| \;\le\; C_0\, h^k \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,, \tag{5.63}$$

and

$$\|(u, \psi, \lambda) - (\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h)\| \;\le\; \widetilde{C}_0\, h^k \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,. \tag{5.64}$$

Proof. We first notice that $\mathrm{H}^1(\Omega) \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\Omega_i) \subseteq \mathcal{C}^0(\overline{\Omega})$, which implies that $I_k^{\mathcal{T}} u$ is meaningful. In addition, we have that $\psi = \gamma(u - w) \in \mathrm{H}^{1/2}(\Gamma) \cap \mathrm{H}_b^{k+1/2}(\Gamma) \subseteq \mathcal{C}^0(\Gamma)$ and $\lambda = \gamma_{\boldsymbol{n}}\big(\nabla(u-w)\big) \in \mathrm{H}^{-1/2}(\Gamma) \cap \mathrm{H}_b^{k-1/2}(\Gamma) \subseteq \mathrm{L}^2(\Gamma)$, whence $\mathcal{L}_k^{\mathcal{E}}\psi$ and $\Pi_{k-1}^{\mathcal{E}}\lambda$ are meaningful as well. It follows that the distances on the right han sides of (5.54) and (5.55) can be bounded as

$$\operatorname{dist}\big((u,\lambda), \mathbb{X}_h^k\big) \;\le\; \|u - I_k^{\mathcal{T}} u\|_{1,\Omega} \,+\, \|\lambda - \Pi_{k-1}^{\mathcal{E}}\lambda\|_{-1/2,\Gamma}$$

and

$$\operatorname{dist}\big((u,\psi,\lambda), \widetilde{\mathbb{X}}_h^k\big) \;\le\; \|u - I_k^{\mathcal{T}} u\|_{1,\Omega} \,+\, \|\psi - \mathcal{L}_k^{\mathcal{E}}\psi\|_{1/2,\Gamma} \,+\, \|\lambda - \Pi_{k-1}^{\mathcal{E}}\lambda\|_{-1/2,\Gamma}\,.$$

In this way, replacing the foregoing estimates back into (5.54) and (5.55), applying the approximation properties of $I_k^{\mathcal{T}}$ (cf. (5.6)), $\Pi_{k-1}^{\mathcal{E}}$ (cf. Lemma 5.1), $\mathcal{L}_k^{\mathcal{E}}$ (cf. Lemma 5.2), $\Pi_k^{\nabla,E}$ (cf. (5.5)), and $\Pi_{k-1}^{\mathcal{T}}$ (cf. (5.4)), and employing the trace inequalities given by

$$\|\psi\|_{k+1/2,b,\Gamma} \;\lesssim\; \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i} \quad \text{and} \quad \|\lambda\|_{k-1/2,b,\Gamma} \;\lesssim\; \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,, \tag{5.65}$$

we are led to (5.63) and (5.64), thus finishing the proof. $\square$

6 The discrete VEM/BEM scheme in 3D

In this section we follow the approach from [8, Section 4] to introduce and analyze a three-dimensional VEM/BEM scheme for the modified Costabel & Han coupling procedure (cf. Section 3.3) as applied to the present Helmholtz equation (cf. (3.25)). As explained in [8, Section 4.2], the original Costabel & Han coupling (cf. Section 3.2) is not suitable for a VEM/BEM scheme in 3D since the trace of a VEM function on the boundary of a given element is not a polynomial but a virtual function as well (see (6.1) below).

6.1 Preliminaries

We now let $\{\mathcal{T}_h\}_h$ be a family of partitions of $\overline{\Omega}$ into polyhedral elements E of diameter $h_E \leq h$, and assume, as in Section 5.1, that the meshes $\{\mathcal{T}_h\}_h$ are aligned with each subdomain Ω_i, $i = 1, \ldots, I$. The boundary ∂E of each $E \in \mathcal{T}_h$ is then subdivided into planar faces denoted by F, so that, in particular, we let $\mathcal{F}_h$ be the set of faces of $\mathcal{T}_h$ that are contained in Γ. In addition, we assume that there exists a constant $\rho \in (0,1)$ with which the family $\{\mathcal{T}_h\}_h$ satisfies the following conditions:

(B1) each E of $\{\mathcal{T}_h\}_h$ is star-shaped with respect to a ball B_E of radius ρh_E,

(B2) for each E of $\{\mathcal{T}_h\}_h$, the diameters h_F of all its faces $F \subseteq \partial E$ satify $h_F \geq \rho h_E$,

(B3) the faces F of each $E \in \{\mathcal{T}_h\}_h$, seen as 2-dimensional elements, satisfy the properties **(A1)** and **(A2)** (cf. Section 5.1) with the same ρ.

Then, given an integer $k \geq 1$ and $E \in \mathcal{T}_h$, we set

$$X_h^k(\partial E) := \Big\{ v \in \mathcal{C}^0(\partial E) : \quad v|_F \in X_h^k(F) \quad \forall\, F \subseteq \partial E \Big\}, \tag{6.1}$$

with $X_h^k(F)$ defined by (5.2) (with F instead of E there), and introduce the local and global virtual element spaces

$$W_h^k(E) := \Big\{ v \in \mathrm{H}^1(E) : \ v|_{\partial E} \in X_h^k(\partial E),\ \Delta v \in \mathcal{P}_k(E),\ \Pi_k^E v - \Pi_k^{\nabla,E} v \ \in \ \mathcal{P}_{k-2}(E) \Big\}, \tag{6.2}$$

and

$$W_h^k := \Big\{ v \in X : \quad v|_E \in W_h^k(E) \quad \forall\, E \in \mathcal{T}_h \Big\}, \tag{6.3}$$

respectively, where Π_k^E is the $\mathrm{L}^2(E)$–orthogonal projection onto $\mathcal{P}_k(E)$, and $\Pi_k^{\nabla,E} : \mathrm{H}^1(E) \to \mathcal{P}_k(E)$ is the operator given by (5.1). In addition, the degrees of freedom of $W_h^k(E)$ consist of:

i) the values at the vertices of E,

ii) the moments of order $\leq k-2$ on the edges e of E,

iii) the moments of order $\leq k-2$ on the faces F of E, and

iv) the moments of order $\leq k-2$ on E,

which uniquely define the corresponding local interpolation operator $I_k^E : \mathrm{H}^2(E) \to W_h^k(E)$, whose associated global operator is denoted $I_k^{\mathcal{T}} : \mathrm{H}^2(\Omega) \to W_h^k$. In turn, we let $\Pi_k^{\mathcal{T}}$ be the global version of Π_k^E, that is the $\mathrm{L}^2(\Omega)$–orthogonal projection onto $\mathcal{P}_k(\mathcal{T}_h)$. The approximation properties of Π_k^E, $\Pi_k^{\nabla,E}$ and I_k^E are given again by (5.4), (5.5), and (5.6), respectively.

Furthermore, we need to introduce the simplicial submesh $\mathfrak{F}_h$ of Γ obtained by subdividing each face $F \in \mathcal{F}_h$ into the set of triangles T that arise after joining each vertex of F with the midpoint of the disc with respect to which F is star-shaped. It readily follows, thanks to the conditions **(A1)** and **(A2)** satisfied by the faces of the meshes, that the triangles of $\mathfrak{F}_h$ have a shape ratio that is uniformly bounded with respect to h. Hence, in order to approximate the non-virtual boundary unknowns λ and ψ of the modified Costabel & Han coupling method, we now introduce the analogue spaces of (5.8) and (5.9), that is

$$\Lambda_h^{k-1} := \Big\{ \mu_h \in \mathrm{L}^2(\Gamma) : \quad \mu_h|_T \in P_{k-1}(T) \quad \forall\, T \in \mathfrak{F}_h \Big\} \tag{6.4}$$

and

$$\Psi_h^k \,:=\, \Big\{ \varphi_h \in \mathcal{C}^0(\Gamma): \quad \varphi_h|_T \in \mathcal{P}_k(T) \quad \forall\, T \in \mathfrak{F}_h \Big\} \,\cap\, \mathrm{H}_0^{1/2}(\Gamma)\,. \tag{6.5}$$

In this way, we let $\Pi_{k-1}^{\mathfrak{F}} : \mathrm{L}^2(\Gamma) \to \Lambda_h^{k-1}$ and $\mathcal{L}_k^{\mathfrak{F}} : \mathcal{C}^0(\Gamma) \to \Psi_h^k$ be the orthogonal projection and the corresponding global Lagrange interpolation operator, respectively. Then, denoting by $\big\{\Gamma_1, ..., \Gamma_J\big\}$ the open polygons, contained in different hyperplanes of $\mathbb{R}^3$, such that $\Gamma = \cup_{j=1}^J \bar{\Gamma}_j$, we recall from [15] that the approximation properties of $\Pi_{k-1}^{\mathfrak{F}}$ and $\mathcal{L}_k^{\mathfrak{F}}$ are exactly those stated in Lemmas 5.1 and 5.2 (certainly, with $\mathfrak{F}$ instead of $\mathcal{E}$). In addition, for each $F \in \mathcal{F}_h$ we let Π_k^F be the $\mathrm{L}^2(F)$-orthogonal projection onto $\mathcal{P}_k(F)$, and denote by $\Pi_k^{\mathcal{F}}$ its global extension to $\mathrm{L}^2(\Gamma)$, which is assembled cellwise. The approximation property of Π_k^F (and hence of $\Pi_k^{\mathcal{F}}$) is exactly that given by (5.4).

6.2 The VEM/BEM scheme

According to the finite dimensional subspaces given by (6.3), (6.4), and (6.5), we now redefine $\widetilde{\mathbb{X}}_h^k$ as $\widetilde{\mathbb{X}}_h^k := W_h^k \times \Psi_h^k \times \Lambda_h^{k-1}$, and consider, instead of (5.21), the following discrete formulation for the 3D version of (3.25): Find $(\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h) \in \widetilde{\mathbb{X}}_h^k$ such that

$$\widetilde{\mathbb{A}}_{\kappa,h}\big((\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h), (v_h, \varphi_h, \mu_h)\big) \,=\, \widetilde{\mathbb{F}}_h(v_h, \varphi_h, \mu_h) \qquad \forall\, (v_h, \varphi_h, \mu_h) \in \widetilde{\mathbb{X}}_h^k\,, \tag{6.6}$$

where

$$\begin{aligned} \widetilde{\mathbb{A}}_{\kappa,h}\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) \,&:=\, \mathbf{A}_{\kappa,h}\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) \,+\, \big\langle W_\kappa \phi_h, \varphi_h \big\rangle \\ &+\, \big\langle \mu_h, V_\kappa \xi_h \big\rangle \,+\, \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K_\kappa)\phi_h \big\rangle \,-\, \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K_\kappa)\varphi_h \big\rangle \end{aligned} \tag{6.7}$$

for all (z_h, ϕ_h, ξ_h), $(v_h, \varphi_h, \mu_h) \in \widetilde{\mathbb{X}}_h^k$, with

$$\begin{aligned} \mathbf{A}_{\kappa,h}\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) \,&:=\, a_{\kappa,h}(z_h, v_h) \,-\, \int_\Gamma \xi_h \, \Pi_{k-1}^{\mathcal{F}}\big(\gamma v_h - \varphi_h\big) \\ &+\, \int_\Gamma \mu_h \, \Pi_{k-1}^{\mathcal{F}}\big(\gamma z_h - \phi_h\big)\,, \end{aligned} \tag{6.8}$$

and

$$\widetilde{\mathbb{F}}_h(v_h, \varphi_h, \mu_h) \,=\, \int_\Gamma \frac{\partial w}{\partial \boldsymbol{n}} \, \Pi_{k-1}^{\mathcal{F}} \gamma v_h \,+\, \int_\Gamma \Pi_{k-1}^{\mathcal{F}} \mu_h \, \gamma w \quad \forall\, (v_h, \varphi_h, \mu_h) \in \widetilde{\mathbb{X}}_h^k\,. \tag{6.9}$$

Note here that $\mathbf{A}_{\kappa,h}$ (and hence $\widetilde{\mathbb{A}}_{\kappa,h}$) differs from the corresponding 2D definition (5.23) (resp. (5.22)) due to the need of replacing $\langle \xi_h, \gamma v_h - \varphi_h \rangle$ and $\langle \mu_h, \gamma z_h - \phi_h \rangle$ by the calculable expressions

$$\int_\Gamma \xi_h \, \Pi_{k-1}^{\mathcal{F}}\big(\gamma v_h - \varphi_h\big) \,=\, \sum_{F \in \mathcal{F}_h} \int_F \xi_h \, \Pi_{k-1}^F\big(\gamma v_h - \varphi_h\big)$$

and

$$\int_\Gamma \mu_h \, \Pi_{k-1}^{\mathcal{F}}\big(\gamma z_h - \phi_h\big) \,=\, \sum_{F \in \mathcal{F}_h} \int_F \mu_h \, \Pi_{k-1}^F\big(\gamma z_h - \phi_h\big)\,,$$

respectively. Similarly, $\widetilde{\mathbb{F}}$ (cf. (3.25)) is replaced now by the approximate, but calculable, functional $\widetilde{\mathbb{F}}_h$, for which we have assumed, for simplicity, that the normal derivative of the datum w lies in $\mathrm{L}^2(\Gamma)$. In addition, while the elements of the explicit finite element subspace Λ_h^{k-1} are certainly calculable, in the definition of $\widetilde{\mathbb{F}}_h$ we have replaced μ_h by $\Pi_{k-1}^{\mathcal{F}} \mu_h$ for technical reasons that will become clear later on in the a priori error analysis of our scheme (see Theorem 6.3 in Section 6.4 for details). On the other hand, we stress that, following [8, Section 4.3], the stabilizing bilinear form S_h^E (cf. (5.10)), which gives rise to $a_{0,h}^E$ (cf. (5.14)) and then to $a_{\kappa,h}^E$ (cf. (5.15)) and $a_{\kappa,h}$ (cf. (5.17)), requires to be slightly modified in this 3D case. Indeed, denoting by $\mathcal{E}(E)$ and $\mathcal{F}(E)$ the set of edges and faces, respectively, of a given $E \in \mathcal{T}_h$, we define (cf. [8, eq.(4.16)])

$$S_h^E(z,v) := \sum_{e\in\mathcal{E}(E)} \int_e \pi_k^e z\, \pi_k^e v \,+\, h_E^{-1} \sum_{F\in\mathcal{F}(E)} \int_F \Pi_{k-2}^F z\, \Pi_{k-2}^F v \qquad \forall\, z,\, v \in W_h^k(E)\,,$$

where Π_{k-2}^F is the $\mathrm{L}^2(F)$-orthogonal projector onto $\mathcal{P}_{k-2}(F)$. In any case, the 3D version of (5.11) holds true (cf. [3, Section 5.5]) and hence we have the corresponding three-dimensional counterparts of (5.13), (5.27), and (5.30) as well. In addition, an analogue remark to the one provided right after (5.13) is also valid here.

6.3 Solvability analysis

In this section we address the solvability analysis of the VEM/BEM scheme (6.6). In this regard, we notice in advance that some of the proofs , being similar to those for the 2D case, are either simplified or omitted, so that we refer to the preprint version of this work (cf. [9]) for all the corresponding details. In what follows we consider, besides the continuous bilinear form $\widetilde{\mathbb{A}}_\kappa$ (cf. (3.26)), suitable modifications of $\widetilde{\mathbb{A}}_0$ (cf. (4.2)) and $\widetilde{\mathbb{A}}_{0,h}$ (cf. (5.25)). In particular, as already announced in Sections 4 and 5.3, the terms $\langle\xi,1\rangle\,\langle\mu,1\rangle$ and $\langle\xi_h,1\rangle\,\langle\mu_h,1\rangle$ can be dropped from their definitions in the present 3D case since they are not needed anymore. In addition, our fully calculable discrete bilinear form $\widetilde{\mathbb{A}}_{\kappa,h}$ (cf. (6.7) - (6.8)) induces corresponding changes in the definition of $\widetilde{\mathbb{A}}_{0,h}$. More precisely, we now introduce, instead of (4.2) and (5.25), the following 3D versions of the bilinear forms $\widetilde{\mathbb{A}}_0$ and $\widetilde{\mathbb{A}}_{0,h}$

$$\begin{aligned}\widetilde{\mathbb{A}}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) \,&=\, \mathbf{A}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) + \left(\int_\Gamma z\right)\left(\int_\Gamma v\right) \,+\, \big\langle W_0\phi,\varphi\big\rangle \\ &\quad + \big\langle\mu, V_0\xi\big\rangle \,+\, \big\langle\mu,(\frac{\mathrm{id}}{2}-K_0)\phi\big\rangle \,-\, \big\langle\xi,(\frac{\mathrm{id}}{2}-K_0)\varphi\big\rangle\end{aligned} \tag{6.10}$$

for all (z,ϕ,ξ), $(v,\varphi,\mu) \in \widetilde{\mathbb{X}}$, with (cf. (4.4))

$$\mathbf{A}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) \,:=\, a_0(z,v) \,-\, \big\langle\xi,\gamma v-\varphi\big\rangle \,+\, \big\langle\mu,\gamma z-\phi\big\rangle\,, \tag{6.11}$$

and

$$\begin{aligned}\widetilde{\mathbb{A}}_{0,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \,&:=\, \mathbf{A}_{0,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \,+\, \big\langle W_0\phi_h,\varphi_h\big\rangle \\ &\quad + \big\langle\mu_h, V_0\xi_h\big\rangle + \big\langle\mu_h,(\frac{\mathrm{id}}{2}-K_0)\phi_h\big\rangle - \big\langle\xi_h,(\frac{\mathrm{id}}{2}-K_0)\varphi_h\big\rangle\end{aligned} \tag{6.12}$$

for all (z_h,ϕ_h,ξ_h), $(v_h,\varphi_h,\mu_h) \in \widetilde{\mathbb{X}}_h^k$, with

$$\begin{aligned}\mathbf{A}_{0,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big) \,&:=\, a_{0,h}(z_h,v_h) + \left\{\int_\Gamma z_h\right\}\left\{\int_\Gamma v_h\right\} \\ &\quad - \int_\Gamma \xi_h\, \Pi_{k-1}^{\mathcal{F}}\big(\gamma v_h - \varphi_h\big) \,+\, \int_\Gamma \mu_h\, \Pi_{k-1}^{\mathcal{F}}\big(\gamma z_h - \phi_h\big)\,,\end{aligned} \tag{6.13}$$

where a_0 and $a_{0,h}$ are defined by (4.3) and (5.16), respectively.

We begin the analysis by observing that the boundedness of $\widetilde{\mathbb{A}}_0$ (cf. (4.6)) and the ellipticity of $\widetilde{\mathbb{A}}_{0,h}$ (cf. (5.29), Lemma 5.4) remain unchanged with the above new definitions, and hence the operator $\widetilde{\mathcal{R}}_h : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}_h^k$, being characterized by (5.32), is still well-defined and uniformly bounded.

On the other hand, in order to establish the 3D version of (5.34) (cf. Theorem 5.1), we consider the subspace of $\widetilde{\mathbb{X}}$ given by $\widetilde{\mathbb{X}}_0 := \mathrm{H}^1(\Omega) \times \mathrm{H}_0^{1/2}(\Gamma) \times \mathrm{L}^2(\Gamma)$. Then, we have the following result.

Theorem 6.1. *There exists a positive constant $\widetilde{C}$, independent of h, such that*

$$\begin{aligned} \|\widetilde{\mathcal{R}}_h(z,\phi,\xi)-(z,\phi,\xi)\| \,\le\, \widetilde{C}\,\Big\{ \operatorname{dist}\big((z,\phi), W_h^k\times\Psi_h^k\big) \,+\, \Big(\sum_{E\in\mathcal{T}_h} |z-\Pi_k^{\nabla,E} z|_{1,E}^2\Big)^{1/2} \\ +\, h^{-1/2}\,\|\gamma z-\Pi_{k-1}^{\mathcal{F}}\gamma z\|_{0,\Gamma} \,+\, h^{-1/2}\,\|\phi-\Pi_{k-1}^{\mathcal{F}}\phi\|_{0,\Gamma} \\ +\, \|\xi-\Pi_{k-1}^{\mathcal{F}}\xi\|_{-1/2,\Gamma} \,+\, \|\xi-\Pi_{k-1}^{\mathfrak{F}}\xi\|_{-1/2,\Gamma} \Big\} \end{aligned} \tag{6.14}$$

for all $(z,\phi,\xi)\in\widetilde{\mathbb{X}}_0$.

Proof. We proceed as in the second part of the proof of Theorem 5.1, except that, given $(z,\phi,\xi)\in\widetilde{\mathbb{X}}_0$ and $(z_h,\phi_h,\xi_h)\in\widetilde{\mathbb{X}}_h^k$, with $\xi_h=\Pi_{k-1}^{\mathfrak{F}}\xi$, we now get the bound

$$\|\widetilde{\mathcal{R}}_h(z,\phi,\xi)-(z,\phi,\xi)\| \,\lesssim\, \|(v_h,\varphi_h,\mu_h)\| \,+\, \|(z,\phi)-(z_h,\phi_h)\| \,+\, \|\xi-\Pi_{k-1}^{\mathfrak{F}}\xi\|_{-1/2,\Gamma}\,, \tag{6.15}$$

with $(v_h,\varphi_h,\mu_h) \,:=\, \widetilde{\mathcal{R}}_h(z,\phi,\xi)-(z_h,\phi_h,\xi_h)\in\widetilde{\mathbb{X}}_h^k$. For further details we refer to [9, Theorem 6.1]. □

Hence, in spite of the factors $h^{-1/2}$ appearing in (6.14), and similarly as we already did with Theorem 5.1, we stress here that the present Theorem 6.1, together with classical density arguments again, the aproximation properties provided by (5.4), (5.5), (5.6), and Lemmas 5.1 and 5.2, and the uniform boundedness of $\widetilde{\mathcal{R}}_h$, guarantee that

$$\lim_{h\to 0}\|\widetilde{\mathcal{R}}_h(z,\phi,\xi)-(z,\phi,\xi)\| \,=\, 0 \qquad \forall\,(z,\phi,\xi)\in\widetilde{\mathbb{X}}\,. \tag{6.16}$$

Now we are ready to establish the discrete inf-sup condition for $\widetilde{\mathbb{A}}_{\kappa,h}$ (cf. (6.7)). More precisely, the 3D version of (5.42) (cf. Theorem 5.2) is stated as follows.

Theorem 6.2. *Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ. Then, there exist positive constants h_0 and $\widetilde{\alpha}_\kappa$, independent of h, such that for each $h\,\le\,h_0$ there hold*

$$\sup_{\substack{(z_h,\phi_h,\xi_h)\in\widetilde{\mathbb{X}}_h^k\\ (z_h,\phi_h,\xi_h)\neq\mathbf{0}}} \frac{\Big|\widetilde{\mathbb{A}}_{\kappa,h}\big((z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\big)\Big|}{\|(z_h,\phi_h,\xi_h)\|} \,\ge\, \widetilde{\alpha}_\kappa\,\|(v_h,\varphi_h,\mu_h)\| \qquad \forall\,(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbb{X}}_h^k\,. \tag{6.17}$$

Proof. We refer to [9, Theorem 6.2] for all the details. □

Therefore, it follows directly from Theorem 6.2 that for each $h\le h_0$, the VEM/BEM scheme (6.6) has a unique solution $(\widetilde{u}_h,\widetilde{\psi}_h,\widetilde{\lambda}_h)\in\widetilde{\mathbb{X}}_h^k$. In turn, we can define the operator $\widetilde{\Theta}_h:\widetilde{\mathbb{X}}_h^k\to\widetilde{\mathbb{X}}_h^k$, which, given $(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbb{X}}_h^k$, is uniquely characterized by the equation

$$\widetilde{\mathbb{A}}_{\kappa,h}\big((z_h,\phi_h,\xi_h),\widetilde{\Theta}_h(v_h,\varphi_h,\mu_h)\big) \,=\, \langle(z_h,\phi_h,\xi_h),(v_h,\varphi_h,\mu_h)\rangle_{\widetilde{\mathbb{X}}} \qquad \forall\,(z_h,\phi_h,\xi_h)\in\widetilde{\mathbb{X}}_h^k\,, \tag{6.18}$$

which implies

$$\widetilde{\mathbb{A}}_{\kappa,h}\big((v_h,\varphi_h,\mu_h),\widetilde{\Theta}_h(v_h,\varphi_h,\mu_h)\big) \,=\, \|(v_h,\varphi_h,\mu_h)\|^2 \qquad \forall\,(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbb{X}}_h^k\,. \tag{6.19}$$

Moreover, it is clear from (6.18) and the discrete inf-sup condition (6.17) that $\|\widetilde{\Theta}_h\|\,\le\,\frac{1}{\widetilde{\alpha}_\kappa}$.

6.4 Error analysis

Similarly as in Section 5.4, we now establish a Cea-type estimate for the VEM/BEM scheme (6.6).

Theorem 6.3. *Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ, and that both $\frac{\partial w}{\partial \boldsymbol{n}}$ and λ belong to $\mathrm{L}^2(\Gamma)$. In addition, let $h_0 > 0$ be the constant whose existence is guaranteed by Theorem 6.2. Then, there exist a constant $\widetilde{C} > 0$, independent of h, such that for each $h \le h_0$ there holds*

$$\begin{aligned}\|(u,\psi,\lambda) - (\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h)\| \,\le\, \widetilde{C} \Big\{ &\mathrm{dist}\big((u,\psi), W_h^k \times \Psi_h^k\big) + \Big\| \frac{\partial w}{\partial \boldsymbol{n}} - \Pi_{k-1}^{\mathcal{F}}\Big(\frac{\partial w}{\partial \boldsymbol{n}}\Big)\Big\|_{-1/2,\Gamma} \\ &+ \Big(\sum_{E\in\mathcal{T}_h} \big\| u - \Pi_k^{\nabla,E} u \big\|_{1,E}^2 \Big)^{1/2} \,+\, \big\| u - \Pi_{k-1}^{\mathcal{T}} u \big\|_{0,\Omega} \\ &+ \|\lambda - \Pi_{k-1}^{\mathcal{F}} \lambda\|_{-1/2,\Gamma} \,+\, \|\lambda - \Pi_{k-1}^{\mathfrak{F}} \lambda\|_{-1/2,\Gamma} \Big\}.\end{aligned} \tag{6.20}$$

Proof. We omit details and refer to [9]. □

Having established (6.20), the rates of convergence for the solution $(\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h)$ of the VEM/BEM scheme (6.6) follow almost exactly as proved by Theorem 5.4 for the corresponding 2D case (cf. (5.64)). Indeed, assuming that both u and the datum w belong to $\mathrm{H}^1(\Omega) \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\Omega_i)$, and applying the approximation properties of $I_k^{\mathcal{T}}$ (cf. (5.6)), $\mathcal{L}_k^{\mathfrak{F}}$ (cf. Lemma 5.2), $\Pi_{k-1}^{\mathcal{F}}$ (cf. Lemma 5.1), $\Pi_k^{\nabla,E}$ (cf. (5.5)), $\Pi_{k-1}^{\mathcal{T}}$ (cf. (5.4)), and $\Pi_{k-1}^{\mathfrak{F}}$ (cf. Lemma 5.1), and employing the 3D version of the trace inequalities given by (5.65), we conclude the existence of a constant $\widetilde{C}_0 > 0$, independent of h, such that for each $h \le h_0$ there holds

$$\|(u,\psi,\lambda) - (\widetilde{u}_h, \widetilde{\psi}_h, \widetilde{\lambda}_h)\| \,\le\, \widetilde{C}_0\, h^k \sum_{i=1}^{I} \Big\{ \|u\|_{k+1,\Omega_i} \,+\, \|w\|_{k+1,\Omega_i} \Big\}. \tag{6.21}$$

We end this paper by commenting that fully calculable approximations of u for the 2D and 3D schemes introduced and analyzed here, can be defined similarly as done in [8, Section 3.4]. In this way, rates of convergence of exactly the same order as those established by Theorem 5.4 and (6.21), can be obtained. We omit further details.

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