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Directional Stochastic Orders with an Application to Financial Mathematics

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Abstract: Relevant integral stochastic orders share a common mathematical model, they are defined by generators which are made up of increasing functions on appropriate directions. Motivated by the aim to provide a unified study of those orders, we introduce a new class of integral stochastic orders whose generators are composed of functions that are increasing on the directions of a finite number of vectors. These orders will be called directional stochastic orders. Such stochastic orders are studied in depth. In that analysis, the conical combinations of vectors in those finite subsets play a relevant role. It is proved that directional stochastic orders are generated by non-stochastic pre-orders and the class of their preserving mappings. Geometrical characterizations of directional stochastic orders are developed. Those characterizations depend on the existence of non-trivial subspaces contained in the set of conical combinations. An application of directional stochastic orders to the field of financial mathematics is developed, namely, to the comparison of investments with random cash flows.

Keywords: convex cone; investment with random cash flows; non-trivial subspace; orthogonal projection; stochastic order

MSC: Primary 60E15; Secondary 06A06

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1. Introduction

This manuscript is focused on the theory of stochastic orders. A lot of effort has been made on this topic during the last decades due to its importance from the theoretical and applied points of view. Very basically, a stochastic order attempts to order probabilities in some sense. Nowadays, stochastic orders are applied in numerous fields like insurance, economics, decision theory, reliability, quality control, medicine, etc.

An integral stochastic order is a stochastic order which can be characterized by means of the comparison of the integrals of a set of functions with respect to the corresponding probabilities, such a set called generator of the order. Some integral stochastic orders have a common condition in relation to the functions of their generators, these are composed of functions which are increasing on appropriate directions. As examples of such orders, we have

(i) the usual multivariate stochastic order, which has as a generator the set of all increasing functions $f : \mathbb{R}^n \to \mathbb{R}$, that is, the generator is made up of mappings which are increasing on the directions of the vectors in $\{e_1, e_2, \ldots, e_n\}$, where e_i stands for the ith-unit vector of $\mathbb{R}^n$;
(ii) the time value of money stochastic order, a generator of that stochastic order is given by the functions which are increasing on the directions of the vectors in $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{n-1}, \hat{e}_n\}$, with $\hat{e}_i = e_i - e_{i+1}$, $1 \leq i \leq n-1$, and $\hat{e}_n = e_n$;
(iii) the family of strong extremality orders, a generator of the strong extremality order in the direction $u \in S^{n-1}$ (unit sphere of $\mathbb{R}^n$) is made up of functions which are increasing

on the directions given by the vectors in $\{\mathcal{R}_u^{-1}e_1, \mathcal{R}_u^{-1}e_2, \ldots, \mathcal{R}_u^{-1}e_n\}$, where $\mathcal{R}_u$ is a rotation matrix such that $\mathcal{R}_u u = \frac{1}{\sqrt{n}}\mathbf{1}$, with $\mathbf{1} = (1, \ldots, 1)^t \in \mathbb{R}^n$.

In order to develop a general analysis of integral stochastic orders satisfying the above condition, we introduce a class of integral stochastic orders with generators which are made up of increasing functions on the directions given by finite subsets of vectors in $\mathbb{R}^n$. These orders will be called directional stochastic orders. The aim of the manuscript was the study of those orders in depth, developing characterizations and properties, and providing useful results to apply in applied problems.

The results of this work permit to provide a thorough study of any order which belongs to the class of directional orders without the necessity of a particular analysis. Moreover, the class considered in the manuscript permits to introduce new orders (as we develop in Section 6, to approach a problem in financial mathematics), the properties of those orders being immediate to state by the theoretical study of the paper.

Our research plan will be focused on the analysis of the relations between two directional stochastic orders given by two different finite subsets of vectors in $\mathbb{R}^n$ (Section 3), the development of geometrical characterizations of directional orders (Section 4), and the study of relevant properties of those orders (Section 5). Moreover, a new directional stochastic order for the comparison of investments with random cash flows is introduced in Section 6, as an application of the results in the present manuscript.

2. Preliminaries

A binary relation $\preceq_{\mathcal{X}}$ on a set $\mathcal{X}$ which is reflexive, transitive, and antisymmetric is said to be a partial order on $\mathcal{X}$. The pair $(\mathcal{X}, \preceq_{\mathcal{X}})$ is called a partially ordered set. If $\preceq_{\mathcal{X}}$ is reflexive and transitive, $\preceq_{\mathcal{X}}$ is called a pre-order.

Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ be a partially ordered set. A set $U \subset \mathcal{X}$ is said to be an upper set if for any $x \in U$ and any $y \in \mathcal{X}$ with $x \preceq_{\mathcal{X}} y$, $y \in U$.

Given $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ partially ordered sets, a mapping $f : \mathcal{X} \to \mathcal{Y}$ is order-preserving if for any $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq_{\mathcal{X}} x_2$, $f(x_1) \preceq_{\mathcal{Y}} f(x_2)$. A mapping $f : \mathcal{X} \to \mathbb{R}$ is said to be $\preceq_{\mathcal{X}}$-preserving if for any $x_1, x_2 \in \mathcal{X}$ such that $x_1 \preceq_{\mathcal{X}} x_2$, $f(x_1) \leq f(x_2)$.

Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ be partially ordered sets. A mapping $\phi : \mathcal{X} \to \mathcal{Y}$ is said to be an order-isomorphism if ϕ is order-preserving, there exists $\phi^{-1} : \mathcal{Y} \to \mathcal{X}$ inverse of ϕ and ϕ^{-1} is order-preserving, equivalently, ϕ is bijective and for all $x_1, x_2 \in \mathcal{X}$, $x_1 \preceq_{\mathcal{X}} x_2$ if and only if $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$.

Two partially ordered sets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ are said to be order-isomorphic if there exists an order-isomorphism $\phi : \mathcal{X} \to \mathcal{Y}$.

A partial order $\preceq$ on $\mathbb{R}^n$ is said to be closed if the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \preceq y\}$ is closed in the (usual) product topology.

All the above concepts are defined in the case of pre-orders in a similar way.

See for instance [1–3] for an introduction to the theory of ordered sets.

A stochastic order is a pre-order relation on a set of probabilities.

In the present manuscript, we consider stochastic orders which are defined on $\mathcal{P}_n$, the set of probabilities of the measurable space $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, with $\mathcal{B}_{\mathbb{R}^n}$ the usual Borel σ-algebra on $\mathbb{R}^n$.

Given a random vector X, $E(X)$ will denote its expected value and P_X its induced probability.

Let $\preceq$ denote a stochastic order on $\mathcal{P}_n$ and let X and Y be two random vectors, $X \preceq Y$ will mean that $P_X \preceq P_Y$.

A stochastic order $\preceq$ is said to be integral when there exists a set $\mathcal{F}$ of real measurable mappings such that for two random vectors X and Y

$$X \preceq Y \quad \text{if} \quad \int_{\mathbb{R}^n} f \, dP_X \leq \int_{\mathbb{R}^n} f \, dP_Y$$

for any $f \in \mathcal{F}$ such that the integrals exist. The set $\mathcal{F}$ is said to be a generator of the order.

Some integral stochastic orders which will appear in the manuscript are the following. Let X and Y be $\mathbb{R}^n$-valued random vectors,

(i) X is said to be smaller than Y in the usual stochastic order, denoted by $X \preceq_{st} Y$, if $E(f(X)) \leq E(f(Y))$ for all increasing functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist (see, for instance, [4,5]).

(ii) Let $\mathcal{F} = \{ f : \mathbb{R}^n \to \mathbb{R} \mid f(x + \varepsilon_i e_i) \geq f(x + \varepsilon_{i+1} e_{i+1})$ for all $x \in \mathbb{R}^n$, $0 \leq \varepsilon_{i+1} \leq \varepsilon_i$, $1 \leq i \leq n-1$, and $f(x + \varepsilon_n e_n) \geq f(x)$ for all $x \in \mathbb{R}^n$ and $0 \leq \varepsilon_n \}$.
It is said that X is smaller than Y in the time value of money stochastic order, if $E(f(X)) \leq E(f(Y))$ for any $f \in \mathcal{F}$ such that the above expectations exist. This relation will be denoted by $X \preceq_{tvm} Y$ (see [6]).

(iii) Given $u \in S^{n-1}$ (unit sphere of $\mathbb{R}^n$), let $\mathcal{R}_u$ be a rotation matrix such that $\mathcal{R}_u u = \frac{1}{\sqrt{n}}\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)^t \in \mathbb{R}^n$. Let $C_t^u = \{x \in \mathbb{R}^n \mid \mathcal{R}_u(x - t) \geq 0_{\mathbb{R}^n}\}$ where $t \in \mathbb{R}^n$, and $\mathcal{G}^u = \{I_{C_t^u} \mid t \in \mathbb{R}^n\}$ (I_A stands for the indicator function of A, with $A \subset \mathbb{R}^n$). Define the order $\preceq^u$ on $\mathbb{R}^n$ given by $x \preceq^u y$ if $f(x) \leq f(y)$ for all $f \in \mathcal{G}^u$, with $x, y \in \mathbb{R}^n$.
We say that X is smaller than Y in the strong extremality stochastic order in the direction u, denoted by $X \preceq_{SE_u} Y$, if $E(f(X)) \leq E(f(Y))$ for any $\preceq^u$-preserving mapping $f : \mathbb{R}^n \to \mathbb{R}$ such that the above integrals exist (see [7]).

An introduction to the theory of stochastic orderings can be found, for instance, in [4,5,8]. The reader is referred to [4,9] for a precise analysis of integral stochastic orders.

Let P be a probability in $\mathcal{P}_n$, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a measurable mapping, then $P \circ T^{-1}$ will denote the probability on $\mathcal{B}_{\mathbb{R}^m}$ given by $P \circ T^{-1}(B) = P(T^{-1}(B))$ for any $B \in \mathcal{B}_{\mathbb{R}^m}$.

Let a be an element of $\mathbb{R}^n$, δ_a will stand for the degenerate distribution at a.

Let $e_i \in \mathbb{R}^n$ be the ith-unit vector, that is, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with number 1 in position i, $1 \leq i \leq n$. The zero vector of $\mathbb{R}^n$ will be denoted by $0_{\mathbb{R}^n}$.

The usual componentwise order on $\mathbb{R}^n$ will be denoted by $\leq$.

Given $V = \{v_1, \ldots, v_l\}$ a set of vectors in $\mathbb{R}^n$, the convex cone C_V defined by V is the set of all conical combinations of vectors in V, that is, $C_V = \{\sum_{i=1}^{l} \alpha_i v_i \mid \alpha_i \geq 0, 1 \leq i \leq l\}$. Moreover, $\langle V \rangle$ will denote the span of V, that is, $\langle V \rangle$ is the vector space of all linear combinations of the elements in V with scalars in $\mathbb{R}$.

Let S be a vector subspace of $\mathbb{R}^n$, $S^\perp$ will be its orthogonal supplementary subspace in $\mathbb{R}^n$ and π_S will stand for the orthogonal projection onto S.

The Minkowski sum of two subsets A and B of $\mathbb{R}^n$, is the set $A \oplus B = \{a + b \in \mathbb{R}^n \mid a \in A, b \in B\}$. It is well-known that the Minkowski sum of two convex sets is a convex set.

3. Directional Stochastic Orders

In this section, the concept of V-directional stochastic order, where V is a finite subset of $\mathbb{R}^n$, is introduced. Relations between directional orders given by different subsets of $\mathbb{R}^n$ are studied.

The following set of mappings provides the definition of the new class of stochastic orders. Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Consider

$$\mathcal{F}_V = \{f : \mathbb{R}^n \to \mathbb{R} \mid \ f(x + \epsilon v_i) \geq f(x) \text{ for all } x \in \mathbb{R}^n, \epsilon \geq 0,\ v_i \in V\}.$$

Definition 1. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Let X and Y be random vectors. It will be said that X is less than Y in the V-directional stochastic order, if $E(f(X)) \leq E(f(Y))$ for any $f \in \mathcal{F}_V$ such that the above expectations exist. This relation will be denoted by $X \preceq_V Y$.*

Some multivariate stochastic orders are directional orders, as the following examples show.

Example 1. Recall that $e_i \in \mathbb{R}^n$ is the ith-unit vector, $1 \leq i \leq n$.

(i) *Let $V = \{e_1, e_2, \ldots, e_n\}$. Then, $\preceq_V$ is the order $\preceq_{st}$.*

(ii) *Let* $V = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{n-1}, \hat{e}_n\}$*, where* $\hat{e}_i = e_i - e_{i+1}$*,* $1 \le i \le n-1$*, and* $\hat{e}_n = e_n$*. According to Theorem 1 in [6],* $\preceq_V$ *is the order* $\preceq_{tvm}$*.*

(iii) *Let* $V = \{\mathcal{R}_u^{-1}e_1, \mathcal{R}_u^{-1}e_2, \dots, \mathcal{R}_u^{-1}e_n\}$*, where* $u \in S^{n-1}$ *and* $\mathcal{R}_u$ *is a rotation matrix such that* $\mathcal{R}_u u = \frac{1}{\sqrt{n}}\mathbf{1}$*, with* $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{R}^n$*. Then,* $\preceq_V$ *is the order* $\preceq_{SE_u}$ *since the condition* $X \preceq_{SE_u} Y$ *is equivalent to* $R_u X \preceq_{st} R_u Y$ *(see [7]).*

Let us study relations between two directional stochastic orders given by different subsets of $\mathbb{R}^n$.

Proposition 1. *Let* V *and* V' *be finite sets of vectors in* $\mathbb{R}^n$*. Then,* $\preceq_V$ *implies* $\preceq_{V'}$ *if and only if* $\mathcal{F}_{V'} \subseteq \mathcal{F}_V$*.*

Proof. Assume that $\preceq_V$ implies $\preceq_{V'}$ but $\mathcal{F}_{V'} \not\subseteq \mathcal{F}_V$. Then, there exist $f \in \mathcal{F}_{V'}$, $x \in \mathbb{R}^n$, $v \in V$, and $\epsilon \ge 0$ with $f(x + \epsilon v) < f(x)$. Therefore, the relation $\delta_x \preceq_{V'} \delta_{x+\epsilon v}$ is false, but this is a contradiction with the fact that $\preceq_V$ implies $\preceq_{V'}$, because $\delta_x \preceq_V \delta_{x+\epsilon v}$.

The converse is trivial. □

In order to obtain other relations between two directional stochastic orders, we consider the conical combinations of vectors in those subsets. In fact, we will see that the V-directional stochastic order is characterized by means of the cone C_V.

Lemma 1. *Let* $V = \{v_1, \dots, v_l\}$ *be a set of vectors in* $\mathbb{R}^n$*. Then,* C_V *is a convex closed set.*

Proof. The convexity of C_V is trivial.

If $C_V = \{0_{\mathbb{R}^n}\}$, it is closed. Consider $C_V \ne \{0_{\mathbb{R}^n}\}$. Note that $C_V = C_{V \setminus \{0_{\mathbb{R}^n}\}}$. Now, Proposition 1.4.7 in [10] provides that C_V is closed. □

The following results will provide that C_V characterizes $\preceq_V$.

Proposition 2 (Theorem A.3.1 in [11])**.** *Let* A *be a closed convex set in* $\mathbb{R}^n$ *that does not contain the origin. Then, there exists a real linear function* ξ *defined on* $\mathbb{R}^n$ *and* $\alpha > 0$ *such that* $\xi(x) \ge \alpha$ *for all* x *in* A*. In particular, the hyperplane* $\xi(x) = 0$ *does not intersect* A*.*

Lemma 2. *Let* $V = \{v_1, \dots, v_l\}$ *be a set of vectors in* $\mathbb{R}^n$*. Let* $\hat{v} \in \mathbb{R}^n$ *with* $\hat{v} \notin C_V$*. Let* $B = \{\lambda \hat{v} \mid \lambda \ge 1\}$ *and* $C_V - B = \{v - w \mid v \in C_V \text{ and } w \in B\}$*. Then,*

(i) $C_V - B$ *is a convex closed set,*

(ii) $C_V - B$ *does not contain the origin.*

Proof. It is easy to prove that B is convex and closed. As a consequence, so is $-B$. Then, $C_V - B = C_V \oplus (-B)$ is a convex set.

Let $\{x_m\}_m \subseteq C_V - B$ such that $\lim_m x_m = x \in \mathbb{R}^n$. Let us see that $x \in C_V - B$. For all $m \in \mathbb{N}$, $x_m = a_m - b_m$ with $a_m \in C_V$ and $b_m = \lambda_m \hat{v} \in B$.

If $\{b_m\}_m$ is bounded, so is $\{a_m\}_m$. Thus, there are convergent subsequences such that $\lim_k b_{m_k} = b$ and $\lim_k a_{m_k} = a$. Since B and C_V are closed, $b \in B$ and $a \in C_V$. Hence, $x \in C_V - B$.

If $\{b_m\}_m$ is not bounded, then $\{\lambda_m\}_m$ is unbounded, and there exists a subsequence with $\lim_k \lambda_{m_k} = +\infty$. Take $x_{m_k}/\lambda_{m_k} = a_{m_k}/\lambda_{m_k} - b_{m_k}/\lambda_{m_k} = a_{m_k}/\lambda_{m_k} - \hat{v}$.

We obtain that $0_{\mathbb{R}^n} = \lim_k a_{m_k}/\lambda_{m_k} - \hat{v}$, but $\{a_{m_k}/\lambda_{m_k}\}_k \subseteq C_V$ which is closed. As a consequence, $\hat{v} \in C_V$, which is a contradiction.

Now, suppose that $0_{\mathbb{R}^n} \in C_V - B$. Then, there exists $v \in C_V$ such that $v = \lambda \hat{v}$ with $\lambda \ge 1$. Then, $\hat{v} \in C_V$, which is a contradiction. □

Proposition 3. *Let* $V = \{v_1, \dots, v_l\}$ *be a set of vectors in* $\mathbb{R}^n$*. Let* $\hat{v} \in \mathbb{R}^n$ *with* $\hat{v} \notin C_V$*. Then, there exists a linear function* $\xi : \mathbb{R}^n \to \mathbb{R}$ *such that* $\xi(v_i) \ge 0$ *for all* $1 \le i \le l$*, and* $\xi(\hat{v}) < 0$*.*

Proof. According to Lemma 2, $C_V - B$ is a closed convex set which does not contain the origin. Proposition 2 provides that there exits a linear mapping $\xi : \mathbb{R}^n \to \mathbb{R}$ such that $\xi(v) > 0$ for all $v \in C_V - B$. Note that $-\hat{v} \in C_V - B$. Then, $\xi(\hat{v}) < 0$.

Now, consider $v^i = \alpha_i v_i - \hat{v}$, $1 \leq i \leq l$, where $\alpha_i \geq 0$. Clearly, $v^i \in C_V - B$ for all $1 \leq i \leq l$. Then, $\xi(v^i) = \alpha_i \xi(v_i) - \xi(\hat{v}) > 0$ for all $\alpha_i \geq 0$. As a consequence, $\xi(v_i) \geq 0$ for all $1 \leq i \leq l$. $\square$

By means of the previous results, we will see that the functions in $\mathcal{F}_V$ are determined by C_V. As a consequence, the stochastic order $\preceq_V$ is characterized by the convex cone C_V.

Proposition 4. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then, $\mathcal{F}_V = \mathcal{F}_{V \cup \{v\}}$ if and only if $v \in C_V$.*

Proof. Suppose that $v \in C_V$, that is, $v = \sum_{i=1}^{l} \alpha_i v_i$ with $\alpha_i \geq 0$ for all $1 \leq i \leq l$. Clearly, $\mathcal{F}_{V \cup \{v\}} \subseteq \mathcal{F}_V$. Let $f \in \mathcal{F}_V$. For all $\epsilon \geq 0$ and $x \in \mathbb{R}^n$, $f(x + \epsilon v) = f(x + \sum_{i=1}^{l} \epsilon \alpha_i v_i) \geq f(x + \sum_{i=1}^{l-1} \epsilon \alpha_i v_i) \geq \ldots \geq f(x)$. Then, $f \in \mathcal{F}_{V \cup \{v\}}$, and so $\mathcal{F}_V \subseteq \mathcal{F}_{V \cup \{v\}}$.

Conversely, suppose that $\mathcal{F}_V = \mathcal{F}_{V \cup \{v\}}$ and $v \notin C_V$. By Proposition 3, there exists a linear function $\xi : \mathbb{R}^n \to \mathbb{R}$ such that $\xi(v_i) \geq 0$ for all $1 \leq i \leq l$, and $\xi(v) < 0$. Therefore, $\xi \in \mathcal{F}_V$, but $\xi \notin \mathcal{F}_{V \cup \{v\}}$, which is a contradiction. $\square$

Proposition 5. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then, $\preceq_V$ is the same order as $\preceq_{V \cup \{v\}}$ if and only if $v \in C_V$.*

Proof. It follows from Propositions 1 and 4. $\square$

Proposition 6. *Let V and $\hat{V}$ be finite sets of vectors in $\mathbb{R}^n$. Then, $\preceq_{\hat{V}}$ implies $\preceq_V$ if and only if $\hat{V} \subseteq C_V$.*

Proof. Firstly, suppose that $\hat{V} \subseteq C_V$. Clearly, $\mathcal{F}_V \subseteq \mathcal{F}_{\hat{V}}$. Proposition 1 ensures that $\preceq_{\hat{V}}$ implies $\preceq_V$.

Now, suppose that $\preceq_{\hat{V}}$ implies $\preceq_V$ but there exists $\hat{v} \in \hat{V}$ with $\hat{v} \notin C_V$. As a consequence of Proposition 3, there exists a linear function $\xi : \mathbb{R}^n \to \mathbb{R}$ such that $\xi \in \mathcal{F}_V$ with $\xi(\hat{v}) < 0$, but this is a contradiction with $\delta_{0_{\mathbb{R}^n}} \preceq_V \delta_{\hat{v}}$ which must be true because $\delta_{0_{\mathbb{R}^n}} \preceq_{\hat{V}} \delta_{\hat{v}}$. $\square$

Proposition 7. *Let $V = \{v_1, \ldots, v_l\}$ be a subset of $\mathbb{R}^n$ with $\{v_1, \ldots, v_r\}$ linearly independent vectors. Then, there exist $w_{r+1}, \ldots, w_l \in \mathbb{R}^n$ such that $(\mathcal{P}_n, \preceq_V)$ is order-isomorphic to $(\mathcal{P}_n, \preceq_{\hat{V}})$, with $\hat{V} = \{e_1, \ldots, e_r, w_{r+1}, \ldots, w_l\}$.*

Proof. Consider $\{v_1, \ldots, v_r, b_{r+1}, \ldots, b_n\}$ a basis of $\mathbb{R}^n$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map such that $h(v_i) = e_i$, $1 \leq i \leq r$, and $h(b_j) = e_j$, $r+1 \leq j \leq n$. Note that h is a bijective mapping.

Let $\widetilde{h} : \mathcal{P}_n \to \mathcal{P}_n$ given by $\widetilde{h}(P) = P \circ h^{-1}$, for any $P \in \mathcal{P}_n$.

Observe that h is bijective and measurable, as a consequence, so is h^{-1} (see, for instance, [12,13]). That guarantees that $\widetilde{h}$ is bijective.

Let $\hat{V} = \{e_1, \ldots, e_r, h(v_{r+1}), \ldots, h(v_l)\}$. Firstly, let us see that $\mathcal{F}_{\hat{V}} = \{f \circ h^{-1} \mid f \in \mathcal{F}_V\}$. Let $f \in \mathcal{F}_V$. For all $v \in \hat{V}$, $x \in \mathbb{R}^n$ and $\epsilon \geq 0$ it holds that $f \circ h^{-1}(x + \epsilon v) = f(h^{-1}(x) + \epsilon h^{-1}(v)) \geq f(h^{-1}(x))$ because $h^{-1}(v) \in V$. In a similar way, $f \in \mathcal{F}_{\hat{V}}$ implies that $f \circ h \in \mathcal{F}_V$.

Now, suppose that $X \preceq_V Y$. For all $f \in \mathcal{F}_V$, $E(f(X)) \leq E(f(Y))$ holds. Then, $E(f \circ h^{-1}(h(X))) \leq E(f \circ h^{-1}(h(Y)))$. As a consequence, $h(X) \preceq_{\hat{V}} h(Y)$.

The converse can be proved analogously.

Hence, $P_X \preceq_V P_Y$ if and only if $\widetilde{h}(P_X) \preceq_{\hat{V}} \widetilde{h}(P_Y)$. Thus, $(\mathcal{P}_n, \preceq_V)$ is order-isomorphic to $(\mathcal{P}_n, \preceq_{\hat{V}})$. $\square$

4. Geometrical Characterization of Directional Stochastic Orders

In this section, we analyze the way in which the set of conical combinations of vectors in V determines the V-directional stochastic order.

Definition 2. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Let x and y be vectors in $\mathbb{R}^n$. It is said that x is less than y in $\leq_{C_V}$ if $y - x \in C_V$. This relation will be denoted by $x \leq_{C_V} y$.*

Note that $\leq_{C_V}$ is reflexive and transitive, but it is not antisymmetric if C_V contains non-trivial subspaces of $\mathbb{R}^n$. In the case where C_V does not contain non-trivial subspaces of $\mathbb{R}^n$, $(\mathbb{R}^n, \leq_{C_V})$ is a partially ordered set.

Let us consider the class $\mathcal{F}_{\leq_{C_V}}$ of all $\leq_{C_V}$-preserving mappings, that is, $f \in \mathcal{F}_{\leq_{C_V}}$ when for any x and y in $\mathbb{R}^n$ such that $x \leq_{C_V} y$, $f(x) \leq f(y)$ holds.

Definition 3. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Let X and Y be random vectors. It will be said that X is less than Y in the stochastic order $\preceq_{C_V}$ if $E(f(X)) \leq E(f(Y))$ for any $f \in \mathcal{F}_{\leq_{C_V}}$ such that the above expectations exist. This relation will be denoted by $X \preceq_{C_V} Y$.*

Proposition 8. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Then, $\preceq_V$ is the same stochastic order as $\preceq_{C_V}$.*

Proof. Proposition 1 ensures the result if $\mathcal{F}_{\leq_{C_V}} = \mathcal{F}_V$.

We have that for all $x \in \mathbb{R}^n$, $\epsilon \geq 0$ and $v_i \in V$, $x \leq_{C_V} x + \epsilon v_i$. Then, $\mathcal{F}_{\leq_{C_V}} \subseteq \mathcal{F}_V$. Let $f \in \mathcal{F}_V$. Let x and y be vectors in $\mathbb{R}^n$ such that $x \leq_{C_V} y$. Then, $y - x = \sum_{i=1}^{l} \alpha_i v_i$, where $\alpha_i \geq 0$ with $1 \leq i \leq l$. Therefore, $f(y) = f(x + \sum_{i=1}^{l} \alpha_i v_i) \geq f(x + \sum_{i=1}^{l-1} \alpha_i v_i) \geq \ldots \geq f(x + \alpha_1 v_1) \geq f(x)$. Hence, $f \in \mathcal{F}_{\leq_{C_V}}$. As a consequence, $\mathcal{F}_V \subseteq \mathcal{F}_{\leq_{C_V}}$. $\square$

We have seen that directional stochastic orders are generated by pre-orders on $\mathbb{R}^n$ and the corresponding class of preserving mappings. In the case where the pre-orders are orders, those stochastic orders have been studied in mathematical literature. The reader is referred, for instance, to [4,14–17] and their references for this kind of stochastic orders.

Now, we study characterizations of $\preceq_V$ depending on the existence of non-trivial subspaces of $\mathbb{R}^n$ contained in C_V.

Firstly, we consider the case where C_V does not contain non-trivial vector subspaces. Recall that in this case, $(\mathbb{R}^n, \leq_{C_V})$ is a partially ordered set.

Proposition 9. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ such that C_V does not contain non-trivial vector subspaces. Then, $\leq_{C_V}$ is a closed order.*

Proof. We should prove that $D = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \leq_{C_V} y\}$ is closed. Let $\{(x_m, y_m)\}_m \subseteq D$ be a sequence which converges to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Therefore, $y_m - x_m \in C_V$ for all $m \in \mathbb{N}$, and $\{y_m - x_m\}_m$ tends to $y - x$. By Lemma 1, C_V is closed. Then, $y - x \in C_V$ and so $x \leq_{C_V} y$. $\square$

Corollary 1. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ such that C_V does not contain non-trivial vector subspaces. Let X and Y be random vectors. Then, the following conditions are equivalent,*

(i) $X \preceq_V Y$,

(ii) *there are random vectors X' and Y' defined on the same probability space, with the same distributions as X and Y, respectively, such that $X' \leq_{C_V} Y'$ almost surely,*

(iii) *$E(f(X)) \leq E(f(Y))$ for all bounded, continuous, and $\leq_{C_V}$-preserving functions f,*

(iv) *$P(X \in U) \leq P(Y \in U)$ for all upper sets U with respect to $\leq_{C_V}$,*

(v) *$P(X \in U) \leq P(Y \in U)$ for all closed upper sets U with respect to $\leq_{C_V}$.*

Proof. Since the partial order $\leq_{C_V}$ is closed, the result follows from [14]. See also Theorems 2.6.3 and 2.6.4 in [4]. $\square$

Now, we study the case where non-trivial subspaces of $\mathbb{R}^n$ are contained in C_V. Note that in this case, $\leq_{C_V}$ is not an order but a pre-order.

Proposition 10. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ such that there exists a non-trivial subspace S of $\mathbb{R}^n$ with $S \subseteq C_V$. Then, $\preceq_V$ is the same stochastic order as $\preceq_{\hat{V}}$, where $\hat{V} = \{s_1, \ldots, s_r, -s_1, \ldots, -s_r, \pi_{S^\perp}(v_1), \ldots, \pi_{S^\perp}(v_l)\}$, $\{s_1, \ldots, s_r\}$ being a basis of S.*

Proof. Note that $-\pi_S(v_i) \in S \subseteq C_V$, $1 \leq i \leq l$. Then, $\pi_{S^\perp}(v_i) = v_i - \pi_S(v_i) \in C_V$ for all $1 \leq i \leq l$. As a consequence, $\hat{V} \subseteq C_V$. By Proposition 6, $\preceq_{\hat{V}}$ implies $\preceq_V$.

On the other hand, for any $v_j \in V$, $v_j = \pi_S(v_j) + \pi_{S^\perp}(v_j)$. Therefore, $v_j \in C_{\hat{V}}$. Then, $V \subseteq C_{\hat{V}}$. Proposition 6 ensures that $\preceq_V$ implies $\preceq_{\hat{V}}$.

As a consequence, $\preceq_V$ and $\preceq_{\hat{V}}$ are the same stochastic order. $\square$

Proposition 11. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ such that there exists a non-trivial subspace S of $\mathbb{R}^n$ with $S \subseteq C_V$. Let X and Y be random vectors. Then, $X \preceq_V Y$ if and only if $\pi_{S^\perp}(X) \preceq_V \pi_{S^\perp}(Y)$.*

Proof. By Proposition 10, we have that $X \preceq_V Y$ if and only if $X \preceq_{\hat{V}} Y$, where $\hat{V} = \{s_1, \ldots, s_r, -s_1, \ldots, -s_r, \pi_{S^\perp}(v_1), \ldots, \pi_{S^\perp}(v_l)\}$, $\{s_1, \ldots, s_r\}$ being a basis of S. This is the same as $E(f(X)) \leq E(f(Y))$ for all $f \in \mathcal{F}_{\hat{V}}$.

Let $s \in S$, then $-s \in S$. For all $f \in \mathcal{F}_{\hat{V}}$, $\epsilon \geq 0$ and $x \in \mathbb{R}^n$, $f(x) \leq f(x + \epsilon s)$ and $f(x) \leq f(x + \epsilon(-s)) = f(x - \epsilon s)$. As a consequence, any f in $\mathcal{F}_{\hat{V}}$ is constant on the direction of any vector in S. Then, for all $x \in \mathbb{R}^n$, $f(x) = f(\pi_{S^\perp}(x) + \pi_S(x)) = f(\pi_{S^\perp}(x))$.

Now, $E(f(X)) \leq E(f(Y))$ for all $f \in \mathcal{F}_{\hat{V}}$ is equivalent to $E(f(\pi_{S^\perp}(X)) \leq E(f(\pi_{S^\perp}(Y))$ for all $f \in \mathcal{F}_{\hat{V}}$, that is, $\pi_{S^\perp}(X) \preceq_{\hat{V}} \pi_{S^\perp}(Y)$, which is equivalent to $\pi_{S^\perp}(X) \preceq_V \pi_{S^\perp}(Y)$ by Proposition 10. $\square$

Corollary 2. *Let $V = \{v_1, \ldots, v_l\}$ be a subset of $\mathbb{R}^n$ with $\langle e_1, \ldots, e_r \rangle \subseteq C_V$ where $r < n$. Let X and Y be random vectors. Then, $X \preceq_V Y$ if and only if $(0, \ldots, 0, X_{r+1}, \ldots, X_n) \preceq_V (0, \ldots, 0, Y_{r+1}, \ldots, Y_n)$.*

Note that Proposition 9 and Corollary 1 show the behavior of $\preceq_V$ when $(\mathbb{R}^n, \leq_{C_V})$ is a partially ordered set. Proposition 11 ensures that, in the case where the non-stochastic pre-order $\leq_{C_V}$ is not an order, that is, a non-trivial subspace S of $\mathbb{R}^n$ is contained in C_V, the orthogonal projection onto $S^\perp$ characterizes the stochastic order $\preceq_V$. In light of that result, the idea is to consider a maximal subspace S under the above conditions and work "outside" S in order to obtain properties similar to those for the case where $(\mathbb{R}^n, \leq_{C_V})$ is a partially ordered set.

Proposition 12. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$ such that there exists a non-trivial subspace S of $\mathbb{R}^n$ with $S \subseteq C_V$. Then, $X \preceq_V Y$ if and only if $\phi(X) \preceq_{\phi(V)} \phi(Y)$, where $\phi = h \circ \pi_{S^\perp}$ and $h : S^\perp \to \mathbb{R}^{n-dim\,S}$ is any linear bijective mapping.*

Proof. Proposition 10 ensures that $X \preceq_V Y$ if and only if $X \preceq_{\hat{V}} Y$ where $\hat{V} = \{s_1, \ldots, s_r, -s_1, \ldots, -s_r, \pi_{S^\perp}(v_1), \ldots, \pi_{S^\perp}(v_l)\}$, $\{s_1, \ldots, s_r\}$ being a basis of S. Note that $\phi(\hat{V}) = \{0_{\mathbb{R}^{n-dim\,S}}, \phi(v_1), \phi(v_2), \ldots, \phi(v_l)\}$.

Let $f \in \mathcal{F}_{\hat{V}}$. Consider $f \circ h^{-1} : \mathbb{R}^{n-dim\,S} \to \mathbb{R}$. For all $\phi(v_i) \in \phi(\hat{V})$ with $1 \leq i \leq l$, $\epsilon \geq 0$ and $x \in \mathbb{R}^{n-dim\,S}$, $f \circ h^{-1}(x + \epsilon\phi(v_i)) = f(h^{-1}(x) + \epsilon\pi_{S^\perp}(v_i)) \geq f(h^{-1}(x)) = f \circ h^{-1}(x)$. Trivially, for all $\epsilon \geq 0$ and $x \in \mathbb{R}^{n-dim\,S}$, $f \circ h^{-1}(x + \epsilon 0_{\mathbb{R}^{n-dim\,S}}) = f \circ h^{-1}(x)$. Hence, $f \circ h^{-1} \in \mathcal{F}_{\phi(\hat{V})}$.

Let $g \in \mathcal{F}_{\phi(\hat{V})}$. Consider $g \circ \phi : \mathbb{R}^n \to \mathbb{R}$. For all $\epsilon \geq 0$ and $x \in \mathbb{R}^n$, we have that

(1) $g \circ \phi(x + \epsilon\hat{v}) = g(\phi(x) + \epsilon\phi(v_i)) \geq g(\phi(x))$ when $\hat{v} = \pi_{S^\perp}(v_i)$ with $1 \leq i \leq l$,
(2) $g \circ \phi(x + \epsilon\hat{v}) = g(\phi(x))$ when $\hat{v} \in S$.

As a consequence, for all $\hat{v} \in \hat{V}$, $\epsilon \geq 0$ and $x \in \mathbb{R}^n$, $g \circ \phi(x + \epsilon\hat{v}) \geq g(\phi(x)) = g \circ \phi(x)$. Thus, $g \circ \phi \in \mathcal{F}_{\hat{V}}$. Moreover, $(g \circ \phi)_{|_{S^\perp}} = (g \circ h)_{|_{S^\perp}}$ when we apply these maps to vectors in $S^\perp$.

Now, by Proposition 11, $X \preceq_{\hat{V}} Y$ if and only if $E(f(\pi_{S^\perp}(X)) \leq E(f(\pi_{S^\perp}(Y))$ for all $f \in \mathcal{F}_{\hat{V}}$, which is $E(f \circ h^{-1}(\phi(X)) \leq E(f \circ h^{-1}(\phi(Y))$ for all $f \in \mathcal{F}_{\hat{V}}$. Note that the previous condition involves vectors in $S^\perp$. As a consequence, $X \preceq_{\hat{V}} Y$ if and only if $\phi(X) \preceq_{\phi(\hat{V})} \phi(Y)$. We have that $\phi(\hat{V}) = \{0_{\mathbb{R}^{n-\dim S}}, \phi(v_1), \phi(v_2), \ldots, \phi(v_l)\}$, and $\phi(V) = \{\phi(v_1), \phi(v_2), \ldots, \phi(v_l)\}$, with $0_{\mathbb{R}^{n-\dim S}} \in C_{\phi(V)}$. Now, Proposition 5 ensures the result. □

5. Properties of Directional Stochastic Orders

The main properties of directional stochastic orders are analyzed in this section.

Proposition 13. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Then, $X \preceq_V Y$ implies $E(X) \leq E(Y)$, if and only if, $v_i \geq 0_{\mathbb{R}^n}$ for all $1 \leq i \leq l$.*

Proof. Suppose that $v_i \geq 0_{\mathbb{R}^n}$ for all $1 \leq i \leq l$. Let S_j be the span of $\{e_j\}$, where $1 \leq j \leq n$. For all $v_i \in V$, $\epsilon \geq 0$ and $x \in \mathbb{R}^n$, $\pi_{S_j}(x + \epsilon v_i) \geq \pi_{S_j}(x)$ because $v_i \geq 0_{\mathbb{R}^n}$ for all $1 \leq i \leq l$. Then, $\pi_{S_j} \in \mathcal{F}_V$ for all $1 \leq j \leq n$. Therefore, $X \preceq_V Y$ implies that $E(\pi_{S_j}(X)) \leq E(\pi_{S_j}(Y))$ for all $1 \leq j \leq n$, so $E(X) \leq E(Y)$.

Conversely, suppose that for all random vectors X and Y such that $X \preceq_V Y$, $E(X) \leq E(Y)$ holds. Note that for every $f \in \mathcal{F}_V$, $f(0_{\mathbb{R}^n} + 1v_i) \geq f(0_{\mathbb{R}^n})$ for all $1 \leq i \leq l$. Then, $\delta_{0_{\mathbb{R}^n}} \preceq_V \delta_{v_i}$ for all $1 \leq i \leq l$. As a consequence, $0_{\mathbb{R}^n} = E(\delta_{0_{\mathbb{R}^n}}) \leq E(\delta_{v_i}) = v_i$ for all $1 \leq i \leq l$. □

Proposition 14. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Let X and Y be random vectors such that $X \preceq_V Y$. Then, $\alpha X \preceq_V \alpha Y$ for all scalars $\alpha \geq 0$.*

Proof. Let $\alpha \geq 0$ and $f \in \mathcal{F}_V$. Consider $g : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, $g(x) = f(\alpha x)$. For all $x \in \mathbb{R}^n$, $v_i \in V$ and $\epsilon \geq 0$, $g(x + \epsilon v_i) = f(\alpha x + \alpha\epsilon v_i) \geq f(\alpha x) = g(x)$. Then $g \in \mathcal{F}_V$, which proves the result. □

Proposition 15. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Then, $\preceq_V$ is closed under mixtures.*

Proof. Note that $\preceq_V$ is an integral order. □

Proposition 16. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. Then, $\preceq_V$ is closed under convolution.*

Proof. Note that for all $z \in \mathbb{R}^n$ and any $f \in \mathcal{F}_V$, the mapping $g : \mathbb{R}^n \to \mathbb{R}$, with $g(x) = f(x + z)$ for all $x \in \mathbb{R}^n$, is in $\mathcal{F}_V$. □

The behavior of directional stochastic orders under weak convergence is analyzed now.

Proposition 17. *Let $V = \{v_1, \ldots, v_l\}$ be a set of vectors in $\mathbb{R}^n$. The stochastic order $\preceq_V$ is closed under weak convergence.*

Proof. Firstly, consider the case where $V = \{v_1, \ldots, v_l\}$ is a set of vectors in $\mathbb{R}^n$ such that C_V does not contain non-trivial vector subspaces. By Corollary 1, there exits a generator of $\preceq_V$ of bounded continuous functions. Thus, $\preceq_V$ is closed under weak convergence.

In case of non-trivial vector subspaces in C_V, let $S = C_V \cap (-C_V)$. Note that $S \neq \{0_{\mathbb{R}^n}\}$. It is not hard to prove that S is a subspace of $\mathbb{R}^n$ contained in C_V such that any other subspace in C_V is contained in S. By Proposition 12, $X \preceq_V Y$ if and only if $\phi(X) \preceq_{\phi(V)} \phi(Y)$, where $\phi = h \circ \pi_{S^\perp}$ and h is any linear bijective map from $S^\perp$ to $\mathbb{R}^{n-\dim S}$. As a result of the maximality of S, $C_{\phi(V)}$ does not contain non-trivial vector subspaces of $\mathbb{R}^{n-dim\,S}$. Therefore, $\preceq_{\phi(V)}$ is closed under weak convergence. Since ϕ is continuous, by Proposition 12, $\preceq_V$ is also closed under weak convergence. $\square$

6. An Application of Directional Stochastic Orders

An application of directional stochastic orders to financial mathematics is developed in this section.

Consider the comparison of two investments which provide n random cash flows. Let $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ be the corresponding random vectors of cash flows, where X_i and Y_i denote the cash flow at instant t_i of the corresponding investment, with $1 \leq i \leq n$, and $t_1 < t_2 < \ldots < t_n$.

Assume an economic context with negative rates of interest or under negative inflation rates, when financial institutions and companies should pay some rates for the money custody. In that case, a company could prefer to receive earnings streams or cash flows later on. Very basically, the later the flows arrive, the lower the rates the company pays.

Under this framework, a company would prefer to "transfer" the first cash flow to the second cash flow, the second flow to the third flow, and so on. Thus, the comparison of investments could be performed by means of the comparison of the expected benefits of mappings which are increasing in the direction of $e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}$, which reflect that transfer preference. Of course those mappings should be increasing in the direction of e_1, that is, the greater the first flow is, the better the result of the investment.

Let $V = \{e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}\}$. Consider the mappings of $\mathcal{F}_V$. Such functions reflect the preferences of the companies under the considered economic framework.

For instance, in a 2-years investment with flows at the end of each year, the vector of flows $(2, 10)$ is preferred to the vector of flows $(5, 7)$. Note that for any $f \in \mathcal{F}_V$, it holds that $f(5, 7) \leq f(2, 10)$ since $(2, 10) = (5, 7) + 3(e_2 - e_1)$.

Observe that the mappings of $\mathcal{F}_V$ are also increasing in the directions of e_i, with $2 \leq i \leq n$, since $e_i \in C_V$, and so, in accordance with Proposition 4, it holds that $\mathcal{F}_V = \mathcal{F}_{V \cup \{e_i\}}$. From an applied point of view, this means that the greater the flows are, the greater the profit of the investment.

The following criterion can be introduced to compare investments under the above economic situation. The investment associated with the random vector Y is preferred to the investment of random vector X, if $E(f(X)) \leq E(f(Y))$ for any mapping $f \in \mathcal{F}_V$, that is, when $X \preceq_V Y$.

The unified study of directional stochastic orders developed in this manuscript permits to derive properties of the proposed order immediately.

(i) According to Proposition 7, the directional stochastic order $\preceq_V$ is order-isomorphic to the usual stochastic order. The proof of that proposition provides that the order-isomorphism is given by the bijective linear map $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(e_1) = e_1$ and $h(e_{i+1} - e_i) = e_{i+1}$, $1 \leq i \leq n-1$. Note that h is bijective since V is a basis of $\mathbb{R}^n$. Proposition 7 ensures that $X \preceq_V Y$, that is, $P_X \preceq_V P_Y$, if and only if, $\widetilde{h}(P_X) = P_X \circ h^{-1} \preceq_{st} \widetilde{h}(P_Y) = P_Y \circ h^{-1}$, equivalently, $h(X) \preceq_{st} h(Y)$. That is the same as $AX \preceq_{st} AY$, where A is an $n \times n$ real matrix such that $(A)_{ij} = 1$ if $j \geq i$, otherwise 0. Hence, $X \preceq_V Y$ if and only if

$$\begin{aligned} &(X_1 + X_2 + \ldots + X_n, X_2 + X_3 + \ldots + X_n, \ldots, X_{n-1} + X_n, X_n) \\ &\preceq_{st} (Y_1 + Y_2 + \ldots + Y_n, Y_2 + Y_3 + \ldots + Y_n, \ldots, Y_{n-1} + Y_n, Y_n). \end{aligned}$$

It is interesting to note the importance of that characterization of the order $\preceq_V$ by means of the usual multivariate stochastic order $\preceq_{st}$, since there are statistical tests to infer on the order $\preceq_{st}$ in statistical literature (see, for instance, [18]). This permits to apply inferential procedures to tests on the order $\preceq_V$.
That is, the stochastic order $\preceq_V$ is easily characterized by means of the well-known usual stochastic order.

(ii) Since C_V does not contain non-trivial vector subspaces, Corollary 1 provides important characterizations of the order $\preceq_V$, like that based on the construction of random vectors on the same probability space, namely, $X \preceq_V Y$ if and only if there are random vectors X' and Y' on a same probability space, with the probability distributions of X and Y, respectively, such that $X' \leq_{C_V} Y'$ almost surely, that is, $Y' - X' \in \{\sum_{i=1}^{n} \alpha_i(e_i - e_{i-1}) \mid \alpha_i \geq 0, e_0 = 0_{\mathbb{R}^n}\}$ almost surely.

(iii) Since $\{e_1, \ldots, e_n\} \subset C_V$, Proposition 6 assures that the usual stochastic order $\preceq_{st}$ implies the order $\preceq_V$.

(iv) By the results of Section 5, we obtain that the order is not closed with respect to the formation of expectations, but is closed under the product by positive scalars, under mixtures, under convolution, and under weak convergence.

(v) The above-mentioned order-isomorphism simplifies the analysis of the directional stochastic order with normal random vectors.
Let $X \sim N(\mu_X, \Sigma_X)$ and $Y \sim N(\mu_Y, \Sigma_Y)$. Observe that $X \preceq_V Y$ is the same as $h(X) \preceq_{st} h(Y)$, and this is equivalent to $AX \preceq_{st} AY$. Note that $AX \sim N(A\mu_X, A\Sigma_X A^t)$ and $AY \sim N(A\mu_Y, A\Sigma_Y A^t)$. Sufficient and necessary conditions to order normal distributions in the usual multivariate stochastic order can be found, for instance, in Theorem 3.3.13 in [4]. Thus, $X \preceq_V Y$ if and only if $A\Sigma_X A^t = A\Sigma_Y A^t$, and $(A\mu_X)_i \leq (A\mu_Y)_i$ for any $1 \leq i \leq n$.
Note that $(A\mu_X)_i \leq (A\mu_Y)_i$ for any $1 \leq i \leq n$, is the same as $\mu_{X_i} + \mu_{X_{i+1}} + \cdots + \mu_{X_n} \leq \mu_{Y_i} + \mu_{Y_{i+1}} + \cdots + \mu_{Y_n}$ for any $1 \leq i \leq n$, and $A\Sigma_X A^t = A\Sigma_Y A^t$ implies $\Sigma_X = \Sigma_Y$ since A is regular.
Then, $X \preceq_V Y$ if and only if $\mu_{X_i} + \mu_{X_{i+1}} + \cdots + \mu_{X_n} \leq \mu_{Y_i} + \mu_{Y_{i+1}} + \cdots + \mu_{Y_n}$ for any $1 \leq i \leq n$, and $\Sigma_X = \Sigma_Y$.
Thus, the comparison of investments in our framework when random cash flows follow normal distribution, is reduced to the simple comparison of the mean vectors and matrix covariances.

(vi) Consider the case of t multivariate distributions. Let X and Y be t random vectors with freedom degrees ν_X and ν_Y, mean vectors μ_X and μ_Y, and matrices parameter Σ_X and Σ_Y, respectively.
We know that $X \preceq_V Y$ if and only if $h(X) \preceq_{st} h(Y)$, equivalently $AX \preceq_{st} AY$.
It is known that AX and AY have t distribution with freedom degrees ν_X and ν_Y, mean vectors $A\mu_X$ and $A\mu_Y$, and matrices parameter $A\Sigma_X A^t$ and $A\Sigma_Y A^t$, respectively. Now, by Proposition 3.26 in [6], $X \preceq_V Y$ if and only if $\nu_X = \nu_Y$, $(A\mu_X)_i \leq (A\mu_Y)_i$ for any $1 \leq i \leq n$, and $A\Sigma_X A^t = A\Sigma_Y A^t$.
As a consequence, $X \preceq_V Y$ if and only if $\nu_X = \nu_Y$, $\mu_{X_i} + \mu_{X_{i+1}} + \cdots + \mu_{X_n} \leq \mu_{Y_i} + \mu_{Y_{i+1}} + \cdots + \mu_{Y_n}$ for any $1 \leq i \leq n$, and $\Sigma_X = \Sigma_Y$.
Hence, the comparison of investments with cash flows following t distribution, can be performed by the comparison of the mean vectors and matrix covariances.
Observe that this reasoning could be applied to other multivariate distributions.

To conclude, it is interesting to note that the generator of the order $\preceq_V$ satisfies desirable properties for different scenarios which could happen. For instance, suppose that during the period of the investment, there is a risk of non-payment of flows, risk which decreases over time.

In such a case, the random cash flows $X_1, X_2, \ldots, X_n$ are replaced by $p_1X_1, p_2X_2, \ldots, p_nX_n$, respectively, where p_i is the probability of receiving the cash flow X_i, $1 \leq i \leq n$. Note that $p_1 \leq p_2 \leq \ldots \leq p_n$ since the risk is decreasing. It is not hard to see that if $f \in \mathcal{F}_V$, the mapping $f_p : \mathbb{R} \to \mathbb{R}$, with $f_p(x_1, x_2, \ldots, x_n) = f(p_1x_1, p_2x_2, \ldots, p_nx_n)$ for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, is also in $\mathcal{F}_V$, and so, the comparison of investments is unaffected by the new scenario.

7. Conclusions

In this manuscript, we have introduced a unified approach of those integral stochastic orders whose generators are given by mappings which are increasing on the directions of the vectors of a finite set of $\mathbb{R}^n$. Those stochastic orders have been called directional stochastic orders. Several characterizations of directional stochastic orders are provided in the manuscript, like those based on geometrical arguments, or on non-stochastic pre-orders and their preserving mappings. The results of the manuscript have been illustrated with an application to financial mathematics.

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