Axiomatization and construction of orness measures for aggregation functions

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The notion of an orness measure for aggregation functions has been a relevant study subject whose history can be traced back to the early works of Dujmović in 1973. Intuitively, an orness measure quantifies the similarity of an aggregation function to the 'or' function and results in an essential tool for decision engineering, field in which the choice of aggregation function is sometimes restricted to a desired value of orness (orness-directed aggregation). In 1988, Yager presented a particular example of orness measure for OWA functions and initiated a series of contributions aiming at proposing an axiomatic definition of orness measure for OWA functions. In this paper, we go much further and present an axiomatic definition of orness measure for the whole family of aggregation functions. We end by proposing two natural construction methods for an orness measure for aggregation functions. The particular examples of the (discrete) Choquet integral and uninorms are studied in detail.

1 Introduction

Nowadays, intelligent systems need to deal with an increasing amount of information coming from many different digital sources. For this very reason, information aggregation is naturally arising more frequently than ever [1] and the study of aggregation functions [2, 3], which are the tools that serve to perform this information aggregation, is becoming a very active field of research. Although there exist some other classes of aggregation functions [4], there is little doubt that averaging aggregation functions [5] — often referred to as means [6, 7] – are the most prominent family of aggregation functions and its study can even be traced back to Cauchy [8]. Actually, some averaging aggregation functions such as the median and the (weighted) arithmetic mean are known to have been used much earlier and have become essential tools in the field of descriptive statistics. One of the most relevant families of averaging aggregation functions is that of Ordered Weighted Averaging aggregation functions, OWA functions for short, that were described for the first time by Yager [9] and currently arise in many different fields of application (e.g., modelling customized individual semantics [10] and movie ratings aggregation [11]).
According to the historical account presented in [12], the notion of orness was first introduced in the context of graded logic by Dujmović [13] (and further developed in [14, 15, 16]) for measuring the degree of disjunction of an aggregation function. Independently, Yager [9] proposed a specific orness measure for the family of OWA functions that has become a standard nowadays for measuring how similar an OWA function is to the 'or' function (which is the greatest possible OWA function). Interestingly, Fernández Salido and Murakami [17] later proved that Yager's proposal is equivalent to Dujmović's proposal when the latter is restricted to measuring the orness of an OWA function.
Recent work on the topic has followed different directions. On the one hand, some authors have studied orness-directedness aggregation [18], which is the process of selecting the most appropriate aggregation function given a desirable value of orness. This process has been of interest to practitioners even when restricting to a specific family of aggregation functions. For instance, Fullér and Majlender [19] provided an analytic solution for identifying the OWA function with maximal entropy weights for a given value of

orness (a problem that was initially brought to attention in the context of real-time expert systems by O'Hagan [20]). On the other hand, other authors have studied the notion of orness for specific families of functions (e.g., quasi-arithmetic means [21] or lattice OWA functions [22, 23, 24, 25, 26]).
The present work has a twofold goal. Firstly, we aim at providing an axiomatic definition of orness measure for aggregation functions that formalizes in a mathematical manner a notion that has been studied for decades. This line of research is similar to that in [27], where the notion of orness measure for OWA functions is axiomatized. Secondly, we aim at proposing different construction methods for orness measures, thus providing different examples of functions fulfilling the newly-presented axiomatic definition. This line of research is similar to that in [28], where a new orness measure for aggregation functions generalizing Yager's proposal for OWA functions is presented. We pay particular attention to orness measures for the (discrete) Choquet integral and uninorms.
The remainder of the paper is structured as follows. Section 2 presents the basic notions related to aggregation functions and OWA functions. Section 3 is devoted to the axiomatization of an orness measure for OWA functions. Section 4 follows a similar structure, providing an axiomatization of an orness measure for aggregation functions. Obviously, this can be understood as a natural generalization of the axiomatization of an orness measure for OWA functions. In Section 5, two construction methods for an orness measure for aggregation functions are presented. We end with some conclusions in Section 6.

2 Preliminaries

In this section, we recall some preliminary concepts related to aggregation functions [5] and OWA functions [9].

2.1 Aggregation functions

Aggregation functions [5] are tools that allow to combine n values into a single one. Since the study of aggregation functions stemmed from the field of fuzzy logic, these values are often assumed to belong to the unit interval. It is immediate to extend the definition from the unit interval to another compact real interval.

Definition 1. *Consider $n \in \mathbb{N}$. A function $f : [0,1]^n \to [0,1]$ (with $n \geq 2$) is called an aggregation function if*

- *it is increasing, i.e., for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0,1]^n$ such that $x_i \leq y_i$ for any $i \in \{1, \ldots, n\}$, it holds that $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$.*
- *it satisfies the boundary conditions, i.e., $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$.*

Definition 2. *The dual of an aggregation function f is the aggregation function $\overline{f}$ defined as $\overline{f}(x_1, \ldots, x_n) = 1 - f(1 - x_1, \ldots, 1 - x_n)$ for any $(x_1, \ldots, x_n) \in [0,1]^n$.*

We denote by $\mathbb{F}_n$ the set of all aggregation functions $f : [0,1]^n \to [0,1]$. Two relations on $\mathbb{F}_n$ will be used throughout the paper.

Definition 3. *Consider $f, g \in \mathbb{F}_n$. The aggregation function f is said to be smaller than or equal to the aggregation function g, denoted by $f \leq g$, if,*

$$f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n) \, .$$

for any $(x_1, \ldots, x_n) \in [0,1]^n$.

Definition 4. *Consider $f, g \in \mathbb{F}_n$. The aggregation function f is said to be almost surely smaller than or equal to the aggregation function g, denoted by $f \leq_* g$, if it holds that*

$$\mu(\{(x_1, \ldots, x_n) \in [0,1]^n \mid f(x_1, \ldots, x_n) > g(x_1, \ldots, x_n)\}) = 0 \, ,$$

where μ denotes the Lebesgue measure.

Obviously, $\leq \subseteq \leq_*$. Also, note that $\leq$ is an order relation on $\mathbb{F}_n$, whereas $\leq_*$ is a preorder relation on $\mathbb{F}_n$. Therefore, the relation $=_*$ defined by $\leq_* \cap \leq_*^T$ (where $\leq_*^T$ denotes the transpose of $\leq_*$) is an equivalence relation.
A prominent family of aggregation functions is that of averaging aggregation functions [5].

Definition 5. *Consider $n \in \mathbb{N}$. An aggregation function $f : [0,1]^n \to [0,1]$ is called averaging if it is bounded by the minimum and the maximum of its arguments, i.e., it holds that:*

$$\min \leq f \leq \max .$$

Averaging aggregation functions are related to the property of idempotence.

Definition 6. *Consider $n \in \mathbb{N}$. A function $f : [0,1]^n \to [0,1]$ is said to be idempotent if, for any $x \in [0,1]$, it holds that:*

$$f(x, \ldots, x) = x .$$

Note that, since the minimum and maximum functions are idempotent, every averaging aggregation function is idempotent.

2.2 OWA functions

An important family of averaging aggregation functions is the family of the so-called OWA functions [9]. An OWA function is associated with a weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$ such that $w_i \in [0,1]$ for any $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n w_i = 1$. The set of all possible weighting vectors (of dimension n), oftentimes referred to as the (standard) simplex, is denoted by $\mathbb{O}_n$.

Definition 7. *Consider $n \in \mathbb{N}$. Given a weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$, the function $f_{\mathbf{w}} : [0,1]^n \to [0,1]$*

$$f_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i \, x_{(i)} ,$$

where $x_{(i)}$ denotes the i-th largest element among $x_1, \ldots, x_n$, is called the OWA function associated with $\mathbf{w}$.

Note that an OWA function characterizes and is characterized by a unique weighting vector [9]. Therefore, there exists a one-to-one correspondence between the set of all possible weighting vectors and the set of all possible OWA functions. For this very reason, we also use the notation $\mathbb{O}_n$ for referring to the set of all OWA functions.

Remark 1. *Some well-known OWA functions are associated with the following weighting vectors:*

- *The maximum ('or'): $\mathbf{w}_1 = (1, 0, \ldots, 0)$.*
- *The minimum ('and'): $\mathbf{w}_0 = (0, \ldots, 0, 1)$.*
- *The arithmetic mean: $\mathbf{w}_A = (\frac{1}{n}, \ldots, \frac{1}{n})$.*
- *The median: if n is odd, $\mathbf{w}_M$ is defined by $w_i = 0$ for any $i \neq \frac{n+1}{2}$ and $w_{\frac{n+1}{2}} = 1$; if n is even, $\mathbf{w}_M$ is defined by $w_i = 0$ for any $i \notin \{\frac{n}{2}, \frac{n}{2}+1\}$ and $w_{\frac{n}{2}} = w_{\frac{n}{2}+1} = \frac{1}{2}$.*

Note that the median is not always defined as the arithmetic mean of the two middle points in case n is even, but, oftentimes, all functions for which $w_{\frac{n}{2}} + w_{\frac{n}{2}+1} = 1$ are said to be medians. However, this latter definition is not considered throughout this paper in order to guarantee that the function uniquely defines a point.

A weighting vector $\mathbf{w}^d$ is called the dual of another weighting vector $\mathbf{w}$, if $\mathbf{w}^d$ is obtained by reversing $\mathbf{w}$, i.e., if, $w_i^d = w_{n-i+1}$ for any $i \in \{1, \ldots, n\}$. An OWA function $f_{\mathbf{w}'}$ is called the dual of the OWA function $f_{\mathbf{w}}$ (or, simply, $f_{\mathbf{w}'}$ and $f_{\mathbf{w}}$ are called duals) if $\mathbf{w}' = \mathbf{w}^d$. Note that the notion of duality for the weighting vector of an OWA function coincides with the more general notion of duality for aggregation functions when restricted to OWA functions. As an example, the maximum and the minimum are duals. The dual of the arithmetic mean is the arithmetic mean itself. Similarly, the dual of the median is the median itself.

3 Orness for OWA functions

The orness of an OWA function $f_{\mathbf{w}}$ (as defined by Yager [9]), denoted by $\mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}})$, measures how close $f_{\mathbf{w}}$ is to being the 'or' (the maximum) function.

Definition 8. *Consider $n \in \mathbb{N}$. Yagers's* **orness measure** *is the function $\mathcal{O}_{\mathcal{Y}} : \mathbb{O}_n \to [0, 1]$ defined as:*

$$\mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}}) = \frac{1}{n-1} \sum_{i=1}^{n} w_i \, (n-i) \, . \tag{1}$$

The orness of the 'or' function is equal to one and is maximum among all OWA functions, whereas the orness of the 'and' function is equal to zero and is minimum among all OWA functions. The orness of the arithmetic mean and the median is $\frac{1}{2}$.
Kishor *et al.* [27] further studied Yager's orness measure and proposed a more general axiomatic definition.

Definition 9. *Consider $n \in \mathbb{N}$. A $\mathcal{K}$-orness measure for OWA functions is a function $\mathcal{O}_{\mathcal{K}} : \mathbb{O}_n \to [0, 1]$ such that:*

(K1) $\mathcal{O}_{\mathcal{K}}(f_{\mathbf{w}}) = 1$ *if and only if* $\mathbf{w} = \mathbf{w}_1$.

(K2) $\mathcal{O}_{\mathcal{K}}(f_{\mathbf{w}}) = 0$ *if and only if* $\mathbf{w} = \mathbf{w}_0$.

(K3) If $\mathbf{w} = (w_1, \ldots, w_n)$ *and* $\mathbf{w}' = (w_1, \ldots, w_j - \epsilon, \ldots, w_k + \epsilon, \ldots, w_n)$ *with* $\epsilon > 0$ *and* $j < k$*, then* $\mathcal{O}_{\mathcal{K}}(f_{\mathbf{w}'}) < \mathcal{O}_{\mathcal{K}}(f_{\mathbf{w}})$.

(K4) $\mathcal{O}_{\mathcal{K}}(f_{\mathbf{w_A}}) = \frac{1}{2}$.

It is immediate to see that Yager's orness measure fulfils the previous axioms and, therefore, it is an example of $\mathcal{K}$-orness measure for OWA functions. Different orness measures fulfilling the axioms of Definition 9 were proposed in [27].
Note that the somehow-cumbersome notation of Axiom (K3) could be expressed in terms of the majorization order relation [29]. The majorization order relation $\leq_{\mathrm{D}}$ is defined by $\mathbf{w} \leq_{\mathrm{D}} \mathbf{w}'$ if for any $k \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{k} w_i \leq \sum_{i=1}^{k} w_i' \, .$$

This order was already mentioned by Yager in [9] and used for defining an intuitive property of his orness measure: if a weighting vector is greater than or equal to another one, then the orness of the former should be greater than or equal to the orness of the latter.
We propose here a slight modification of the orness measure proposed by Kishor *et al.* [27]. The rationale for introducing this modification is to further extend the dual role played by the 'or' and 'and' functions and reflected by Axiom (K4) to any pair of dual OWA functions. Actually, the special role played by the orness value 0.5, which represents that an OWA function is equally close to being the 'or' and the 'and' functions, was already pointed out by Kishor et al. in [27] for the arithmetic mean (Axiom (K4)). This could be extended to any OWA function with a symmetric weighting vector (as in the case of the arithmetic mean), for which the orness should be equal to 0.5.

Definition 10. *Consider $n \in \mathbb{N}$. A $\mathcal{P}$-orness measure for OWA functions is a function $\mathcal{O}_{\mathcal{P}} : \mathbb{O}_n \to [0,1]$ such that:*

(P1) $\mathcal{O}_{\mathcal{P}}(f_{\mathbf{w}}) = 1$ *if and only if* $\mathbf{w} = \mathbf{w}_1$.

(P2) $\mathcal{O}_{\mathcal{P}}(f_{\mathbf{w}}) \leq \mathcal{O}_{\mathcal{P}}(f_{\mathbf{w}'})$ *if* $\mathbf{w} \leq_D \mathbf{w}'$.

(P3) $\mathcal{O}_{\mathcal{P}}(f_{\mathbf{w}^d}) = 1 - \mathcal{O}_{\mathcal{P}}(f_{\mathbf{w}})$.

Remark 2. *One should note that Yager's orness measure satisfies all axioms above. It is immediate to see that there exist other functions different from that by Yager satisfying the axioms above, e.g., the function $\mathcal{O}^* : \mathbb{O}_n \to [0,1]$ defined by $\mathcal{O}^*(f_{\mathbf{w}_1}) = \mathcal{O}^*(\max) = 1$, $\mathcal{O}^*(f_{\mathbf{w}_0}) = \mathcal{O}^*(\min) = 0$ and $\mathcal{O}^*(f_{\mathbf{w}}) = \frac{1}{2}$ otherwise.*

It is immediate to prove that (P1) is exactly the same as (K1), (P2) is equivalent to (K3) and (K2) is a consequence of (P1) and (P3). Moreover, (P3) implies (K4), even though the converse is not true. Thus, the family of $\mathcal{P}$-orness measures is contained in the family of $\mathcal{K}$-orness measures for which (K4) is further restricted to (P3). From now on, we will restrict our attention only to $\mathcal{P}$-orness measures for OWA functions and we will just refer to them as orness measures for OWA functions.

4 A generalization of the notion of orness to aggregation functions

Inspired by the study of orness for OWA functions, we aim at studying and measuring the orness of any aggregation function. For an OWA function, an orness measure quantifies how close the OWA function is to being equal to the 'or' function. Since the 'or' function is the greatest possible OWA function, this is equivalent to quantifying how close the OWA function is to being greater than or equal to the 'or' function. When we move to the setting of aggregation functions, as long as the aggregation function is not idempotent, both "being equal to the 'or' function" and "being greater than or equal to the 'or' function" are no longer equivalent. We consider the latter understanding for quantifying the orness of an aggregation function.
In the following, we present four intuitive conditions that will be later on used for defining in an axiomatic manner the notion of an orness measure for aggregation functions. Firstly, any intuitive orness measure should assign the greatest possible value of orness to all possible functions that are greater than or equal to the 'or' function (and, in particular, to the 'or' function). These functions (and functions that are equal to them up to a set of Lebesgue measure zero) will be the only ones attaining the greatest possible value of orness. Secondly, it seems natural to require that the greater the aggregation function is, the greater its orness should be. Thirdly, since an aggregation function is as close to being the 'or' function as its dual is to being the 'and' function, it is logical to require the orness of an aggregation function to be related to the orness of its dual, as was represented in Axiom (P3). Fourthly, OWA functions are invariant under variable permutations (i.e., symmetry). Even if we consider an aggregation function that is no longer assured to be symmetric, it still seems natural for an orness measure not to depend on the order of the inputs.
Considering the properties described above, we establish the following axiomatization of an orness measure for aggregation functions. As usual, an aggregation function with a value of orness smaller than or equal to 0.5 will be called a conjunctive aggregation function, whereas an aggregation function with a value of orness greater than or equal to 0.5 will be referred to as a disjunctive aggregation function. In case the extreme values 0 and 1 are reached by an aggregation function different than the minimum or the maximum, the terms hyperconjunctive aggregation function and hyperdisjunctive aggregation function are respectively used. Some authors, see e.g. [30], argue that the orness of a hyperconjunctive aggregation function should be smaller than 0 and that the orness of a hyperdisjunctive aggregation function should be greater than 1, but this point of view is here abandoned since the value of orness is seen as a degree of disjunction (thus a value between 0 and 1).

Definition 11. *Consider $n \in \mathbb{N}$. An* **orness measure** *for aggregation functions is a function $\mathcal{O} : \mathbb{F}_n \longrightarrow [0,1]$ such that:*

(O1) $\mathcal{O}(f) = 1$ *if and only if* $\max \leq_* f$.

(O2) *If* $f \leq g$, *then* $\mathcal{O}(f) \leq \mathcal{O}(g)$.

(O3) $\mathcal{O}(\overline{f}) = 1 - \mathcal{O}(f)$.

(O4) *If there exists a permutation* σ *of* $\{1, \ldots, n\}$ *such that* $f(x_1, \ldots, x_n) = g(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ *for any* $(x_1, \ldots, x_n) \in [0,1]^n$, *then* $\mathcal{O}(f) = \mathcal{O}(g)$.

Note that, similarly to Axiom (O1), the orness of any function smaller than the minimum (up to a set of measure zero) is uniquely determined.

Proposition 1. *Consider* $n \in \mathbb{N}$. *Let* $f : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function and* $\mathcal{O} : \mathbb{F}_n \longrightarrow [0,1]$ *be an orness measure for aggregation functions. It holds that* $\mathcal{O}(f) = 0$ *if and only if* $f \leq_* \min$.

Proof. From Axiom (O3) we have that $\mathcal{O}(f) = 0$ iff $\mathcal{O}(\overline{f}) = 1$. From Axiom (O1), this is equivalent to $\max \leq_* \overline{f}$, which is equivalent to

$$\mu\left(\{(x_1, \ldots, x_n) \in [0,1]^n \,|\, \max(1 - x_1, \ldots, 1 - x_n) > \overline{f}(1 - x_1, \ldots, 1 - x_n)\}\right) = 0\,.$$

Since $\overline{f}(1 - x_1, \ldots, 1 - x_n) = 1 - f(x_1, \ldots, x_n)$ and $1 - \min(x_1, \ldots, x_n) = \max(1 - x_1, \ldots, 1 - x_n)$, the above is equivalent to

$$\mu\big(\big\{(x_1, \ldots, x_n) \in [0,1]^n | 1 - \min(x_1, \ldots, x_n) > 1 - f(x_1, \ldots, x_n)\big\}\big) = 0\,,$$

or, equivalently,

$$\mu\big(\big\{(x_1, \ldots, x_n) \in [0,1]^n | f(x_1, \ldots, x_n) > \min(x_1, \ldots, x_n)\big)\big) = 0\,.$$

□

In the following we prove that an orness measure for aggregation functions in the sense of Definition 11 restricts to an orness measure for OWA functions in the sense of Definition 10 when the aggregation function is an OWA function.

Theorem 1. *Consider* $n \in \mathbb{N}$. *Let* $\mathcal{O} : \mathbb{F}_n \longrightarrow [0,1]$ *be an orness measure for aggregation functions (in the sense of Definition 11). The function* $\mathcal{O}_{|\mathbb{O}_n} : \mathbb{O}_n \longrightarrow [0,1]$, *defined as the restriction of* $\mathcal{O}$ *to* $\mathbb{O}_n$, *is an orness measure for OWA functions (in the sense of Definition 10).*

Proof. (P1) For any OWA function $f_{\mathbf{w}}$, it holds that $\mathbf{w} = \mathbf{w}_1$ iff

$$\mu(\big\{(x_1, \ldots, x_n) \in [0,1]^n | f_{\mathbf{w}}(x_1, \ldots, x_n) < \max(x_1, \ldots, x_n)\big\}) = 0\,.$$

The result then follows straightforwardly from Axiom (O1).

(P2) If $f_{\mathbf{w}}$ and $g_{\mathbf{w}'}$ are two OWA functions with associated weighting vectors $\mathbf{w}$ and $\mathbf{w}'$, then it holds that $f_{\mathbf{w}} \leq g_{\mathbf{w}'}$ if and only if $\mathbf{w} \leq_{\mathrm{D}} \mathbf{w}'$. The result then follows straightforwardly from Axiom (O2).

(P3) Consider $f_{\mathbf{w}}$ an OWA with weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$. As mentioned in the Preliminaries, $\overline{f}_{\mathbf{w}} = f_{\mathbf{w}^d}$. The result then follows straightforwardly from Axiom (O3).

□

As an illustrative example, we end the section by identifying the value of the orness measure for some relevant aggregation functions.

- Any t-conorm S is such that $S(x_1, \ldots, x_n) \geq \max(x_1, \ldots, x_n)$, for any $(x_1, \ldots, x_n) \in [0,1]^n$. Therefore, as a consequence of Axiom (O1), it holds that $\mathcal{O}(S) = 1$.
- Any t-norm T is such that $T(x_1, \ldots, x_n) \leq \min(x_1, \ldots, x_n)$, for any $(x_1, \ldots, x_n) \in [0,1]^n$. Therefore, as a consequence of Proposition 1, it holds that $\mathcal{O}(T) = 0$.
- The dual of the arithmetic mean is the arithmetic mean itself and the dual of the median is the median itself. Therefore, as a consequence of Axiom (O3), it holds that $\mathcal{O}(f_{\mathbf{w}_A}) = \mathcal{O}(f_{\mathbf{w}_M}) = 0.5$.

5 Construction of orness measures for aggregation functions

As discussed by Dujmović [31] (see Conclusions), there exist different orness measures and decision makers can be trained to use any of them. Even though some specific orness measures might be preferred, all orness measures are potentially useful. In this section we develop two construction methods for orness measures for aggregation functions. The particular examples of the (discrete) Choquet integral and uninorms are studied in detail.

5.1 First construction method

5.1.1 General results

The rectified function of an aggregation function f is the 'closest' averaging aggregation function to f. In particular, if f is an averaging aggregation function, the rectified function of f is f itself.

Definition 12. *Consider $n \in \mathbb{N}$. Let $f : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The* **rectified function** *of f is the function $f_r : [0,1]^n \longrightarrow [0,1]$ defined as:*

$$f_r(x_1,\ldots,x_n) = \begin{cases} \min(x_1,\ldots,x_n) & \text{if } f(x_1,\ldots,x_n) < \min(x_1,\ldots,x_n)\,, \\ f(x_1,\ldots,x_n) & \text{if } \min(x_1,\ldots,x_n) \le f(x_1,\ldots,x_n) \le \max(x_1,\ldots,x_n)\,, \\ \max(x_1,\ldots,x_n) & \text{if } \max(x_1,\ldots,x_n) < f(x_1,\ldots,x_n)\,. \end{cases}$$

Obviously, for any function $f \in \mathbb{F}_n$, it holds that $\min \le f_r \le \max$. In addition, f_r is monotone since all f, min and max are monotone. Therefore, f_r is always an averaging aggregation function
Similarly to the classical definition of global orness (see, e.g., [12]), but using the rectified function rather than the aggregation function, it is possible to construct an orness measure for an aggregation function. Intuitively, the use of the rectified function assures that we are measuring the degree in which an aggregation function is greater than or equal to the 'or' function. Formally, this rectification guarantees that the orness measure takes values within the unit interval. Otherwise, we could obtain values of orness greater than one for some aggregation functions, e.g., the value $n/(n-1) > 1$ for the drastic disjunction (defined as $f(x_1,\ldots,x_n) = 0$, if $x_1 = \ldots = x_n = 0$, and $f(x_1,\ldots,x_n) = 1$, if there exists $i \in \{1,\ldots,n\}$ for which $x_i \neq 0$).

Theorem 2. *Consider $n \in \mathbb{N}$. The function $\mathcal{O}_{\mathcal{R}} : \mathbb{F}_n \longrightarrow [0,1]$ defined as*

$$\begin{aligned} \mathcal{O}_{\mathcal{R}}(f) &= \frac{\displaystyle\int_0^1\int_0^1 \cdots \int_0^1 f_r(x_1,\ldots,x_n) - \min(x_1,\ldots,x_n)\, dx_n \ldots dx_1}{\displaystyle\int_0^1\int_0^1 \cdots \int_0^1 \max(x_1,\ldots,x_n) - \min(x_1,\ldots,x_n)\, dx_n \ldots dx_1} \\ &= \frac{(n+1)\displaystyle\int_0^1\int_0^1 \cdots \int_0^1 f_r(x_1,\ldots,x_n)\, dx_n \ldots dx_1 - 1}{n-1}\,, \end{aligned}$$

is an orness measure for aggregation functions.

Proof. We recall that (see, e.g., [32]):

$$\int_0^1\int_0^1 \cdots \int_0^1 \max(x_1,\ldots,x_n) dx_n \ldots dx_1 = \frac{n}{n+1}\,,$$
$$\int_0^1\int_0^1 \cdots \int_0^1 \min(x_1,\ldots,x_n) dx_n \ldots dx_1 = \frac{1}{n+1}\,.$$

Therefore, it is straightforward to see that $\mathcal{O}_{\mathcal{R}}(f) \in [0,1]$, for any $f \in \mathbb{F}_n$, and that $\mathcal{O}_{\mathcal{R}}$ is well-defined. In the following, we prove that the four axioms introduced in Definition 11 are satisfied.

(O1) It holds that $\mathcal{O}_{\mathcal{R}}(f)=1$ is equivalent to

$$\begin{aligned}\frac{n}{n+1} &= \int_0^1\int_0^1\cdots\int_0^1 f_r(x_1,\ldots,x_n)dx_n\ldots dx_1\\ &= \int_0^1\int_0^1\cdots\int_0^1 \max(x_1,\ldots,x_n)dx_n\ldots dx_1\,.\end{aligned}$$

Since $f_r\le\max$, the above is equivalent to $f_r=_*\max$, and, therefore, to $\max\le_* f$.

(O2) Note that $f\le g$ implies that $f_r\le g_r$. The result then follows from the monotonicity of the integral.

(O3) Since $\max(x_1,\ldots,x_n)=1-\min(1-x_1,\ldots,1-x_n)$ and $\min(x_1,\ldots,x_n)=1-\max(1-x_1,\ldots,1-x_n)$, it is easy to check that $(\overline{f})_r=\overline{f_r}$.

Therefore, it holds that

$$\begin{aligned}&\int_0^1\int_0^1\cdots\int_0^1 (\overline{f})_r(x_1,\ldots,x_n)dx_n\ldots dx_1\\ &=\int_0^1\int_0^1\cdots\int_0^1 \Big(1-f_r(1-x_1,\ldots,1-x_n)\Big)dx_n\ldots dx_1\\ &=1-\int_0^1\int_0^1\cdots\int_0^1 f_r(1-x_1,\ldots,1-x_n)dx_n\ldots dx_1\\ &=1-\int_0^1\int_0^1\cdots\int_0^1 f_r(x_1,\ldots,x_n)dx_n\ldots dx_1\,.\end{aligned}$$

We finally conclude that

$$\begin{aligned}\mathcal{O}_{\mathcal{R}}(\overline{f}) &= \frac{(n+1)\Big(1-\int_0^1\int_0^1\cdots\int_0^1 f_r(x_1,\ldots,x_n)dx_n\ldots dx_1\Big)-1}{n-1}\\ &=\frac{n-(n+1)\Big(\int_0^1\int_0^1\cdots\int_0^1 f_r(x_1,\ldots,x_n)dx_n\ldots dx_1\Big)}{n-1}\\ &=1+\frac{1-(n+1)\int_0^1\int_0^1\cdots\int_0^1 f_r(x_1,\ldots,x_n)dx_n\ldots dx_1}{n-1}\\ &=1-\mathcal{O}_{\mathcal{R}}(f)\,.\end{aligned}$$

(O4) The result follows from the fact that variable permutation does not affect the integral computation.

□

Since the rectified function of an averaging aggregation function is the averaging aggregation function itself, it is immediate to see that the orness measure above coincides with Dujmović's global orness for an averaging aggregation function. Also note that this construction method returns Yager's orness measure when restricted to OWA functions (as expected from the fact that Dujmović's global orness is equivalent to Yager's orness measure for OWA functions [17]).

Theorem 3. *Consider $n\in\mathbb{N}$. Let $f_{\mathbf{w}}:[0,1]^n\longrightarrow[0,1]$ be an OWA function. It follows that*

$$\mathcal{O}_{\mathcal{R}}(f_{\mathbf{w}})=\mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}})\,.$$

Proof. Each OWA function is an averaging aggregation function and therefore $(f_{\mathbf{w}})_r=f_{\mathbf{w}}$. Thus, to obtain $\mathcal{O}_{\mathcal{R}}(f_{\mathbf{w}})$, we must compute

$$\int_0^1\int_0^1\cdots\int_0^1 (w_1x_{(1)}+\ldots+w_nx_{(n)})dx_n\ldots dx_1\,.$$

Over the domain $x_1 > \ldots > x_n$, it holds that

$$\begin{aligned}
&\int_0^1 \int_0^1 \cdots \int_0^1 (w_1 x_{(1)} + \ldots + w_n x_{(n)}) dx_n \ldots dx_1 \\
\underset{x_1>\ldots>x_n}{=} &\int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} (w_1 x_1 + \ldots + w_n x_n) dx_n \ldots dx_1 \\
= &\sum_{i=1}^n w_i \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} x_i \, dx_n \ldots dx_1 \\
= &\sum_{i=1}^n \frac{(n-i+1)\, w_i}{(n+1)!} \,.
\end{aligned}$$

Similarly, we can distinguish $n!$ domains corresponding to all possible ways of ordering the values $\{x_1, \ldots, x_n\}$. All these domains are disjoint and add up to the full domain (bearing in mind that the cases in which two values coincide have measure zero). Therefore, it suffices to compute the integral over the domain $x_1 > \ldots > x_n$ and multiply by $n!$. Thus, it holds that

$$\begin{aligned}
\int_0^1 \int_0^1 \cdots \int_0^1 (w_1 x_{(1)} + \ldots + w_n x_{(n)}) dx_n \ldots dx_1 &= n! \sum_{i=1}^n \frac{(n-i+1)\, w_i}{(n+1)!} \\
&= \frac{1}{n+1} + \sum_{i=1}^n \frac{(n-i)\, w_i}{(n+1)} \,.
\end{aligned}$$

We conclude that

$$\mathcal{O}_{\mathcal{R}}(f_{\mathbf{w}}) = \frac{(n+1)\left(\dfrac{1}{n+1} + \displaystyle\sum_{i=1}^n \frac{(n-i)\, w_i}{(n+1)}\right) - 1}{n-1} = \frac{1}{n-1} \sum_{i=1}^n (n-i) w_i = \mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}}) \,.$$

□

5.1.2 The case of the (discrete) Choquet integral

In the following, we study the particular example of the (discrete) Choquet integral. We recall that a (regular) fuzzy measure (or capacity) $\nu : \mathcal{C}_n \to [0,1]$ on $\{1, 2, \ldots, n\}$ is a function such that $\mathcal{C}_n$ is the powerset of $\{1, 2, \ldots, n\}$, $\nu(\emptyset) = 0$, $\nu(\{1, 2, \ldots, n\}) = 1$ and $A \subseteq B \subseteq \{1, 2, \ldots, n\}$ implies $\nu(A) \leq \nu(B)$. The (discrete) Choquet integral based on ν is the function $C_\nu : [0,1]^n \to [0,1]$ defined as

$$C_\nu(x_1, \ldots, x_n) = x_{(n)} \nu\big(\{(n)\}\big) + \sum_{i=1}^{n-1} x_{(i)} \left(\nu\big(\{(i), \ldots, (n)\}\big) - \nu\big(\{(i+1), \ldots, (n)\}\big)\right) ,$$

where $x_{(1)} \leq \ldots \leq x_{(n)}$ represents an increasing permutation of the values $x_1, \ldots, x_n$ and (i) represents the i-th index in the permutation. The Choquet integral is an averaging aggregation function independently of the chosen ν. For more details on the Choquet integral and fuzzy measures, we refer to [33].

Proposition 2. *Consider $n \in \mathbb{N}$ and a fuzzy measure $\nu : \mathcal{C}_n \to [0,1]$ on $\{1, 2, \ldots, n\}$. Let $C_\nu : [0,1]^n \to [0,1]$ be the Choquet integral based on ν. It follows that*

$$\mathcal{O}_{\mathcal{R}}(C_\nu) = \frac{1}{n-1} \left(\sum_{i=1}^{n-1} \frac{1}{\binom{n}{i}} \left(\sum_{a_1 < \cdots < a_i} \nu(a_1, \ldots, a_i) \right) \right) .$$

Proof. Since C_ν is an averaging aggregation function, the result follows from Theorem 6.2.1 in [34] and the fact that the orness measure above coincides for an averaging aggregation function with Dujmović's global orness.
□

A prominent type of fuzzy measures is that of symmetric fuzzy measures, for which only the cardinality of the subsets of $\{1,\ldots,n\}$ matters, i.e., $\nu(A)=\nu(B)$ if $|A|=|B|$.

Corollary 1. *Consider $n\in\mathbb{N}$ and a symmetric fuzzy measure $\nu:\mathcal{C}_n\to[0,1]$ on $\{1,2,\ldots,n\}$. Let $C_\nu:[0,1]^n\to[0,1]$ be the Choquet integral based on ν. It follows that*

$$\mathcal{O}_{\mathcal{R}}(C_\nu)=\frac{1}{n-1}\left(\sum_{i=1}^{n-1}\nu(1,\ldots,i)\right).$$

The corollary above corresponds to intuition since there exists a correspondence between Choquet integrals based on a symmetric fuzzy measure and OWA functions [33]. More precisely, the weights of the OWA function are given by $w_i=\nu(1,\ldots,i)-\nu(1,\ldots,i-1)$, and, conversely, the symmetric fuzzy measure is given by $\nu(1,\ldots,i)=\sum_{j=n-i+1}^{n}w_j$. It is then immediate to conclude that the orness obtained in Corollary 1 for the special case of Choquet integrals based on a symmetric fuzzy measure coincides with that of Yager for OWA functions.

5.1.3 The case of uninorms

Uninorms were introduced by Yager and Rybalov [35] as a generalization of both t-norms and t-conorms. Formally, a uninorm is a commutative, associative and increasing binary operator $U:[0,1]^2\to[0,1]$ with a neutral element $e\in[0,1]$. The extension from $n=2$ to $n>2$ arguments follows immediately from the associativity property. In case $e=1$ we recover the definition of a t-norm, and in case $e=0$ we recover the definition of a t-conorm.
It has been discussed in Section 4 that the orness of any t-norm equals zero and the orness of any t-conorm equals one. The case of uninorms is not immediate and only some bounding results in terms of the neutral element can be provided.

Proposition 3. *Consider $n\in\mathbb{N}$. Let $U:[0,1]^n\to[0,1]$ be a uninorm with neutral element $e\in[0,1]$. It follows that*

$$\mathcal{O}_{\mathcal{R}}(U)\in\left[(1-e)^{n+1},1-e^{n+1}\right].$$

Proof. On the one hand, for any interval $[a,b]\subseteq[0,1]$, it holds that

$$\int_a^b\int_a^b\cdots\int_a^b\max(x_1,\ldots,x_n)dx_n\ldots dx_1=\frac{(b-a)^n(a+b\,n)}{n+1},$$
$$\int_a^b\int_a^b\cdots\int_a^b\min(x_1,\ldots,x_n)dx_n\ldots dx_1=\frac{(b-a)^n(a\,n+b)}{n+1}.$$

On the other hand, for any uninorm, the rectified function is such that $U_r(x_1,\ldots,x_n)=\min(x_1,\ldots,x_n)$ for any $(x_1,\ldots,x_n)\in[0,e]^n$ and $U_r(x_1,\ldots,x_n)=\max(x_1,\ldots,x_n)$ for any $(x_1,\ldots,x_n)\in[e,1]^n$, with $U_r(x_1,\ldots,x_n)\in[\min(x_1,\ldots,x_n),\max(x_1,\ldots,x_n)]$ elsewhere.
Both results above together imply that

$$\begin{aligned}\int_0^1\int_0^1\cdots\int_0^1 U_r(x_1,\ldots,x_n)dx_n\ldots dx_1 &\geq \int_e^1\int_e^1\cdots\int_e^1\max(x_1,\ldots,x_n)dx_n\ldots dx_1\\ &\quad+\int_0^1\int_0^1\cdots\int_0^1\min(x_1,\ldots,x_n)dx_n\ldots dx_1\\ &\quad-\int_e^1\int_e^1\cdots\int_e^1\min(x_1,\ldots,x_n)dx_n\ldots dx_1\\ &=\frac{(n-1)(1-e)^{n+1}+1}{n+1},\end{aligned}$$

and

$$\int_0^1\int_0^1\cdots\int_0^1 U_r(x_1,\ldots,x_n)dx_n\ldots dx_1 \le \int_0^e\int_0^e\cdots\int_0^e \min(x_1,\ldots,x_n)dx_n\ldots dx_1$$
$$+\int_0^1\int_0^1\cdots\int_0^1 \max(x_1,\ldots,x_n)dx_n\ldots dx_1$$
$$-\int_0^e\int_0^e\cdots\int_0^e \max(x_1,\ldots,x_n)dx_n\ldots dx_1$$
$$=\frac{n-(n-1)e^{n+1}}{n+1}\,.$$

Finally, if one substitutes the two values above in the formula of Theorem 2, then the following results are obtained:

$$\mathcal{O}_{\mathcal{R}}(U)=\frac{(n+1)\displaystyle\int_0^1\int_0^1\cdots\int_0^1 U_r(x_1,\ldots,x_n)\,dx_n\ldots dx_1-1}{n-1}\ge(1-e)^{n+1}\,,$$
$$\mathcal{O}_{\mathcal{R}}(U)=\frac{(n+1)\displaystyle\int_0^1\int_0^1\cdots\int_0^1 U_r(x_1,\ldots,x_n)\,dx_n\ldots dx_1-1}{n-1}\le 1-e^{n+1}\,.$$

□

After some trivial computations, it is concluded from the proposition above that a uninorm is assured to be conjunctive (orness smaller than or equal to 0.5) if $e \ge 0.5^{\frac{1}{n+1}}$ and disjunctive (orness greater than or equal to 0.5) if $e \le 1-0.5^{\frac{1}{n+1}}$.

5.2 Second construction method

5.2.1 General results

Consider an aggregation function $f\in\mathbb{F}_n$. For any $i\in\{1,\ldots,n-1\}$, consider $f_{(i)}$ to be the arithmetic mean of the function f evaluated in all possible vectors formed by i ones and $n-i$ zeros. Formally,

$$f_{(i)}=\frac{i!\,(n-i)!}{n!}\sum_{\substack{\mathbf{x}\in\{0,1\}^n\\ \sum_{j=1}^n x_j=i}} f(\mathbf{x})\,.$$

Note that, in case f is symmetric, it holds that

$$f_{(i)}=f(\underbrace{1,\ldots,1}_{i\text{ times}},\underbrace{0,\ldots,0}_{n-i\text{ times}})\,.$$

The values above together with an aggregation function satisfying some minimal conditions (satisfied by, e.g., the arithmetic mean) allow us to construct an orness measure for aggregation functions.

Lemma 1. *Consider $n\in\mathbb{N}$ and an aggregation function $A:[0,1]^{n-1}\to[0,1]$ such that:*

- *A satisfies the strict upper boundary condition, i.e., $A(\mathbf{x})=1$ if and only if $\mathbf{x}=(1,\ldots,1)$.*
- *A is increasing.*
- *$A(x_1,\ldots,x_{n-1})+A(1-x_{n-1},\ldots,1-x_1)=1$.*

The function $\mathcal{O}_A:\mathbb{F}_n\longrightarrow[0,1]$ defined as

$$\mathcal{O}_A(f)=A(f_{(1)},\ldots,f_{(n-1)})\,,$$

satisfies Axioms (O2), (O3), (O4) and the right to left implication of Axiom (O1) of an orness measure for aggregation functions.

Proof. It is straightforward to see that $\mathcal{O}_A(f) \in [0,1]$, for any $f \in \mathbb{F}_n$. Therefore, $\mathcal{O}_A$ is well-defined.

(O1) If $\max \leq_* f$, then it holds that $f(\mathbf{x}) = 1$ for any $\mathbf{x}$ with at least a one. Therefore, $f_{(i)} = 1$ for any $i \in \{1, \ldots, n-1\}$. This implies that $A(f_{(1)}, \ldots, f_{(n-1)}) = 1$. Finally, we conclude that $\mathcal{O}_A(f) = 1$.

(O2) If $f \leq g$, then $f_{(i)} \leq g_{(i)}$ for any $i \in \{1, \ldots, n-1\}$. The result then follows from the monotonicity of A.

(O3) Note that $(\overline{f})_{(i)} = 1 - f_{(n-i)}$. Therefore,

$$\begin{aligned}\mathcal{O}_A(\overline{f}) &= A((\overline{f})_{(1)}, \ldots, (\overline{f})_{(n-1)}) \\ & A(1 - f_{(n-1)}, \ldots, 1 - f_{(1)}) \\ &= 1 - A(f_{(1)}, \ldots, f_{(n-1)}) \\ &= 1 - \mathcal{O}_A(f)\,,\end{aligned}$$

by applying the fact that $A(x_1, \ldots, x_{n-1}) + A(1 - x_{n-1}, \ldots, 1 - x_1) = 1$.

(O4) The result follows from the fact that variable permutation does not affect the computation of the values $f_{(i)}$.

□

Note that, Axiom (O1) is not satisfied by $\mathcal{O}_A$ since there can be functions with orness 1 that are not greater than or equal to the maximum. For instance, the function

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \max(\mathbf{x}) = 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

satisfies that $\mathcal{O}_A(f) = 1$. However, it holds that for any $\mathbf{x} = (x_1, \ldots, x_n)$ such that $0 < x_i < 1$ for any $i \in \{1, \ldots, n\}$, $f(\mathbf{x}) = 0 < \max(\mathbf{x})$, therefore, it does not hold that $\max \leq_* f$.
If one wants to overcome this problem, it suffices to consider a small margin δ by which we reduce the orness measure in case it does not hold that $\max \leq_* f$.

Theorem 4. *Consider $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$. Consider an aggregation function $A : [0,1]^{n-1} \to [0,1]$ such that:*

- *A satisfies the strict upper boundary condition, i.e., $A(\mathbf{x}) = 1$ if and only if $\mathbf{x} = (1, \ldots, 1)$.*
- *A is increasing.*
- *$A(x_1, \ldots, x_{n-1}) + A(1 - x_{n-1}, \ldots, 1 - x_1) = 1$.*

For any $\delta \in (0, \frac{1}{4})$, the function $\mathcal{O}_{A,\delta} : \mathbb{F}_n \longrightarrow [0,1]$ defined as

$$\mathcal{O}_{A,\delta}(f) = \begin{cases} 0 & \text{if } f \leq_* \min\,, \\ 1 & \text{if } \max \leq_* f\,, \\ \dfrac{\delta}{2} + (1-\delta) A(f_{(1)}, \ldots, f_{(n-1)}) & \text{otherwise}\,, \end{cases}$$

is an orness measure for aggregation functions.

Proof. Note that, since $0 \leq f_{(i)} \leq 1$ for any $i \in \{1, \ldots, n-1\}$, it holds that

$$0 \leq A(f_{(1)}, \ldots, f_{(n-1)}) \leq 1\,.$$

Therefore, it holds that

$$0 \leq (1-\delta) A(f_{(1)}, \ldots, f_{(n-1)}) \leq (1-\delta)\,,$$

and, then, $0 < \delta/2 \leq \mathcal{O}_{A,\delta}(f) \leq 1 - \delta/2 < 1$ whenever it does not hold that $f \leq_* \min$, nor $\max \leq_* f$. The remainder of the proof follows from Lemma 1. □

The following theorem states that $\mathcal{O}_A$ and $\mathcal{O}_{A,\delta}$ are as close as required.

Theorem 5. *Consider $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$. Let $f : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. It follows that*

$$|\mathcal{O}_{A,\delta}(f) - \mathcal{O}_A(f)| \leq \frac{\delta}{2}\,.$$

Proof. We distinguish three different cases:

- If $\max \leq_* f$, then $\mathcal{O}_{A,\delta}(f) = \mathcal{O}_A(f) = 1$.
- If $f \leq_* \min$, then $\mathcal{O}_{A,\delta}(f) = \mathcal{O}_A(f) = 0$.
- Otherwise, it holds that

$$\begin{aligned}|\mathcal{O}_{A,\delta}(f) - \mathcal{O}_A(f)| &= |\frac{\delta}{2} + (1-\delta)A(f_{(1)}, \ldots, f_{(n-1)}) - A(f_{(1)}, \ldots, f_{(n-1)})| \\ &= |\frac{\delta}{2} - \delta A(f_{(1)}, \ldots, f_{(n-1)})| \leq \frac{\delta}{2}\,.\end{aligned}$$

□

Note that this construction methods returns Yager's orness measure when restricted to OWA functions.

Theorem 6. *Consider $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$. Let $f_{\mathbf{w}} : [0,1]^n \longrightarrow [0,1]$ be an OWA function and let $A : [0,1]^{n-1} \longrightarrow [0,1]$ be the arithmetic mean. It follows that*

$$\mathcal{O}_{\mathcal{A}}(f_{\mathbf{w}}) = \mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}})\,,$$

and

$$|\mathcal{O}_{A,\delta}(f_{\mathbf{w}}) - \mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}})| \leq \frac{\delta}{2}\,.$$

Proof. We distinguish three different cases:

- If $f_{\mathbf{w}} = \max$, then it holds that $\max \leq_* f$, and, therefore, $\mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}}) = \mathcal{O}_{A,\delta}(f_{\mathbf{w}}) = \mathcal{O}_A(f_{\mathbf{w}}) = 1$.
- If $f_{\mathbf{w}} = \min$, then it holds that $f \leq_* \min$, and, therefore, $\mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}}) = \mathcal{O}_{A,\delta}(f_{\mathbf{w}}) = \mathcal{O}_A(f_{\mathbf{w}}) = 0$.
- Otherwise, it holds that $f_{(i)} = \sum_{j=1}^{i} w_j$. Therefore, it holds that

$$\mathcal{O}_A(f_{\mathbf{w}}) = \frac{1}{n-1}\sum_{i=1}^{n} f_{(i)} = \frac{1}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{i} w_j = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}})\,.$$

The fact that

$$|\mathcal{O}_{A,\delta}(f_{\mathbf{w}}) - \mathcal{O}_{\mathcal{Y}}(f_{\mathbf{w}})| \leq \frac{\delta}{2}\,,$$

follows from the result above and Theorem 5.

□

5.2.2 The case of the (discrete) Choquet integral

Again, we study the particular example of the Choquet integral.

Proposition 4. *Consider $n \in \mathbb{N}$, $\delta \in (0, \frac{1}{4})$, a fuzzy measure $\nu : \mathcal{C}_n \to [0,1]$ on $\{1, 2, \ldots, n\}$ and let $A : [0,1]^{n-1} \longrightarrow [0,1]$ be the arithmetic mean. Let $C_\nu : [0,1]^n \to [0,1]$ be the Choquet integral based on ν. The following results hold:*

(i) If $\nu(A) = 0$ for any $A \neq \{1, \ldots, n\}$, it holds that $\mathcal{O}_{A,\delta}(C_\nu) = 0$.

(ii) If $\nu(A) = 1$ for any $A \neq \emptyset$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(C_\nu) = 1$.

(iii) Otherwise, it holds that

$$\mathcal{O}_{\mathcal{A},\delta}(C_\nu) = \frac{\delta}{2} + (1-\delta)\mathcal{O}_{\mathcal{R}}(C_\nu)\,.$$

Proof. First, it should be noted that (i) and (ii) represent the two extreme cases in which the Choquet integral equals the minimum and the maximum, respectively. The result then follows straightforwardly from the fact that

$$(C_\nu)_{(i)} = \frac{i!\,(n-i)!}{n!} \sum_{\substack{\mathbf{x}\in\{0,1\}^n \\ \sum_{j=1}^n x_j = i}} C_\nu(\mathbf{x}) = \frac{1}{\binom{n}{i}} \sum_{a_1<\cdots<a_i} \nu(a_1,\ldots,a_i)\,.$$

□

As a result of Corollary 1, the result above can be further simplified in the case of symmetric fuzzy measures.

Corollary 2. *Consider $n \in \mathbb{N}$, $\delta \in (0, \frac{1}{4})$, a symmetric fuzzy measure $\nu : \mathcal{C}_n \to [0,1]$ on $\{1,2,\ldots,n\}$ and let $A : [0,1]^{n-1} \longrightarrow [0,1]$ be the arithmetic mean. Let $C_\nu : [0,1]^n \to [0,1]$ be the Choquet integral based on ν. The following results hold:*

(i) If $\nu(A) = 0$ for any $A \neq \{1,\ldots,n\}$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(C_\nu) = 0$.

(ii) If $\nu(A) = 1$ for any $A \neq \emptyset$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(C_\nu) = 1$.

(iii) Otherwise, it holds that

$$\mathcal{O}_{\mathcal{A},\delta}(C_\nu) = \frac{\delta}{2} + \frac{1-\delta}{n-1}\left(\sum_{i=1}^{n-1} \nu(1,\ldots,i)\right)\,.$$

5.2.3 The case of uninorms

For any uninorm it either holds that $U(0,1) = 0$ or $U(1,0) = 1$ [36, 37]. Uninorms fulfilling the first condition are called conjunctive uninorms, whereas uninorms fulfilling the second condition are called disjunctive uninorms. In the following, we see that these terms correspond to intuition with regard to the second construction method for an orness measure.

Proposition 5. *Consider $n \in \mathbb{N}$, $\delta \in (0, \frac{1}{4})$ and let $A : [0,1]^{n-1} \longrightarrow [0,1]$ be the arithmetic mean. Let $U : [0,1]^n \to [0,1]$ be a uninorm with neutral element $e \in [0,1]$. The following results hold:*

(i) If $e = 1$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(U) = 0$.

(ii) If $e \in]0,1[$ and $U(0,1) = 0$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(U) = \frac{\delta}{2}$.

(iii) If $e \in]0,1[$ and $U(0,1) = 1$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(U) = 1 - \frac{\delta}{2}$.

(iv) If $e = 0$, it holds that $\mathcal{O}_{\mathcal{A},\delta}(U) = 1$.

Proof. Cases (i) and (iv) are trivial since they represent the two extreme cases in which U is a t-norm and a t-conorm, respectively. Case (ii) follows from the fact that $U_{(i)} = 0$ for any $i \in \{1,\ldots,n-1\}$ in case $U(0,1) = 0$. Case (iii) follows from the fact that $U_{(i)} = 1$ for any $i \in \{1,\ldots,n-1\}$ in case $U(0,1) = 1$. □

e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\mathcal{O}_{\mathcal{R}}(U_{e,T,S})$	1	0.729	0.512	0.343	0.216	0.125	0.064	0.027	0.008	0.001	0
$\mathcal{O}_{\mathcal{R}}(U_{T,S,e})$	1	0.999	0.992	0.973	0.936	0.875	0.784	0.657	0.488	0.271	0
$\mathcal{O}_{\mathcal{A},\delta}(U_{e,T,S})$	1	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	$\frac{\delta}{2}$	0
$\mathcal{O}_{\mathcal{A},\delta}(U_{T,S,e})$	1	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	$1-\frac{\delta}{2}$	0

Table 1: Comparison of the orness measures $\mathcal{O}_{\mathcal{R}}$ and $\mathcal{O}_{\mathcal{A},\delta}$ for the uninorms $U_{e,T,S}$ and $U_{T,S,e}$ for different values of $e \in [0,1]$.

5.3 Further discussion on uninorms

In this section, we provide a comparison of both construction methods for two prominent uninorms. In particular, the two considered uninorms are the following:

$$U_{e,T,S}(x,y) = \begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2\,, \\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2\,, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

$$U_{T,S,e}(x,y) = \begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2\,, \\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2\,, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

where $e \in [0,1]$ is the neutral element, T is a t-norm and S is a t-conorm. By convention, we consider $U_{0,T,S}(x,y) = U_{T,S,0}(x,y) = S(x,y)$ and $U_{1,T,S}(x,y) = U_{T,S,1}(x,y) = T(x,y)$.
Table 1 presents the values of $\mathcal{O}_{\mathcal{R}}$ and $\mathcal{O}_{\mathcal{A},\delta}$ (with A being the arithmetic mean) for both uninorms considering different values of $e \in [0,1]$. Note that both $\mathcal{O}_{\mathcal{R}}(U_{e,T,S})$ and $\mathcal{O}_{\mathcal{R}}(U_{T,S,e})$ respectively reach the lower bound and the upper bound given in Proposition 3. It is also interesting to note that, in terms of the literature on uninorms, the uninorm $U_{e,T,S}$ is conjunctive for any $e \neq 0$ (since $U_{e,T,S}(0,1) = 0$) and the uninorm $U_{T,S,e}$ is disjunctive for any $e \neq 1$ (since $U_{T,S,e}(0,1) = 1$). However, as can be seen in Table 1, $\mathcal{O}_{\mathcal{R}}(U_{e,T,S})$ might be greater than 0.5 (thus, for an appropriate $e \in]0,1[$, $U_{e,T,S}$ is a disjunctive aggregation function according to $\mathcal{O}_{\mathcal{R}}$) and $\mathcal{O}_{\mathcal{R}}(U_{T,S,e})$ might be smaller than 0.5 (thus, for an appropriate $e \in]0,1[$, $U_{T,S,e}$ is a conjunctive aggregation function according to $\mathcal{O}_{\mathcal{R}}$). This implies that, even though approaches similar to the first construction method have been more popular when studying the orness of an aggregation function, the second construction method leads to a notion of orness that is more consistent with the terminology on uninorms.

6 Conclusions

In this paper, we have provided an axiomatization of the notion of an orness measure for aggregation functions beyond its current restriction to OWA functions. Furthermore, two different methods for constructing an orness measure for aggregation functions have been provided. It has also been proven that, in case these aggregation functions are OWA functions, we recover the orness measure for OWA functions originally proposed by Yager. Obviously, all results here presented can be easily adapted in order to measure the andness of an aggregation function.
Recent studies on the notion of orness (see, e.g., [12]) abandon the interpretation of orness as the degree of disjunction in order to accommodate values outside the unit interval. This re-interpretation allows to compare, for instance, two t-conorms in terms of their degree of (hyper)disjunction. An example of such orness is the so-called global orness. This global orness does not fulfill the axioms presented in Definition 11 but coincides with the orness measure presented in Subsection 5.1 for averaging aggregation functions (i.e., in case the rectified function coincides with the averaging aggregation function itself). A potential solution for accommodating global orness as an orness measure for aggregation functions could

be to define an orness measure as a function $\mathcal{O} : \mathbb{F}_n \longrightarrow \mathbb{R}$ satisfying Axioms (O1'), (O2), (O3) and (O4), with Axiom (O1') $\mathcal{O}(\max) = 1$. Nevertheless, the authors favour a future study concerning the definition of a degree of hyperdisjunction covering the whole spectrum between the 'or' function and the drastic disjunction as a more interesting solution.

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References

[1] R. R. Yager, J. Kacprzyk, G. Beliakov, editors, *Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice*, Springer-Verlag, Berlin, Heidelberg, **2011**.

[2] G. Beliakov, A. Pradera, T. Calvo, *Aggregation Functions: A Guide for Practitioners*, volume 221 of *Studies in Fuzziness and Soft Computing*, Springer, Berlin, Heidelberg, **2007**.

[3] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation Functions*, Cambridge University Press, Cambridge, **2009**.

[4] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation functions: Constructions methods, conjunctive, disjunctive and mixed classes, *Information Sciences* **2011**, *181*, 23–43.

[5] G. Beliakov, H. Bustince, T. Calvo, *A Practical Guide to Averaging Functions*, Studies in Fuzziness and Soft Computing. Springer-Verlag, Berlin, Heidelberg, Berlin, **2016**.

[6] P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and Their Inequalities*, D. Reidel, Dordrecht, **1988**.

[7] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation functions: Means, *Information Sciences* **2011**, *181*, 1–22.

[8] A. L. Cauchy, *Cours d'Analyse de l'Ecole Royale Polytechnique*, volume 1 of *Analyse Algébraique*, Debure, Paris, **1821**.

[9] R. R. Yager, On Ordered Weighted Averaging aggregation operators in multicriteria decisionmaking, *IEEE Transactions on Systems, Man, and Cybernetics* **1988**, *18*, 183–190.

[10] Z.-S. Chen, K.-W. Chin, L. Martínez, K.-L. Tsui, Customizing semantics for individuals with attitudinal HFLTS possibility distributions, *IEEE Transactions on Fuzzy Systems* **2018**, *26*, 3452–3466.

[11] Z.-S. Chen, C. Yu, K.-W. Chin, L. Martínez, An enhanced ordered weighted averaging operators generation algorithm with applications for multicriteria decision making, *Applied Mathematical Modelling* **2019**, *71*, 467–490.

[12] J. Dujmović, *Soft Computing Evaluation Logic*, Wiley, Hoboken, **2018**.

[13] J. Dujmović, A generalization of some functions in continuous mathematical logic, In *Proceedings of the Informatica Conference (in Serbo-Croatan)*. Bled, **1973**, paper d27.

[14] J. Dujmović, Mixed Averaging by Levels (MAL) – A system and computer evaluation method, In *Proceedings of the Informatica Conference (in Serbo-Croatan)*. Bled, **1973**, paper d28.

[15] J. Dujmović, New results in the development of the "Mixed Averaging by Levels" method for system evaluation, In *Proceedings of the Informatica Conference (in Serbo-Croatan)*. Bled, **1974**, paper 4.36.

[16] J. Dujmović, Weighted conjunctive and disjunctive means and their application in system evaluation, *Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika* **1974**, 461/497, 147–158.

[17] J. M. Fernández Salido, S. Murakami, Extending Yager's orness concept for the OWA aggregators to other mean operators, *Fuzzy Sets and Systems* **2003**, *139*, 515–542.

[18] V. Torra, Andness directedness for operators of the OWA and WOWA families, *Fuzzy Sets and Systems, in press, https://doi.org/10.1016/j.fss.2020.09.004* **2020**.

[19] R. Fullér, P. Majlender, An analytic approach for obtaining maximal entropy OWA operator weights, *Fuzzy sets and Systems* **2001**, *124*, 53–57.

[20] M. O'Hagan, Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic, In *Proceedings of the 22nd Asilomar Conference on Signals, Systems and Computers*, volume 2. Pacific Grove, **1988**, 681–689.

[21] X. Liu, An orness measure for quasi-arithmetic means, *IEEE Transactions on Fuzzy Systems* **2006**, *14*, 837–848.

[22] I. Lizasoain, C. Moreno, OWA operators defined on complete lattices, *Fuzzy Sets and Systems* **2013**, *224*, 36–52.

[23] L. De Miguel, D. Paternain, I. Lizasoain, G. Ochoa, H. Bustince, Some characterizations of lattice OWA operators, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **2017**, *25*, 5–17.

[24] L. De Miguel, D. Paternain, I. Lizasoain, G. Ochoa, H. Bustince, Orness measurements for lattice m-dimensional interval-valued OWA operators, *Information Fusion* **2018**, *339*, 63–80.

[25] G. Ochoa, I. Lizasoain, D. Paternain, H. Bustince, N. R. Pal, From quantitative to qualitative orness for lattice OWA operators, *International Journal of General Systems* **2017**, *46*, 640–669.

[26] D. Paternain, G. Ochoa, I. Lizasoain, H. Bustince, R. Mesiar, Quantitative orness for lattice OWA operators, *Information Fusion* **2016**, *30*, 27–35.

[27] A. Kishor, A. K. Singh, N. R. Pal, Orness measure of OWA operators: A new approach, *IEEE Transactions on Fuzzy Systems* **2014**, *22*, 1039–1045.

[28] L. Legarreta, I. Lizasoain, I. Mardones-Pérez, Orness for idempotent aggregation functions, *Axioms* **2017**, *6*, paper 25.

[29] B. C. Arnold, *Majorization and the Lorenz Order: A Brief Introduction*, Springer-Verlag, Berlin, **1987**.

[30] J. Dujmović, Graded logic for decision support systems, *International Journal of Intelligent Systems* **2019**, *34*, 2900–2919.

[31] J. Dujmović, A comparison of andness/orness indicators, In *Proceedings of the 11th Information Processing and Management of Uncertainty international conference (IPMU 2006)*. Paris, **2006**, 691–698.

[32] J. Dujmović, Two integrals related to means, *Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika* **1973**, 412/460, 231–232.

[33] G. Beliakov, S. James, J.-Z. Wu, *Discrete Fuzzy Measures: Computational Aspects*, Studies in Fuzziness and Soft Computing. Springer, Berlin, **2019**.

[34] J.-L. Marichal, Aggregation operators for multicriteria decision aid, Ph.D. thesis, University of Liège, Liège, Belgium, **1998**.

[35] R. R. Yager, A. Rybalov, Uninorm aggregation operators, *Fuzzy sets and systems* **1996**, *80*, 111–120.

[36] J. Fodor, B. De Baets, Uninorm basics, In *Fuzzy Logic*, 49–64. Springer, **2007**.

[37] J. C. Fodor, R. R. Yager, A. Rybalov, Structure of uninorms, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **1997**, *5*, 411–427.