A unified view of different axiomatic measures defined on L-fuzzy sets

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***Abstract*—The recent literature contains a multitude of extensions of (axiomatic) notions from the context of ordinary fuzzy sets to more general contexts. Using the language of lattices, we provide a general and compact formulation encompassing a large number of those notions and their potential extensions to even more complex frameworks. The new formulation offers a unifying perspective of the different measures and operations between (generalised) fuzzy sets and has a potential impact on the simplification of the redundancy mathematical proofs concerning the formal relations between the different notions, and the properties of certain particular constructive definitions.**

***Index Terms*—Interval-valued fuzzy sets, Atanassov intuitionistic fuzzy sets, L-fuzzy sets**

I. Introduction

We have very recently studied [1] a vast collection of axiomatic definitions in the context of fuzzy sets, and their corresponding extensions to more general frameworks such as interval-valued [2] and Atanassov intuitionistic fuzzy sets [3]. We adopted a completely formal perspective and we concluded that many of those extensions followed very similar formal procedures. We observed that all those axiomatic extensions could be separated into two categories (real-valued vs set-valued extensions).

For each of those two categories, we managed to provide a general formulation encompassing many (apparently different) procedures of extensions found in the recent literature. Regarding set-valued extensions, the general procedure involved the transformation of every "property" $\mathbf{P}$ included in the original list of axioms into its corresponding "interval extension". Something analogous could be said about "point-valued" or "scalar extensions".

Apart from the need to separate our extension procedures into those two categories, they could only be applied to real-valued measures, but not to other kinds of functions like operations between fuzzy sets (union, intersection, complement, etc.) So, in the end, we had found two general (and a bit convoluted) procedures respectively generalising set-valued and point-valued extensions of real-valued measures from fuzzy sets to interval-valued and Atanassov intuitionistic fuzzy sets.

Now we have a more ambitious purpose: we aim at finding a single general formulation that covers not only set-valued and point-valued extensions of numerical-valued measures altogether, but also extensions of operations between fuzzy sets.

Furthermore, our formulation will not only apply to interval-valued or Atanassov intuitionistic fuzzy sets (IVF and AIF sets, for short), but also to other even more general frameworks such as type 2 fuzzy sets (T2F sets) [2] or hesitant fuzzy sets (HF sets) [4], among others. This will be possible by means of formulating everything in terms of lattices: Let us recall that (ordinary) fuzzy sets, IVF sets, AIF sets or T2F sets (among many other extensions of fuzzy sets encountered in the literature, see [5]) can be regarded as particular instances of L-fuzzy sets (where L stands for "lattice", see [6] for further details), by means of considering the different partial orderings associated to the different notions of "inclusion" in those contexts.

Let us furthermore notice that, if we consider an arbitrary lattice L and a universe U, the set of functions L^U (characterising the family of L-fuzzy subsets of U) also inherits the properties of lattices. In fact, an analysis of the convolution operator in this general setting was done in [7]. Let us finally observe that any bounded interval of the real line, together with the usual ordering between numbers, can be also regarded as a lattice. Thus, we can easily observe that both real-valued measures and operations between fuzzy sets fit the general formulation $M : L^U \times \stackrel{(k)}{\ldots} \times L^U \to L'$, where the lattice L' is either a subset of the real line or the family of fuzzy subsets of U, L^U.

In the rest of the paper, we will elaborate on this idea, and provide a general and very compact formulation encompassing many recent axiomatic notions introduced in the context of fuzzy sets together with their corresponding extensions to more general contexts such as IVF sets, AIF sets, T2F sets and HF sets, among others. In fact, the recent literature is full of cases of different extensions of the same notion to different contexts. To give an example, the axiomatic notion of "divergence" between fuzzy sets ([8]) has been recently extended to the case of AIF sets ([9]) and HF sets ([10]). According to the general formulation proposed in the present manuscript, those separate definitions can be seen as particular instances of a single general one. Furthermore, Montes et al. ([8]) had proved that some additional properties could be derived from the axioms of divergence measure, in the particular context of fuzzy sets. Recently [9] and [10] extended some of those results to the context of AIF and HF sets, by means of offering separate proofs in the respective contexts. With our general formulation in terms of lattices, those separate proofs would not be necessary anymore. A single proof in the general context of lattices would suffice to derive the same mathematical result in every particular context. This can be

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applied to a multitude of mathematical results already proved in the context of fuzzy set that have not been extended to more general contexts.

II. Preliminaries

Let $(L, \leq_L)$ be a lattice. Let L^U denote the collection of functions from U to L. Let us consider the following ordering on L^U:

$$A \leq_{L^U} B \text{ if } A(x) \leq_L B(x) \, \forall\, x \in U.$$

We observe that $(L^U, \leq_{L^U})$ is, in turn, a new lattice. In fact, given $A, B \in L^U$ we can easily check that:

$$\inf\{A, B\}(x) = \inf\{A(x), B(x)\}, \quad \text{and}$$

$$\sup\{A, B\}(x) = \sup\{A(x), B(x)\}, \forall\, x \in U.$$

We will respectively denote them as $A \wedge B = \inf\{A, B\}$ and $A \vee B = \sup\{A, B\}$.

Let us now consider the collection of intervals of elements of L:

$$\mathbb{L} = \{[a, b] \,:\, a, b \in L \text{ with } a \leq b\}$$

and the following (partial) order defined on it:

$$[a, b] \leq_{\mathbb{L}} [c, d] \text{ if } \;\; a \leq c \;\; \text{and} \;\; b \leq d.$$

Then the pair $(\mathbb{L}, \leq_{\mathbb{L}})$ is also a lattice. More than that, from the above statements we can derive that $(\mathbb{L}^U, \leq_{\mathbb{L}^U})$ is in turn a lattice. Let us furthermore consider the collection $\mathbb{I}(L^U)$ denoting the collection of intervals of L^U of the form:

$$[A, B] = \{X \in L^U \,:\, A \leq_{L^U} \;\; X \leq_{L^U} B\},$$

$\forall\, A, B \in L^U$ with $A \leq_{L^U} B$. The interval $[A, A]$ denotes the singleton $\{A\}$, that we can identify with the element $A \in L^U$. Thus, the set $\mathbb{I}(L^U)$ can be seen as a superset of L^U.

On the other hand, every element $[A, B] \in \mathbb{I}(L^U)$ can be identified with an $\mathbb{L}$-valued mapping $[A, B] : U \to \mathbb{L}$ defined as follows:

$$[A, B](x) = [A(x), B(x)], \;\forall\, x \in U.$$

Thus, the set of intervals of elements of L^U, $\mathbb{I}(L^U)$, can be identified with the set of functions from U to $\mathbb{L}$, $\mathbb{L}^U$. Let us furthermore notice that the collection of functions L^U represents the family of L-fuzzy subsets of U and $\mathbb{I}(L^U) \equiv \mathbb{L}^U$ can be identified with the L-interval-valued fuzzy subsets of U.

When L is in particular the unit interval (with the usual ordering over the real line), we can identify the set L^U with the collection of (ordinary) fuzzy subsets of U. In this work, we will consider mappings of the form $M : L^U \times \overset{(k)}{\cdots} \times L^U \to L'$, where the ordered set $(L', \leq_{L'})$ is also assumed to be a lattice. In many situations, L' represents, in particular, either a bounded interval of the real line ($\mathbb{R}$) or the set of functions L^U. The first case ($L' = \mathbb{R}$) corresponds to the case of numerical-valued measures (similarity measures, distance measures, etc.), while the second case corresponds to operations between L-fuzzy sets, such as the intersection, the union or the complement.

As we have mentioned in the Introduction, we have very recently examined the different procedures used in the literature in order to extend those measures and operations from the context of fuzzy sets to more general frameworks, such as interval-valued fuzzy sets and Atanassov intuitionistic fuzzy sets. In the case of numerical-valued measures ($L' = \mathbb{R}$), their corresponding extended versions can be either point-valued or set-valued (see [11], [1] and references therein for an overview). According to the above notation, those generalised measures respectively fit the following formulations $M : \mathbb{L}^U \times \overset{(k)}{\cdots} \times \mathbb{L}^U \to L'$ and $M : \mathbb{L}^U \times \overset{(k)}{\cdots} \times \mathbb{L}^U \to \mathbb{L}'$. In words, in each of those cases, the final space is either the original lattice L' or the family of intervals of the original one $\mathbb{L}'$.

Alternatively, in the case of extensions of fuzzy-sets operations, M takes the form $M : L^U \times \overset{(k)}{\cdots} \times L^U \to L^U$. (i.e., its final space is L^U) and furthermore its extension to interval-valued fuzzy sets takes the form $ext(M) : \mathbb{L}^U \times \overset{(k)}{\cdots} \times \mathbb{L}^U \to \mathbb{L}^U$.

Most axiomatic definitions of measures and operations were originally formulated in the context of (ordinary) fuzzy sets. Notwithstanding, most of those definitions can be easily generalised to the case where L represents an arbitrary lattice, not necessarily the unit interval. For the sake of simplicity and shortness, let L^U_k denote the initial space of the mapping M, i.e., the Cartesian product of k components of the form L^U. Let the reader notice that L^U_k together with the product (component-wise) order, $\leq_{L^U_k}$, also satisfies the properties of lattices.

We can in turn consider n-tuples of elements of L^U_k and define a component-wise ordering over the set $L^U_{k*n} = L^U_k \times \overset{(n)}{\cdots} \times L^U_k$, that we will denote as $\mathcal{L}$ from now on. The pair $(\mathcal{L}, \leq_{\mathcal{L}}) = (L^U_{k*n}, \leq_{L^U_{k*n}})$ also satisfies the properties of lattices. We can additionally consider the set $2^{\mathcal{L}}$ of mappings from $\mathcal{L}$ to the binary set $\{0, 1\}$, representing the power set of $\mathcal{L}$, together with the partial ordering associated to the inclusion relation. The pair $(2^{\mathcal{L}}, \subseteq)$ also satisfies the properties of lattices. Furthermore, the set $\mathbb{L}^U_{k*n}$ together with the component-wise ordering defined on it is also a lattice. Let us denote it $\boldsymbol{\mathcal{L}}$. We will also consider its power set $2^{\boldsymbol{\mathcal{L}}}$ and the lattice structure $(2^{\boldsymbol{\mathcal{L}}}, \subseteq)$. Table I summarises the information provided in this section.

TABLE I
DIFFERENT LATTICES INVOLVED IN THIS GENERAL FORMULATION.

GENERIC ELEMENT	SET	ORDER
a	L	$\leq_L$
$[a, b]$ with $a, b \in L,\;\; a \leq_L b$	$\mathbb{L}$	$\leq_{\mathbb{L}}$
$A : U \to L$	L^U	$\leq_{L^U}$
$A : U \to \mathbb{L}$	$\mathbb{L}^U$	$\leq_{\mathbb{L}^U}$
$\vec{A} = (A, B, \ldots)$, with $A, B, \overset{(k)}{\ldots} \in L^U$	L^U_k	$\leq_{L^U_k}$
$\mathbf{A} = (\vec{A}_1, \ldots, \vec{A}_n)$ with $\vec{A}_i \in L^U_k$	$\mathcal{L} = L^U_{k*n}$	$\leq_{\mathcal{L}} = \leq_{L^U_{k*n}}$
$\mathcal{A}$	$2^{\mathcal{L}}$	$\subseteq$
$\mathbf{A} = (\vec{A}_1, \ldots, \vec{A}_n)$ with $\vec{A}_i \in \mathbb{L}^U_k$	$\boldsymbol{\mathcal{L}} = \mathbb{L}^U_{k*n}$	$\leq_{\boldsymbol{\mathcal{L}}} = \leq_{\mathbb{L}^U_{k*n}}$
$\boldsymbol{\mathcal{A}}$	$2^{\boldsymbol{\mathcal{L}}}$	$\subseteq$

We will make use of the above notation in order to present our general formulation in the next section.

III. Axioms of L'-valued measures and operations between L-fuzzy sets

As we have mentioned in the Introduction, we have formally analysed in [1] and [11] an extensive collection of procedures of generalisation of numerical-valued measures from fuzzy sets to the contexts of IVF sets and AIF sets. We have distinguished between "scalar extensions" and "set-valued extensions".

In every "axiomatic" extension, the authors consider some particular notion originally defined in the context of fuzzy sets by means of a list of properties, and then adapt or extend each of those properties to the case of interval-valued or Atanassov intuitionistic fuzzy sets. We have reviewed in [1] a number of extensions and encompassed them into two different general extension procedures (respectively for set-valued measures and for numerical-valued extensions) Our study was restricted to the case of measures of the type $M : [0,1]^U \times \overset{(k)}{\cdots} \times [0,1]^U \to \mathbb{R}$ and did apply to other cases such as operations between fuzzy sets, that can be expressed as mappings of the form $M : [0,1]^U \times \overset{(k)}{\cdots} \times [0,1]^U \to [0,1]^U$.

In this paper, we go a step forward and, by means of using the language of lattices, we manage to provide a single general formulation that simultaneously encompasses all the previous extensions (set-valued and numerical-valued extensions of numerical measures, plus extensions of operations between fuzzy sets). Furthermore, this new approach encompasses in a single formula the original definition (in the fuzzy-sets context) and its corresponding extensions to the case of interval-valued fuzzy sets, as well as further extensions to even more complex structures such as type 2 fuzzy sets.

We will therefore refer in general to L'-valued mappings of the form $M : L_k^U \times \overset{(n)}{\cdots} \times L_k^U \to L'$. For the sake of simplicity, we will consider finite universes, and without loss of generality, we will identify each of them with the finite collection of natural numbers, $[m] = \{1, \ldots, m\}$ where m represents the corresponding (finite) cardinality. Thus, $L_k^{[m]}$ will denote the collection of L-fuzzy subsets of a universe of cardinality $m \in \mathbb{N}$. We will show that many axiomatic definitions from the literature follow a similar formal pattern that can be described as follows. Every definition is expressed in terms of a list of axioms, each of them matches the following general formula:

Definition 1: Consider a pair of lattices, L and L'. Given a particular tuple of mappings (F, G, H_1, H_2, T):

- $F : L_{k*n}^{[m]} \to L_{k*n}^{[m]}$, $G : L_{k*n}^{[m]} \to L_{k*n}^{[m]}$,
- $H_1 : L_{k*n}^{[m]} \to L_k^{[m]}$, $H_2 : L_{k*n}^{[m]} \to L_k^{[m]}$,
- $T : L_{k*n}^{[m]} \to L'$,

and a vector of scalars $(\lambda_1, \lambda_2, \lambda_3) \in \{-1, 0, 1\}^3$ with $-1 \leq \lambda_1 + \lambda_2 + \lambda_3 \leq 1$, we say that the mapping $M : \cup_{m \in \mathbb{N}} L_k^{[m]} \to L'$ satisfies the *lattice-fuzzy property* $\mathbf{P} = P_{(F,G,H_1,H_2,T,\lambda_1,\lambda_2,\lambda_3,\triangle,\triangledown,\bigcirc)}$ if:

$$\{\mathbf{A} : F(\mathbf{A}) \triangle G(\mathbf{A})\} \bigcirc \{\mathbf{A} : \sum_{i=1}^{2} \lambda_i M(H_i(\mathbf{A})) + \lambda_3 T(\mathbf{A}) \triangledown 0_{L'}\}, \quad (1)$$

Where

- $\triangle$ represents either $\leq_{L_{k*n}^{[m]}}$, $\geq_{L_{k*n}^{[m]}}$, or $=_{L_{k*n}^{[m]}}$,
- $\triangledown$ represents either $\leq_{L'}$, $\geq_{L'}$ or $=_{L'}$, and
- $\bigcirc$ either represents either $\subseteq$, $\supseteq$ or $=$

($\subseteq$ denotes the set inclusion -lattice ordering- defined on $2^{\mathcal{L}}$, as mentioned in the set of preliminaries).

Remark 1: In the above equation, $\bigcirc$ either represents a set-inclusion or an equality. The distinction between both of them is related to the notions of "property of type 1" and "property of type 2" considered in [1].

Remark 2: For the sake of shortness, from now on we will refer to lattice-fuzzy properties as *L-fuzzy properties*.

Remark 3: The above expression $\sum_{i=1}^2 \lambda_i M(H_i(\mathbf{A})) + \lambda_3 T(\mathbf{A}) \triangledown 0_{L'}$ included in Equation (1) needs a clarification, as we have not assumed the existence of the sum or the multiplication by a scalar on L'. It is a compact expression encompassing the following options, each of them in accordance with a particular selection of the tuple $(\lambda_1, \lambda_2, \lambda_3, \triangledown)$. Thus, it should be interpreted as:

- $M(H_1(\mathbf{A})) \geq_{L'} M(H_2(\mathbf{A}))$,
  for $(\lambda_1, \lambda_2, \lambda_3, \triangledown) \in \{(-1, 1, 0, \leq_{L'}), (1, -1, 0, \geq_{L'})\}$,
- $M(H_1(\mathbf{A})) \leq_{L'} M(H_2(\mathbf{A}))$,
  for $(\lambda_1, \lambda_2, \lambda_3, \triangledown) \in \{(-1, 1, 0, \geq_{L'}), (1, -1, 0, \leq_{L'})\}$,
- $M(H_1(\mathbf{A})) \geq_{L'} T(\mathbf{A})$
  when $(\lambda_1, \lambda_2, \lambda_3, \triangledown) \in \{(-1, 0, 1, \leq_{L'}), (1, 0, -1, \geq_{L'})\}$,

and

- $M(H_1(\mathbf{A})) \leq_{L'} T(\mathbf{A})$
  when $(\lambda_1, \lambda_2, \lambda_3, \triangledown) \in \{(-1, 0, 1, \geq_{L'}), (1, 0, -1, \leq_{L'})\}$.

Example 1: Suppose for instance that H_1 and H_2 are defined on $L_{2*2}^{[m]}$ as follows:

$$H_1((A, B), (C, D)) = (A \cap C, B \cap C),$$

$$H_2((A, B), (C, D)) = (A, B),$$

$\forall\, ((A, B), (C, D)) \in L_{2*2}^{[m]}$, $\forall\, m \in \mathbb{N}$, and consider the tuple $(\lambda_1, \lambda_2, \lambda_3, \triangledown) = (1, -1, 0, \geq_{L'})$. In this particular case, the above expression, $\sum_{i=1}^2 \lambda_i M(H_i(\mathbf{A})) + \lambda_3 T(\mathbf{A}) \triangledown 0_{L'}$, is interpreted as:

$$M(A \cap C, B \cap C) \geq_{L'} M(A, B).$$

(This expression appears in the definition of divergence between (interval-valued) fuzzy sets and their extensions (see the original definition in [8] and its extension to IVF sets in [9]).

Remark 4: We assume, for the sake of generality, the existence of arbitrary mappings F, G, H_1, H_2 and T satisfying Equation (1). Notwithstanding we should notice that they fulfill some additional restrictions in most of the cases: in particular, in most of the examples to be reviewed in this manuscript, F is made up of one or two components ($n = 1$ or $n = 2$). In the first case, F is a mapping from $L_k^{[m]}$ to $L_k^{[m]}$. In the second case, it can be written as a pair (F_1, F_2) with $F_i : L_{k*2}^{[m]} \to L_k^{[m]}$, $i = 1, 2$. In the first case ($n = 1$), the coefficient λ_2 is usually null and H_1 coincides with F. In the second case, H_1 and H_2 usually coincide respectively with F_1 and F_2.

On the other hand, F and G are usually expressed in terms of projections, permutations, and operations between L-fuzzy sets. With respect to the mapping T, we can distinguish two different kinds of formulations: in all the reviewed cases where the original fuzzy set definition corresponds to the case $L' = \mathbb{R}$, T is defined as a constant while, in those cases where $L' = L^U$, it is defined in terms of projections, permutations and/or operations, similarly to what happens with F, G, H_1 and H_2. This fact allows us to include every original axiomatic definition and its potential extensions to more complex structures in a single (encompassing) formulation.

Once we have established the general formulation of the properties to be listed in further axiomatic definitions, we can introduce the concept of "L-fuzzy notion", characterised by a finite sequence of those "L-fuzzy properties":

Definition 2: Consider a pair of lattices, L and L' and a finite sequence of L-fuzzy properties $\mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(r)}$, where

$$P^{(i)} = P_{(F^{(i)}, G^{(i)}, H_1^{(i)}, H_2^{(i)}, T^{(i)}, \lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)}, \triangle^{(i)}, \triangledown^{(i)}, \bigcirc^{(i)})},$$

for a certain sequence of mappings and operators, for all $i = 1, \ldots, r$. A mapping $M : \cup_{m\in\mathbb{N}} L_k^{[m]} \to L'$ is said to satisfy the *L-fuzzy notion* $\mathbf{N} = N(\mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(l)})$ if it satisfies all the above r L-fuzzy properties.

In the next subsections, we will show some examples of well-known axiomatic definitions in the context of fuzzy sets, together with their extensions to more general structures, and prove that all of them can be seen as particular cases of Definition 2.

A. Operations between fuzzy sets

In 1965, Zadeh [12] proposed the well known (constructive) definitions of union, intersection and complement of fuzzy sets. Later on in 1982, Klement [13] suggested to generalise them, by means of providing three separate lists of axioms that those three operations should satisfy in general (and that Zadeh's particular constructive definitions did satisfy). To give an example of those axiomatic definitions, let us recall his notion of intersection:

Definition 3: A mapping $M : [0, 1]^U \times [0, 1]^U \to [0, 1]^U$ is called an *intersection* if and only if the following properties are satisfied:

(I) Neutral element.- $M(A, 1_U) = A$, where 1_U denotes the constant membership function $1_U(x) = 1,\ \forall\, x \in U$,

(II) Monotonicity.- If $C \le A$ and $D \le B$ then $M(C, D) \le M(A, B)$,

(III) Symmetry.- $M(A, B) = M(B, A)$, $\forall\, A, B \in [0, 1]^U$,

(IV) Associativity.- $M(M(A, B), C) = M(A, M(B, C))$, $\forall\, A, B, C \in [0, 1]^U$.

If we replace $[0, 1]$ by a generic lattice L and we consider the collection of all finite universes of the form $U = [m] = \{1, \ldots, m\}$, Klement's definition of intersection can be alternatively formulated as follows:

Definition 4: Coinsider a lattice L. A mapping $M : \cup_{m\in\mathbb{N}} L_2^{[m]} \to \cup_{m\in\mathbb{N}} L^{[m]}$ is called an *L-fuzzy intersection* if and only if the following properties are satisfied, for every $m \in \mathbb{N}$:

(I) Neutral element.- $M(A, [m]) =_{L^{[m]}} A$, where $1_{L^{[m]}}$ denotes the top element of $(L^{[m]}, \le_{L^{[m]}})$

(II) Monotonicity.- If $C \le_{L^{[m]}} A$ and $D \le_{L^{[m]}} B$ then $M(C, D) \le_{L^{[m]}} M(A, B)$,

(III) Symmetry.- $M(A, B) =_{L^{[m]}} M(B, A)$, $\forall\, A, B \in L^{[m]}$,

(IV) Associativity.- $M(M(A, B), C) =_{L^{[m]}} M(A, M(B, C))$, $\forall\, A, B, C \in L^{[m]}$.

Let us now prove that Definition 4 corresponds to an L-fuzzy notion, i.e., it is a particular instance of Definition 2. In other words, let us check that the conditions (I)-(IV) are particular instances of L-fuzzy properties, and therefore, each of them fits the formulation established in Equation (1),

Proposition 1: The notion of L-fuzzy intersection is a particular case of L-fuzzy notion.

Proof: Let us prove that each of the four conditions established in Definition 4 admits an alternative formulation in terms of Equation (1). We can check that a mapping $M : \cup_{m\in\mathbb{N}} L_2^{[m]} \to \cup_{m\in\mathbb{N}} L^{[m]}$ is and L-fuzzy intersection if and only if the following properties are satisfied:

(I) $$\{(A, B) \in L_2^{[m]} : F^{(\mathrm{I})}(A, B) \ge_{L_2^{[m]}} G^{(\mathrm{I})}(A, B)\} \subseteq \{(A, B) \in L_2^{[m]} : M(H_1^{(\mathrm{I})}(A, B)) = T^{(\mathrm{I})}(A, B)\},$$

where $F^{(\mathrm{I})}$, $G^{(\mathrm{I})}$, $H_1^{(\mathrm{I})}$ and $T^{(\mathrm{I})}$ are respectively defined on $L_2^{[m]}$ as follows:

- $F^{(\mathrm{I})}(A, B) = H_1^{(\mathrm{I})}(A, B) = (A, B)$,
- $G^{(\mathrm{I})}(A, B) = (A, 1_{[m]})$ and
- $T^{(\mathrm{I})}(A, B) = A,\ \forall\, (A, B) \in L_2^{[m]}$,

where $1_{[m]} : [m] \to L$ is the constant mapping $1_{[m]}(x) = 1_L,\ \forall\, x \in [m],\ \forall\, m \in \mathbb{N}$.

(Let the reader notice that $F^{(\mathrm{I})}(A, B) \ge_{L_2^{[m]}} G^{(\mathrm{I})}(A, B)$ if and only if $B = 1_{[m]}$, and therefore we can easily deduce that the above formulation in terms of $F^{(\mathrm{I})}$, $G^{(\mathrm{I})}$, $H_1^{(\mathrm{I})}$ and $T^{(\mathrm{I})}$ is equivalent to the above original formulation included in Definition 4.

(II) $$\{((A, B), (C, D)) \in L_{2*2}^{[m]} : F^{(\mathrm{II})}((A, B), (C, D)) \le_{L_{2*2}^{[m]}} G^{(\mathrm{II})}((A, B), (C, D))\} \subseteq \{((A, B), (C, D)) \in L_{2*2}^{[m]} : M(H_1^{(\mathrm{II})}((A, B), (C, D))) \ge_{L_2^{[m]}} M(H_2^{(\mathrm{II})}((A, B), (C, D)))\},$$

where

- $F^{(\mathrm{II})}((A, B), (C, D)) = ((A, B), (C, D))$,
- $G^{(\mathrm{II})}((A, B), (C, D)) = ((A, B), (A \wedge C, B \wedge D)))$,
- $H_1^{(\mathrm{II})}((A, B), (C, D)) = (A, B)$, and
- $H_2^{(\mathrm{II})}((A, B), (C, D)) = (C, D)$,

$\forall\, A, B, C, D \in L^{[m]}$, and $\forall\, m \in \mathbb{N}$.
(Let us notice that
$F^{(\mathrm{II})}((A,B),(C,D)) \leq_{L^{[m]}_{2*2}} G^{(\mathrm{II})}((A,B),(C,D))$
if and only if $C \leq_{L^{[m]}} A$ and $D \leq_{L^{[m]}} B$).

(III) $\{((A,B),(C,D)) \in L^{[m]}_{2*2} :$
$F^{(\mathrm{III})}((A,B),(C,D)) \leq_{L^{[m]}_{2*2}}$
$G^{(\mathrm{III})}((A,B),(C,D))\} \subseteq$
$\{((A,B),(C,D)) \in L^{[m]}_{2*2} :$
$M(H_1^{(\mathrm{III})}((A,B),(C,D))) \leq_{L^{[m]}_2}$
$M(H_2^{(\mathrm{III})}((A,B),(C,D)))\}$,
where

- $F^{(\mathrm{III})}((A,B),(C,D)) = ((D,C),(B,A))$,
- $G^{(\mathrm{III})}((A,B),(C,D)) = ((A,B),(C,D))$,
- $H_1^{(\mathrm{III})}((A,B),(C,D)) = (D,C)$,
- $H_2^{(\mathrm{III})}((A,B),(C,D)) = (B,A)$,

$\forall\, A, B, C, D \in L^{[m]}$, and $\forall\, m \in \mathbb{N}$.
Let us check that the above condition means that M is symmetric. In fact, we observe that the condition $F^{(\mathrm{III})}((A,B),(C,D)) \leq_{L^{[m]}_{2*2}} G^{(\mathrm{III})}((A,B),(C,D))$ is equivalent to the condition $(C,D) = (B,A)$. Let us also notice that M is symmetric if and only if $M(A,B) \geq_{L^{[m]}} M(B,A),\ \forall\,(A,B) \in L^{[m]}_2$, i.e., the condition $M(A,B) \geq_{L^{[m]}} M(B,A),\ \forall\,(A,B) \in L^{[m]}_2$ is equivalent to the -initially supposedly stronger- condition $M(A,B) = M(B,A),\ \forall\,(A,B) \in L^{[m]}_2$.

(IV) $\{((A,B),(C,D)) \in L^{[m]}_{2*2} :$
$F^{(\mathrm{IV})}((A,B),(C,D)) \leq_{L^{[m]}_{2*2}}$
$G^{(\mathrm{IV})}((A,B),(C,D))\} \subseteq$
$\{(A,B),(C,D) \in L^{[m]}_{2*2} :$
$M(H_1^{(\mathrm{IV})}((A,B),(C,D)) =$
$M(H_2^{(\mathrm{IV})}((A,B),(C,D))\}$,
where

- $F^{(\mathrm{IV})}((A,B),(C,D)) = ((M(A,B),C),(A,M(B,C)))$,
- $G^{(\mathrm{IV})}((A,B),(C,D)) = ((1_{[m]},1_{[m]}),(1_{[m]},1_{[m]}))$,
- $H_1^{(\mathrm{IV})}((A,B),(C,D)) = (M(A,B),C)$,
- $H_2^{(\mathrm{IV})}((A,B),(C,D)) = (A,M(B,C))$,

$\forall\, A, B, C, D \in L^{[m]}$, $\forall\, m \in \mathbb{N}$.
The above condition means that M is associative: First of all, let us notice that every tuple $((A,B),(C,D)) \in L^{[m]}_{2*2}$ straightforwardly satisfies the condition

$$F^{(\mathrm{IV})}((A,B),(C,D)) \leq_{L^{[m]}_{2*2}} G^{(\mathrm{IV})}((A,B),(C,D)).$$

Thus, the above condition is equivalent to the following one:

$$M(H_1^{(\mathrm{IV})}((A,B),(C,D)) = M(H_2^{(\mathrm{IV})}((A,B),(C,D)),$$

for every tuple $((A,B),(C,D)) \in L^{[m]}_{2*2}$, i.e., it is equivalent to the condition:

$$M(M(A,B),C) = M((A,M(B,C))),$$

$\forall\, A, B, C \in L^{[m]}$. □

Remark 5: The general formulation considered in Equation (1) admits plenty of possibilities for H_1 and H_2 in relation with F and G. Let us nevertheless notice that they may coincide in many cases with the components of the mapping F, as it happens in all the previous examples.

Remark 6: We can easily check that Definition 4 comes down to Klement's definition of intersection (Definition 3) for $L = [0,1]$. On the other hand, when $(L, \leq_L)$ is considered to be the family of subintervals of $[0,1]$, together with the usual partial interval ordering, then it can be seen as generalization of Klement's definition to the case of interval-valued fuzzy sets. We could even assume that the lattice L coincides with the family of fuzzy subsets of the unit interval $[0,1]^{[0,1]}$. In that case, Definition 4 would be an axiomatic definition of the notion of intersection between type 2 fuzzy sets. Our definition also generalizes Torra's definition of intersection between hesitant fuzzy sets, as we check in the next remark. Thus, the formulation established in Equation (1) simultaneously encompasses Klement's axioms and their corresponding generalizations to IVF, T2F and HF sets. Something analogous can be said about the axiomatic definitions of the operations of union and complement.

Remark 7: The notions of intersection, union and complement have been also considered in other fuzzy-related contexts such as hesitant fuzzy sets ([4]). According to Torra, a (typical) hesitant fuzzy set is characterised by a finite-set-valued membership function, i.e, by mapping $A : U \to \wp^F([0,1])$, where $\wp^F([0,1])$ denotes the collection of finite non-empty subsets of $[0,1]$. Let $HFS(U)$ denote the family of typical hesitant fuzzy subsets of U. Torra [4] introduced constructive definitions of intersection, union and complement of hesitant sets, extending Zadeh's original definition to this context. The intersection of two hesitant fuzzy sets $A, B \in HFS(U)$ is defined as the hesitant fuzzy set $A \cap B \in HFS(U)$ defined as follows:

$$A \cap B(x) = \{c \in A(x) \cup B(x) \,:\, c \leq \min\{\max A(x), \max B(x)\}\}.$$

We can easily check that the above expression is equivalent to the following one:

$$A \cap B(x) = \{\min\{a,b\} \,:\, a \in A(x), b \in B(x)\}.$$

If we consider the component-wise ordering between finite subsets of the unit interval defined by Farhadinia in [14], and the inclusion of hesitant sets based on it $\leq_H$, then the pair $(HFS(U), \leq_H)$ satisfies the properties of lattices. (Please see [15] for a detailed explanation.) Taking this fact into account, we can easily observe that Torra's definition of intersection can be also regarded as a particularisation of the general notion of intersection introduced in Definition 4, and therefore, according to Proposition 1, also a particularisation of Definition 2.

B. Real-valued measures

During the last decades, many authors have introduced different axioms of measures of similarity, dissimilarity, distance, divergence, etc. between fuzzy sets (see [16] and [17] for an overview). Some of those definitions have been also recently extended to more general contexts such as Atanassov intuitionistic fuzzy sets or interval-valued fuzzy sets (see [18], [19], [9]) or type 2 fuzzy sets (see definition provided in [20] also recalled in [21]). As an example of this kind of definitions, we will provide below an axiomatic definition of "distance" initially defined in the context of fuzzy sets and later extended by Dengfeng and Chuntian [18] to the context of interval-valued fuzzy sets:

Definition 5: A mapping $M : [0,1]^U \times [0,1]^U \to \mathbb{R}$ is called a *measure of similarity* between fuzzy sets if it satisfies the following axioms:

(V) $0 \leq M(A,B),\ \forall (A,B) \in [0,1]^U \times [0,1]^U$.
(VI) $M(A,B) \leq 1,\ \forall (A,B) \in [0,1]^U \times [0,1]^U$.
(VII) $M(A,B) = M(B,A),\ \forall (A,B) \in [0,1]^U \times [0,1]^U$.
(VIII) If $A \subseteq B \subseteq C \Rightarrow M(A,C) \leq M(A,B)$.
(IX) If $A \subseteq B \subseteq C \Rightarrow M(A,C) \leq M(B,C)\}$.

The above definition can be extended to the case of L'-valued similarity measures between pairs of L-fuzzy sets as follows:

Definition 6: Consider a pair of lattices L and L'. A mapping $M : \cup_{m\in\mathbb{N}} L_2^{[m]} \to L'$ is called an *L-fuzzy similarity* if the following properties are satisfied:

(V) $0_{L'} \leq_{L'} M(A,B),\ \forall (A,B) \in L_2^{[m]}$.
(VI) $M(A,B) \leq_{L'} 1_{L'},\ \forall (A,B) \in L_2^{[m]}$.
(VII) $M(A,B) =_{L'} M(B,A),\ \forall (A,B) \in L_2^{[m]}$.
(VIII) If $A \leq_{L^{[m]}} B \leq_{L^{[m]}} C \Rightarrow M(A,C) \leq_{L'} M(A,B)$.
(IX) If $A \leq_{L^{[m]}} B \leq_{L^{[m]}} C \Rightarrow M(A,C) \leq_{L'} M(B,C)\}$.

Remark 8: Strictly speaking the property of Symmetry included in Definition 4 (Property (VII)) does not exactly coincide with Property (III) from Definition 4, as the respective codomains, $\cup_{m\in\mathbb{N}} L_2^{[m]}$ and L' of the corresponding mappings do not coincide. In fact, Property (III) can be seen as a particular instance of Property (VII).

Similarly to what we have done in the previous section, we will show that everyone of the above axioms can be seen as a particular case of Equation (1).

Proposition 2: The notion of L-fuzzy similarity is a particular case of L-fuzzy notion.

Proof: We can easily observe that a mapping $M : \cup_{m\in\mathbb{N}} L_2^{[m]} \to L'$ is an L-fuzzy similarity if and only if the following properties are satisfied:

(V) $\{(A,B) \in L_2^{[m]} : F^{(\mathrm{V})}(A,B) \leq_{L_2^{[m]}} G^{(\mathrm{V})}(A,B)\} \subseteq \{(A,B) \in L_2^{[m]} : M(H_1^{(\mathrm{V})}(A,B)) \geq T^{(\mathrm{V})}(A,B)\}$, where
- $F^{(\mathrm{V})}(A,B) = (A,B)$,
- $G^{(\mathrm{V})}(A,B) = (1_{[m]}, 1_{[m]})$
- $H_1^{(\mathrm{V})} = F^{(\mathrm{V})}$, and
- $T^{(\mathrm{V})}(A,B) = 0_{L'}$,

$\forall (A,B) \in L_2^{[m]},\ \forall m \in \mathbb{N}$.

(VI) $\{(A,B) \in L_2^{[m]} : F^{(\mathrm{VI})}(A,B) \leq_{L_2^{[m]}} G^{(\mathrm{VI})}(A,B)\} \subseteq \{(A,B) \in L_2^{[m]} : M(H_1^{(\mathrm{VI})}(A,B)) \leq T^{(\mathrm{VI})}(A,B)\}$, where
- $F^{(\mathrm{VI})}(A,B) = (A,B)$,
- $G^{(\mathrm{VI})}(A,B) = (1_{[m]}, 1_{[m]})$
- $H_1^{(\mathrm{VI})} = F$,
- $T^{(\mathrm{VI})}(A,B) = 1_{L'}$,

$\forall (A,B) \in L_2^{[m]},\ \forall m \in \mathbb{N}$.

(VII) $\{((A,B),(C,D)) \in L_{2*2}^{[m]} : F^{(\mathrm{III})}((A,B),(C,D)) \leq_{L_{2*2}^{[m]}} G^{(\mathrm{III})}((A,B),(C,D))\} \subseteq \{((A,B),(C,D)) \in L_{2*2}^{[m]} : M(H_1^{(\mathrm{III})}((A,B),(C,D))) \leq_{L'} M(H_2^{(\mathrm{III})}((A,B),(C,D)))\}$, where $F^{(\mathrm{III})}$, $G^{(\mathrm{III})}$, $H_1^{(\mathrm{III})}$, and $H_2^{(\mathrm{III})}$ are the ones defined in Proposition 1. $\forall\, A,B,C,D \in L^{[m]}$, and $\forall\, m \in \mathbb{N}$. (The only thing that changes wrt Part (III) in Proposition 1 is the codomain of the mapping M.)

(VIII) $\{(A,B) \in L_2^{[m]} : F^{(\mathrm{V})}((A,B),(C,D)) \leq_{L_2^{[m]}} G^{(\mathrm{V})}((A,B),(C,D))\} \subseteq \{(A,B) \in L_2^{[m]} : M(H_1^{(\mathrm{V})}((A,B),(C,D))) \geq M(H_2^{(\mathrm{V})}((A,B),(C,D)))\}$, where
- $F^{(\mathrm{VIII})}$ is the identity, i.e.

$$F^{(\mathrm{VIII})}((A,B),(C,D)) = ((A,B),(C,D)),\ \forall ((A,B),(C,D)) \in L_{2*2}^{[m]}$$

and
- $G^{(\mathrm{VIII})}((A,B),(C,D)) = ((C,B),(A,B)), \forall ((A,B),(C,D)) \in L_{2*2}^{[m]}$.
- $H_1^{(\mathrm{VIII})}((A,B),(C,D)) = F_1^{(\mathrm{VIII})}((A,B),(C,D)) = (A,B)$
- $H_2^{(\mathrm{VIII})} = F_2^{(\mathrm{VIII})}((A,B),(C,D)) = (C,D)$

(We can easily observe that $F^{(\mathrm{VIII})}((A,B),(C,D)) \leq G^{(\mathrm{VIII})}((A,B),(C,D))$ if and only if $A = C$ and $D \subseteq B$). □

Some authors (see [22]) add the following (boundary) conditions to be satisfied by similarity measures between fuzzy sets:

(X) If $A = B$, then $M(A,B) = 1$.
(XI) If $A \in 2^U$ and $B = A^c$ then $M(A,B) = 0$, where $A^c : U \to [0,1]$ denotes the complement, i.e. $A^c(x) = 1 - A(x),\ \forall x \in U$.

These conditions can also be expressed as particular instances of Equation (1), as we prove below.

Proposition 3: Conditions (X) and (XI) are L-fuzzy properties.

Proof: We can easily observe that they can be written alternatively as follows:

(X) $\{(A,B) \in L_2^{[m]} : F^{(\mathrm{X})}(A,B) \leq_{L_2^{[m]}} G^{(\mathrm{X})}(A,B)\} \subseteq \{(A,B) \in L_2^{[m]} : M(H_1^{(\mathrm{X})}(A,B)) = T^{(\mathrm{X})}(A,B)\}$, where
- $F^{(\mathrm{X})}(A,B) = (A,B)$
- $G^{(\mathrm{X})}(A,B) = (B,A)$
- $H_1^{(\mathrm{X})}(A,B) = (A,B)$,
- $T^{(\mathrm{X})}(A,B) = 1_{L'}$,

$\forall (A,B) \in L_2^{[m]},\ \forall m \in \mathbb{N}$.

(XI) $\{(A,B) \in L_2^{[m]} : F^{(\mathrm{XI})}(A,B) \leq_{L_2^{[m]}} G^{(\mathrm{XI})}(A,B)\} \subseteq \{(A,B) \in L_2^{[m]} : M(H_1^{(\mathrm{XI})}(A,B)) = T^{(\mathrm{XI})}(A,B)\}$, where $F^{(XI)}$ and $G^{(XI)}$ $H_1^{(XI)}$ and $T^{(XI)}$ are respectively defined on $L_2^{[m]}$ as follows:

- $F^{(XI)}(A,B) = (A \wedge B, U)$,
- $G^{(XI)}(A,B) = (\emptyset, A \vee B)$,
- $H_1^{(XI)} = F^{(XI)}(A,B) = (A \wedge B, U)$,
- $T^{(\mathrm{XI})}(A,B) = 0_{L'}$,

$\forall\,(A,B) \in L_2^{[m]},\ \forall\, m \in \mathbb{N}$.

We can observe that $F^{(XI)}(A,B) \leq_{L_2^{[m]}} G^{(XI)}(A,B)$ if and only if $A(x) \wedge B(x) = 0'_L$ and $A(x \vee B(x) = 1'_L$ for all $x \in [m]$, something that is satisfied in particular when $(A(x), B(x)) \in \{(0_L, 1_L), (1_L, 0_L)\}$ for all $x\ n[m]$. □

Similarity measures considered by Dengfeng and Chuntian are numerical-valued, even when computing the similarity between two AIF sets. But, as we have mentioned in the Introduction, other authors have considered set-valued extensions of the notion of similarity to the cases of IVF and AIF sets. To give an example, the axiomatic definition introduced by Galar et al. in [23] considers interval-valued similarity measures between IVF sets. Their definition also matches Definition 2. More concretely, it particularises Axioms V-XI together with Axiom XII (below) to the case $L = L' = \mathbb{I}([0,1])$:

(XII) $M(A,B) = M(A^c, B^c)$, for all $(A,B) \in L_2^{[m]}$,

The above condition can be alternative expressed as an L-fuzzy property as follows:

(XII) $\{((A,B),(C,D)) \in L_{2*2}^U :$
$F^{(\mathrm{XII})}((A,B),(C,D)) = G^{(\mathrm{XII})}((A,B),(C,D))\} =$
$\{(A,B) \in L_{2*2}^U : M(H_1^{(\mathrm{XII})}((A,B),(C,D))) =$
$M(H_2^{(\mathrm{XII})}((A,B),(C,D)))\}$,

where

- $F^{(\mathrm{XII})}((A,B),(C,D)) = ((A,B),(C,D))$,
- $G^{(\mathrm{XII})}((A,B),(C,D)) = ((C^c, D^c),(A,B))$,
- $H_1^{(\mathrm{XII})}((A,B),(C,D)) = (A,B)$,
- $H_2^{(\mathrm{XII})}((A,B),(C,D)) = (C,D)$,

$\forall\,((A,B),(C,D)) \in L_{2*2}^{[m]},\ \forall\, m \in \mathbb{N}$.

Thus, we can can deduce that the notion of interval-valued similarity measure proposed by Galar et al. [23] can be also generalized as an L-fuzzy notion.

IV. Concluding remarks and further perspectives

In summary, we had proposed in [1] two general notions encompassing, as particular cases, many extensions of axiomatic definitions from the context of (ordinary) fuzzy sets to the cases of IVF and AIF sets. In this paper we have introduced a single formulation that does not only include all those extensions of numerical-valued measures, but also additional extensions of other types of functions, such as operations between fuzzy sets.

In Subsections III-A and III-B, we have only mentioned some examples to illustrate this fact, but many additional extensions to different contexts of well-known "fuzzy" notions encountered in the literature can be also regarded as particularisations of Definition 2. Some of them could be framed into the pair of general definitions proposed in [1]: In this regard, some examples of set-valued axiomatic extensions to the frameworks of IVF and IF-sets, such as the "set-valued inclusion" by Cornelis and Kerre in [24] (originally defined by Sinha-Dougherty [25] in the context of fuzzy sets), of some set-valued similarity measures independently introduced by Stachowiak-Dyczkowski in [26] and Galar et al. in [23], can be seen as particular instances of Definition 5 in [1]. On the other hand, some numerical-valued extensions like the notions of similarity between two AIF sets (two different variants have been respectively proposed in [18] and [19]), distance [19], dissimilarity [9] or divergence [9] can be regarded as particular examples of Definition 11 in [1].

All the above examples can be framed in turn into the general setting proposed here, and seen as particular instances of the general definition of L-fuzzy notion. But with this new general formulation, we do not restrict ourselves to the contexts of IVF and AIF sets, and therefore we can cite additional examples from the literature that could not be included in our initial formulation proposed in [1], like the notions of inclusion and similarity between T2F sets introduced by Yang and Lin in [27], the notions of similarity and distance between T2F sets considered by Hung and Yang [28], different axiomatisations of similarity, distance and entropy between hesitant fuzzy sets (see [14], [29], for instance) inclusion measures between hesitant fuzzy sets [29], divergences between hesitant fuzzy sets [10]. Further examples in different generalised contexts can be found. Notwithstanding, an exhaustive list of the axiomatic definitions that can be regarded as particular instances of Definition 2 would fall outside the scope of this manuscript.

One of the main features of this new formulation is that we do not need separate equations for every particular framework, but the same formula applies in different contexts, each of them referring to a particular instance of the lattice L. Thus, a single general L-fuzzy notion encompasses the original fuzzy notion, together with its possible extensions to more complex frameworks.

Another feature of this general formulation is that we do not need to distinguish between scalar and interval-valued extensions of the original notions in the fuzzy sets context. We can use the same formulation for both of them, by means of referring to a general lattice L', that can be instantiated as a bounded interval, the set of subintervals of a given interval, etc. In our previous paper [1] we just referred to numerical and interval-valued generalisations, but this more general formulation, including a general lattice L' opens the door to consider other extensions with different types of outcomes (like fuzzy-valued extensions of numerical-valued measures, for instance).

According to the general formula proposed in Equation (1), many axiomatic fuzzy notions can be reformulated as L-fuzzy notions. Taking all these facts into account would contribute to a unified view about the study of different measures and operations. It could furthermore result in a reduction in the number of mathematical proofs, counterexamples and definitions, as they would not need to be separately formulated in every particular context. Furthermore, in some cases, the general formulation in terms of lattices could reduce the complexity of some results, as some particular lattices considered in the literature (let us think about complex structures such as interval-valued hesitant sets, for instance) involve the treatment of a multitude of parameters.

Notwithstanding, a particular formulation in a specific context can be justified by the need to emphasise an specific interpretation of that notion in that particular framework.

In the near future, we plan to evaluate different constructive extension procedures from the literature under the light of this general formulation, and see whether they can be encompassed in a single general one. In particular, we will try to check whether the three constructive extension methods proposed in [11] for set-valued extended measures (which encompasses in turn a multitude of particular constructions, see [30], [31], [32], [33], [34], [35], [36] for some examples) and the three additional extension methods proposed at the end of [1] for numerical-valued extensions (encompassing many constructive proposals such as [37], [38], [39], [40], [41], [42], [43]) can be regarded as particular instances of a more general formulation.

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