

BUCKLING OF LAMINATED GLASS PLATES USING THE EFFECTIVE THICKNESS CONCEPT

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ABSTRACT

Laminated glass is a sandwich material composed of two or more glass layers and one or more polymeric interlayers. The mechanical behaviour of glass is commonly considered linear-elastic whereas most laminated interlayers exhibit a viscoelastic behaviour. The mechanical behaviour of laminated glass elements is governed by the material properties of the interlayer, which means that the critical buckling loads of laminated glass panels are time and temperature dependent. In this paper, a simplified method to calculate the buckling critical load of rectangular laminated glass plates is presented, which uses the analytical solutions of elastic monolithic plates, the quasi-elastic approximation and the effective thickness concept. The analytical solutions are validated by numerical simulations on simply supported laminated glass plates subject to different loading conditions.

KEYWORDS

Laminated glass, structural composites, PVB, bucking, structural stability, viscoelasticity.

NOMENCLATURE

$$D_1 = \frac{EH_1^3}{12(1 - \nu_1^2)}$$

$$D_3 = \frac{EH_3^3}{12(1 - \nu_3^2)}$$

$$D_T = D_1 + D_3$$

$$D_{TOT=} = D_T(1 + Y_p)$$

E_{eff} Effective Young modulus

E Glass Young modulus of glass layers

$E_2(t)$ Viscoelastic relaxation tensile modulus for polymeric interlayer

$G_2(t)$ Viscoelastic relaxation shear modulus for the polymeric interlayer

H_1 Thickness of glass layer 1 in laminated glass

H_2 Thickness of polymeric layer in laminated glass

H_3 Thickness of glass layer 3 in laminated glass

$$H_{TOT} = H_1 + H_2 + H_3$$

$$H_{13} = H_2 + \left(\frac{H_1 + H_3}{2} \right)$$

I Second moment of area

$$I_1 = \frac{bH_1^3}{12}$$

$$I_3 = \frac{bH_3^3}{12}$$

$$I_T = I_1 + I_3 = b \frac{H_1^3 + H_3^3}{12}$$

$$I_{TOT} = I_T(1 + Y)$$

$K_2(t, T)$ Viscoelastic bulk modulus

L Length of a glass beam

$P(t, T)$ Critical load

T Temperature

T_0 Reference temperature

$$Y = \frac{H_{13}^2 H_1}{I_T (H_1 + H_3)}$$

$$Y_P = \frac{H_{13}^2 \frac{E H_1}{(1 - \nu^2)} \frac{E H_3}{(1 - \nu^2)}}{(D_1 + D_3) \left(\frac{E H_1}{(1 - \nu^2)} + \frac{E H_3}{(1 - \nu^2)} \right)} = \frac{12 H_1 H_3 H_{13}^2}{(H_1 + H_3)(H_1^3 + H_3^3)}$$

LOWERCASE LETTERS

a_T Shift factor

$g(x)$ Shape function in beams

$g(x, y)$ Shape function in plates

t Time

w Deflection

GREEK LETTERS

η_2 Loss factor of the polymeric interlayer of laminated glass

ν Poisson ratio of the glass layers

$\nu_2(t, T)$ Viscoelastic Poisson ratio of the polymeric interlayer

1 INTRODUCTION

Laminated glass is a sandwich or layered material which consists of two or more plies of monolithic glass with one or more interlayers of a polymeric material which usually show a viscoelastic behaviour, i.e., their mechanical properties are time (or frequency) and temperature dependent [1]. Polyvinyl butyral (PVB) is the most used interlayer material, however the new ionoplastic interlayers improve the mechanical properties of laminated glass for a large range of temperatures [1].

Structural stability is one of the design requirements in laminated glass beams and plates due to their slenderness and brittleness. Due to the fact that the stiffness of the interlayer is temperature and time dependent, the same can be said about the critical load, i.e., the critical load of a laminated glass plate subject to constant compressive loading decreases with time. Most of the analytical models for predicting the buckling of laminated glass plates have been derived from classical sandwich theory formulations [2, 3, 4] where it is generally assumed that the materials exhibit a linear-elastic behavior and shear deformations occur in the core. The equations for sandwich beams and plates with linear-elastic interlayers can be easily extended to viscoelastic interlayers using the correspondence principle [5, 6].

In order to facilitate the calculation of laminated glass elements, the viscoelastic solution can be simplified using the quasi-elastic method, which consists of describing the viscoelastic behaviour of the interlayer by an elastic behaviour with parameters that depend on the load duration and temperature [7, 8, 9]. This means that the memory effect of the viscoelastic material is neglected and that the mechanical properties are linear-elastic but dependent on time [10].

The concept of effective thickness has been proposed in recent years [6, 7, 8] based on the quasi-elastic solution, which consists of calculating the thickness (time and temperature dependent) of a monolithic element with bending properties equivalent to those of the laminated one. The effective thickness can then be used in analytical equations and simplified finite element models in place of the layered laminated-glass element [11, 12, 13, 14]. An effective Young modulus [13] can also be formulated from the models developed by Benninson et al [10, 14] and Galuppi and Royer-Carfagni [11],

which is more attractive to be used in numerical models because a monolithic model with constant thickness is defined, whereas the Young modulus is time and temperature dependent. The effective thickness and the effective Young modulus concepts can be used interchangeably with the same accuracy.

Several analytical models have been proposed for determining the critical load of a simply supported laminated glass beams [15, 16, 17, 18]. Galuppi and Royer-Carfagni [17] developed an analytical model for the buckling of a simply supported laminated glass column with viscoelastic core under a compressive load $P(t)$, which can be time dependent. A full viscoelastic solution is developed but a simpler model was derived using the quasi-elastic approximation [17].

The effect of the boundary conditions in monolithic beams is considered through the buckling ratio β (or alternatively with the effective length L_{eff}), whereas the stiffness EI is constant. Aenlle et al [19] proposed a simplified method to calculate critical loads in laminated glass beams with different boundary conditions using the Euler theory [20] of monolithic beams, the quasi-elastic solution [9,10] and the effective stiffness, and the effect of the boundary conditions are taken into account through the buckling ratios β derived for elastic monolithic beams [20, 21]. The authors demonstrated that that the effective stiffness, the effective thickness and the effective Young's modulus also depend on the boundary conditions and their effect can also be taken into account through the buckling ratio β . The model was validated by experimental tests and numerical models.

In this paper a simplified method to calculate critical loads in laminated glass plates using the solutions of elastic monolithic plates [20, 21], the quasi-elastic solution [9] and the effective stiffness concept [19] is proposed. In order to validate the model, the critical load of several simply supported laminated glass plates, made of annealed glass plies and PVB interlayer, subject to different loading conditions, were predicted using the simplified model proposed in this paper and validated by numerical simulations carried out with the finite element software ABAQUS [22]. It is concluded that the simplified analytical model predict the critical load of simply supported laminated glass plates with an error less than 10% for all the cases considered in the paper.

2. STATE OF THE ART

2.1 Sandwich beams with thin faces under static loading

The displacement of a simply supported sandwich beam with thin faces (see Figure 1) of equal thickness $H_1 = H_3 = H$ and Young modulus E , core with thickness H_2 and shear modulus G_2 , span L and with a central point load P can be found superimposing the bending displacement w_1 and the shear displacement w_2 [3], i.e.:

$$w = w_1 + w_2 = w_1 \left(1 + \frac{w_2}{w_1} \right) \quad (1)$$

where

$$w_1 = \frac{PL^3}{48EI} \quad (2)$$

$$w_2 = \frac{PLH_2}{4bG_2H_{13}^2} \quad (3)$$

$$EI = \frac{EH^3}{6} + \frac{EHH_{13}^2}{2} \quad (4)$$

$$H_{13} = H + H_2 \quad (5)$$

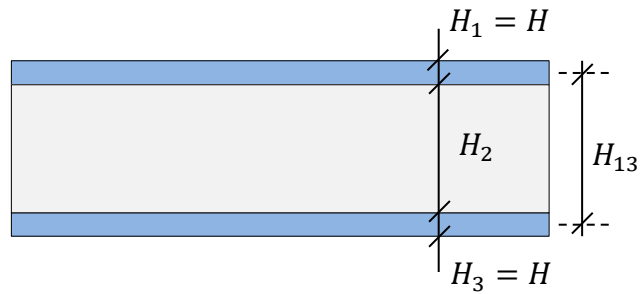


Figure 1. Sandwich beam with two external faces of thicknesses $H = H_1 = H_2$ and one interlayer with thickness H_2 .

In order to facilitate the calculation of displacements in sandwich beams, an effective stiffness EI_{eff} to be used with bending monolithic models, can be derived from Eq. (1), i.e:

$$\frac{PL^3}{48EI_{eff}} = \frac{PL^3}{48EI} \left(1 + \frac{\frac{PLH_2}{4bG_2H_{13}^2}}{\frac{PL^3}{48EI}} \right) \quad (6)$$

which results in:

$$EI_{eff} = EI \left(\frac{1}{1 + \frac{EI \cdot H_2}{G_2bH_{13}^2} \psi_{thin}} \right) \quad (7)$$

Where ψ_{thin} is a parameter which depends on the loading and boundary conditions. Expressions for the effective stiffness of any statically-determinate symmetrically-loaded sandwich beam can be formulated superimposing the bending and the shear deflections [3]. For a simply supported beam with a central point load $\psi_{thin} = 12/L^2$, whereas $\psi_{thin} = 9.6/L^2$ for a simply supported beam under distributed loading.

2.2 Sandwich beams with thick faces under static loading

In the case of sandwich beams with thick faces, the effective stiffness is derived using the same procedure as that used in the previous section and it is given by:

$$EI_{eff} = EI \left(\frac{1}{1 + \frac{EI \cdot H_2}{G_2bH_{13}^2} \psi_{thick}} \right) \quad (8)$$

Where:

$$\psi_{thick} = \psi_{thin} \left(1 - \frac{I_f}{I} \right)^2 \gamma \quad (9)$$

In Eq. (9), I_f is the inertia of one of the faces with respect to its own centroidal axis. The expressions of parameters ψ_{thin} and γ for simply supported beams under mid-point concentrated loading and uniformly distributed loading are presented in Table 1, where the parameter θ is given by [3]:

$$\theta = \frac{L}{2} \sqrt{\frac{G_2 b H_{12}^2}{H_2 E I_f \left(1 - \frac{I_f}{I}\right)}} \quad (10)$$

Table 1. Parameters ψ_{thin} and γ for simply supported beams.

Beam	ψ_{thin}	γ
Central point load	$12/L^2$	$1 - \frac{\sinh \theta + \frac{(1 - \cosh \theta)}{\tanh \theta + 1/\tanh \theta}}{\theta}$
Distributed load	$9.6/L^2$	$1 + \frac{2}{\theta^2 \cosh \theta} (1 - \cosh \theta)$

If $I_f \rightarrow 0$, from Eq. (9) it is easily derived that:

$$\psi_{thick} = \psi_{thin} \quad (11)$$

2.3 Buckling of a monolithic beam

The critical load of a linear-elastic monolithic beam with constant cross section and stiffness EI , according to the Euler theory, is given by [20, 21]:

$$N_{crit} = \frac{\pi^2}{(\beta L)^2} EI \quad (12)$$

Where β is the buckling ratio.

If the critical load is calculated by the Energy method [20, 21] with an approximate deflection curve $g(x)$, the critical load is obtained from:

$$N_{crit} = \frac{\int_0^L EI g''(x)^2 dx}{\int_0^L g'(x)^2 dx} \quad (13)$$

From which is inferred that:

$$\frac{\pi^2}{(\beta L)^2} = \frac{\int_0^L g''(x)^2 dx}{\int_0^L g'(x)^2 dx} \quad (14)$$

For a simply supported beam the mode shapes are sinusoidal in shape, i.e.:

$$g(x) = A \sin\left(\frac{n\pi x}{L}\right) \quad (15)$$

Substitution of Eq. (15) in Eq. (14) gives for $n = 1$:

$$\frac{\pi^2}{(\beta L)^2} = \frac{\pi^2}{L^2} \quad (16)$$

From which it is obtained that $\beta = 1$. This result coincides with buckling ratio of a simply supported beam.

2.4 Buckling of sandwich beams with thin faces.

The critical load of a simply supported sandwich beam with thin faces can be estimated using the expression [3]:

$$N_{crit} = \frac{\pi^2}{L^2} EI_{eff} \quad (17)$$

where

$$EI_{eff} = EI \left(\frac{1}{1 + \frac{EI \cdot H_2}{G_2 b H_{13}^2} \psi_{thin}} \right) \quad (18)$$

and

$$\psi_{thin} = \frac{\pi^2}{L^2} \quad (19)$$

It can be observed that Eqs. (7) and (18) coincide, i.e. the expression of the effective stiffness is the same for static loads and for buckling but with the special feature that the parameters ψ_{thin} are different in statics and in buckling, for the same boundary conditions.

2.5 Buckling of sandwich beams with thick faces

The critical load of a sandwich beam with thin faces can be estimated with Eq. (17) but with a different effective stiffness given by:

$$EI_{eff} = EI \left(\frac{1}{1 + \frac{EI \cdot H_2}{G_2 b H_{13}^2} \psi_{thick}} \right) \quad (20)$$

where

$$\psi_{thick} = \psi_{thin} \left(\frac{\left(1 - \left(\frac{I_f}{I}\right)\right)^2}{1 + \frac{EI_f \cdot H_2 \pi^2}{G_2 b H_{12}^2 L^2} \left(1 - \frac{I_f}{I}\right)} \right) \quad (21)$$

From Eq. (21) it is inferred that:

$$\psi_{thick} = \psi_{thin} \quad (22)$$

when $I_f \rightarrow 0$.

2.6 Buckling of rectangular monolithic plates

The critical load of a simply supported rectangular monolithic plate with dimensions $a \times b$, thickness H , and subject to uniaxial compression in the “x” direction (see Figure 2) is given by:

$$N_{xcrit} = \frac{\pi^2}{b^2} kD \quad (23)$$

Where D is the stiffness of the plate:

$$D = \frac{EH^3}{12(1-\nu^2)} \quad (24)$$

and k is a parameter dependent on the boundary conditions, which is expressed as:

$$k = \left(\frac{mb}{a} + \frac{an^2}{mb} \right)^2 \quad (25)$$

where the parameters m and n are the number of half-waves that the plate buckles into the x and y directions, respectively.

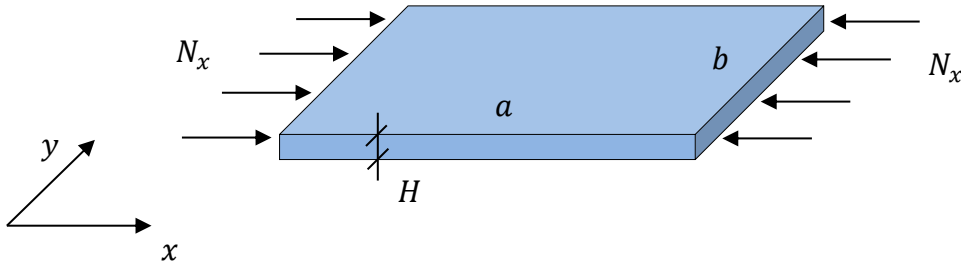


Figure 2. Rectangular monolithic plate subject to a compressive uniaxial loading in the x direction.

On the other hand, the critical loading of a rectangular plate subject to normal loadings N_x , N_y and N_{xy} (see Figure 3), can also be calculated by the Energy method assuming an approximate buckling mode shape $g(x, y)$, which satisfies the boundary conditions [20, 21]. The total potential energy of the system consists of two parts, the strain energy due to bending U_b , which can be calculated by means of the expression:

$$U_b = \frac{D}{2} \iint_0^a \int_0^b \left[\left(\frac{\partial^2 g(x, y)}{\partial x^2} + \frac{\partial^2 g(x, y)}{\partial y^2} \right)^2 - 2(1-\nu) \left(\frac{\partial^2 g(x, y)}{\partial x^2} \frac{\partial^2 g(x, y)}{\partial y^2} - \left(\frac{\partial^2 g(x, y)}{\partial x \partial y} \right)^2 \right) \right] dx dy \quad (26)$$

and the potential energy of the external loads U_N which is given by:

$$U_N = -\frac{1}{2} \iint_0^a \int_0^b \left[N_x \left(\frac{\partial g(x, y)}{\partial x} \right)^2 + N_y \left(\frac{\partial g(x, y)}{\partial y} \right)^2 + 2N_{xy} \frac{\partial g(x, y)}{\partial x} \frac{\partial g(x, y)}{\partial y} \right] dx dy \quad (27)$$

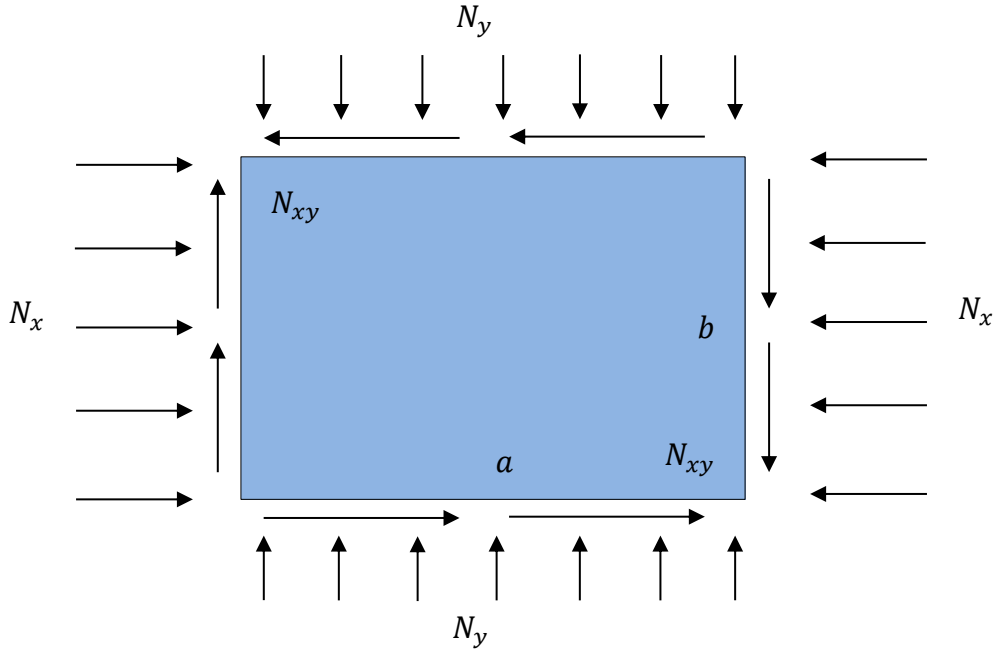


Figure 3: Rectangular monolithic plate subject to loadings N_x , N_y and N_{xy} .

Using the notation:

$$\begin{aligned}
 \frac{\partial^2 g(x, y)}{\partial x^2} &= g_x'' \\
 \frac{\partial^2 g(x, y)}{\partial y^2} &= g_y'' & \frac{\partial g(x, y)}{\partial x} &= g_x' \\
 \frac{\partial^2 g(x, y)}{\partial x \partial y} &= g_{xy}'' & \frac{\partial g(x, y)}{\partial y} &= g_y'
 \end{aligned} \tag{28}$$

Eqs. (26) is expressed as:

$$U_b = \frac{D}{2} \iint_0^a \int_0^b \left[(g_x'' + g_y'')^2 - 2(1 - \nu) (g_x'' g_y'' - (g_{xy}'')^2) \right] dx dy \tag{29}$$

and Eq. (27) as:

$$U_N = -\frac{1}{2} \int_0^a \int_0^b \left[N_x (g'_x)^2 + N_y (g'_y)^2 + 2N_{xy} g'_x g'_y \right] dx dy \quad (30)$$

The critical values of the forces can be found minimizing the total energy Π , i.e. minimizing the equation:

$$\Pi = U_b + U_N \quad (31)$$

2.7 Buckling of sandwich plates with thin faces

The critical load of a rectangular sandwich plate simply supported along the four edges, with thin faces of thickness H , core with shear modulus G_2 and thickness H_2 , dimensions $a \times b$ and compressed in the x direction is given by [3]:

$$N_{xcrit} = \frac{\pi^2}{b^2} k D_{eff} \quad (32)$$

where

$$k = \left(\frac{mb}{a} + \frac{an^2}{mb} \right)^2 \quad (33)$$

$$D_{eff} = \frac{D}{1 + \frac{EH_2H}{2G_2(1-\nu^2)} \psi_{thin}} \quad (34)$$

$$D = \frac{EHH_{13}^2}{2(1-\nu^2)} \quad (35)$$

$$\psi_{thin} = \frac{\pi^2}{b^2} \left(n^2 + m^2 \frac{b^2}{a^2} \right) \quad (36)$$

2.8 Buckling of sandwich plates with thick faces

In the case of a rectangular sandwich plate with thick faces, with the same geometry as that described in section 2.7, and compressed in the x direction, the critical load can be predicted with Eq. (32) but using the following effective stiffness given by:

$$D_{eff} = D \left(\frac{1}{1 + \frac{EH_2H}{2G_2(1-\nu^2)} \psi_{thick}} + \frac{H^2}{3H_{12}^2} \right) \quad (37)$$

Where the parameters k , D are given by Eqs. (33) and (35), respectively, and $\psi_{thick} = \psi_{thin}$.

2.9 Laminated glass beams under static loading

The concept of effective thickness for the calculation of deflections in laminated glass beams under static loads was proposed by Benninson et al. [1, 14] based on a previous work of Wölfel [13]. From the model proposed by Benninson et al. [1, 6], an effective stiffness for a laminated-glass beam subject to a static loading can be formulated, which is expressed as:

$$EI(t,T)_{eff} = EI_T \left(1 + \frac{Y}{1 + \frac{E H_1 H_2 H_3}{G_2(t,T) (H_1 + H_3) L^2} \gamma} \right) \quad (38)$$

Benninson et al. assume $\gamma = 9.6$ for all the boundary conditions although in the Wölfel's formulation this is associated to a simply supported beam under uniformly distributed load. The Parameter Y in Eq. (38) is a constant coefficient that relates the monolithic limit EI_{TOT} and the layered limit EI_T of the effective stiffness $EI(t,T)_{eff}$, by means of the expression:

$$1 + Y = \frac{EI_{TOT}}{EI_T} \quad (39)$$

Later, Galuppi and Royer-Carfagni [11] derived new equations for the deflection effective thickness using a variational approach and assuming that the deflection shape of the laminated glass beam coincide with that of a monolithic beam with the same load and boundary conditions, i.e. ,the deflection of the beam is assumed as:

$$w(x, t, T) = -\frac{g(x)}{EI(t, T)_{eff}} \quad (40)$$

where $g(x)$ is a shape function that takes the form of the elastic deflection of a monolithic beam, with constant cross section under the same load and boundary conditions as the laminated glass beam, and $EI(t)_{eff}$ is the bending stiffness of the laminated glass beam given by:

$$EI(t, T)_{eff} = EI_T \left(1 + \frac{Y}{1 + \frac{E H_1 H_2 H_3}{G_2(t, T) (H_1 + H_3)} \psi_B} \right) \quad (41)$$

where the parameter ψ_B is calculated with the equation:

$$\psi_B = \frac{\int_0^L g''(x)^2 dx}{\int_0^L g'(x)^2 dx} ; \quad 0 \leq x \leq L \quad (42)$$

The authors derived $\psi_B = 10/L^2$ for a beam under concentrated load and $\psi_B = \frac{168}{17L^2} = \frac{9.882}{L^2}$ for a beam under distributed load. These parameters ψ_B coincide with those proposed for sandwich beams (ψ_{thin}) under the same load and same boundary conditions.

2.10 Buckling of laminated glass beams

Blaauwendraad [15] proposed a formula for the buckling force of a simply supported laminated glass column, which is expressed as:

$$N_{crit}(t, T) = \frac{\pi^2 EI_T}{L^2} + \frac{H_0^2}{\frac{L^2}{\pi^2 E H_1} + \frac{L^2}{\pi^2 E H_3} + \frac{H_2}{G_2(t, T)}} \quad (43)$$

As it is pointed out by the author, this equation coincides with the equation derived by Satler and Stein [16] for sandwich panels.

Feldmann et al. [18] proposed to calculate the critical load of a simply supported beam with the Euler Theory but using an effective stiffness $EI(t, T)_{eff}$, i.e.:

$$N_{crit}(t, T) = \frac{\pi^2 EI(t, T)_{eff}}{L^2} \quad (44)$$

where

$$EI(t, T)_{eff} = EI_T \left(1 + \frac{Y}{1 + \frac{E H_1 H_2 H_3}{G_2(t, T) (H_1 + H_3) L^2} \frac{\pi^2}}{L^2} \right) \quad (45)$$

Galuppi and Royer-Carfagni [17] developed an analytical model for the buckling of simply supported laminated three-layered composite beams with viscoelastic core and under a compressive load $P(t)$, which can be time dependent. The authors derived a simple model using the quasi-elastic approximation which neglects the memory effect and assuming that the deflection of the beam is given by:

$$w(x, t, T) = a(t, T) \sin\left(\frac{\pi x}{L}\right) \quad (46)$$

The critical load is given by:

$$P_{crit}(t, T) = \frac{\pi^2}{L^2} EI_T \left(1 + \frac{Y}{1 + \frac{E H_1 H_2 H_3}{G_2(t, T) (H_1 + H_3) L^2} \frac{\pi^2}}{L^2} \right) \quad (47)$$

It can be easily demonstrated that Eq. (47) coincides with Eq. (44) developed by Blaauwendraad [15] and with Eq. (43) derived by Feldmann et al. [18].

Aenlle et al. [19] proposed to calculate the critical load of a laminated glass beam using the equation of the critical load of a linear-elastic monolithic beam according to the Euler theory, but using an effective stiffness, i.e.:

$$N_{crit}(t, T) = \frac{\pi^2}{\beta^2 L^2} EI(t, T)_{eff} \quad (48)$$

where $EI(t, T)_{\text{eff}}$ is given by Eq. (41). Due to the fact that the deflection of a beam in buckling differs from that of a beam under static loading, new parameters ψ_B have to be calculated for buckling. In Table 2, the values of ψ_B for the buckling of the most common boundary conditions are presented where it has been assumed that the buckling mode shapes of the laminated glass beams are equal to those of an elastic monolithic beam with the same boundary conditions.

Table 2. ψ_B for the first buckling mode shape of a beam

Boundary condition	$g(x)$	ψ_B
Simply supported	$A \sin\left(\frac{\pi x}{L}\right)$	$\frac{\pi^2}{L^2}$
Cantilever	$A \left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)$	$\frac{\pi^2}{4L^2}$
Fixed-pinned	$A \left(\frac{x}{L} + 1.02 \cdot \sin\left(\frac{\pi x}{0.699L}\right)\right)$	$\frac{\pi^2}{0.4886L^2}$
Fixed-fixed	$A \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right)$	$\frac{\pi^2}{0.25L^2}$

From Table 2, it can be easily inferred that the parameter ψ_B is related to the buckling ratio β by means of the equation:

$$\psi_B = \frac{\pi^2}{(\beta L)^2} \quad (49)$$

and Eq. (48) can be expressed as:

$$N_{\text{crit}}(t, T) = \psi_B EI(t, T)_{\text{eff}} \quad (50)$$

Eqs. (48) and (50) are general equations to calculate the critical load of a laminated glass beam, and they can be considered a generalization of Eq. (47), derived for simply supported beams to any boundary condition through the buckling ratio β .

2.11 Buckling of laminated glass Plates

Amadio and Bedon [24, 25] developed an analytical model for the buckling of laminated glass plates based on the Euler theory and using an effective stiffness $D(t, T)_{eff}$:

$$N_{crit} = \frac{\pi^2}{b^2} k D(t, T)_{eff} \quad (51)$$

where $D(t, T)_{eff}$ is given by:

$$D(t, T)_{eff} = \frac{E t_{eq}^3(t, T)}{12(1 - \nu^2)} \quad (52)$$

In Eq. (52) $t_{eq}(t, T)$ is an equivalent thickness derived from Eq. (38). Due to the fact that Eq. (38) provides the effective stiffness for a laminated glass beam, it is corrected through a parameter β dependent on the ratio a/b .

The accuracy of Eq. (51) was studied in laminated glass panels in-plane compressed in one direction [24] and subject to in-plane shear loading [25].

3 CRITICAL LOAD OF RECTANGULAR LAMINATED GLASS PLATES

In this paper, it is proposed to calculate the critical load of a laminated glass plate using the equations of the critical load of a linear-elastic monolithic thin plate, but with an effective stiffness $D(t, T)_{eff}$ i.e.:

$$N_{crit} = \frac{\pi^2}{b^2} k D(t, T)_{eff} \quad (53)$$

Where k is a parameter dependent on the boundary conditions and on the ratio $\gamma = \frac{a}{b}$ [20, 21].

With Respect to the effective stiffness $D(t, T)_{eff}$, the expression formulated by Galuppi and Royer Carfagni for rectangular plates under static loadings is considered, i.e.:

$$D(t, T)_{eff} = D_T \left(1 + \frac{Y_P}{1 + \frac{12 D_1 D_3 H_2}{G_2(t, T)(D_1 H_3^2 + D_3 H_1^2)} \psi_P} \right) \quad (54)$$

where Y_P is a constant parameter dependent on the material properties and thicknesses of the laminated plate layers through the expression:

$$Y_P = \frac{H_{13}^2 \frac{E H_1}{(1 - \nu^2)} \frac{E H_3}{(1 - \nu^2)}}{(D_1 + D_3) \left(\frac{E H_1}{(1 - \nu^2)} + \frac{E H_3}{(1 - \nu^2)} \right)} \quad (55)$$

and the parameter ψ_P is given by:

$$\psi_P = \frac{\iint_{0 \ 0}^a \ b \left[(g_x'' + g_y'')^2 - 2(1 - \nu) (g_x'' g_y'' - (g_{xy}'')^2) \right] dx dy}{\iint_{0 \ 0}^a \ b \left[(g_x')^2 + (g_y')^2 \right] dx dy} \quad (56)$$

where $g(x, y)$ is the buckling mode shape of an elastic monolithic plate with constant cross section under the same boundary conditions of the problem at hand.

With respect to parameter k , values for the most common boundary conditions can be found in the literature [20, 21].

If the expressions of D_1 and D_3 are substituted in Eqs. (54) and (55), they become:

$$D(t, T)_{eff} = D_T \left(1 + \frac{Y_P}{1 + \frac{E}{(1 - \nu^2)} \frac{H_1 H_3 H_2}{G_2(t, T)(H_1 + H_3)} \psi_P} \right) \quad (57)$$

and

$$Y_P = \frac{12H_1H_3H_{12}^2}{(H_1 + H_3)(H_1^3 + H_3^3)} \quad (58)$$

respectively.

3.1 Methodology

The critical load of a laminated glass plate can be calculated with Eq. (53) using appropriate parameters k and ψ_P . We assume that the parameter k consider in Eq. (53) is the same as that consider in monolithic plates, which means that the buckling mode shape is the same for the monolithic and the laminated glass plate. This also means that the critical load of a laminated glass plate with dimensions $a \times b$ can also be calculated by means of the equation:

$$N_{crit}(t, T) = N_{crit-MON} \frac{D(t, T)_{eff}}{D} \quad (59)$$

Where $N_{crit-MON}$ is the buckling load of a monolithic plate with same dimensions $a \times b$ and stiffness D . The following procedure can be followed:

- 1) To assemble a finite element monolithic model with thickness H_{TOT} , dimensions $a \times b$ and material properties E and ν .
- 2) To calculate the buckling load of the finite element monolithic model. Alternatively, $N_{crit-MON}$ can be calculated using the analytical equations proposed in the literature [20, 21] for elastic monolithic plates.
- 3) To calculate the effective stiffness $D(t, T)_{eff}$. Previously, the parameter ψ_P must be calculated or taking from the literature [3].
- 4) To predict the critical load by means of Eq. (59)

If the monolithic and the laminated glass models have the same total thickness H_{TOT} , the critical load can also be calculated with the equation:

$$N_{crit}(t, T) = N_{crit-MON} \frac{E(t, T)_{eff}}{E} \quad (60)$$

where:

$$E(t, T)_{\text{eff}} = \frac{E(H_1^3 + H_3^3)}{H_{\text{Tot}}^3} \left(1 + \frac{Y_P}{1 + \frac{EH_1H_2H_3}{G_2(t, T)(H_1 + H_3)} \psi_P} \right) \quad (61)$$

is an effective Young modulus.

Eqs. (59) and (60) show that the critical load of a laminated glass plate can be predicted using a monolithic model with the same dimensions $a \times b$ and the same boundary conditions as the laminated plate. Therefore, the assembly of a layered model can be avoided.

An alternative consists of assembling a monolithic model with constants thickness but defining a time and temperature dependent Young modulus given by $E(t, T)_{\text{eff}}$.

4. EXAMPLES OF APPLICATION

Three different cases of rectangular laminated glass plates simply supported along the fourth edges, are now examined using different configurations of loading and dimensions a and b . The elastic properties used for both glass and PVB layers are presented in Table 3. The mechanical behaviour of the PVB interlayers was considered viscoelastic in terms of Prony series and the data used in this paper was obtained in previous works [27].

Table 3. Material properties for glass and PVB [27].

Glass ($H_1 = H_3 = 4 \text{ mm}$)			PVB ($H_2 = 0.38 \text{ mm}$)					
E	ν	ρ	G_0	K	ν	ρ	C_1	C_2
(Young's Modulus)	(Poisson's ratio)	(Density)	(Instantaneous shear modulus)	(Bulk Modulus)	(Poisson's ratio)	(Density)	(WLF: Tref=20 C)	
[GPa]		[kg/m ³]	[GPa]	[GPa]		[kg/m ³]		
72	0.22	2500	0.3696	2	0.44	1030	12.60	74.46

The critical load predicted with the analytical Eq. (53) is always compared with the results obtained with a finite element model (FEM) assembled in ABAQUS.

In the FEM, 3D linear shell continuum elements (SC8R) were used for the glass layers whereas the PVB layers were meshed with 3D linear hexahedral elements (C3D8R). This meshing technique has been demonstrated to be adequate to reproduce the laminated glass behaviour with a relatively low computational time [27, 28]. The material properties considered in the simulations were the same as those used in the analytical predictions (Table 3) but modelling the PVB interlayer as a linear elastic material. The buckling load was calculated iteratively for each considered time $t = t_i$ being the interlayer Young's modulus $E_t = E_t(t_i, T)$ and the Poisson ratio $\nu_t = 0.44$ [29]. The same temperature $T = 20^\circ C$ was considered for both the numerical simulations and the analytical predictions.

4.1 Simply supported plate under uniaxial compression.

The first case studied in this paper is a rectangular simply supported laminated glass plate compressed in the x direction with the load N_x (force per unit length) i.e. $N_y = 0$ and $N_{xy} = 0$ (see Figure 2). In the case of elastic simply supported plates the buckling mode shape can be represented by the double series [20, 21]:

$$g(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (62)$$

Using the Energy method [20, 21] with an approximate buckling mode shape $g(x, y)$ and considering $N_y = 0$ and $N_{xy} = 0$ in Eq. (31), the critical load can be obtained minimizing the equation:

$$N_{xcrit} = D(t, T)_{\text{eff}} \frac{\iint_{0 \ 0}^a \ b \left[(g''_x + g''_y)^2 - 2(1 - \nu) (g''_x g''_y - (g''_{xy})^2) \right] dx dy}{\iint_{0 \ 0}^a \ b (g'_x)^2 dx dy} \quad (63)$$

which results in:

$$N_{xcrit} = \pi^2 b^2 D(t, T)_{\text{eff}} \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left(\frac{m^2 b}{a} + \frac{an^2}{b} \right)^2}{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 A_{mn}^2} \quad (64)$$

Eq. (64) becomes a minimum [20, 21] if all coefficients A_{mn} , except one, are taken equal to zero. This means that each mode shape is given by:

$$g(x, y) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (65)$$

Identifying Eq. (53) and Eq. (64) it is derived that:

$$\frac{\pi^2 k}{b^2} = \frac{\iint_0^a \int_0^b \left[(g_x'' + g_y'')^2 - 2(1 - \nu) (g_x'' g_y'' - (g_{xy}'')^2) \right] dx dy}{\iint_0^a \int_0^b (g_x')^2 dx dy} \quad (66)$$

If Eq. (65) is substituted in Eq. (66), the later becomes:

$$\frac{\pi^2 k}{b^2} = \frac{\pi^2 \left(\frac{m^2 b}{a} + \frac{a n^2}{b} \right)^2}{b^2 m^2} \quad (67)$$

and parameter k is given by:

$$k = \left(\frac{mb}{a} + \frac{an^2}{mb} \right)^2 \quad (68)$$

On the other hand, substitution of Eq. (65) in Eq. (56) leads to:

$$\psi_P = \frac{\pi^2}{b^2} \left(n^2 + \frac{m^2 b^2}{a^2} \right) \quad (69)$$

From Eqs. (68) and (69) is inferred that the parameters $\pi^2 k/b^2$ and ψ_P are different (except for some particular cases), contrary to laminated glass beams where

$$\psi_B = \frac{\pi^2}{(\beta L)^2} \quad (70)$$

It must also be remarked that the parameter ψ_P given by Eq. (69) for a simply supported laminated glass plate loaded in the “x” direction is equal to the parameter ψ_{thin} (Eq. (36)) for a simply supported sandwich plate subjected to the same loading.

If $D(t, T)_{eff}$ is expressed as:

$$D(t, T)_{eff} = \frac{D_{TOT}}{1 + Y_P} \left(1 + \frac{Y_P}{1 + \frac{12 D_1 D_3 H_2}{G_2(t, T)(D_1 H_3^2 + D_3 H_1^2)} \psi_P} \right) \quad (71)$$

The critical load can also be expressed as [3]:

$$N_{xcrit} = \frac{\pi^2}{b^2} k_1 D_{TOT} \quad (72)$$

Where the new parameter $k_1(t, T)$ depends on the ratio $\gamma = a/b$ through the parameters k and ψ_P , and on the thicknesses and material properties of the glass and viscoelastic layers, and it is expressed as [3]:

$$k_1(t, T) = \frac{k}{1 + Y_P} \left(1 + \frac{Y_P}{1 + \frac{12 D_1 D_3 H_2}{G_2(t, T)(D_1 H_3^2 + D_3 H_1^2)} \psi_P} \right) \quad (73)$$

If Eqs. (68) and (69) are substituted in Eq. (73), the later becomes:

$$k_1(t, T) = \frac{\left(\frac{m b}{a} + \frac{a n^2}{m b} \right)^2}{1 + Y_P} \left(1 + \frac{Y_P}{1 + \frac{12 D_1 D_3 H_2}{G_2(t, T)(D_1 H_3^2 + D_3 H_1^2)} \frac{\pi^2}{b^2} \left(n^2 + \frac{m^2 b^2}{a^2} \right)} \right) \quad (74)$$

Since the critical value of N_x is the smallest value that satisfy Eq. (64), the values of m and n that minimize Eq. (64) must be determined. The value of N_x increases as n increases and therefore $n = 1$, i.e., the plate buckles in a single half wave in the y direction. Since a simply supported plate must buckle into a whole number of half-waves, m must be an integer [20, 21]. To determine the buckling pattern in the x direction the variation of k_1 with the ratio a/b for different values of m must be consider. Moreover, due to the fact that the shear stiffness $G_2(t, T)$ is not constant but it depends on time and temperature, the parameter k_1 will also vary with $G_2(t, T)$. This means that the number of half-waves

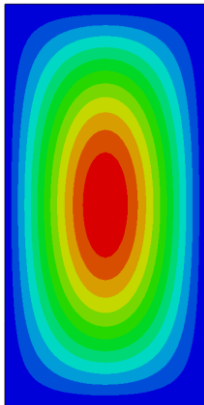
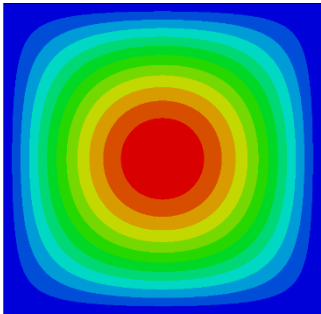
m in which the plate buckles into may not be constant but it might change with time and this fact has to be verified before predicting the critical loading of a plate [3].

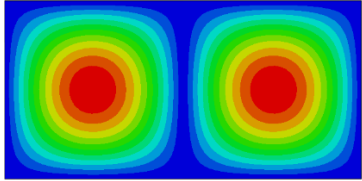
In order to validate the equations and parameters presented in this section, the critical load N_{xcrit} of a simply supported rectangular laminated glass plate with three different configurations $a = b$, $a = 0.5b$ and $a = 2b$, respectively were predicted with Eq. (53). The parameters, m , n , k and ψ_P corresponding to the three configurations are shown in Table 4, together with the corresponding mode shape obtained with a finite element model. The parameter ψ_P can also be expressed as:

$$\psi_P = \frac{\pi^2}{b^2} k_\psi \quad (75)$$

The values of k_ψ are presented in Table 4.

Table 4. Parameters k and ψ_P for a simply supported laminated glass plate with different ratios a/b .

a [m]	b [m]	$\frac{a}{b}$	m	n	k	ψ_P	k_ψ	Mode shape
1	2	.5	1	1	6.25	12.34	5	
2	2	1	1	1	4	4.93	2	

2	1	2	2	1	4	19.74	2	
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The parameter k_1 for this simply supported laminated glass plate considering $n = 1$, $m = 1, 2, 3$ and ratios $0.3 < a/b < 3$ is shown in Figure 4. The figure confirms that the values taken for m in Table 4 are correct, i.e., they are valid for all the range of the shear parameter G_2 (see Figure 5) [27]) where $2 \cdot 10^5 < G_2 < 4 \cdot 10^8$ MPa. The case $a = 2$ m and $b = 1$ m is not covered in Figure 4 (only valid for $b = 2$ m). In this case, the plate buckle with 2 or 3 half-waves depending of the values of $G_2(t, T)$. In table 4, the mode shape with two half-waves is presented.

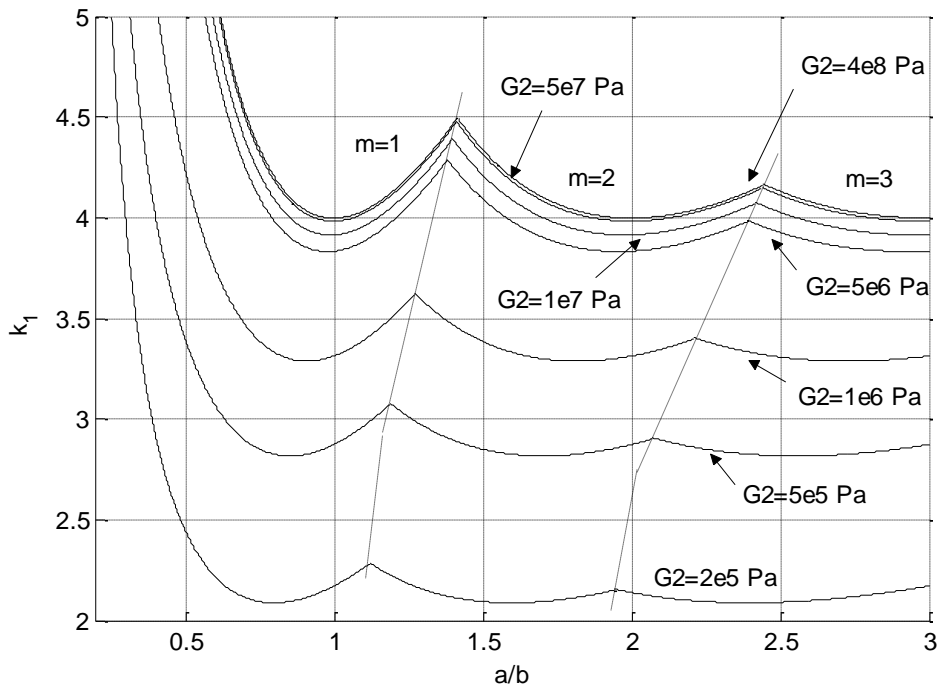


Figure 4. Parameter k_1 for a simple-supported laminated glass plate loaded in the x direction (curves valid for plates with $b = 2$ m).

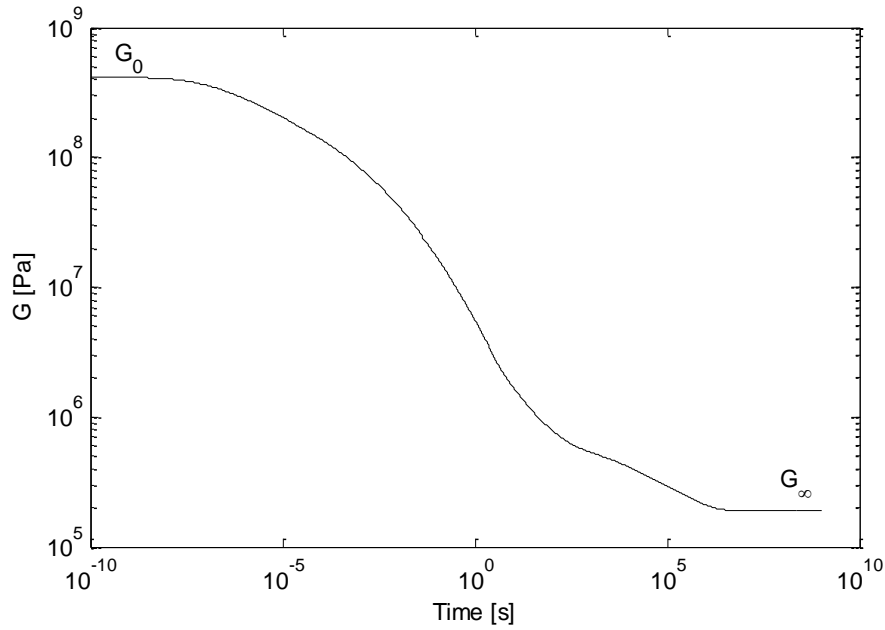


Figure 5. Shear modulus, G_2 , of the PVB interlayer.

The predicted critical loads N_{xcrit} using Eq. (53) and the parameters presented in Table 3 are shown in Figure (6) for $a = 1\text{ m}, b = 2\text{ m}$, in Figure (7) for $a = 2\text{ m}, b = 2\text{ m}$ and in Figure (8) for $a = 1\text{ m}, b = 2\text{ m}$. The critical load of each case, obtained with the FEM, are also presented in Figures (6) to (8), respectively. It is inferred that Eq. (53) provides a good accuracy the error being less than 7.5% for the three cases considered in this section.

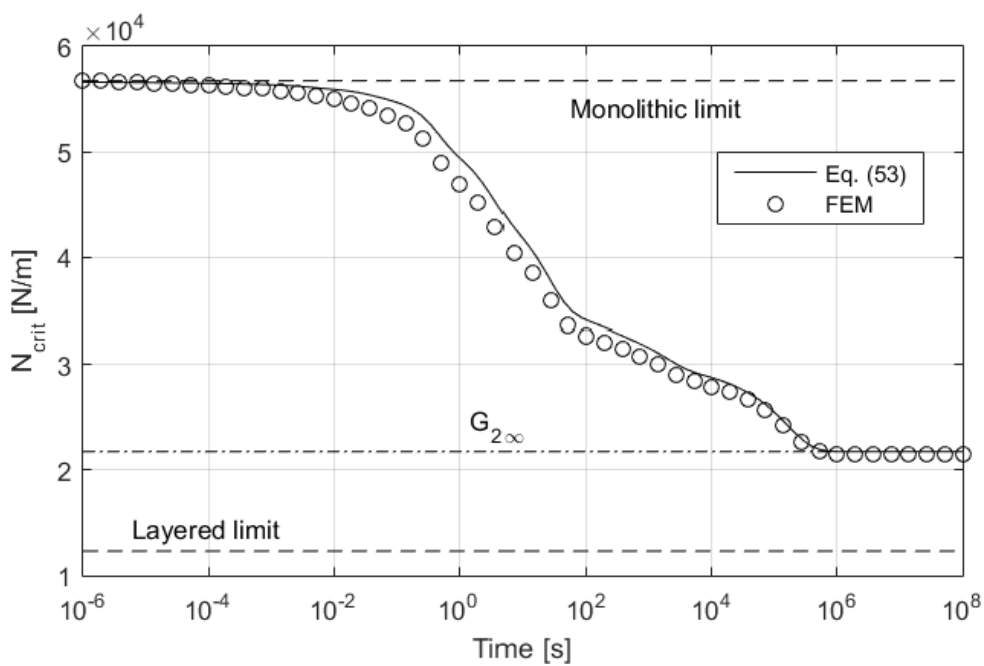


Figure 6. Critical load N_{xcrit} of a simply-supported laminated glass plate uniaxially loaded in the x direction for Temperature $T = 20^\circ C$ ($a = 1 m$, $b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

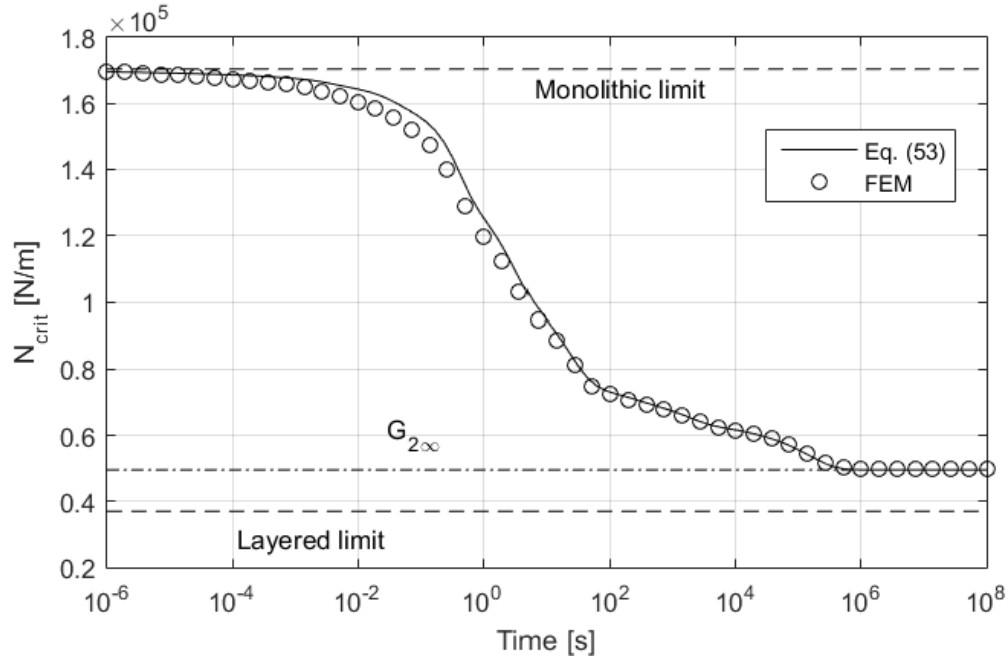


Figure 7. Critical load N_{xcrit} of a simply-supported laminated glass plate uniaxially loaded in the x direction for Temperature $T = 20^\circ C$ ($a = 2 m$, $b = 1 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

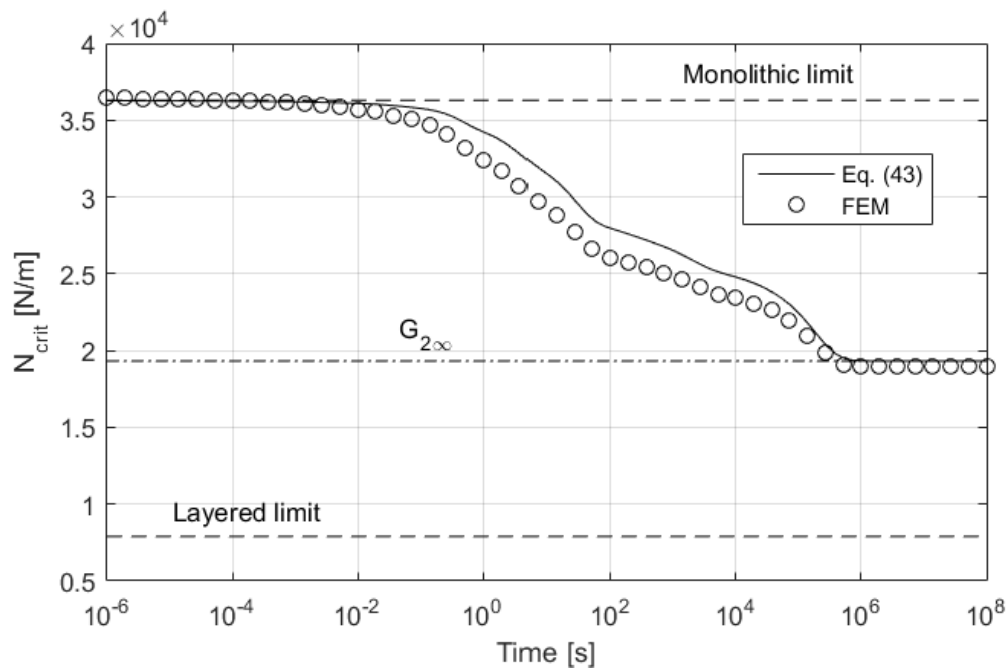


Figure 8. Critical load N_{xcrit} of a simply-supported laminated glass plate uniaxially loaded in the x direction for Temperature $T = 20^\circ C$ ($a = 2 m$, $b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

4.2. Simply supported plate under biaxial compression

For the case of a rectangular simply supported laminated glass plate subject to biaxial compression with loads N_x and N_y ($N_{xy} = 0$), the same expression for the deflection shape given by Eq. (63) can be used. Moreover, it can be proved again that only one term of the series should be considered in calculating the critical values of N_x and N_y [20, 21]. Thus, the buckling mode shape is given by:

$$g(x, y) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (76)$$

Applying the Energy method [20, 21] with the approximate buckling mode shape $g(x, y)$ the bending energy is given by:

$$U_b = \frac{D_{eff}(t, T)}{2} \iint_0^a \int_0^b \left[(g''_x + g''_y)^2 - 2(1 - \nu) (g''_x g''_y - (g''_{xy})^2) \right] dx dy \quad (77)$$

which after substitution of Eq. (76) results in:

$$U_b = \frac{\pi^4 D_{eff}(t, T)}{8ab} A_{mn}^2 \left(\frac{m^2 b}{a} + \frac{an^2}{b} \right)^2 \quad (78)$$

The potential energy of the external loads is given by:

$$U_N = -\frac{1}{2} \iint_0^a \int_0^b \left[N_x (g'_x)^2 + N_y (g'_y)^2 \right] dx dy \quad (79)$$

which leads to:

$$U_N = -\frac{\pi^2}{8} A_{mn}^2 \left(N_x \frac{m^2 b}{a} + N_y \frac{an^2}{b} \right) \quad (80)$$

Minimization of the total energy gives:

$$N_x m^2 + N_y \cdot n^2 \left(\frac{a}{b}\right)^2 = \frac{\pi^2}{b^2} D_{eff}(t, T) \left(\frac{m^2 b}{a} + \frac{a n^2}{b}\right)^2 \quad (81)$$

If the load N_y is expressed as:

$$N_y = \beta N_x \quad (82)$$

the critical load N_{xcrit} can be obtained from:

$$N_{xcrit} = \frac{\pi^2 \left(\frac{mb}{a} + \frac{an^2}{mb}\right)^2}{b^2 \left(1 + \beta \frac{a^2 n^2}{b^2 m^2}\right)} D_{eff}(t, T) \quad (83)$$

from which is obtained that the parameter k is given by:

$$k = \frac{\left(\frac{mb}{a} + \frac{an^2}{mb}\right)^2}{\left(1 + \beta \frac{a^2 n^2}{b^2 m^2}\right)} \quad (84)$$

With respect to parameter ψ_P , substitution of eq. (76) in eq. (56) gives:

$$\psi_P = \frac{\pi^2}{b^2} k = \frac{\pi^2 \left(\frac{mb}{a} + \frac{an^2}{mb}\right)^2}{b^2 \left(1 + \frac{a^2 n^2}{b^2 m^2}\right)} \quad (85)$$

The number of half waves in which the plate buckles into depends also on the parameter β . The parameter k_1 for a plate with $b = 2m$, and ratios $\beta = 0.2$ and $\beta = 1$ is presented in Figures 9 and 10, respectively, for different ratios a/b .

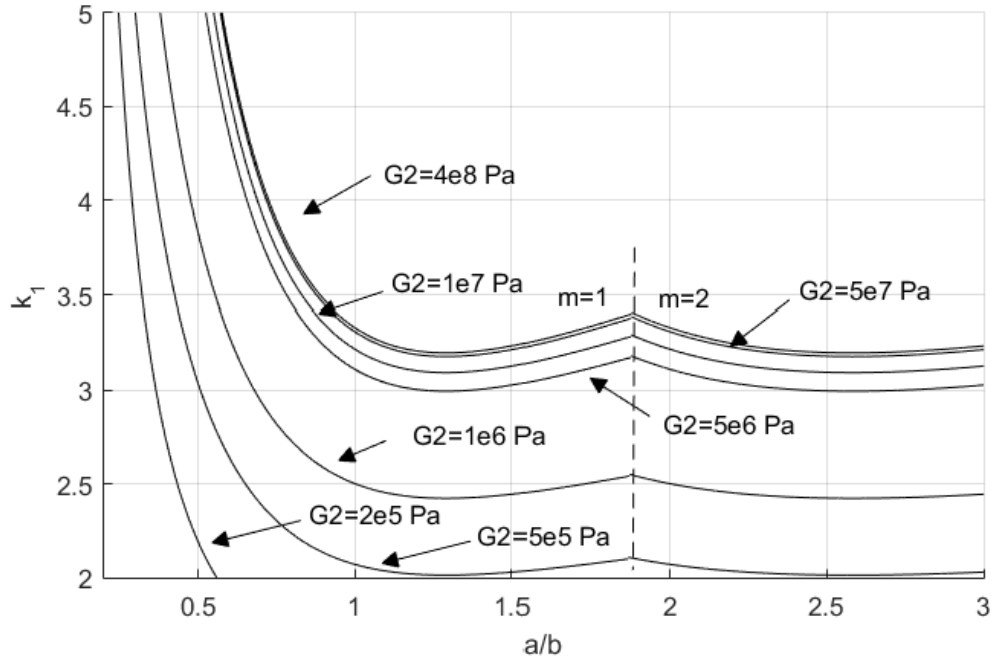


Figure 9. Parameter k_1 for a simply-supported laminated glass plate biaxially loaded with N_x and $N_y = 0.2 N_x$ (curves valid for $b = 2 m$).

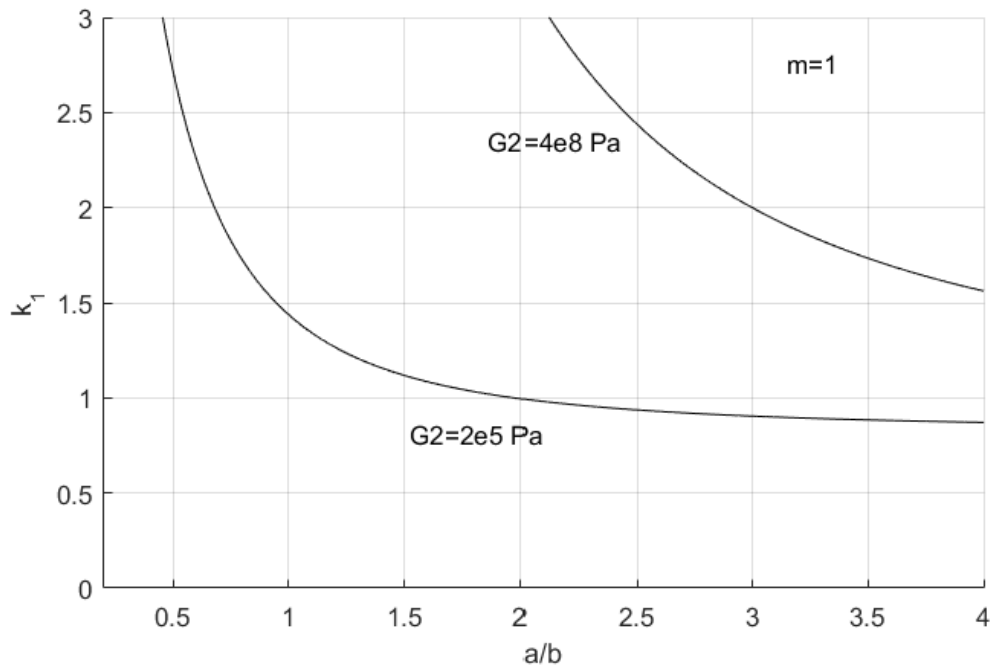


Figure 10. Parameter k_1 for a simply-supported laminated glass plate biaxially loaded with $N_x = N_y$ (curves valid for $b = 2 m$).

The critical load of a square plate with $b = 2 m$, $N_x = N_y$, $\gamma = 1$ and $m = n = 1$ is predicted with Eq. (53) and shown in Figure (11). The critical load for this example was

calculated with a FEM (see Figure (11)), following the same steps described previously. The maximum error between the numerical and the predicted critical load is 8.5%.

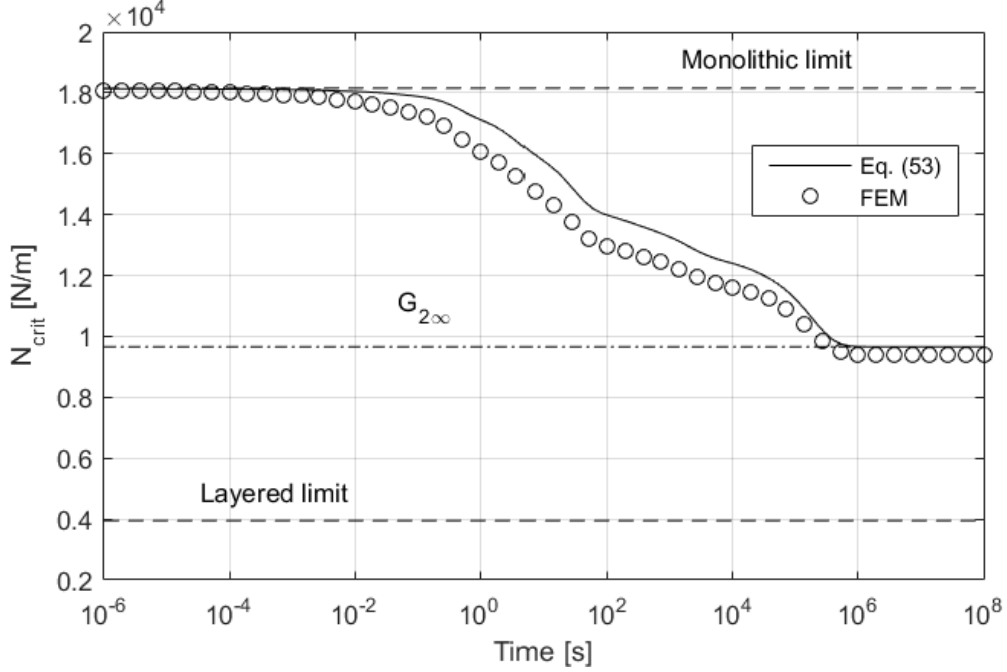


Figure 11. Critical load N_{xcrit} of a simply-supported laminated glass plate byaxially loaded for Temperature $T = 20^{\circ} C$ ($a = b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

4.3. Simply supported plate under in-plane shear

We consider in this section the case of a simply supported plated subject to shearing forces N_{xy} uniformly distributed along the four edges (see Figure 3). The boundary conditions are satisfied by taking for the deflection of the buckled plate the double series expression given by Eq. (63) [20, 21].

Using the Energy method with an approximate buckling mode shape $g(x, y)$ and considering $N_x = 0$ and $N_y = 0$ in eq. (30), the critical load can be obtained minimizing the equation:

$$N_{xycrit} = \frac{D(t, T)_{\text{eff}} \iint_{0 \ 0}^{a \ b} \left[(g''_x + g''_y)^2 - 2(1 - \nu) (g''_x g''_y - (g''_{xy})^2) \right] dx dy}{2 \iint_{0 \ 0}^{a \ b} (g'_x g'_y) dx dy} \quad (86)$$

from which:

$$\frac{\pi^2 k}{b^2} = \frac{\frac{1}{2} \iint_{0,0}^{a,b} \left[(g''_x + g''_y)^2 - 2(1-\nu) (g''_x g''_y - (g''_{xy})^2) \right] dx dy}{\iint_{0,0}^{a,b} (g'_x g'_y) dx dy} \quad (87)$$

If eq. (62) is substituted in eq. (87), the later becomes:

$$\frac{\pi^2 k}{b^2} = \frac{ab}{32} \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left(\frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} \right)^2}{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{mn} A_{pq} \frac{mnpq}{(m^2 - p^2)(q^2 - n^2)}} \quad (88)$$

where m, n, p and q are such integers that $m \pm p$ and $n \pm q$ are odd numbers.

For short elastic monolithic plates ($a/b < 2$) [21] the minimum value N_{xycrit} is obtained when $m + n$ are even numbers. With respect to the parameter ψ_p , it can be calculated from:

$$\psi_p = \frac{\iint_{0,0}^{a,b} \left[(g''_x + g''_y)^2 - 2(1-\nu) (g''_x g''_y - (g''_{xy})^2) \right] dx dy}{\iint_{0,0}^{a,b} \left[(g'_x)^2 + (g'_y)^2 \right] dx dy} \quad (89)$$

Substitution of Eq. (62) in eq. (89), leads to:

$$\psi_p = \frac{\frac{ab}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left(\frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} \right)^2}{\frac{1}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left(\pi^2 m^2 \frac{b}{a} + \pi^2 n^2 \frac{a}{b} \right)} \quad (91)$$

In monolithic elastic plates [21] a good accuracy is obtained considering the first five terms ($m + n$ even numbers), i.e. $A_{11}, A_{22}, A_{13}, A_{31}$ and A_{33} . With this assumption, the buckling deflection shape is given by:

$$g(x, y) = \sum_{m=1,3} \sum_{n=1,3} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (92)$$

where [21]:

$$A_{11} = -\frac{4}{9} \frac{\gamma^2}{\lambda(1 + \gamma^2)^2} A_{22} \quad (93)$$

$$A_{22} = -\frac{25\lambda(9+9\gamma^2)^2}{36\gamma^2}A_{33}$$

$$A_{13} = \frac{4}{5}\frac{\gamma^2}{\lambda(1+9\gamma^2)^2}A_{22}$$

$$A_{13} = \frac{4}{5}\frac{\gamma^2}{\lambda(9+\gamma^2)^2}A_{22}$$

$$\lambda^2 = \frac{\gamma^4}{81(1+\gamma^2)^4} \left(1 + \frac{81}{625} + \frac{81}{25} \left(\frac{1+\gamma^2}{1+9\gamma^2} \right)^2 + \frac{81}{25} \left(\frac{1+\gamma^2}{9+\gamma^2} \right)^2 \right)$$

The parameters k and k_ψ are shown in Figure 12 for different ratios $0.5 < a/b < 2$ whereas the parameter k_1 is shown in Figure 13 for different ratios a/b and different values of $G_2(t, T)$.

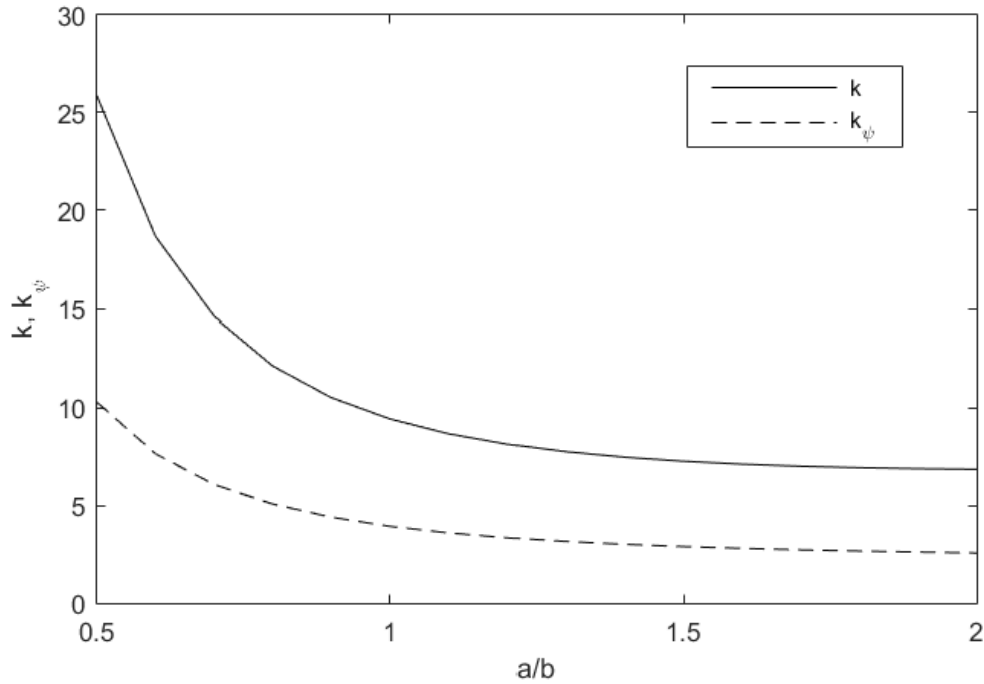


Figure 12. Parameters k and k_ψ for a simply-supported laminated glass subject to in-plane shear.

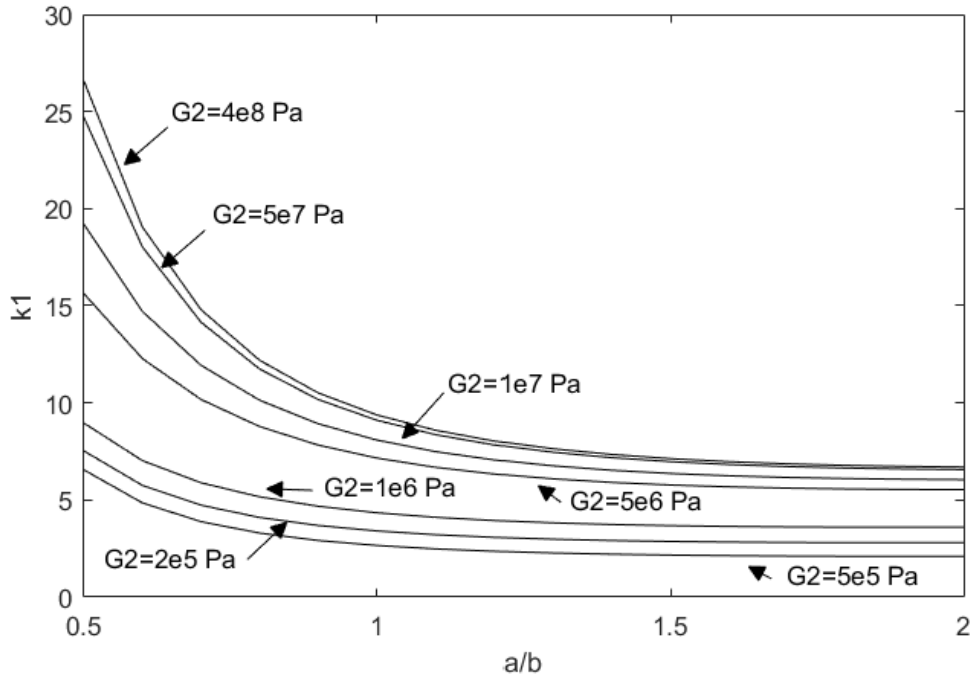


Figure 13. Parameters k_1 for a simply-supported laminated glass subject to in-plane shear (valid for $b = 2\text{ m}$).

The critical load of a square plate with $a = b = 2\text{ m}$ was predicted with Eq. (63) using the parameters k and k_ψ shown in Figure 12 and it is presented in Figure 14 together with critical load obtained with a FEM. The maximum error between the numerical and the predicted is less than 10%.

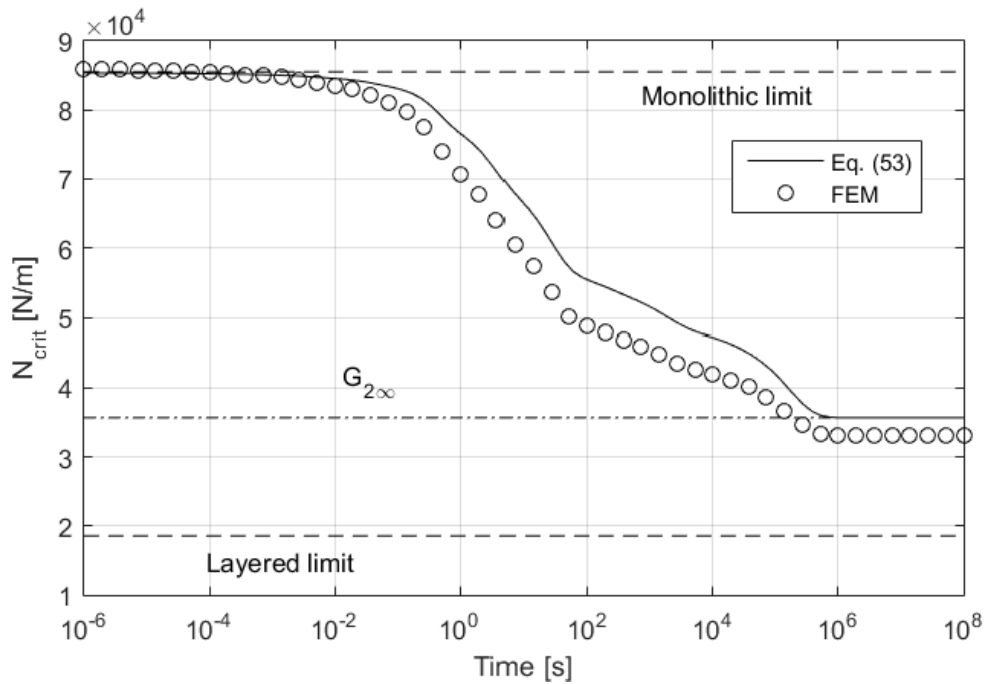


Figure 14. Critical load N_{xcrit} of a simply-supported laminated glass plate subjected to in-shear loading for Temperature $T = 20^{\circ} C$ ($a = b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

4 CONCLUSIONS

Structural stability is a design criterion to be considered when laminated glass elements are subject to compressive loads. The critical load of a laminated glass plate is time and temperature dependent due to the viscoelastic behavior exhibited by the polymeric interlayers. However, in order to avoid failures due to buckling, the compressive load must be less than the critical load given by the long-term limit of the interlayer shear modulus $G_{2\infty}$, which means that the quasi-elastic solution [10, 17] can be used advantageously to calculate safe critical loads in laminated glass elements.

In this paper, an analytical expression (Eq. (53)) to predict the critical load of laminated glass plates subject to different compressive loading conditions has been proposed, which has been derived extending the buckling theory of isotropic monolithic plates [20, 21] to laminated glass plates, using the concept of effective stiffness. This effective stiffness takes into account the time and temperature dependency of the polymeric interlayers.

The effective stiffness derived by Galuppi and Royer Carfagni [26] for laminated glass plates has been used with Eq. (53), which is dependent on the boundary conditions through parameter ψ_P . However, the concepts of effective stiffness, effective thickness and effective Young's modulus can be used interchangeably with the same accuracy [13].

Moreover, it has been assumed that the buckling mode shapes of the laminated glass plate is equal to that of a monolithic plate subject to the same loading and with the same boundary conditions.

When using Eq. (53), two parameters, k and ψ_P , dependent of the boundary and loading conditions, have to be known. With respect to parameter k , the same expressions used in monolithic plates have been considered, which can be found in the classical books of structural stability [20, 21] for the most common boundary and loading conditions. As regards parameter ψ_P , the analytical expression derived by Galuppi and Royer Carfagni

[26] has been used to determine the values of this parameter corresponding to the boundary and loading conditions considered in the paper.

Parameters k and ψ_p are dependent on the number of half-waves in which the plate buckles into through the parameters m and n . Due to the fact that the shear modulus of the interlayer $G_2(t, T)$ is time and temperature dependent, when a plate is subject to a buckling, the number of half-waves of the buckling mode shape can change also with time and temperature. This can be observed in Figures 4 and 9, where for some specific ratios a/b , the number of half-waves depend on the magnitude of the shear modulus of the interlayer.

The accuracy provided by Eq. (53) has been validated by numerical simulations. A finite element model of simply-supported laminated glass plates were assembled in ABAQUS. The plates were subject to uniaxial, biaxial and in plane shear loadings at a temperature $T = 20^\circ C$. It has been proved that the discrepancies between the numerical and the analytical results are less than 10% for all the boundary and loading conditions considered in the simulations, which demonstrates that Eq. (53) can predict with a good accuracy the buckling critical load of a laminated glass plates. Moreover, the monolithic limit and the long-term limit, corresponding to the interlayer shear modulus G_2^∞ , are predicted with a good accuracy the error being less than 5%

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Figure Captions

Figure 1. Sandwich beam with two external faces of thicknesses $H = H_1 = H_2$ and one interlayer with thickness H_2 .

Figure 2. Rectangular monolithic plate subject to a compressive uniaxial loading in the x direction.

Figure 3: Rectangular monolithic plate subject to loadings N_x , N_y and N_{xy} .

Figure 4. Parameter k_1 for a simple-supported laminated glass plate loaded in the x direction (curves valid for plates with $b = 2 m$).

Figure 5. Shear modulus, G_2 , of the PVB interlayer.

Figure 6. Critical load N_{xcrit} of a simply-supported laminated glass plate uniaxially loaded in the x direction for Temperature $T = 20^\circ C$ ($a = 1 m$, $b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

Figure 7. Critical load N_{xcrit} of a simply-supported laminated glass plate uniaxially loaded in the x direction for Temperature $T = 20^\circ C$ ($a = 2 m$, $b = 1 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

Figure 8. Critical load N_{xcrit} of a simply-supported laminated glass plate uniaxially loaded in the x direction for Temperature $T = 20^\circ C$ ($a = 2 m$, $b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

Figure 9. Parameter k_1 for a simply-supported laminated glass plate biaxially loaded with N_x and $N_y = 0.2 N_x$ (curves valid for $b = 2 m$).

Figure 10. Parameter k_1 for a simply-supported laminated glass plate biaxially loaded with $N_x = N_y$ (curves valid for $b = 2 m$).

Figure 11. Critical load N_{xcrit} of a simply-supported laminated glass plate byaxially loaded for Temperature $T = 20^\circ C$ ($a = b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

Figure 12. Parameters k and k_ψ for a simply-supported laminated glass subject to in-plane shear.

Figure 13. Parameters k_1 for a simply-supported laminated glass subject to in-plane shear (valid for $b = 2 m$).

Figure 14. Critical load N_{xcrit} of a simply-supported laminated glass plate subjected to in-shear loading for Temperature $T = 20^{\circ} C$ ($a = b = 2 m$, $H_1 = H_3 = 4 mm$, $H_2 = 0.38 mm$).

Table 1. Parameters ψ_{thin} and γ for simply supported beams.

Beam	ψ_{thin}	γ
Central point load	$12/L^2$	$1 - \frac{\sinh \theta + \frac{(1 - \cosh \theta)}{\tanh \theta + 1/\tanh \theta}}{\theta}$
Distributed load	$9.6/L^2$	$1 + \frac{2}{\theta^2 \cosh \theta} (1 - \cosh \theta)$

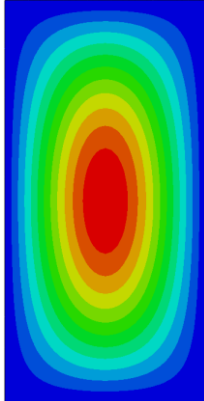
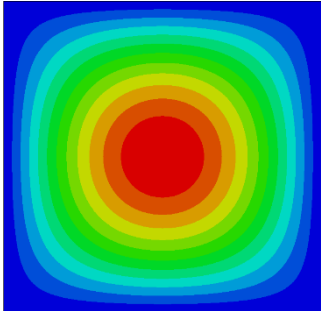
Table 2. ψ_B for the first buckling mode shape of a beam

Boundary condition	$g(x)$	ψ_B
Simply supported	$A \sin\left(\frac{\pi x}{L}\right)$	$\frac{\pi^2}{L^2}$
Cantilever	$A \left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)$	$\frac{\pi^2}{4L^2}$
Fixed-pinned	$A \left(\frac{x}{L} + 1.02 \cdot \sin\left(\frac{\pi x}{0.699L}\right)\right)$	$\frac{\pi^2}{0.4886L^2}$
Fixed-fixed	$A \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right)$	$\frac{\pi^2}{0.25L^2}$

Table 3. Material properties for glass and PVB [27].

Glass				PVB				
$(H_1 = H_3 = 4 \text{ mm})$				$(H_2 = 0.38 \text{ mm})$				
E	ν	ρ	G_0	K	ν	ρ	C_1	C_2
(Young's Modulus)	(Poisson's ratio)	(Density)	(Instantaneous shear modulus)	(Bulk Modulus)	(Poisson's ratio)	(Density)	(WLF: Tref=20 C)	
[GPa]		[kg/m ³]	[GPa]	[GPa]		[kg/m ³]		
72	0.22	2500	0.3696	2	0.44	1030	12.60	74.46

Table 4. Parameters k and ψ_P for a simply supported laminated glass plate with different ratios a/b .

a [m]	b [m]	$\frac{a}{b}$	m	n	k	ψ_P	k_ψ	Mode shape
1	2	.5	1	1	6.25	12.34	5	
2	2	1	1	1	4	4.93	2	
2	1	2	2	1	4	19.74	2	