J. Math. Anal. Appl. ••• (••••) ••••••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Strong solutions of evolution equations with $p(x,t)$-Laplacian: Existence, global higher integrability of the gradients and second-order regularity

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ARTICLE INFO

Article history:
Received 10 May 2020
Available online xxxx
Submitted by D. Repovš

Keywords:
Singular and degenerate parabolic equation
Variable nonlinearity
Strong solution
Higher regularity

ABSTRACT

We study the homogeneous Dirichlet problem for the degenerate parabolic equation

$$u_t - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) = f(x,t) \quad \text{in } Q_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$. The variable exponent $p(x,t)$ is a given Lipschitz-continuous function, $p(x,t) : Q_T \mapsto [p^-, p^+]$ with some constants $\frac{2N}{N+2} < p^- \leq p^+ < \infty$. We derive conditions on f and u_0 sufficient for the existence of a unique strong solution with the following global regularity properties:

$$u_t \in L^2(Q_T), \quad |\nabla u|^{q(x,t)} \in L^\infty(0,T;L^1(\Omega)) \text{ with } q(x,t) = \max\{2, p(x,t)\},$$

$$D^2_{x_i x_j} u \in L^{p(\cdot)}_{loc}(Q_T \cap \{p(x,t) < 2\}).$$

If $N \geq 3$, or $N = 2$ and $p^- > \frac{6}{5}$, then $D_{x_j}\left(|\nabla u|^{\frac{p(x,t)-2}{2}} D_{x_i} u\right) \in L^2(Q_T)$. Moreover, the solution possesses the property of global higher integrability of the gradient:

$$|\nabla u|^{p(x,t)+\delta} \in L^1(Q_T) \quad \text{for every } 0 < \delta < \frac{4p^-}{p^-(N+2)+2N}.$$

This property turns out to be crucial for the proof of existence of strong solutions in the case when $p(x,t) > 2$ on a part of the domain Q_T. The same results are proved for the equation with the regularized fluxes $(\epsilon^2 + |\nabla u|^2)^{\frac{p(x,t)-2}{2}}\nabla u$, $\epsilon > 0$.
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[1] The first author was funded by the Energy and Environment Solutions (E2S), UPPA under International Doctoral Mobility grants programme 2019.
[2] The second author acknowledges the support of the Research Grant MTM2017-87162-P, Spain.

https://doi.org/10.1016/j.jmaa.2020.124506
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Please cite this article in press as: R. Arora, S. Shmarev, Strong solutions of evolution equations with $p(x,t)$-Laplacian: Existence, global higher integrability of the gradients and second-order regularity, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124506

1. Introduction

In this article, we study the Dirichlet problem for the class of parabolic equations with variable nonlinearity

$$\begin{cases} \partial_t u - \operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right) = f(x,t) \quad \text{in } Q_T = \Omega \times (0,T), \\ u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$. The exponent $p(x,t)$ is a given function whose properties will be described later.

Equation (1.1) falls into the class of equations with variable nonlinearity or nonstandard growth, which have been intensively studied in the last decades. On one hand, the interest in such equations is motivated by their applications in the mathematical modelling of various real-world processes, such as the flows of electrorheological or thermorheological fluids [9,10,21], the problem of thermistor [28], or processing of digital images [13]. On the other hand, their theoretical study is very interesting and challenging from a purely mathematical point of view.

If $p(x,t) \not\equiv 2$, equation (1.1) becomes degenerate or singular at the points where $|\nabla u| = 0$, which prevents one from expecting the existence of classical solutions. The solution of problem (1.1) is understood in the weak sense (see Definition 2.1). We refer to [3,4] for the results on existence and uniqueness of weak solutions for a single equation of the type (1.1), to [16] for systems of equations with the homogeneous Dirichlet boundary conditions, and to [18] for the case of the nonhomogeneous boundary conditions. The weak solution exists if $u_0 \in L^2(\Omega)$, $f \in L^2(Q_T)$, $p(x,t)$ is continuous in $\overline{Q}_T$ with a logarithmic modulus of continuity, and $\partial\Omega \in \text{Lip}$.

Much attention has been paid to the study of the regularity of weak solutions. Let $\Omega' \Subset \Omega$, $\epsilon \in (0,T)$, $Q' = \Omega' \times (\epsilon, T)$, and let u be a weak solution of equation (1.1). It is known that u possesses the property of higher integrability of the gradient: for every Ω' and $\epsilon > 0$ there exists a constant $\delta > 0$ such that $|\nabla u|^{p(x,t)+\delta} \in L^1(Q')$ and $\||\nabla u|^{p+\delta}\|_{1,Q'} \leq C$ with a constant C depending on ϵ and the distance between $\partial\Omega$ and $\partial\Omega'$ - see [1,7,11,29] and [27] for global estimates in Reifenberg domains. The weak solutions are locally Hölder-continuous, provided that $p(x,t)$ is log-continuous [25,12,2]. Moreover, if $p(x,t)$ is Hölder-continuous, then ∇u is locally Hölder-continuous and $u \in C^{1,1/2}_{x,t}(Q')$ - [12,26]. These local regularity properties are intrinsic for every weak solution of equation (1.1) and are completely defined by the nonlinear structure of the equation.

The issues of local higher regularity of solutions of systems of parabolic equations with nonstandard growth have been addressed for the first time in paper [1]. Among other results, it was proven that the solutions of a system of equations with $p(x,t)$-growth conditions with the exponent p Hölder-continuous in t and Lipschitz-continuous in x possess the property of local higher integrability and Hölder-continuity of the spatial gradient, as well as the property of local higher differentiability of the solutions.

In the present work, we are interested in the existence of strong solutions of problem (1.1) and their global regularity properties. Although these questions have already been addressed in a number of works, all known results refer to the singular equation (1.1) with $\frac{2N}{N+2} \leq p(x,t) \leq 2$, or to the equations with $p(x,t)$ nonincreasing in t. It is known [8,6,22] that the weak solution becomes a strong solution with $u_t \in L^2(Q_T)$ and $|\nabla u|^{p(x,t)} \in L^\infty(0,T; L^1(\Omega))$, provided that $|\nabla u_0|^{p(x,0)} \in L^1(\Omega)$, $f \in L^2(Q_T)$, $p_t \in L^\infty(Q_T)$ and either $p_t \leq 0$ a.e. in Q_T, or $|p_t| \leq C$ a.e. in Q_T and $p \leq 2$. Further, if $|\nabla p| + |p_t| \leq C$ a.e. in Q_T, $u_0 \in W^{1,2}_0(\Omega)$ and $p \leq 2$ in Q_T, then $|D^2_{x_i x_j} u|^{p(x,t)} \in L^1(Q_T)$ [8,6], or $D^2_{x_i x_j} u \in L^2(\Omega \times (\epsilon, T))$ for every $\epsilon \in (0,T)$ [6,5]. The strong solution may be Hölder or even Lipschitz continuous in t in the cylinders $\Omega \times (\epsilon, T)$ with $\epsilon > 0$, [22,24]. It is proven in [6] that if the initial function possesses a second-order regularity with respect to x and satisfies certain compatibility conditions, $|f|^{p'(x,t)} \in L^1(Q_T)$ and $f_t \in L^2(Q_T)$, then the singular equation with the Lipschitz-continuous exponent $p(x,t) \le 2$ in a convex C^2 domain has a unique strong solution such that

$$|u_t|^{p'(x,t)} \in L^1(Q_T), \quad |\nabla u_t|^{p(x,t)} \in L^1(Q_T), \quad |D^2_{x_ix_j}u|^{p(x,t)} \in L^1(Q_T), \quad p' = \frac{p}{p-1}.$$

Stronger global regularity properties are known in the case of constant $p > 1$. It is shown in [14] that if $f \in L^2(Q_T)$, $u_0 \in W^{1,p}_0(\Omega)$, and $\partial\Omega$ is subject to minimal regularity assumptions, then

$$u_t \in L^2(Q_T), \quad |\nabla u|^{p-2}\nabla u \in (L^2(0,T;W^{1,2}(\Omega)))^N, \quad u \in L^\infty(0,T;W^{1,p}_0(\Omega))$$

and the corresponding norms are bounded through the norms of the data. The authors of [14] show that problem (1.1) with $p = const$ admits an approximable solution, i.e., a solution obtained as the limit of the sequence of smooth solutions of the same problem with smooth right-hand sides and initial data. The approximable solution inherits the regularity properties of the smooth approximations. We refer to [14] for a review of the previous results on the global regularity in the case of constant p.

The study of higher regularity of solutions usually involves "differentiation" of the equation. In the case of nonconstant p this leads to appearance of the term $|\nabla u|^{p(x,t)} \ln |\nabla u|$, which can not be controlled through the usual energy estimates for the weak solution of equation (1.1) unless $p(x,t) \le 2$. This is the main issue of the present work: we get rid of the restriction $p(x,t) \le 2$ in the proof of existence of strong solutions and in the study of their higher regularity. We derive a special interpolation inequality that yields the global integrability of $|\nabla u|^{p(x,t)+\delta}$ with some $\delta > 0$ independent of u, and provides an estimate on the term with the logarithmic growth. The interpolation inequality is also used in the proof of $W^{1,2}(Q_T)$-regularity of the flux in the degenerate problem (1.1) and the counterpart problems with regularized fluxes.

We prove that if $f \in L^2(0,T;W^{1,2}_0(\Omega))$, $|\nabla u_0|^{\max\{2,p(x,0)\}} \in L^1(\Omega)$, $p(x,t)$ is Lipschitz-continuous in $\overline{Q}_T$ with the values in the interval $[p^-,p^+]$, $\frac{2N}{N+2} < p^- \le p^+ < \infty$, then problem (1.1) has a unique strong solution $u(x,t)$ such that

$$\begin{aligned}
&u_t \in L^2(Q_T), \quad |\nabla u|^{\max\{2,p(x,t)\}} \in L^\infty(0,T;L^1(\Omega)),\\
&|\nabla u|^{p(x,t)+\delta} \in L^1(Q_T) \text{ with every } 0 < \delta < \frac{4p^-}{p^-(N+2)+2N},\\
&D_{x_j}\left(|\nabla u|^{\frac{p(x,t)-2}{2}} D_{x_i}u\right) \in L^2(Q_T) \quad \text{if either } N \ge 3, \text{ or } N = 2 \text{ and } p^- > \frac{6}{5},\\
&|D^2_{x_ix_j}u|^{p(x,t)} \in L^1_{\text{loc}}(Q_T \cap \{p(x,t) < 2\}).
\end{aligned}$$

The same existence and regularity results are obtained for the solution of problem (1.1) with the regularized flux function $(\epsilon^2 + |\nabla u|^2)^{\frac{p(x,t)-2}{2}}\nabla u$, $\epsilon > 0$.

2. Assumptions and results

Prior to formulating the results, we introduce the variable Lebesgue and Sobolev space. We limit ourselves to collecting the most basic facts of the theory and refer to [15] for a detailed insight, see also [4, Ch.1] and [16].

2.1. Function spaces

Let Ω be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ and $p : \Omega \to [p^-,p^+] \subset (1,\infty)$ be a measurable function. Let us define the functional

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$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx$$

(the modular). The set

$$L^{p(\cdot)}(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable on } \Omega, A_{p(\cdot)}(f) < \infty\}$$

equipped with the Luxemburg norm

$$\|f\|_{p(\cdot),\Omega} = \inf\left\{\lambda > 0 : A_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

is a reflexive and separable Banach space and $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$. The modular $A_{p(\cdot)}(f)$ is lower semicontinuous. By the definition of the norm

$$\min\{\|f\|^{p^-}_{p(\cdot),\Omega}, \|f\|^{p^+}_{p(\cdot),\Omega}\} \le A_{p(\cdot)}(f) \le \max\{\|f\|^{p^-}_{p(\cdot),\Omega}, \|f\|^{p^+}_{p(\cdot),\Omega}\}.$$

The dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ with the conjugate exponent $p' = \dfrac{p}{p-1}$. For $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, the generalized Hölder inequality holds:

$$\int_\Omega |fg| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|f\|_{p(\cdot),\Omega}\|g\|_{p'(.),\Omega} \le 2\|f\|_{p(.),\Omega}\|g\|_{p'(\cdot),\Omega}.$$

Let p_1, p_2 are two bounded measurable functions in Ω such that $1 < p_1(x) \le p_2(x)$ a.e. in Ω, then $L^{p_1(\cdot)}(\Omega)$ is continuously embedded in $L^{p_2(\cdot)}(\Omega)$ and

$$\forall\, u \in L^{p_2(\cdot)}(\Omega) \qquad \|u\|_{p_1(\cdot),\Omega} \le C(|\Omega|, p_1^\pm, p_2^\pm)\|u\|_{p_2(\cdot),\Omega}.$$

The variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the set of functions

$$W_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \in L^{p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot),\Omega} + \|\nabla u\|_{p(\cdot),\Omega}.$$

It is known that $C_c^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and the Poincaré inequality holds if $p \in C_{\log}(\overline{\Omega})$, *i.e.*, the exponent p is continuous in $\overline{\Omega}$ with the logarithmic modulus of continuity:

$$|p(x_1) - p(x_2)| \le \omega(|x_1 - x_2|), \tag{2.1}$$

where $\omega(\tau)$ is a nonnegative function satisfying the condition

$$\limsup_{\tau\to 0^+} \omega(\tau)\ln\left(\frac{1}{\tau}\right) = C < \infty.$$

By $W'(\Omega)$ we denote the dual of $W_0^{1,p(\cdot)}(\Omega)$, which is the set of bounded linear functionals over $W_0^{1,p(\cdot)}(\Omega)$: $\Phi \in W'(\Omega)$ iff there exist $\Phi_0 \in L^{p'(\cdot)}(\Omega)$, $\Phi_i \in L^{p'(\cdot)}(\Omega)$, $i = 1, \ldots, N$, such that for all $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\langle \Phi, u\rangle = \int_\Omega \left(u\Phi_0 + \sum_{i=1}^N u_{x_i} \cdot \Phi_i \right)\, dx.$$

For the study of parabolic problem (1.1), we need the spaces of functions depending on $(x,t) \in Q_T$. Let us define the spaces:

$$V_t(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u|^{p(x,t)} \in L^1(\Omega)\}, \quad t \in (0,T),$$
$$W(Q_T) = \{u : (0,T) \to V_t(\Omega) \mid u \in L^2(Q_T), |\nabla u|^{p(x,t)} \in L^1(Q_T)\}.$$

The dual $W'(Q_T)$ of the space $W(Q_T)$ is defined as follows: $\Phi \in W'(Q_T)$ iff there exists $\Phi_0 \in L^2(Q_T)$, $\Phi_i \in L^{p'(x,t)}(\Omega)$, $i = 1, \dots, N$, such that for all $u \in W(Q_T)$

$$\langle \Phi, u\rangle = \int_{Q_T} \left(u\Phi_0 + \sum_{i=1}^N u_{x_i} \Phi_i \right)\, dx\ dt.$$

Let $C_{\log}(\overline{Q}_T)$ be the set of functions satisfying condition (2.1) in the closure of the cylinder Q_T. If $u \in W(Q_T)$, $u_t \in W'(Q_T)$ and $p(x,t) \in C_{\log}(\overline{Q}_T)$, then

$$\int_{Q_T} u u_t\, dz = \frac{1}{2} \int_\Omega u^2(x,t)\, dx \Big|_{t=0}^{t=T}. \tag{2.2}$$

2.2. Statement of main results

We will distinguish between the weak and strong solutions of problem (1.1) defined as follows.

Definition 2.1. A function u is called **weak solution** of problem (1.1), if

1. $u \in W(Q_T)$, $u_t \in W'(Q_T)$,
2. for every $\psi \in W(Q_T)$ with $\psi_t \in W'(Q_T)$

$$\int_{Q_T} u_t\psi\ dx\ dt + \int_{Q_T} |\nabla u|^{p(x,t)-2}\nabla u \cdot \nabla\psi\ dx\ dt = \int_{Q_T} f\psi\, dx\, dt, \tag{2.3}$$

3. for every $\phi \in C_0^1(\Omega)$

$$\int_\Omega (u(x,t) - u_0(x))\phi\ dx \to 0 \quad \text{as } t \to 0,$$

4. the weak solution u is called **strong solution** of problem (1.1) if

$$u_t \in L^2(Q_T), \quad |\nabla u| \in L^\infty(0,T; L^{p(\cdot)}(\Omega)).$$

The existence of a unique weak solution to problem (1.1) can be proven under the minimal requirements on the regularity of the data.

Proposition 2.1 *([3,4,16]). Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the Lipschitz-continuous boundary. Assume that $p : Q_T \to \mathbb{R}$ satisfies the conditions*

$$\frac{2N}{N+2} < p^- \le p(x,t) \le p^+, \qquad p \in C_{\log}(\overline{Q}_T), \tag{2.4}$$

where $p^- := \min_{Q_T} p(x,t)$ *and* $p^+ := \max_{Q_T} p(x,t)$. *Then for every* $f \in L^2(Q_T)$ *and* $u_0 \in L^2(\Omega)$ *problem* (1.1) *has a unique weak solution* $u \in C^0([0,T]; L^2(\Omega)) \cap W(Q_T)$ *with* $u_t \in W'(Q_T)$. *The solution satisfies the estimate*

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|u\|_{2,\Omega} + \int\limits_{Q_T} |\nabla u|^{p(x,t)}\,dx\,dt \le C \tag{2.5}$$

with a constant C *depending only on* N, $p^\pm$, $\|f\|_{2,Q_T}$ *and* $\|u_0\|_{2,\Omega}$.

We are interested in the global regularity of weak solutions in the case when the problem data, f, u_0, p, Ω, possess better regularity properties. The main result of this work is given in the following theorem.

Theorem 2.1. *Let* $\Omega \subset \mathbb{R}^N$, $N \ge 2$, *be a bounded domain with the boundary* $\partial\Omega \in C^2$. *Assume that* $p(x,t)$ *satisfies conditions* (2.4) *and*

$$\operatorname*{ess\,sup}_{Q_T} |\nabla p| \le C_* < \infty, \qquad \operatorname*{ess\,sup}_{Q_T} |p_t| \le C^*$$

with nonnegative finite constants C_*, C^*. *Let*

$$f \in L^2(0,T; W_0^{1,2}(\Omega)), \quad u_0 \in L^2(\Omega) \cap W_0^{1,q_0(\cdot)}(\Omega) \text{ with } q_0(x) = \max\{2, p(x,0)\}.$$

(i) *The weak solution* $u(x,t)$ *of problem* (1.1) *is a strong solution. The function* $u(x,t)$ *satisfies estimate* (2.5) *and*

$$\|u_t\|^2_{2,Q_T} + \operatorname*{ess\ sup}_{(0,T)} \int\limits_\Omega |\nabla u|^{q(x,t)}\,dx \le C \tag{2.6}$$

with the exponent $q(x,t) = \max\{2, p(x,t)\}$ *and a constant* $C = C(N, \partial\Omega, T, p^\pm, C_*, C^*, \|u_0\|, \|f\|)$.

(ii) *The solution* $u(x,t)$ *possesses the property of higher integrability of the gradient:*

$$\int\limits_{Q_T} |\nabla u|^{p(x,t)+\delta}\,dxdt \le C_\delta \qquad \text{for every } 0 < \delta < \frac{4p^-}{p^-(N+2)+2N} \tag{2.7}$$

with a finite constant C_δ *depending on* δ *and the same quantities as the constant* C *in* (2.6).

(iii) *Moreover,*

$$D^2_{x_i x_j} u \in L^{p(\cdot)}_{loc}(Q_T \cap \{(x,t) : p(x,t) < 2\}), \quad \text{if } N \ge 2,$$
$$D_{x_i}\left(|\nabla u|^{\frac{p(x,t)-2}{2}} D_{x_j} u\right) \in L^2(Q_T) \quad \text{if } N \ge 3, \text{ or } N = 2 \text{ and } p^- > \frac{6}{5},$$

$i,j = 1,2,\dots,N$, *and the corresponding norms are bounded by constants depending only on the data.*

Let us give an outline of the rest of the paper. The solution of problem (1.1) is constructed as the limit of the sequence of solutions of the regularized problems (3.1) wherein the flux $|\nabla u|^{p-2}\nabla u$ is substituted by $(\epsilon^2 + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u$, $\epsilon \in (0,1)$. For every ϵ, a solution of problem (3.1) is obtained as the limit of the sequence of finite-dimensional Galerkin's approximations $\{u_\epsilon^{(m)}\}$, $m \in \mathbb{N}$. In Section 3 we formulate the regularized problem, pose the system of ordinary differential equations for defining the coefficients of the approximate solutions, and derive the basic a priori estimates. This is where the difference between the cases of constant and variable exponent p becomes obvious: in the latter case the estimates involve the expression $|\nabla p|(\epsilon^2+|\nabla u|^2)^{\frac{p}{2}}|\ln(\epsilon^2+|\nabla u|^2)|$, not included into the basic energy estimate (2.5).

In Section 4 we establish the interpolation inequality which entails the global higher integrability of the gradients of the finite-dimensional approximations: instead of the natural order of integrability $p(z)$ prompted by the equation, the gradients are integrable in Q_T with the power $p(z)+\delta$ (estimate (2.7)). We also derive the trace-interpolation inequality used to estimate the traces of $|\nabla u|^{p(z)}$ on the lateral boundary of the cylinder Q_T.

In Section 5 we use the property of global higher integrability of the gradients to complete derivation of the uniform a priori estimates of the type (2.6) for Galerkin's approximations.

In Theorem 6.1 we prove that problem (3.1) has a unique weak solution u_ϵ with $\partial_t u_\epsilon \in L^2(Q_T)$. This stems from the uniform estimates of the type (2.5) and (2.6) for $\partial_t u_\epsilon^{(m)}$, the weak convergence of the sequence $\{\nabla u_\epsilon^{(m)}\}$, and the monotonicity of the flux $\gamma_\epsilon((x,t),\mathbf{s})\mathbf{s}=(\epsilon^2+|\mathbf{s}|^2)^{\frac{p(x,t)-2}{2}}\mathbf{s}$ in (3.1). However, these properties do not ensure that $\gamma_\epsilon((x,t),\nabla u_\epsilon^{(m)})\nabla u_\epsilon^{(m)} \to \gamma_\epsilon((x,t),\nabla u_\epsilon)\nabla u_\epsilon$ a.e. in Q_T, even in the case of constant p. In Theorem 6.2 we prove a.e. convergence of the sequence of $\nabla u_\epsilon^{(m)}$ to ∇u_ϵ, which yields a.e. convergence of fluxes. The proof relies on the convexity of the function $\gamma_\epsilon((x,t),\mathbf{s})|\mathbf{s}|^2$ with respect to $\mathbf{s}$, the weak convergence of the sequence $\nabla u_\epsilon^{(m)}$ to ∇u_ϵ, and the convergence of the integrals of $\gamma_\epsilon((x,t),\nabla u_\epsilon^{(m)})|\nabla u_\epsilon^{(m)}|^2$ to the integral of $\gamma_\epsilon((x,t),\nabla u_\epsilon)|\nabla u_\epsilon|^2$. The pointwise convergence of fluxes of Galerkin's approximations and the uniform a priori estimates allow one to show that the limit of the sequence of regularized fluxes $(\epsilon^2+|\nabla u_\epsilon^{(m)}|^2)^{\frac{p(x,t)-2}{4}}\nabla u_\epsilon^{(m)}$ belongs to $(L^2(0,T;W^{1,2}(\Omega)))^N$. The difference between the cases $N\geq 3$ and $N=2$ is explained by the convexity properties of the function $\gamma_\epsilon((x,t),\mathbf{s})|\mathbf{s}|^2$ with $\epsilon>0$. It is strictly convex with respect to $\mathbf{s}$ if $p>\frac{6}{5}$, which is true for $N\geq 3$ because $p^->\frac{2N}{N+2}$, but in the case $N=2$ leads to the additional restriction.

The proof of Theorem 2.1 is given in Section 7. It is based on the same ideas as the proofs in the case of the regularized problems (3.1). The difference in the arguments is due to the necessity of passing to the limit with respect to ϵ, which changes the nonlinear structure of the equation.

The existence and global regularity properties of strong solutions are obtained simultaneously by means of approximation of the equation by a family of equations with regularized fluxes, derivations of a priori estimates on the solutions of the regularized problems, and the limit passage in these estimates with the use of specific interpolation inequalities. Such an approach has been successfully applied in a number of works beginning with [1], but the results on the higher regularity were either local or required restrictions on the range and properties of the exponent p. The novelty of the present work is that the new interpolation inequality derived in Section 4 allows one to obtain global higher regularity and prove the existence of a strong solution without superfluous restrictions on p.

Notation. Throughout the text, the symbol C represents the constants which can be calculated or estimates using the known quantities, but whose exact value is not crucial for the argument and may change from line to line even inside the same formula. We use the notation z for the points of the cylinder Q_T: $z=(x,t)\in\Omega\times(0,T)=Q_T$. The notation D_i is used for the spatial derivative with respect to x_i. We also use the shorthand notation

$$|u_{xx}|^2 \equiv \sum_{i,j=1}^{N} |D_{ij}^2 u|^2$$

and omit the arguments of the variable exponent p wherever it does not cause confusion.

3. Regularized problem

Given a parameter $\epsilon > 0$, let us consider the family of regularized nondegenerate parabolic problems

$$\begin{cases} \partial_t u - \operatorname{div}((\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} \nabla u) = f(z) \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega. \end{cases} \tag{3.1}$$

3.1. Galerkin's approximations

For every fixed ϵ, a solution of problem (3.1) can be constructed as the limit of the sequence of finite-dimensional Galerkin's approximations $\{u_\epsilon^{(m)}\}$. The functions $u_\epsilon^{(m)}(x,t)$ are sought in the form

$$u_\epsilon^{(m)}(x,t) = \sum_{j=1}^{m} u_j^{(m)}(t)\phi_j(x), \tag{3.2}$$

where $\phi_j \in W_0^{s,2}(\Omega)$ and $\lambda_j > 0$ are the eigenfunctions and the corresponding eigenvalues of the problem

$$(\nabla\phi_j, \nabla\psi)_{W^{s,2}(\Omega)} = \lambda(\phi_j, \psi)_{2,\Omega} \qquad \forall \psi \in W^{s,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

The number $s \in \mathbb{N}$, $s \geq 2$, is taken so large that $W_0^{s,2}(\Omega) \subset W_0^{1,p^+}(\Omega)$. The systems $\{\phi_j\}$ and $\{\lambda_j^{-\frac{1}{2}}\phi_j\}$ form the orthogonal bases in $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$. The coefficients $u_j^{(m)}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary differential equations

$$\begin{aligned} &(u_j^{(m)})'(t) = -\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla\phi_j \, dx + \int_\Omega f\phi_j \, dx, \\ &u_j^{(m)}(0) = (u_0, \phi_j)_{2,\Omega}, \quad j = 1, 2, \ldots, m, \end{aligned} \tag{3.3}$$

where the functions

$$u_0^{(m)} = \sum_{j=1}^{m} (u_0, \phi_j)_{2,\Omega}\phi_j \in \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_m\},$$

are chosen so that

$$u_0^{(m)} \to u_0 \text{ in } W_0^{1,q(x,0)}(\Omega), \quad q(x,0) = \max\{2, p(x,0)\}.$$

By the Peano Theorem, for every finite m system (3.3) has a solution $(u_1^{(m)}, u_2^{(m)}, \ldots, u_m^{(m)})$ on an interval $(0, T_m)$. This solution can be continued on the arbitrary interval $(0,T)$ because of the uniform estimate $\sup_{(0,T_m)} \|\nabla u_\epsilon^{(m)}(\cdot,t)\|_{q(\cdot),\Omega} \leq M$ with $q(x,t) = \max\{2, p(x,t)\}$, which follows from (5.3) and (5.7).

3.2. Basic a priori estimates

Lemma 3.1. *Let Ω and p satisfy the conditions of Theorem 2.1. If $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, then $u_\epsilon^{(m)}$ satisfy the estimates*

$$\sup_{(0,T)} \|u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 \, dz \leq e^T(\|f\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2) := L_0. \tag{3.4}$$

Proof. Multiplying jth equation of (3.3) by $u_j^{(m)}(t)$ and summing up the results for $j = 1, 2, \dots, m$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\epsilon^{(m)}\|_{2,\Omega}^2 &= \sum_{j=1}^m u_j^{(m)}(t)(u_j^{(m)})'(t)\\ &= -\sum_{j=1}^m \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j u_j^{(m)}(t)\, dx + \sum_{j=1}^m \int_\Omega f\phi_j u_j^{(m)}(t)\, dx\\ &= -\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2\, dx + \int_\Omega f u_\epsilon^{(m)}\, dx.\end{aligned}$$

Applying the Cauchy inequality to the last term of the right-hand side we transform this inequality into the form

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon^{(m)}\|_{2,\Omega}^2 + \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2\, dx \le \frac{1}{2}\|f_0\|_{2,\Omega}^2 + \frac{1}{2}\|u_\epsilon^{(m)}\|_{2,\Omega}^2.$$

The last inequality can be written as

$$\frac{1}{2}\frac{d}{dt}\left(\mathrm{e}^{-t}\|u_\epsilon^{(m)}\|_{L^2(\Omega)}^2\right) + \mathrm{e}^{-t}\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2\, dx \le \frac{\mathrm{e}^{-t}}{2}\|f_0\|_{2,\Omega}^2.$$

Integration of the last inequality in t gives

$$\sup_{(0,T)} \|u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + \int_{Q_T} \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2\right)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2\, dx\, dt \le C\mathrm{e}^T\left(\|f_0\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2\right)$$

with a constant C which does not depend on $u_\epsilon^{(m)}$. □

Corollary 3.1. *Let $\epsilon \in (0,1)$. Under the conditions of Lemma 3.1*

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)}\, dz \le \int_{Q_T} \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2\right)^{\frac{p(z)}{2}} dz \le L_1 \tag{3.5}$$

with a constant L_1 independent of ϵ and m.

Proof. The assertion immediately follows from (3.4) and the inequalities

$$|\nabla u_\epsilon^{(m)}|^{p(z)} \le \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2\right)^{\frac{p(z)}{2}} \le \begin{cases} 2\left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2\right)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 & \text{if } |\nabla u_\epsilon^{(m)}| \ge \epsilon,\\ (2\epsilon^2)^{\frac{p(z)}{2}} \le 2^{\frac{p^+}{2}} & \text{otherwise.} \end{cases} \tag{3.6}$$

□

Let us denote $\mathbf{n}$ by the exterior normal vector to $\partial\Omega$.

Lemma 3.2. *Let $\partial\Omega \in C^2$, $p(z)$ satisfies (2.4) and*

$$\operatorname*{ess\,sup}_{Q_T} |\nabla p| \le C_* < \infty, \qquad u_0 \in W_0^{1,2}(\Omega), \qquad f \in L^2(0,T; W_0^{1,2}(\Omega)).$$

Then the following inequality holds: for a.e. $t \in (0,T)$ and any $\delta > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + (\min\{p^-, 2\} - 1 - \delta)\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2\,dx \\
&\qquad \le C_0 \int_\Omega |\nabla u_\epsilon^{(m)}|^2 (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)\,dx \\
&\qquad - \int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_\epsilon^{(m)} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\epsilon^{(m)} \cdot \nabla(\nabla u_\epsilon^{(m)} \cdot \mathbf{n})\right) dx \\
&\qquad + C_1 \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + C_2 \|f(t)\|^2_{W_0^{1,2}(\Omega)}
\end{aligned} \tag{3.7}$$

with constants C_i, $i = 0, 1, 2$, depending on the data and δ, but independent of m and ϵ.

Proof. Multiplying each of equations in (3.3) by $\lambda_j u_j^{(m)}$, $j = 1, 2, \ldots, m$, and summing up the results we obtain the equality

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u_\epsilon^{(m)}\|_{2,\Omega}^2 &= \sum_{j=1}^m \lambda_j (u_j^{(m)})'(t) u_j^{(m)}(t) \\
&= \sum_{j=1}^m \lambda_j u_j^{(m)} \int_\Omega \operatorname{div}((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)})\ \phi_j\,dx + \sum_{j=1}^m \lambda_j u_j^{(m)} \int_\Omega f(x,t)\phi_j\,dx \\
&= -\int_\Omega \operatorname{div}((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)})\ \Delta u_\epsilon^{(m)}\,dx + \int_\Omega f \Delta u_\epsilon^{(m)}\,dx.
\end{aligned} \tag{3.8}$$

Since $u_\epsilon^{(m)} \in C^3(\Omega)$ and $\partial\Omega \in C^2$, the first term on the right-hand can be transformed by means of the Green formula:

$$\begin{aligned}
&-\int_\Omega \operatorname{div}\left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)}\right) \Delta u_\epsilon^{(m)}\,dx \\
&\quad = -\int_\Omega \left(\sum_{k=1}^N (u_\epsilon^{(m)})_{x_k x_k}\right) \left(\sum_{i=1}^N \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (u_\epsilon^{(m)})_{x_i}\right)_{x_i}\right) dx \\
&\quad = -\int_{\partial\Omega} \Delta u_\epsilon^{(m)} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n})\,dS \\
&\qquad + \int_\Omega \sum_{k,i=1}^N (u_\epsilon^{(m)})_{x_k x_k x_i} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (u_\epsilon^{(m)})_{x_i}\,dx \\
&\quad = -\int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \sum_{k,i=1}^N \left((u_\epsilon^{(m)})_{x_k x_k} (u_\epsilon^{(m)})_{x_i} n_i - (u_\epsilon^{(m)})_{x_k x_i} (u_\epsilon^{(m)})_{x_i} n_k\right) dS \\
&\qquad - \int_\Omega \sum_{k,i=1}^N (u_\epsilon^{(m)})_{x_k x_i} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (u_\epsilon^{(m)})_{x_i}\right)_{x_k} dx \\
&\quad = -\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2\,dx + J_1 + J_2 + J_{\partial\Omega}
\end{aligned}$$

where

$$J_1 := \int_\Omega (2-p(z))(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}-1} \left(\sum_{k=1}^N \left(\nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_{x_k} \right)^2 \right) dx,$$

$$J_2 = -\int_\Omega \sum_{k,i=1}^N (u_\epsilon^{(m)})_{x_k x_i} (u_\epsilon^{(m)})_{x_i} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \frac{p_{x_k}}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)\, dx,$$

$$J_{\partial\Omega} = -\int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_\epsilon^{(m)} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\epsilon^{(m)} \cdot \nabla (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) \right) dS.$$

Substitution into (3.8) leads to the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|\nabla u_\epsilon^{(m)}\|_{2,\Omega}^2 + \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2\, dx \\ &\qquad = J_1 + J_2 + J_{\partial\Omega} - \int_\Omega \nabla f \cdot \nabla u_\epsilon^{(m)}\, dx \\ &\qquad \le J_1 + J_2 + J_{\partial\Omega} + \frac{1}{2} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + \frac{1}{2} \|f(t)\|^2_{W_0^{1,2}(\Omega)}. \end{aligned}$$

The term J_1 is absorbed in the left-hand side because

$$\begin{aligned} J_1 &= \int_{\{x\in\Omega:\ p(z)\ge 2\}} (2-p(z)) \ldots + \int_{\{x\in\Omega:\ p(z)<2\}} (2-p(z)) \ldots \\ &\le \int_{\{x\in\Omega:\ p(z)<2\}} (2-p(z))(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}-1} \left(\sum_{k=1}^N \left(\nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_{x_k} \right)^2 \right) dx, \end{aligned}$$

and

$$|J_1| \le \max\{0, 2-p^-\} \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2\, dx.$$

The term J_2 is estimated in the following way: by the Cauchy inequality, for every $\delta > 0$

$$\begin{aligned} |J_2| &\le \|\nabla p\|_{\infty, Q_T} \int_\Omega \left(\sum_{i,k=1}^N |(u_\epsilon^{(m)})_{x_i x_k}| (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{4}} \right) \\ &\qquad \times \left(|\nabla u_\epsilon^{(m)}| (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{4}} |\ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)| \right) dx \\ &\le \delta \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2\, dx \\ &\quad + C \int_\Omega |\nabla u_\epsilon^{(m)}|^2 (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)\, dx \end{aligned}$$

with a constant $C = C(C^*, N, \delta)$. Choosing $\delta \in (0,1)$ so small that $\min\{2, p^-\} > 1 + \delta$ and collecting in the right-hand side all terms which contain $(u^{(m)})_{xx}$ we obtain (3.7) because

$$1-\delta-\max\{0,2-p^-\}=\begin{cases}1-\delta & \text{if } p^-\ge 2,\\ p^--1-\delta & \text{if } p^-<2\end{cases}=\min\{p^-,2\}-1-\delta. \quad \square$$

4. Interpolation inequalities

In this section, we derive first the interpolation inequality which yields the property of higher integrability of the gradient of the finite-dimensional approximations $u_\epsilon^{(m)}$ of the solutions of problems (3.1). We prove next an estimate on the trace of $\nabla u_\epsilon^{(m)}$ on $\partial\Omega$, which turns out to be useful in the study of the nonconvex domains. Both estimates will be applied to obtain upper bounds for the terms on the right-hand side of (3.7).

With certain abuse of notation, throughout the section we denote by $p(x)$ or $p(x,t)$ given exponents defined on Ω or Q_T and not related to the exponent in equation (1.1). Let us accept the notation

$$\begin{aligned}&\beta_\epsilon(\mathbf{s})=\epsilon^2+|\mathbf{s}|^2,\quad \epsilon>0,\quad \mathbf{s}\in\mathbb{R}^N,\quad x\in\Omega,\\ &\gamma_\epsilon(x,\mathbf{s})=\beta_\epsilon^{\frac{p(x)-2}{2}}(\mathbf{s})\equiv(\epsilon^2+|\mathbf{s}|^2)^{\frac{p(x)-2}{2}}.\end{aligned}$$

Lemma 4.1. *Let $\partial\Omega\in C^1$, $u\in C^2(\overline{\Omega})$ and $u=0$ on $\partial\Omega$. Assume that*

$$\begin{aligned}&p:\Omega\mapsto[p^-,p^+],\quad p^\pm=const,\\ &\frac{2N}{N+2}<p^-,\quad p(\cdot)\in C^0(\overline{\Omega}),\qquad \operatorname*{ess\,sup}_{\Omega}|\nabla p|=L,\\ &\int_\Omega\gamma_\epsilon(x,\nabla u)|u_{xx}|^2\,dx<\infty,\qquad \int_\Omega u^2\,dx=M_0,\qquad \int_\Omega|\nabla u|^{p(x)}\,dx=M_1.\end{aligned}\tag{4.1}$$

Then for every

$$\frac{2}{N+2}=:r_*<r<r^*:=\frac{4p^-}{p^-(N+2)+2N}\tag{4.2}$$

and every $\delta\in(0,1)$

$$\int_\Omega\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx\le\delta\int_\Omega\gamma_\epsilon(x,\nabla u)|u_{xx}|^2\,dx+C\left(1+\int_\Omega|\nabla u|^{p(x)}\,dx\right)\tag{4.3}$$

with an independent of u constant $C=C(\partial\Omega,\delta,p^\pm,N,r,M_0,M_1)$.

Proof. Let us fix some $r\in(r_*,r^*)$. By the Green formula

$$\begin{aligned}\int_\Omega\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx&=\int_\Omega\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)\ \nabla u\cdot\nabla u\,dx\\ &=\int_{\partial\Omega}u\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)\ \nabla u\cdot\mathbf{n}\,dS-\int_\Omega u\operatorname{div}(\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)\nabla u)\,dx\\ &=-\int_\Omega u\operatorname{div}(\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)\nabla u)\,dx=:-J,\end{aligned}$$

where $\mathbf{n}$ stands for the outer normal to $\partial\Omega$. A straightforward computation leads to the representation

$$
\begin{aligned}
J = &\int_\Omega u\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)\Delta u\,dx \\
&+ \int_\Omega (p(x)+r-2)u\beta_\epsilon^{\frac{p(x)+r-2}{2}-1}(\nabla u)\sum_{i=1}^n\left(u_{x_i}\sum_{j=1}^n u_{x_j}u_{x_ix_j}\right)dx \\
&+ \int_\Omega u\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)\ln(\beta_\epsilon(\nabla u))\nabla u\cdot\nabla p\,dx,
\end{aligned}
$$

whence

$$
|J| \le C\int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|u_{xx}|\,dx + L\int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-1}{2}}(\nabla u)|\ln\beta_\epsilon(\nabla u)|\,dx \tag{4.4}
$$

with $C = C(n,p^\pm,r)$ and $|u_{xx}|^2 = \sum_{i,j=1}^n |D^2_{ij}u|^2$. For every constant $0 < \rho < \min\{1, p^- + r - 1\}$ and $0 < \nu < 1$, the integrand of the last term in (4.4) admits the estimate

$$
\begin{aligned}
\beta_\epsilon^{\frac{p(x)+r-1}{2}}(\nabla u)|\ln\beta_\epsilon(\nabla u)| &\le \begin{cases} \beta_\epsilon^{\frac{p(x)+r-1-\rho}{2}}(\nabla u)\left(\beta_\epsilon^{\frac{\rho}{2}}(\nabla u)|\ln\beta_\epsilon(\nabla u)|\right) & \text{if } \beta_\epsilon(\nabla u)\le 1, \\ \beta_\epsilon^{\frac{p(x)+r-1+\nu}{2}}(\nabla u)\left(\beta_\epsilon^{-\frac{\nu}{2}}(\nabla u)|\ln\beta_\epsilon(\nabla u)|\right) & \text{if } \beta_\epsilon(\nabla u) > 1, \end{cases} \\
&\le C(\rho) + C(\nu)\beta_\epsilon^{\frac{p(x)+r-1+\nu}{2}}(\nabla u),
\end{aligned} \tag{4.5}
$$

which allows one to continue (4.4) as follows:

$$
\begin{aligned}
|J| &\le C\int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|u_{xx}|\,dx + C'\left(\int_\Omega |u|\,dx + \int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-1+\nu}{2}}(\nabla u)\,dx\right) \\
&\le C\int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|u_{xx}|\,dx + C'\int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-1+\nu}{2}}(\nabla u)\,dx + M_0^{1/2} + C'' =: I.
\end{aligned}
$$

Using Young's inequality we finally estimate: for every $\delta \in (0,1)$

$$
\begin{aligned}
|I| &= C\int_\Omega \left(|u|\beta_\epsilon^{\frac{p(x)+r-1}{2}-\frac{p(x)}{4}}(\nabla u)\right)\left(\beta_\epsilon^{\frac{p(x)-2}{4}}(\nabla u)|u_{xx}|\right)dx + C'\int_\Omega |u|\beta_\epsilon^{\frac{p(x)-1+r+\nu}{2}}(\nabla u)\,dx + \widehat{C} \\
&\le \delta\int_\Omega \gamma_\epsilon(x,\nabla u)|u_{xx}|^2\,dx + C_\delta\int_\Omega u^2\beta_\epsilon^{\frac{p(x)+2r-2}{2}}(\nabla u)\,dx + C'\int_\Omega |u|\beta_\epsilon^{\frac{p(x)+r-1+\nu}{2}}(\nabla u)\,dx + \widehat{C} \\
&\equiv \delta I_0 + C_\delta I_1 + C' I_2 + \widehat{C}.
\end{aligned}
$$

Let $\{\Omega_i\}_{i=1}^K$ be a finite cover of Ω such that

$$
\Omega_i \subset \Omega, \quad \partial\Omega_i \in C^2, \quad p_i^+ = \max_{\Omega_i} p(x), \quad p_i^- = \min_{\Omega_i} p(x).
$$

For any $r_* < r < r^*$, the continuity of $p(x)$ allows us to choose Ω_i so small that for every $i = 1, 2, \dots, K$

$$
p_i^+ - p_i^- + r\left(1 + \frac{2N}{p^-(N+2)}\right) < \frac{4}{N+2}. \tag{4.6}
$$

To estimate the terms I_1 and I_2 we represent them in the form

$$I_j = \sum_{i=1}^{K} I_j^{(i)}, \quad I_1^{(i)} = \int_{\Omega_i} u^2 \beta_\epsilon^{\frac{p(x)+2(r-1)}{2}}(\nabla u)\,dx, \quad I_2^{(i)} = \int_{\Omega_i} |u| \beta_\epsilon^{\frac{p(x)+r-1+\nu}{2}}(\nabla u)\,dx.$$

Recall that $\nu \in (0,1)$. By the Young inequality, for any $\lambda > 0$

$$\begin{aligned}
I_2^{(i)} &\le \lambda \int_{\Omega_i} \beta_\epsilon^{\frac{p(x)+r}{2}}(\nabla u)\,dx + C_\lambda \int_{\Omega_i} |u|^{\frac{p(x)+r}{1-\nu}}\,dx \\
&\le \lambda \int_{\Omega_i} \beta_\epsilon^{\frac{p(x)+r}{2}}(\nabla u)\,dx + C_\lambda \left(1 + \int_{\Omega_i} |u|^{\frac{p_i^+ + r}{1-\nu}}\,dx\right) \\
&= \lambda \left(\int_{\Omega_i \cap \{|\nabla u|>1\}} \beta_\epsilon^{\frac{p(x)+r}{2}}(\nabla u)\,dx + \int_{\Omega_i \cap \{|\nabla u|\le 1\}} \dots \right) + C_\lambda \left(1 + \int_{\Omega_i} |u|^{\frac{p_i^+ + r}{1-\nu}}\,dx\right).
\end{aligned}$$

For $\epsilon \in (0,1)$

$$\beta_\epsilon^{\frac{p(x)+r}{2}}(\nabla u) = \beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)(\epsilon^2 + |\nabla u|^2) \le \begin{cases} 2\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2 & \text{if } |\nabla u| > 1, \\ (1+\epsilon^2)^{\frac{p^+ + r}{2}} & \text{otherwise,} \end{cases}$$

which entails the estimate

$$I_2^{(i)} \le 2\lambda \int_{\Omega} \beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx + C_\lambda \int_{\Omega_i} |u|^{\frac{p_i^+ + r}{1-\nu}}\,dx + C.$$

The second integral on the right-hand side is estimated by the Gagliardo-Nirenberg inequality:

$$\|u\|_{\sigma,\Omega_i}^{\sigma} \le C_1 \|\nabla u\|_{p_i^-,\Omega_i}^{\sigma\theta} \|u\|_{2,\Omega_i}^{\sigma(1-\theta)} + C_2 \|u\|_{2,\Omega_i}^{\sigma} \le C_1' \|\nabla u\|_{p_i^-,\Omega_i}^{\sigma\theta} + C_2 M_0^{\frac{\sigma}{2}}, \qquad C_1' = C_1 M_0^{\frac{\sigma}{2}(1-\theta)}, \tag{4.7}$$

with

$$\sigma = \frac{p_i^+ + r}{1-\nu} > p_i^+ + r > p_i^-, \qquad \theta = \frac{p_i^-}{\sigma} \in (0,1), \qquad \frac{1}{\sigma} = \left(\frac{1}{p_i^-} - \frac{1}{N}\right)\theta + \frac{1-\theta}{2}.$$

Such a choice of the parameters σ, θ is possible if

$$\nu = 1 - \frac{p_i^+ + r}{p_i^-} \frac{N}{N+2} \quad \text{with } r_* < r < r^*.$$

Gathering the estimates on $I_2^{(i)}$ and using the Young inequality we finally obtain: for every $\lambda \in (0,1)$

$$\begin{aligned}
I_2 &\le 2\lambda K \int_{\Omega} \beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx + C_\lambda \sum_{i=1}^{K} \int_{\Omega_i} |\nabla u|^{p_i^-}\,dx + C \\
&\le 2\lambda K \int_{\Omega} \beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx + C_\lambda' \int_{\Omega} |\nabla u|^{p(x)}\,dx + C' \\
&= 2\lambda K \int_{\Omega} \beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx + C'', \quad C'' = C''(N, \lambda, p_i^\pm, r, |\Omega|, M_0, M_1).
\end{aligned}$$

To estimate $I_1^{(i)}$ we first use the Young inequality: since $\frac{2N}{N+2} < p_i^-$ by assumption, then for every $\widetilde{\lambda} \in (0,1)$

$$I_1^{(i)} \leq C_{\tilde{\lambda}} \int_{\Omega_i} |u|^{p_i^- \frac{N+2}{N}} \, dx + \widetilde{\lambda} \int_{\Omega_i} \beta_\epsilon^{\frac{\kappa}{2}}(\nabla u) \, dx, \quad \kappa = (p(x) + 2(r-1)) \frac{p_i^- \frac{N+2}{2N}}{p_i^- \frac{N+2}{2N} - 1}. \tag{4.8}$$

To estimate the second integral, let us claim that $0 < \kappa < p(x) + r$ on Ω_i, i.e.,

$$0 < p(x) + 2(r-1) < \frac{p(x)+r}{p_i^-} \left(p_i^- - \frac{2N}{N+2} \right).$$

In this double inequality the first one is fulfilled by the choice of r:

$$0 = \frac{2N}{N+2} + 2(r_* - 1) < p^- + 2(r-1) \leq p(x) + 2(r-1).$$

The second inequality is fulfilled if

$$p_i^+ + 2(r-1) < \frac{p_i^- + r}{p_i^-} \left(p_i^- - \frac{2N}{N+2} \right) \quad \Leftrightarrow \quad (p_i^+ - p_i^-) + r < 2 - \frac{p_i^- + r}{p_i^-} \frac{2N}{N+2},$$

which is true because of (4.6) and the condition $r < r^*$. By the Young inequality

$$\beta_\epsilon^{\frac{\kappa}{2}}(\nabla u) \leq 1 + \beta_\epsilon^{\frac{p(x)+r}{2}}(\nabla u) \leq 1 + \begin{cases} (2\epsilon^2)^{\frac{p(x)+r}{2}} & \text{if } |\nabla u| \leq \epsilon, \\ 2\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2 & \text{if } |\nabla u| > \epsilon \end{cases}$$
$$\leq C + 2\beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2.$$

It remains to estimate the first integral in (4.8). By the Gagliardo-Nirenberg inequality

$$\int_{\Omega_i} |u|^{p_i^- \frac{N+2}{N}} \, dx \leq C_1(M_0) \|\nabla u\|_{p_i^-, \Omega_i}^{\theta p_i^- \frac{N+2}{N}} + C_2(M_0)$$

with

$$\theta = \frac{\frac{1}{2} - \frac{N}{p_i^-(N+2)}}{\frac{N+2}{2N} - \frac{1}{p_i^-}} = \frac{N}{N+2} \in (0,1),$$

whence

$$I_1 \leq 2\widetilde{\lambda} \int_\Omega \beta_\epsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2 \, dx + C' \int_\Omega |\nabla u|^{p(x)} \, dx + C''.$$

Gathering the estimates of I_i for $|I|$ and choosing $\lambda, \widetilde{\lambda}$ so small that $2\lambda K + 2\widetilde{\lambda} < 1$, we arrive at the desired estimate (4.3). $\square$

The assertion of Lemma 4.1 easily extends to functions defined on the cylinder Q_T. Let us recall the notation $z = (x,t) \in Q_T = \Omega \times (0,T)$ and re-define

$$\gamma_\epsilon(z, \mathbf{s}) = \beta_\epsilon^{\frac{p(z)-2}{2}}(\mathbf{s}) \equiv (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}}, \quad \epsilon > 0, \quad \mathbf{s} \in \mathbb{R}^N.$$

Theorem 4.1. *Let* $\partial\Omega \in C^1$, $u \in C^1([0,T]; C^2(\overline{\Omega}))$ *and* $u = 0$ *on* $\partial\Omega \times [0,T]$. *Assume that*

$$\begin{aligned}
&p(z) : Q_T \mapsto [p^-, p^+], \quad p^{\pm} = const,\\
&p(\cdot) \in C^0(\overline{Q}_T) \text{ with the modulus of continuity } \omega,\\
&\frac{2N}{N+2} < p^-, \qquad \operatorname*{ess\,sup}_{Q_T} |\nabla p| = L,\\
&\int\limits_{Q_T} \gamma_\epsilon(z, \nabla u)|u_{xx}|^2\,dz < \infty, \qquad \sup_{(0,T)} \|u(t)\|^2_{2,\Omega} = M_0, \qquad \int\limits_{Q_T} |\nabla u|^{p(z)}\,dz = M_1.
\end{aligned} \tag{4.9}$$

Then for every

$$\frac{2}{N+2} = r_* < r < r^* = \frac{4p^-}{p^-(N+2) + 2N}$$

and every $\delta \in (0,1)$ *the function* u *satisfies the inequality*

$$\int\limits_{Q_T} \beta_\epsilon^{\frac{p(z)+r-2}{2}}(\nabla u)|\nabla u|^2\,dz \le \delta \int\limits_{Q_T} \gamma_\epsilon(z, \nabla u)|u_{xx}|^2\,dz + C\left(1 + \int\limits_{Q_T} |\nabla u|^{p(z)}\,dz\right), \tag{4.10}$$

with an independent of u *constant* $C = C(N, \partial\Omega, T, \delta, p^{\pm}, \omega, r, M_0, M_1)$.

Proof. Since the exponent $p(z)$ is uniformly continuous in $\overline{Q}_T$, then for any $r_* < r < r^*$ there exists a finite cover of Q_T composed of the cylinders $Q^{(i)} = \Omega_i \times (t_{i-1}, t_i)$, $i = 1, 2, \dots, K$, such that

$$\begin{aligned}
&t_0 = 0, \quad t_K = T, \quad t_i - t_{i-1} = \rho, \quad Q_T \subset \bigcup_{i=1}^K Q^{(i)}, \quad \partial\Omega_i \in C^2,\\
&p_i^+ = \max_{Q^{(i)}} p(z), \quad p_i^- = \min_{Q^{(i)}} p(z),\\
&p_i^+ - p_i^- + r\left(1 + \frac{2N}{p_-(N+2)}\right) < \frac{4}{N+2}, \quad i = 1, 2, \dots, K.
\end{aligned}$$

For a.e. $t \in (0,T)$ the function $u(x,t)$ satisfies inequality (4.3). Integrating this inequality over the interval (t_{i-1}, t_i) and summing the results gives (4.10). $\square$

Remark 4.1. If $p = const > \dfrac{2N}{N+2}$ and $u(z)$ satisfies conditions (4.9), then inequalities (4.6) and (4.10) hold for every $r_* < r < r^*$.

Lemma 4.2. *Let* $\partial\Omega$ *be a Lipschitz-continuous surface and* $\|\nabla p\|_{\infty,\Omega} = L$. *There exists a constant* $\delta = \delta(\partial\Omega)$ *such that for every* $u \in W^{1,p(\cdot)}(\Omega)$

$$\delta \int\limits_{\partial\Omega} |u|^{p(x)}\,dS \le C \int\limits_{\Omega} \left(|u|^{p(x)-1}|\nabla u| + |u|^{p(x)}|\ln|u|| + |u|^{p(x)}\right)\,dx \tag{4.11}$$

with a constant $C = C(p^+, L, N, \partial\Omega)$.

Proof. By [19, Lemma 1.5.1.9] there exists $\delta > 0$ and $\mu \in (C^\infty(\overline{\Omega}))^N$ such that $\mu \cdot \mathbf{n} \ge \delta$ a.e. on $\partial\Omega$. By the Green formula

$$
\begin{aligned}
\delta \int_{\partial\Omega} |u|^{p(x)}\, dS &\leq \int_{\partial\Omega} |u|^{p(x)}(\mu\cdot\mathbf{n})\, dS = \int_{\Omega} \operatorname{div}(|u|^{p(x)}\mu)\, dx \\
&= \int_{\Omega} \Big(p(x)|u|^{p(x)-2}u(\nabla u\cdot\mu) + |u|^{p(x)}\ln|u|(\nabla p\cdot\mu) + |u|^{p(x)}\operatorname{div}\mu\Big)\, dx \\
&\leq p^+ \max_{\Omega}|\mu| \int_{\Omega} |u|^{p(x)-1}|\nabla u|\, dx + \|\nabla p\|_{\infty,\Omega}\max_{\Omega}|\mu| \int_{\Omega} |u|^{p(x)}|\ln|u||\, dx \\
&\quad + \max_{\Omega}|\operatorname{div}\mu| \int_{\Omega} |u|^{p(x)}\, dx \\
&\leq C(p^+, L, N, \Omega) \int_{\Omega} \Big(|u|^{p(x)-1}|\nabla u| + |u|^{p(x)}|\ln|u|| + |u|^{p(x)}\Big)\, dx. \quad \square
\end{aligned}
$$

Lemma 4.3. *Under the conditions of Lemma 4.2, for every $\lambda\in(0,1)$ and $\epsilon\in(0,1)$*

$$
\int_{\partial\Omega} |u|^{p(x)}\, dS \leq \lambda \int_{\Omega} (\epsilon^2+|u|^2)^{\frac{p(x)-2}{2}}|\nabla u|^2 + C_1\int_{\Omega}|u|^{p(x)}\, dx + C_2\int_{\Omega}|u|^{p(x)}|\ln|u||\, dx + C_3 \tag{4.12}
$$

with constants C_i, $i = 1,2,3$, depending on $N, p^{\pm}, L, \partial\Omega, \lambda$, but independent of u.

Proof. By the Cauchy inequality, for every $\lambda\in(0,1)$

$$
\begin{aligned}
|u|^{p-1}|\nabla u| &= \Big(\lambda(\epsilon^2+|u|^2)^{\frac{p-2}{2}}|\nabla u|^2\Big)^{\frac12}\Big(\lambda^{-1}(\epsilon^2+|u|^2)^{\frac{2-p}{2}}|u|^{2(p-1)}\Big)^{\frac12} \\
&\leq \lambda(\epsilon^2+|u|^2)^{\frac{p-2}{2}}|\nabla u|^2 + \frac{1}{\lambda}(\epsilon^2+|u|^2)^{1-\frac{p}{2}}|u|^{2(p-1)} \\
&\leq \lambda(\epsilon^2+|u|^2)^{\frac{p-2}{2}}|\nabla u|^2 + \frac{1}{\lambda}(\epsilon^2+|u|^2)^{1-\frac{p}{2}+(p-1)} \\
&= \lambda(\epsilon^2+|u|^2)^{\frac{p-2}{2}}|\nabla u|^2 + \frac{1}{\lambda}(\epsilon^2+|u|^2)^{\frac{p}{2}} \\
&\leq \lambda(\epsilon^2+|u|^2)^{\frac{p-2}{2}}|\nabla u|^2 + C(1+|u|^p).
\end{aligned}
$$

Inequality (4.12) follows now from (4.11). $\square$

Lemma 4.4. *Let $\partial\Omega\in C^2$. Assume that the functions $p(x)$ and $u(x)$ satisfy the conditions of Lemma 4.1. Then for every $\lambda\in(0,1)$*

$$
\int_{\partial\Omega} |\nabla u|^{p(x)}\, dS \leq \lambda\int_{\Omega}(\epsilon^2+|\nabla u|^2)^{\frac{p(x)-2}{2}}|u_{xx}|^2\, dx + C\left(1+\int_{\Omega}|\nabla u|^{p(x)}\, dx\right) \tag{4.13}
$$

with a constant C depending on λ and the constants $p^{\pm}$, L, M_0, M_1 in (4.1) and $\partial\Omega$, but independent of u.

Proof. Applying (4.12) to u_{x_i} we obtain

$$
\begin{aligned}
\int_{\partial\Omega} |\nabla u|^{p(x)}\, dS &\leq \lambda\int_{\Omega}(\epsilon^2+|\nabla u|^2)^{\frac{p(x)-2}{2}}|u_{xx}|^2\, dx \\
&\quad + C_1\int_{\Omega}|\nabla u|^{p(x)}\, dx + C_2\int_{\Omega}|\nabla u|^{p(x)}\ln|\nabla u||\, dx + C_3
\end{aligned} \tag{4.14}
$$

with independent of u constants M, L, K. For every $0 < \theta < p^-$ and r from inequality (4.3)

$$|\nabla u|^{p(x)}|\ln|\nabla u|| \leq \begin{cases} |\nabla u|^{p^- -\theta}\left(|\nabla u|^{\theta}|\ln|\nabla u||\right) \leq C'(p^-,\theta) & \text{if } |\nabla u| \leq 1, \\ |\nabla u|^{p(x)+r}\left(|\nabla u|^{-r}|\ln|\nabla u||\right) \leq C''(p^-,r)\ |\nabla u|^{p(x)+r} & \text{if } |\nabla u| \geq 1. \end{cases}$$

Thus, there exists a constant C such that

$$|\nabla u|^{p(x)}|\ln|\nabla u|| \leq C(1 + |\nabla u|^{p(x)+r}) \text{ in } \Omega$$

and (4.13) follows from (4.14), (3.6) and (4.3). □

Theorem 4.2. *Let us assume that $p(z)$ and $u(z)$ satisfy the conditions of Theorem 4.1. Then for every $\lambda \in (0,1)$*

$$\int_{\partial\Omega\times(0,T)} |\nabla u|^{p(z)}\, dSdt \leq \lambda \int_{Q_T} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |u_{xx}|^2\, dz + C\left(1 + \int_{Q_T} |\nabla u|^{p(z)}\, dz\right)$$

with an independent of u constant $C = C(\lambda, N, p^{\pm}, \partial\Omega, T, L, M_0, M_1)$.

Corollary 4.1. *Under the conditions of Theorem 4.2*

$$\int_{\partial\Omega\times(0,T)} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2\, dSdt \leq \lambda \int_{Q_T} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |u_{xx}|^2\, dz + C\left(1 + \int_{Q_T} |\nabla u|^{p(z)}\, dz\right)$$

with an independent of u constant C.

Proof. The inequality is an immediate byproduct of Theorem 4.2 and the inequality

$$(\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2 \leq (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)}{2}} \leq C(1 + |\nabla u|^{p(z)}). \quad \square$$

5. A priori estimates

We are in position to estimate every term on the right-hand side of (3.7).

(a) By (4.5) and Lemma 4.1

$$\begin{aligned} \int_{\Omega} |\nabla u_\epsilon^{(m)}|^2 \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)\, dx &\leq C\left(1 + \int_{\Omega} \beta_\epsilon^{\frac{p(z)+r-2}{2}}(\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2\, dx\right) \\ &\leq \delta_1 \int_{\Omega} \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2\, dx + C\left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)}\, dx\right) \end{aligned} \tag{5.1}$$

with an arbitrary $\delta_1 > 0$.

(b) The term

$$I_{\partial\Omega} = -\int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_\epsilon^{(m)} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\epsilon^{(m)} \cdot \nabla(\nabla u_\epsilon^{(m)} \cdot \mathbf{n})\right)$$

is estimated with the use of Lemma 4.3 and the following known assertion.

Please cite this article in press as: R. Arora, S. Shmarev, Strong solutions of evolution equations with $p(x,t)$-Laplacian: Existence, global higher integrability of the gradients and second-order regularity, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124506

Lemma 5.1 *(Lemma A.1, [8]). If $\partial\Omega \in C^2$ and $u \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, then*

$$|I_{\partial\Omega}| \leq L \int_{\partial\Omega} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2 \, dS$$

with a constant $L = L(\partial\Omega)$. Moreover, $I_{\partial\Omega} \geq 0$ if $\partial\Omega$ is convex.

Gathering Lemmas 4.4 and 5.1 we arrive at the following estimate: for a.e. $t \in (0, T)$

$$\int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 \, dS \leq \delta_2 \int_\Omega \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}|) |u_{\epsilon xx}^{(m)}|^2 \, dx + C \left(1 + \int_\Omega |\nabla u_\epsilon^{(m)}|^{p(z)} \, dx \right) \tag{5.2}$$

with an arbitrary $\delta_2 > 0$ and a constant C independent of ϵ and m.

Lemma 5.2. *Under the conditions of Lemma 3.2*

$$\begin{aligned} \sup_{(0,T)} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 &+ \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 \, dz \\ &\leq C\mathrm{e}^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right) \end{aligned} \tag{5.3}$$

and

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} \, dz \leq C'' \quad \text{for any } 0 < r < \frac{4p^-}{p^-(N+2) + 2N} \tag{5.4}$$

with constants C, C', C'' independent of m and ϵ.

Proof. Substitution of estimates (5.1), (5.2) into (3.7) leads to the differential inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + (\min\{2, p^-\} - \delta - \delta_1 - \delta_2 - 1) \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 \\ &\quad \leq C_0 + C_1 \int_\Omega |\nabla u_\epsilon^{(m)}|^{p(z)} \, dx + C_2 \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + C_3 \|f(t)\|_{W_0^{1,2}(\Omega)}^2 \end{aligned} \tag{5.5}$$

with constants C_i, $i = 0, 1, 2, 3$, depending on the data but independent of m and ϵ, and arbitrary positive δ_1, δ_2. Choosing δ_i so small that $\min\{2, p^-\} - (1 + \delta + \delta_1 + \delta_2) = \mu > 0$, multiplying by $\mathrm{e}^{-2C_2 t}$ and dropping the second term on the left-hand side, we transform (5.5) into the differential inequality for $\|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2$:

$$\frac{d}{dt} \left(\mathrm{e}^{-2C_2 t} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 \right) \leq C\mathrm{e}^{-2C_2 t} \left(1 + \int_\Omega |\nabla u_\epsilon^{(m)}|^{p(z)} \, dx + \|f(t)\|_{W_0^{1,2}(\Omega)}^2 \right).$$

Integrating in t and using (3.4) and (3.5), we finally obtain: for every $t \in (0, T)$

$$\begin{aligned} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 &\leq C\mathrm{e}^{2C_2 T} \left(\|\nabla u_0\|_{2,\Omega}^2 + \mathrm{e}^T \left(1 + \|u_0\|_{2,\Omega}^2 + \|f\|_{2,Q_T}^2 \right) + \|\nabla f\|_{2,Q_T}^2 \right) \\ &\leq C\mathrm{e}^{C'T} \left(1 + \|u_0\|_{W_0^{1,2}(\Omega)}^2 + \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right). \end{aligned}$$

Now we substitute this estimate into (5.5) and integrate the result in t. Plugging (3.5), we arrive at the inequality

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 \, dz \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f\|^2_{L^2(0,T;W_0^{1,2}(\Omega))}\right).$$

Estimate (5.4) follows then from Theorem 4.1. It is sufficient to prove (5.4) for $r \in (r_*, r^*)$ with r_*, r^* defined in (4.2). Fix some $r \in (r_*, r^*)$, define $Q_T^+ = Q_T \cap \{p(z) + r \geq 2\}$, $Q_T^- = Q_T \cap \{p(z) + r < 2\}$ and represent

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p+r} \, dz = \int_{Q_T^+} |\nabla u_\epsilon^{(m)}|^{p+r} \, dz + \int_{Q_T^-} \ldots \equiv I_+ + I_-.$$

Then

$$I_+ \leq \int_{Q_T^+} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 \, dz \leq \int_{Q_T} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 \, dz$$

and estimate on I_+ follows. To estimate I_-, set $B_+ = Q_T^- \cap \{z : |\nabla u_\epsilon^{(m)}| \geq \epsilon\}$, $B_- = Q_T^- \cap \{z : |\nabla u_\epsilon^{(m)}| < \epsilon\}$. Then

$$\begin{aligned} I_- &= \int_{B_+ \cup B_-} |\nabla u_\epsilon^{(m)}|^{p+r} \, dz = \int_{B_+} (|\nabla u_\epsilon^{(m)}|^2)^{\frac{p+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 \, dz + \int_{B_-} \epsilon^{p+r} \, dz \\ &\leq 2^{\frac{2-r-p^-}{2}} \int_{B_+} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 \, dz + \epsilon^{p^- + r} T |\Omega| \\ &\leq C \left(1 + \int_{Q_T} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 \, dz\right). \end{aligned}$$

Gathering these estimates and applying Theorem 4.1 we obtain (5.4) with $r \in (r_*, r^*)$. The case $r \in (0, r_*]$ follows then by the Young inequality. □

Remark 5.1. Inequality (5.4) entails the inequality

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r}{2}} \, dz \leq C, \quad \epsilon \in (0,1), \quad t \in (0, r^*), \tag{5.6}$$

with an independent of ϵ constant C.

Lemma 5.3. *Let the conditions of Lemma 3.2 be fulfilled and*

$$\operatorname*{ess\,sup}_{Q_T} |p_t| \leq C^* < \infty.$$

Then the following estimate holds:

$$\|(u_\epsilon^{(m)})_t\|_{2,Q_T}^2 + \sup_{(0,T)} \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} \, dx \leq C \left(1 + \int_\Omega |\nabla u_0|^{p(x,0)} \, dx\right) + \|f\|_{2,Q_T}^2 \tag{5.7}$$

with an independent of m and ϵ constant C.

Proof. Multiplying (3.3) with $(u_j^{(m)})_t$ and summing over $j = 1, 2, \ldots, m$ we obtain the equality

$$\int_\Omega (u_\epsilon^{(m)})_t^2\,dx + \int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(x,t)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_t\,dx = \int_\Omega f (u_\epsilon^{(m)})_t\,dx. \tag{5.8}$$

It is straightforward to check that

$$\begin{aligned}(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_t &= \frac{d}{dt}\left(\frac{(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p(z)}\right) \\ &\quad + \frac{p_t(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2(z)}\left(1 - \frac{p(z)}{2}\ln((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2))\right).\end{aligned}$$

With the use of this identity we rewrite (5.8) in the form

$$\begin{aligned}&\int_\Omega (u_\epsilon^{(m)})_t^2\,dx + \frac{d}{dt}\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}\,dx \\ &\quad = -\int_\Omega \frac{p_t(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2(z)}\left(1 - \frac{p(z)}{2}\ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)\right) + \int_\Omega f (u_\epsilon^{(m)})_t\,dx.\end{aligned} \tag{5.9}$$

The terms on the right-hand side of (5.9) are estimated separately. For the first term, we use (3.5) and (5.1):

$$\begin{aligned}&\left|\int_\Omega \frac{p_t(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2}\left(1 - \frac{p}{2}\ln((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2))\right)\right| \le C_1\left(1 + \int_\Omega |\nabla u_\epsilon^{(m)}|^{p(z)}\,dx\right) \\ &\quad + C_2\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}\ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)\,dx.\end{aligned} \tag{5.10}$$

The second term is estimated by the Cauchy inequality:

$$\left|\int_\Omega f (u_\epsilon^{(m)})_t\,dx\right| \le \frac{1}{2}\|(u_\epsilon^{(m)})_t\|_{2,\Omega}^2 + \frac{1}{2}\|f\|_{2,\Omega}^2. \tag{5.11}$$

Estimate (5.7) follows after substitution of (5.10), (5.11) into (5.9) and integration of the resulting inequality in t: for every $t \in (0, T)$

$$\|(u_\epsilon^{(m)})_t\|_{2,Q_t}^2 + 2\int_\Omega (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}\,dx \le C\left(1 + \int_\Omega (\epsilon^2 + |\nabla u_0^{(m)}(x)|^2)^{\frac{p(x,0)}{2}}\,dx\right) + \|f\|_{2,Q_t}^2. \qquad \square$$

6. Strong solution of the regularized problem

In this section, we prove that the regularized problem (3.1) has a unique strong solution. We show first the existence of a weak solution with $u_{\epsilon t} \in L^2(Q_T)$ and then prove that this solution possesses extra regularity properties and, thus, is the strong solution.

Please cite this article in press as: R. Arora, S. Shmarev, Strong solutions of evolution equations with $p(x,t)$-Laplacian: Existence, global higher integrability of the gradients and second-order regularity, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124506

6.1. Existence and uniqueness of weak solution

Theorem 6.1. *Let u_0, f, p and $\partial\Omega$ satisfy the conditions of Theorem 2.1. Then for every $\epsilon \in (0,1)$ problem (3.1) has a unique solution u_ϵ which satisfies the estimates*

$$\begin{aligned}&\|u_\epsilon\|_{W(Q_T)} \leq C_0,\\ &\operatorname*{ess\,sup}_{(0,T)} \|u_\epsilon(t)\|^2_{2,\Omega} + \|u_{\epsilon t}\|^2_{2,Q_T} + \operatorname*{ess\,sup}_{(0,T)} \int_\Omega |\nabla u_\epsilon|^{q(z)}\,dx \leq C_0, \qquad q(z) = \max\{2, p(z)\},\end{aligned} \tag{6.1}$$

with a constant C_0 depending on the data but not on ϵ. Moreover, u_ϵ possesses the property of global higher integrability of the gradient: for every

$$\delta \in (0, r^*), \qquad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

there exists a constant $C = C\left(\partial\Omega, N, p^\pm, \delta, \|u_0\|_{W_0^{1,q(\cdot,0)}(\Omega)}, \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}\right)$ such that

$$\int_{Q_T} |\nabla u_\epsilon|^{p(z)+\delta}\,dz \leq C. \tag{6.2}$$

Remark 6.1. Due to the fact that estimate (6.2) is global in time and space, it is new even in the case of constant p. We refer to [17] for a detailed insight into this issue, in particular, to [17, Lemma 5.4].

Let $\epsilon > 0$ be a fixed parameter, Ω be a bounded domain with the boundary $\partial\Omega \in C^2$ boundary, and let $u_\epsilon^{(m)}$ be the sequence of Galerkin approximations defined in (3.2). Under the assumptions

$$u_0 \in W_0^{1,2}(\Omega), \quad f \in L^2((0,T); W_0^{1,2}(\Omega)), \quad \|\nabla p\|_{\infty,Q_T} \leq C_*, \quad \|p_t\|_{\infty,Q_T} \leq C^*$$

the functions $u_\epsilon^{(m)}$ exist and satisfy estimates (3.4), (3.5), (5.3), (5.4) and (5.7). These uniforms in m and ϵ estimates allow one to choose a subsequence $u_\epsilon^{(m)}$ (for which we keep the same notation), and functions u_ϵ, η_ϵ such that

$$\begin{aligned}&u_\epsilon^{(m)} \to u_\epsilon \quad \star\text{-weakly in } L^\infty(0,T; L^2(\Omega)),\\ &u_{\epsilon t}^{(m)} \rightharpoonup u_{\epsilon t} \text{ in } L^2(Q_T),\\ &\nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon \text{ in } (L^{p(\cdot)}(Q_T))^N,\\ &(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \rightharpoonup \eta_\epsilon \text{ in } (L^{p'(\cdot)}(Q_T))^N\end{aligned} \tag{6.3}$$

The assumption $p^- > \frac{2N}{N+2}$ yields the inclusions

$$W_0^{1,p(\cdot,t)}(\Omega) \subset W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega).$$

Since $u_\epsilon^{(m)}$ and $(u_\epsilon^{(m)})_t$ are uniformly bounded in $L^\infty(0,T; W_0^{1,p^-}(\Omega))$ and $L^\infty(0,T; L^2(\Omega))$, it follows from the compactness lemma [23, Sec.8, Corollary 4] that the sequence $\{u_\epsilon^{(m)}\}$ is relatively compact in $C([0,T]; L^2(\Omega))$, *i.e.*, there exists a subsequence $\{u_\epsilon^{(m_k)}\}$, which we assume coinciding with $\{u_\epsilon^{(m)}\}$, such that $u_\epsilon^{(m)} \to u_\epsilon$ in $C([0,T]; L^2(\Omega))$ and a.e. in Q_T.

Let us define

$$\mathcal{P}_m = \left\{ \psi : \psi = \sum_{i=1}^{m} \psi_i(t)\phi_i(x),\ \psi_i \in C^1[0,T] \right\}.$$

Fix some $m \in \mathbb{N}$. By the method of construction $u_\epsilon^{(m)} \in \mathcal{P}_m$. Since $\mathcal{P}_k \subset \mathcal{P}_m$ for $k < m$, then for every $\xi_k \in \mathcal{P}_k$ with $k \le m$

$$\int_{Q_T} u_{\epsilon t}^{(m)} \xi_k \, dz + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \xi_k \, dz = \int_{Q_T} f \xi_k \, dz. \tag{6.4}$$

Let $\xi \in W(Q_T)$. Take a sequence $\{\xi_k\}$ such that $\xi_k \in \mathcal{P}_k$ and $\xi_k \to \xi \in W(Q_T)$. Passing to the limit as $m \to \infty$ with a fixed k, and then letting $k \to \infty$, from the above equality we infer that

$$\int_{Q_T} u_{\epsilon t} \xi \, dz + \int_{Q_T} \eta_\epsilon \cdot \nabla \xi \, dz = \int_{Q_T} f \xi \, dz \tag{6.5}$$

for all $\xi \in W(Q_T)$. To identify the limit vector η_ϵ we use the classical argument based on monotonicity of the function $\gamma_\epsilon(z, \mathbf{s})\mathbf{s} \equiv (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}} \mathbf{s} : \mathbb{R}^N \mapsto \mathbb{R}^N$.

Lemma 6.1. *For all $z \in Q_T$, $\xi, \zeta \in \mathbb{R}^N$, $(\zeta \neq \xi)$ and $\epsilon > 0$*

$$(\gamma_\epsilon(z,\zeta)\zeta - \gamma_\epsilon(z,\xi)\xi) \cdot (\zeta - \xi) \ge 0. \tag{6.6}$$

Proof. Let $\zeta \neq \xi$. The straightforward computation shows that

$$\begin{aligned}
(\gamma_\epsilon(z,\xi)\xi - \gamma_\epsilon(z,\zeta)\zeta) \cdot (\xi - \zeta) &= \int_0^1 \frac{d}{d\theta} (\epsilon^2 + |\theta\xi + (1-\theta)\zeta|^2)^{\frac{p(z)-2}{2}} (\theta\xi + (1-\theta)\zeta) \, d\theta \cdot (\xi - \zeta) \\
&= \int_0^1 (\epsilon^2 + |\theta\xi + (1-\theta)\zeta|^2)^{\frac{p(z)-2}{2}} \left[(p(z)-2)\cos^2(\widehat{\mu,\nu}) + 1 \right] d\theta |\xi - \zeta|^2 \\
&\ge |\xi - \zeta|^2 \begin{cases} \epsilon^{p(z)-2} & \text{if } p(z) \ge 2, \\ (p(z)-1) \displaystyle\int_0^1 (\epsilon^2 + |\theta\xi + (1-\theta)\zeta|^2)^{\frac{p(z)-2}{2}} \, d\theta & \text{if } p(z) \in (1,2), \end{cases}
\end{aligned}$$

where μ, ν are the unit vectors $\mu = \dfrac{\xi - \zeta}{|\xi - \zeta|}$, $\nu = \dfrac{\zeta + \theta(\xi - \zeta)}{|\zeta + \theta(\xi - \zeta)|}$. □

Equality (6.4) is true for $\xi_k = u_\epsilon^{(m)}$. By virtue of (6.6), for every $\psi \in \mathcal{P}_k$ with $k \le m$

$$\begin{aligned}
0 &= \int_{Q_T} (u_\epsilon^{(m)})_t u_\epsilon^{(m)} \, dz + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 \, dz - \int_{Q_T} f u_\epsilon^{(m)} \, dz \\
&\ge \int_{Q_T} (u_\epsilon^{(m)})_t u_\epsilon^{(m)} \, dz + \int_{Q_T} (\epsilon^2 + |\nabla \psi|^2)^{\frac{p(z)-2}{2}} \nabla \psi \cdot \nabla (u_\epsilon^{(m)} - \psi) \, dz \\
&\quad + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \psi \, dz - \int_{Q_T} f u_\epsilon^{(m)} \, dz.
\end{aligned}$$

Let us pass to the limit as $m \to \infty$. Using the limit relations (6.3), the fact that $u_\epsilon^{(m)}(u_\epsilon^{(m)})_t \rightharpoonup u_\epsilon u_{\epsilon t}$ as the product of weakly and strongly convergent sequences, and substituting (6.5) into the resulting inequality, we find that for every $\psi \in \mathcal{P}_k$

$$\begin{aligned} 0 &\geq \int\limits_{Q_T} u_\epsilon u_{\epsilon t}\,dz + \int\limits_{Q_T} (\epsilon^2 + |\nabla\psi|^2)^{\frac{p(z)-2}{2}} \nabla\psi \cdot \nabla(u_\epsilon - \psi)\,dz + \int\limits_{Q_T} \eta_\epsilon \cdot \nabla\psi\,dz - \int\limits_{Q_T} f u_\epsilon\,dz \\ &= \int\limits_{Q_T} \left((\epsilon^2 + |\nabla\psi|^2)^{\frac{p(z)-2}{2}} \nabla\psi - \eta_\epsilon\right) \cdot \nabla(u_\epsilon - \psi)\,dz. \end{aligned}$$

Since $\bigcup_{k=1}^\infty \mathcal{P}_k$ is dense in $W(Q_T)$, the last inequality also holds for every $\psi \in W(Q_T)$. Let us take $\psi = u_\epsilon + \lambda\xi$ with $\lambda > 0$ and an arbitrary $\xi \in W(Q_T)$. Then

$$\lambda \int\limits_{Q_T}\int\limits_{Q_T} \left((\epsilon^2 + |\nabla(u_\epsilon + \lambda\xi)|^2)^{\frac{p(z)-2}{2}} \nabla(u_\epsilon + \lambda\xi) - \eta_\epsilon\right) \cdot \nabla\xi\,dz \leq 0.$$

Simplifying and letting $\lambda \to 0$ we find that

$$\int\limits_{Q_T} \left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon - \eta_\epsilon\right) \cdot \nabla\xi\,dz \leq 0 \quad \forall \xi \in W(Q_T),$$

which is possible only if

$$\int\limits_{Q_T} \left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon - \eta_\epsilon\right) \cdot \nabla\xi\,dz = 0 \quad \forall \xi \in W(Q_T).$$

Thus, the limit function u_ϵ satisfies identity (2.3) with the regularized flux $(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon$. The initial condition for u_ϵ is fulfilled by continuity because $u_\epsilon \in C^0([0,T]; L^2(\Omega))$.

Uniqueness of the weak solution is an immediate byproduct of monotonicity of the function $\gamma_\epsilon(z, \mathbf{s})\mathbf{s}$. Let u_1, u_2 be two different strong solutions of problem (3.1). Combining equalities (2.3) for u_i with the test-function $u_1 - u_2$, using (6.6) and the formula of integration by parts (2.2) we find that

$$\|u_1 - u_2\|_{2,\Omega}^2(t) \leq 0 \quad \text{for a.e. } t \in (0,T),$$

whence $u_1 = u_2$ a.e. in Q_T.

Let us prove estimates (6.1), (6.2). The uniform with respect to ϵ estimates (5.3) and (5.7) allow us to choose a subsequence of $\{u_\epsilon^{(m)}\}$ which satisfies (6.3) and also $|\nabla u_\epsilon^{(m)}|^{q(x,t)} \to |\nabla u_\epsilon|^{q(x,t)}$ $\star$-weakly in $L^\infty(0,T; L^1(\Omega))$, $q(x,t) = \max\{p(x,t), 2\}$. Estimate (6.1) follows now from the lower semicontinuity of the norm and the modular $\rho_{r(\cdot)}(s) = \int\limits_\Omega |s|^{r(x)}\,dx$ with $r(x) \in C^0(\overline{\Omega})$, $r(x) \in [1, r^+]$, $r^+ < \infty$ (see [15, Th. 3.2.9]). For every $\delta \in (0, r^*)$, inequality (6.2) follows in the same way from the uniform estimate (5.4).

6.2. Second-order regularity

Lemma 6.2. *If* $p^- \geq \max\left\{\dfrac{2N}{N+2}, \dfrac{6}{5}\right\}$, *the function* $h(\mathbf{s}) = \gamma_\epsilon(z, \mathbf{s})|\mathbf{s}|^2$ *is strictly convex with respect to* $\mathbf{s}$.

Proof. Fix two points $\xi, \zeta \in \mathbb{R}^N$, $\xi \neq \zeta$, and consider the function

$$F(\tau) = \gamma_\epsilon(z, \tau\xi + (1-\tau)\zeta)|\tau\xi + (1-\tau)\zeta|^2, \quad \tau \in [0,1].$$

Let us accept the notation $\sigma = |\tau\xi + (1-\tau)\zeta|^2$ and $\eta = \dfrac{\xi-\zeta}{|\xi-\zeta|}$. The straightforward computation gives

$$F''(\tau) = |\xi-\zeta|^2(\epsilon^2+\sigma)^{\frac{p-2}{2}-2}\left[(p\sigma+2\epsilon^2)(\sigma+\epsilon^2)+(p-2)(p\sigma+4\epsilon^2)(\tau\xi+(1-\tau)\zeta,\eta)^2\right].$$

Obviously, $F''(\tau) > 0$ if $p(z) \geq 2$. Let $1 < p(z) < 2$. Since $(\tau\xi+(1-\tau)\zeta,\eta)^2 \leq \sigma$, we obtain:

$$\begin{aligned} F''(\tau) &\geq |\xi-\zeta|^2(\epsilon^2+\sigma)^{\frac{p-2}{2}-2}\left[(p\sigma+2\epsilon^2)(\sigma+\epsilon^2)+(p-2)(p\sigma+4\epsilon^2)\sigma\right] \\ &= |\xi-\zeta|^2(\epsilon^2+\sigma)^{\frac{p-2}{2}-2}\left[p(p-1)\sigma^2+(5p-6)\sigma\epsilon^2+2\epsilon^4\right], \end{aligned}$$

whence $F'' > 0$ for all $\xi \neq \zeta$ and $\epsilon \geq 0$, provided that $p^- \geq \dfrac{6}{5}$. □

The proof of stronger convergence properties of the sequence $\nabla u_\epsilon^{(m)}$ stems from the following general result on the convergence of sequences of functionals. For convenience, we formulate it in the form already adapted to our problem.

Proposition 6.1 *(Theorem 2.1, Corollary 2.1, [20]). Let $\mathcal{F}_m(z,\mathbf{s}) : Q_T \times \mathbb{R}^N \to \mathbb{R}$ be a sequence of nonnegative functions, convex with respect to $\mathbf{s}$ for every $z \in Q_T$ and locally uniformly convergent to a function $\mathcal{F}_0(z,\mathbf{s})$ as $m \to \infty$, which is essentially convex with respect to $\mathbf{s}$ for every $z \in Q_T$. Assume that $\mathcal{F}_m(z,\mathbf{s}) \geq a(|\mathbf{s}|^\alpha + 1)$ with some constants $a > 0$, $\alpha > 1$. If $v_m \in (L^s(Q_T))^N$, $v_m \rightharpoonup v_0$ in $(L^s(Q_T))^N$, $s > 1$, and*

$$\int_{Q_T} \mathcal{F}_m(z, v_m)\,dz \to \int_{Q_T} \mathcal{F}_0(z, v_0)\,dz < \infty,$$

then

$$\int_{Q_T} |v_m - v_0|^\alpha\,dz \to 0 \quad \text{as } m \to \infty.$$

Theorem 6.2. *Let the conditions of Theorem 6.1 be fulfilled.*

(i) *If $N \geq 3$ or $N = 2$ and $p^- > \frac{6}{5}$, then*

$$\nabla u_\epsilon^{(m)} \to \nabla u_\epsilon \text{ a.e. in } Q_T.$$

(ii) *Under the conditions of item (i) $\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon)D_i u_\epsilon \in L^2(0,T;W^{1,2}(\Omega))$, $i = 1,2,\dots,N$, and*

$$\|\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon)D_i u_\epsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leq M, \quad i = 1,2,\dots,N,$$

with an independent of ϵ constant M.

(iii) *If $N \geq 2$ and $p^- > \dfrac{2N}{N+2}$, then $D_{ij}^2 u_\epsilon \in L_{loc}^{p(\cdot)}(Q_T \cap \{z : p(z) < 2\})$, $i,j = 1,2,\dots,N$, and*

$$\sum_{i,j=1}^N \|D_{ij}^2 u_\epsilon\|_{p(\cdot),Q_T\cap\{z:\,p(z)<2\}} \leq M'$$

with an independent of ϵ constant M'.

Proof. (i) It is already shown that $\nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon$ in $L^{p(\cdot)}(Q_T)$ as $m \to \infty$. By Lemma 6.2 the function $\gamma_\epsilon(z,\mathbf{s})|\mathbf{s}|^2$ is strictly convex with respect to $\mathbf{s}$. According to (3.6)

$$\left(1+2^{\frac{p^+}{2}}\right)+\gamma_\epsilon(z,\mathbf{s})|\mathbf{s}|^2 \geq 1+|\mathbf{s}|^{p(z)} \geq |\mathbf{s}|^{p^-}. \tag{6.7}$$

By virtue of the energy equalities (6.4), (6.5) and the limit relations (6.3)

$$\begin{aligned}\int\limits_{Q_T}\gamma_\epsilon(z,\nabla u_\epsilon^{(m)})|\nabla u_\epsilon^{(m)}|^2\,dz &= -\int\limits_{Q_T} u_{\epsilon t}^{(m)}u_\epsilon^{(m)}\,dz+\int\limits_{Q_T} f u_\epsilon^{(m)}\,dz\\ &\to -\int\limits_{Q_T} u_{\epsilon t}u_\epsilon\,dz+\int\limits_{Q_T} f u_\epsilon\,dz=\int\limits_{Q_T}\gamma_\epsilon(z,\nabla u_\epsilon)|\nabla u_\epsilon|^2\,dz \qquad \text{as } m\to\infty.\end{aligned}$$

Now we apply Proposition 6.1 with $\mathcal{F}_m(z,\mathbf{s})=\gamma_\epsilon(z,\mathbf{s})|\mathbf{s}|^2+M$ and a sufficiently large positive constant M. It follows that $\nabla u_\epsilon^{(m)} \to \nabla u_\epsilon$ a.e. in Q_T, whence

$$\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon^{(m)})\nabla u_\epsilon^{(m)} \to \gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon)\nabla u_\epsilon \quad \text{a.e. in } Q_T. \tag{6.8}$$

(ii) According to (5.3) and (5.6), for every $i,j=1,2,\ldots,N$

$$\begin{aligned}&\left\|D_i\left(\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon^{(m)})D_j u_\epsilon^{(m)}\right)\right\|_{2,Q_T}^2\\ &\leq C\left(\int\limits_{Q_T}\gamma_\epsilon(z,\nabla u_\epsilon^{(m)})|u_{\epsilon xx}^{(m)}|^2\,dz+\int\limits_{Q_T}(\epsilon^2+|\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}|\ln(\epsilon^2+|\nabla u_\epsilon^{(m)}|^2)|\,dz\right)\\ &\leq C'\left(1+\int\limits_{Q_T}(\epsilon^2+|\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}|\ln(\epsilon^2+|\nabla u_\epsilon^{(m)}|^2)|\,dz\right)\\ &\leq C\int\limits_{Q_T}\gamma_\epsilon(z,\nabla u_\epsilon^{(m)})|u_{\epsilon xx}^{(m)}|^2\,dz+C''\left(1+\int\limits_{Q_T}|\nabla u_\epsilon^{(m)}|^{p(z)+\mu}\,dz\right)\\ &\leq M, \qquad M=M(\|u_0\|_{W_0^{1,2}(\Omega)},\|f\|_{L^2(0,T;W_0^{1,2}(\Omega))},N,p^\pm,\omega,\partial\Omega),\end{aligned}$$

whence the existence of a subsequence $\{u_\epsilon^{(m_k)}\}$ (we may assume that it coincides with the whole sequence) such that

$$D_i\left(\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon^{(m)})D_j u_\epsilon^{(m)}\right) \rightharpoonup \eta_{ij}\in L^2(Q_T) \quad \text{as } m\to\infty.$$

By (5.4) there exists $\delta>0$ such that

$$\|\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon^{(m)})D_j u_\epsilon^{(m)}\|_{2+\delta,Q_T} \leq \|(\epsilon^2+|\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{4}}\|_{2+\delta,Q_T}\leq C$$

with a constant C independent of m and ϵ. Since $\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon^{(m)})D_j u_\epsilon^{(m)}$ are uniformly bounded in $L^{2+\delta}(Q_T)$ and converge pointwise due to (6.8), it follows from the Vitali convergence theorem that

$$\gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon^{(m)})D_j u_\epsilon^{(m)} \to \gamma_\epsilon^{\frac{1}{2}}(z,\nabla u_\epsilon)D_j u_\epsilon \text{ in } L^2(Q_T).$$

For every $\phi \in C^\infty(Q_T)$ with $\operatorname{supp}\phi \Subset Q_T$ and $i, j = 1, \dots, N$

$$\left(D_i\left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)}\right), \phi\right)_{2,Q_T} = -\left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)}, D_i\phi\right)_{2,Q_T}$$
$$\to -\left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon) D_j u_\epsilon, D_i\phi\right)_{2,Q_T} \quad \text{as } m \to \infty.$$

Thus, it is necessary that

$$\eta_{ij} = D_i\left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{4}} D_j u_\epsilon\right) \in L^2(Q_T) \quad \text{and} \quad \|\eta_{ij}\|_{2,Q_T}^2 \le M.$$

(iii) Let us denote $Q_T^- = Q_T \cap \{z : p(z) < 2\}$. By Young's inequality, (3.5) and (5.3), for every $D \Subset Q_T^-$

$$\int_D |D_{ij}^2 u_\epsilon^{(m)}|^{p(z)}\, dz = \int_D \left(\gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |D_{ij}^2 u_\epsilon^{(m)}|^2\right)^{\frac{p(z)}{2}} \gamma_\epsilon^{-\frac{p(z)}{2}}(z, \nabla u_\epsilon^{(m)})\, dz$$
$$\le \int_D \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |D_{ij}^2 u_\epsilon^{(m)}|^2\, dz + \int_D (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}\, dz \le C$$

with a constant C independent of ϵ, m and D. It follows that there exists $\chi \in L^{p(\cdot)}(Q_T^-)$ such that $D_{ij}^2 u_\epsilon^{(m)} \rightharpoonup \chi$ in $L^{p(\cdot)}(D)$ (up to a subsequence). Since $\nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon$ in $L^{p(\cdot)}(Q_T)$, for every $\phi \in C^\infty(Q_T^-)$ with $\operatorname{supp}\phi \Subset Q_T^-$

$$(\chi, \phi)_{2,Q_T} = \lim_{m\to\infty} (D_{ij}^2 u_\epsilon^{(m)}, \phi)_{2,Q_T} = -\lim_{m\to\infty} (D_i u_\epsilon^{(m)}, D_j\phi)_{2,Q_T} = (D_i u_\epsilon, D_j\phi)_{2,Q_T}.$$

It is necessary that $\chi = D_{ij}^2 u_\epsilon$, and $\|D_{ij}^2 u_\epsilon\|_{p(\cdot),D} \le C$ by the lower semicontinuity of the modular. □

Remark 6.2. Let u_ϵ be a solution of problem (3.1). The regularity of the regularized flux

$$D_{x_j}\left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{4}} D_{x_i} u_\epsilon\right) \in L^2(Q_T)$$

leads to the local fractional differentiability of ∇u, see [17, Ch.6] for the case of constant p.

7. Strong solution of the degenerate problem. Proof of Theorem 2.1

7.1. Existence and uniqueness of strong solutions

Let $\{u_\epsilon\}$ be the family of strong solutions of the regularized problems (3.1). The uniform in ϵ estimates (6.1) allow us to choose a sequence $\{u_{\epsilon_k}\}$ and functions $u \in W(Q_T)$, $u_t \in L^2(Q_T)$, $\eta \in (L^{p'(\cdot)}(Q_T))^N$ with the following properties:

$$\begin{aligned}
& u_{\epsilon_k} \to u \quad \star\text{-weakly in } L^\infty(0, T; L^2(\Omega)), \\
& u_{\epsilon_k t} \rightharpoonup u_t \text{ in } L^2(Q_T), \\
& \nabla u_{\epsilon_k} \rightharpoonup \nabla u \text{ in } (L^{p(\cdot)}(Q_T))^N, \\
& \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \rightharpoonup \eta \text{ in } (L^{p'(\cdot)}(Q_T))^N.
\end{aligned}$$

Moreover, $u \in C([0, T]; L^2(\Omega))$. Each of u_{ϵ_k} satisfies the identity

$$\int_{Q_T} u_{\epsilon_k t}\xi\,dz + \int_{Q_T} \gamma_{\epsilon_k}(z,\nabla u_{\epsilon_k})\nabla u_{\epsilon_k}\cdot\nabla\xi\,dz = \int_{Q_T} f\xi\,dz \qquad \forall \xi\in W(Q_T), \tag{7.1}$$

which yields

$$\int_{Q_T} u_t\xi\,dz + \int_{Q_T} \eta\cdot\nabla\xi\,dz = \int_{Q_T} f\xi\,dz \qquad \forall \xi\in W(Q_T). \tag{7.2}$$

To identify η, we use the monotonicity argument. Take $\xi = u_{\epsilon_k}$ in (7.1):

$$\int_{Q_T} u_{\epsilon_k t}u_{\epsilon_k}\,dz + \int_{Q_T} \gamma_{\epsilon_k}(z,\nabla u_{\epsilon_k})\nabla u_{\epsilon_k}\cdot\nabla u_{\epsilon_k}\,dz = \int_{Q_T} f u_{\epsilon_k}\,dz. \tag{7.3}$$

By virtue of monotonicity, for every $\phi\in W(Q_T)$

$$\begin{aligned}
\int_{Q_T} \gamma_{\epsilon_k}(z,\nabla u_{\epsilon_k})\nabla u_{\epsilon_k}\cdot\nabla u_{\epsilon_k}\,dz &\geq \int_{Q_T} \gamma_{\epsilon_k}(z,\nabla\phi)\nabla\phi\cdot\nabla(u_{\epsilon_k}-\phi)\,dz + \int_{Q_T} \gamma_{\epsilon_k}(z,\nabla u_{\epsilon_k})\nabla u_{\epsilon_k}\cdot\nabla\phi\,dz \\
&= \int_{Q_T} (\gamma_{\epsilon_k}(z,\nabla\phi)-|\nabla\phi|^{p-2})\nabla\phi\cdot\nabla(u_{\epsilon_k}-\phi)\,dz + \int_{Q_T} \gamma_{\epsilon_k}(z,\nabla u_{\epsilon_k})\nabla u_{\epsilon_k}\cdot\nabla\phi\,dz \\
&+ \int_{Q_T} |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(u_{\epsilon_k}-\phi)\,dz \equiv J_1+J_2+J_3,
\end{aligned}$$

where

$$J_2 \to \int_{Q_T} \eta\cdot\nabla\phi\,dz, \qquad J_3 \to \int_{Q_T} |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(u-\phi)\,dz \quad \text{as } k\to\infty.$$

Since $\left|(\gamma_{\epsilon_k}(z,\nabla\phi)-|\nabla\phi|^{p-2})\nabla\phi\right| \to 0$ a.e. in Q_T as $k\to\infty$, and the integrand of J_1 has the majorant

$$\left|(\gamma_{\epsilon_k}(z,\nabla\phi)-|\nabla\phi|^{p-2})\nabla\phi\right| \leq 2(1+|\nabla\phi|^2)^{\frac{p(z)}{2}} \leq C\left(1+|\nabla\phi|^{p(z)}\right),$$

$J_1\to 0$ by the dominated convergence theorem. Combining (7.2) with (7.3) and letting $k\to\infty$ we find that for every $\phi\in W(Q_T)$

$$\int_{Q_T} \left(|\nabla\phi|^{p(z)-2}\nabla\phi-\eta\right)\cdot\nabla(u-\phi)\,dz \geq 0.$$

Choosing $\phi = u+\lambda\zeta$ with $\lambda>0$ and $\zeta\in W(Q_T)$, simplifying and letting $\lambda\to 0^+$, we obtain the inequality

$$\int_{Q_T} \left(|\nabla u|^{p(z)-2}\nabla u-\eta\right)\cdot\nabla\zeta\,dz \geq 0 \quad \forall\zeta\in W(Q_T),$$

which means that in (7.2) η coincides with $|\nabla u|^{p(z)-2}\nabla u$. Since $u\in C([0,T];L^2(\Omega))$, the initial condition is fulfilled by continuity.

By virtue of (6.1), (6.2), the subsequence convergent to the solution may be chosen so that $|\nabla u_{\epsilon_k}|^{q(x,t)} \to |\nabla u|^{q(x,t)}$ $\star$-weakly in $L^\infty(0,T;L^1(\Omega))$. Estimate (2.6) follows then from the lower semicontinuity of the modular exactly as in the proof of (6.1).

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Uniqueness of the constructed strong solution of problem (1.1) stems from the monotonicity of the mapping $\gamma_0(z, \mathbf{s})\mathbf{s}$ and the formula of integration by parts.

7.2. Higher integrability of the gradient

Let us fix $\delta \in (0, r^*)$. According to (6.2) $\|\nabla u_{\epsilon_k}\|_{p(\cdot)+\delta, Q_T} \leq C_\delta$ with an independent of ϵ_k constant C, which allows one to choose a subsequence (for which we use the same notation), such that

$$\nabla u_{\epsilon_k} \rightharpoonup \nabla u \text{ in } L^{p(\cdot)+\delta}(Q_T).$$

By the property of lower semicontinuity of the modular

$$\int_{Q_T} |\nabla u|^{p(z)+\delta}\, dz \leq \liminf_{k\to\infty} \int_{Q_T} |\nabla u_{\epsilon_k}|^{p(z)+\delta}\, dz \leq C_\delta$$

with the constant C_δ from (6.2).

7.3. Second-order regularity of strong solutions

Let us assume that $p^- > \max\left\{\frac{2N}{N+2}, \frac{6}{5}\right\}$ and show that $\nabla u_\epsilon \to \nabla u$ a.e. in Q_T. Consider the sequence of nonnegative functions

$$F_{\epsilon_k}(z, \mathbf{s}) = \gamma_{\epsilon_k}(z, \mathbf{s})|\mathbf{s}|^2.$$

$F_\epsilon(x, \mathbf{s})$ are strictly convex with respect to $\mathbf{s}$ (by Lemma 6.2) and satisfy inequality (6.7). It is already shown that $\nabla u_{\epsilon_k} \rightharpoonup \nabla u$ in $L^{p(\cdot)}(Q_T)$. According to (7.1), (7.2)

$$\int_{Q_T} F_{\epsilon_k}(z, \nabla u_{\epsilon_k})\, dz \to \int_{Q_T} F_0(z, \nabla u)\, dz \text{ as } k \to \infty$$

and $F_\epsilon(z, \mathbf{s}) \to F_0(z, \mathbf{s}) = |\mathbf{s}|^p$ as $\epsilon \to 0$ locally uniformly with respect to $(z, \mathbf{s}) \in Q_T \times \mathbb{R}^N$. Indeed:

$$\begin{aligned}
\left|(\epsilon^2 + |\mathbf{s}|^2)^{\frac{p-2}{2}}|\mathbf{s}|^2 - |\mathbf{s}|^p\right| &= |\mathbf{s}|^2 \left|\int_0^1 \frac{d}{d\theta}(\theta\epsilon^2 + |\mathbf{s}|^2)^{\frac{p-2}{2}}\, d\theta\right| = |\mathbf{s}|^2\epsilon^2\frac{|p-2|}{2}\int_0^1 (\theta\epsilon^2 + |\mathbf{s}|^2)^{\frac{p-4}{2}}\, d\theta \\
&\leq \frac{|p-2|}{2}\begin{cases} \epsilon^2(1+|\mathbf{s}|^2)^{\frac{p^+-2}{2}} & \text{if } p \geq 2, \\ \dfrac{2\epsilon^{p^-}}{p^-} & \text{if } 1 < p < 2. \end{cases}
\end{aligned}$$

By Proposition 6.1, $\nabla u_{\epsilon_k} \to \nabla u$ a.e. in Q_T.

Let us fix $i, j \in \{1, 2, \ldots, N\}$. By Theorem 6.2

$$\|D_j\left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k})D_i u_{\epsilon_k}\right)\|_{2,Q_T} \leq C$$

uniformly in ϵ_k, therefore there exists $\eta_{ij} \in L^2(Q_T)$ such that $D_j\left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k})D_i u_{\epsilon_k}\right) \rightharpoonup \eta_{ij}$ in $L^2(Q_T)$. The pointwise convergence $\nabla u_{\epsilon_k} \to \nabla u$ yields the pointwise convergence $\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k})\nabla u_{\epsilon_k} \to |\nabla u|^{\frac{p(z)-2}{2}}\nabla u$, by virtue of (6.2) $\|\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k})\nabla u_{\epsilon_k}\|_{2+\delta,Q_T}$ is uniformly bounded for some $\delta > 0$. It follows from the

Please cite this article in press as: R. Arora, S. Shmarev, Strong solutions of evolution equations with $p(x, t)$-Laplacian: Existence, global higher integrability of the gradients and second-order regularity, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124506

Vitali convergence theorem that $\gamma_{\epsilon_k}^{\frac{1}{2}}(z,\nabla u_{\epsilon_k})\nabla u_{\epsilon_k} \to |\nabla u|^{\frac{p(z)-2}{2}}\nabla u$ in $L^2(Q_T)$. It follows that $\eta_{ij} = D_j\left(|\nabla u|^{\frac{p(z)-2}{2}} D_i u\right)$: for every $\phi \in C^\infty(\overline{Q}_T)$, $\operatorname{supp}\phi \Subset Q_T$,

$$-(\eta_{ij},\phi)_{2,Q_T} = -\lim_{k\to\infty}\left(D_j\left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z,\nabla u_{\epsilon_k})D_i u_{\epsilon_k}\right),\phi\right)_{2,Q_T}$$
$$= \lim_{k\to\infty}\left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z,\nabla u_{\epsilon_k})D_i u_{\epsilon_k}, D_j\phi\right)_{2,Q_T} = \left(|\nabla u|^{\frac{p(z)-2}{2}}D_i u, D_j\phi\right)_{2,Q_T}.$$

Let $N \geq 2$ and $p^- > \frac{2N}{N+2}$. Assume that $p^- < 2$ and, thus, $Q_T^- = Q_T \cap \{z : p(z) < 2\} \neq \emptyset$. Arguing as in the proof of Theorem 6.2 we find that for every $D \Subset Q_T^-$

$$\int_D |D_{ij}^2 u_\epsilon|^{p(z)}\,dz = \int_D \left(\gamma_\epsilon(z,\nabla u_\epsilon)|D_{ij}^2 u_\epsilon|^2\right)^{\frac{p(z)}{2}} \gamma_\epsilon^{-\frac{p(z)}{2}}(z,\nabla u_\epsilon^{(m)})\,dz$$
$$\leq \int_D \gamma_\epsilon(z,\nabla u_\epsilon)|D_{ij}^2 u_\epsilon|^2\,dz + \int_D (\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)}{2}}\,dz \leq C$$

with a constant C independent of ϵ and D. It follows that $D_{ij}^2 u_{\epsilon_k} \rightharpoonup \zeta \in L^{p(\cdot)}(D)$ (up to a subsequence). Because of the weak convergence $\nabla u_{\epsilon_k} \rightharpoonup \nabla u$ in $L^{p(\cdot)}(Q_T)$, it is necessary that $\zeta = D_{ij}^2 u$. The estimate $\|D_{ij}^2 u\|_{p(\cdot),D} \leq C$ follows from the uniform estimate on $D_{ij}^2 u_\epsilon$.

Acknowledgments

The authors would like to thank the anonimous referees for valuable comments and suggestions that led to improvement of this work.

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