Infinitesimal relative position field for observer in a reference frame. Application to relevant spacetimes.

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Abstract

We analyze the concept of infinitesimal relative position vector fiel betwenn "infinitesimally nearby" observers showing the equivalence between different definitions. Through Fermi-Walker derivative of infinitesimal position vector fields along the observer in a reference frame, we characterize several relevant class of spcetimes.

Keywords: Leibnizian and Galilean structures, irrotational vector fields, conformal Killing vector fields, spatially conformal Killing fields of observers.

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1 Introduction

The concept of observer has had an important role in the Physics history. Nevertheless the case of General Relativity with its intrinsic general covariance, has sometimes been able to develop the false belief that all observers are physically equivalent. As a consequence of this error, the concept of observer has been deprived of part of its meaning. So, if we think that all observer are physically equivalent, the concept of observer the concept is irrelevant in the description of a physical scenario.

The reality is that the covariant character of the General Relativity allows that its fundamental equations hold for all observer. In the cae of several relativistic model, the description of the physical scenario is jointed to the election of a distinguished congruence of observers. So for example, the Friedmann Robertson-Walker model, whose fiber is flat represent a perfect fluid solution for the called commoving observer, which are at rest respect to the molecules of the fluid. But, this model also can be exact solutions of the field equation for a vicous fluid, with or without an electromagnetic field, as seen by observers moving relative to the previously mentioned commoving observers and that these solutions can be physically acceptable (see [10]).

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In relativistic spacetimes, each observer can establish local coordinates around him, via the exponential map, however they cannot extract measurable quantities from a test particle or compare frame dependent information with another observer unless they meet at the same point or come close enough so that the spacetime can be considered, effectively, as flat and then a Special Relativity like situation is recovered. In general relativity, the Lorentz transformation between two frames is only possible locally. This is quite a different situation than in Special Relativity. And indeed this limits the role of observer notion in general relativity and cosmology. That is the reason why in General Relativity the meaningful physical quantities are those which are observer independent, like, the Lorentzian metric, proper time and the other tensorial quantities (electromagnetic strength tensor F, etc.). Nevertheless, this is quite a realistic situation; because real life observers can only measure physical quantities locally. The measurable quantites such as electric field strength, magnetic field strength, and in general energy-momentum tensor are local quantities.

The different nature between observers from Special to General Relativity is a purely mathematical question, since a differentaible manifold, in general, do not have vector space structure. The position vector and hence the coordinates asociated to him, only make sense in a vector space, but in a manifold the position vector loses its meaning. In a generaldifferentiable manifold, to talk about coordinates we need to focus on a localised region of the manifold, the chart, dipheomorphic to $\mathbb{R}^n$, being n the dimension of the manifold. Thus observes cannot set up reference frames which explore the whole spacetime, making the role of the observer strongly local and can measure only local observations.

Therefore, observers's measurements are local, but this does not imply that global inferences are impossible. The key here is the concept of simmetrry. If the quantities which we are interested in, follow a pattern then the whole spacetime needs not be explored, a study over a local region can be extrapolated to figure out the global structure of the spacetime. From the information of metric tensor, one can also find out, if possible, the conformal space transformation and study a great deal about the global properties of spacetime.

More precisely, In general relativity, symmetry is usually based on the assumption of the existence of a one-parameter group of transformations generated by a Killing or, more generally, conformal vector field. In fact, a usual simplification for the search of exact solutions to the Einstein equation is to assume the existence a priori of such an infinitesimal symmetry (see [11], [14] for instance). A complete general approach to symmetries in general relativity can be found in [24] (see also [13] and references therein). Although the same causal character for the infinitesimal symmetry is not always assumed, the timelike choice is natural, since the integral curves of such a timelike infinitesimal symmetry provide a privileged class of observers or test particles in the spacetime. Moreover, this choice is supported by very well-known examples of exact solutions.

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A spacetime M admitting a timelike Killing vector field is called stationary. It can be easily seen that if a spacetime M has a timelike conformal vector field, then it is globally conformal to a stationary spacetime. This is a reason to call M conformally stationary (CS). Clearly, a CS spacetime is time orientable. In general, the orthogonal distribution defined by a timelike conformal vector field K in a spacetime is not integrable. If the 1-form metrically equivalent to the vector field is closed, or equivalently, if K is locally the gradient of some function, then this distribution is integrable and provides the spacetime with a distinguished foliation by spacelike hypersurfaces. The presence of such a vector field is not enough to prevent the existence of closed nonspacelike curves, i.e. CS spacetimes fail to be causal, in general. However, if the timelike vector field K is globally the gradient of some smooth function (K is then called a gradient vector field), then the (clearly noncompact) spacetime admits a global time function. Therefore, it is stably causal [16], i.e. there is a fine C0 neighborhood of the original metric of the spacetime such that any of its Lorentzian metrics is causal[2]. The existence of a gradient conformal vector field in a spacetime has been used to study certain cosmological models [20] and plays a relevant role for vacuum and perfect fluid spacetimes (see [11]). Spacetime admitting a timelike gradient conformal vector field, have been widely studied in [6].

An interesting subclass of GCS spacetimes is the family of generalized Robertson-Walker (GRW) spacetimes. A GRW spacetime is a warped product, with base a negatively defined line, fiber a general Riemannian manifold and arbitrary warping function. Note that, in this definition, the fiber is not assumed to be of constant sectional curvature, in general. When this assumption holds and the dimension of the spacetime is 3, the GRW spacetime is a (classical) Robertson-Walker spacetime. Thus, GRW spacetimes widely extend Robertson-Walker spacetimes, and they include, for instance, the Einstein?de Sitter spacetime, Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime. Observe that conformal changes of the metric of a GRW spacetime, with a conformal factor which only depends on universal time, produce new GRW spacetimes. Moreover, small deformations of the metric on the fiber of Robertson-Walker spacetimes also fit into the class of GRW spacetimes. Thus, a GRW spacetime is not necessarily spatially homogeneous, as in the classical cosmological models. Recall that spatial homogeneity seems appropriate just as a rough approach to consider the universe in large. However, in order to consider it in a more accurate scale, this assumption is not realistic. Thus, GRW spacetimes could be suitable spacetimes to model universes with inhomogeneous spacelike geometry [19].

In this work we study two diiferent ways to stablish in a accurate mathematical form the concept of the infinitesimal relative position vector field respect to fixed observer in a congruence or vector field of observers (Section 3). Moreover, we show it mathematical equivalence (see Theorem 4). The chage of position taht this observer measures for other nearby observer is given mathematically thank to the Fermi-Walker dervative. At this point, we ask ourselves when the observer measures that the nearby observers (for a given congruence), do not rotate. It is clear, that this situation hold when the Fermi-Walker derivative of the infinetesimal relative vector fields is colinear to these vector fields. In Proposition 7, we characterize this condition via the Lie derivative of the metric and Curl tensors. As aconsequence of this characterization it is clear that in a conformaly stationary spacetime, the position vector fileds stablished by the obsevers determined by the timelike conformal vector field satisfy the colinearity condition.

A natural mathematical question arises, if the colineary condition holds asociated to a field of observers, when the spacetime will be conformaly stationary. This question is widely studied in Theorem 8 and Theorem 9.

Finaly, we applied our results to characterize the relevant class of cosmological model called generalized Robertson-walker spacetimes, under the existence of a distinguished family of observers (see Theorem 11).

2 Preliminaries: relativistic spacetimes, synchronizability and Fermi-Walker connection

A relativistic spacetime is an oriented $(n+1)-$dimensional Lorentzian manifold (M, g), $n \geq 1$, endowed with a fixed time orientation, [18]. Along this paper the signature of a Lorentzian metric is considered to be $(-,+,...,+)$. The points of M are also named *events*. A tangent vector $v \in T_pM$ is named *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *lightlike* otherwise. A hypersurface in M is called *spacelike* if its tangent vectors are spacelike, i.e., the induced tensor metric from (M, g) is Riemannian. An observer in M is mathematically represented by a (smooth) curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ such that its velocity $\gamma'(t)$ is future pointing and $g(\gamma'(t), \gamma'(t)) = -1$ for any $t \in I$, (see [21]). The parameter t is called the *proper time* of the observer.

The Levi-Civita connection ∇ of (M, g), induces a connection along the observer γ, such that its corresponding covariant derivative is given by $\dfrac{DY}{dt} = \nabla_{\gamma'(t)}Y \in \mathfrak{X}(\gamma)$ for $Y \in \mathfrak{X}(\gamma)$, where $\mathfrak{X}(\gamma)$ denotes the space of smooth vector fields along γ. The covariant derivative $\dfrac{D\gamma'}{dt}$ of γ', is understood as the (proper) *acceleration* of the observer γ. When $\dfrac{D\gamma'}{dt} \equiv 0$, γ is a timelike geodesic in M, and the observer γ is in free falling.

For each event $\gamma(t)$ the tangent space $T_{\gamma(t)}M$ splits as

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

where $T_t = \mathrm{Span}\{\gamma'(t)\}$ and $R_t = T_t^{\perp}$. Endowed with the restriction of g, R_t is a spacelike hyperplane of $T_{\gamma(t)}M$. It is interpreted as the instantaneous physical space observed by γ at t and it is called the *infinitesimal rest space of the observer at* $\gamma(t)$, (see[18]). Clearly, the observer γ is able to compare spatial directions at t. In order to compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ could use, as a first attempt, the parallel transport along γ defined by the Levi-Civita covariant derivative along γ,

$$P^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M.$$

This linear isometry satisfies $P^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$ if γ is free falling (i.e., it has null proper acceleration). But, unfortunately, this property does not remain true for any general observer. In order to solve this difficulty, for each $Y \in \mathfrak{X}(\gamma)$ put Y_t^T, Y_t^R the orthogonal projections of Y_t on T_t and R_t, respectively, i.e., $Y_t^T = -g(Y_t, \gamma'(t))\,\gamma'(t)$ and $Y_t^R = Y_t - Y_t^T$. In this way, we define $Y^T, Y^R \in \mathfrak{X}(\gamma)$. We have, [21, Prop. 2.2.1],

Proposition 1 *There exists a unique connection* $\widehat{\nabla}$ *along* γ *such that*

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R,$$

for any $X \in \mathfrak{X}(\gamma)$ *and* $Y \in \mathfrak{X}(\gamma)$.

This connection $\widehat{\nabla}$ is called the *Fermi-Walker connection of* γ. It has the suggestive property that if $Y \in \mathfrak{X}(\gamma)$ satisfies $Y = Y^R$, then $\left(\widehat{\nabla}_X Y\right)_t \in R_t$ for any t.

Denote by $\dfrac{\widehat{D}}{dt}$ the covariant derivative corresponding to $\widehat{\nabla}$. Then, we have [21, Prop. 2.2.2],

$$\frac{\widehat{D}Y}{dt} = \frac{DY}{dt} + g(\gamma', Y)\,\frac{D\gamma'}{dt} - g(\frac{D\gamma'}{dt}, Y)\,\gamma', \tag{1}$$

for any $Y \in \mathfrak{X}(\gamma)$. Note that $\dfrac{\widehat{D}}{dt} = \dfrac{D}{dt}$ if and only if γ is free falling.

The acceleration $\dfrac{D\gamma'}{dt}$ satisfies $\dfrac{D\gamma'}{dt}(t) \in R_t$, for any t. Therefore, it is observed by γ whereas the velocity γ' is not.

A field of observers or reference frame on (M, g) (see [18] or [21]), is a unit timelike vector field $Z \in \mathfrak{X}(M)$ pointing to the future. Note that each integral curve of Z is an observer in M. If, in addition, Z is a geodesic vector field, i.e. $\nabla_Z Z \equiv 0$, then every observer in Z will be in free falling. A field of observers Z induces a smooth distribution in M, denoted by $Z^{\perp}$, which is given by the kernel of the 1-form $Z^b = g(Z, \cdot)$ metrically equivalent to Z. When this distribution is integrable, i.e., $Z^b \wedge dZ^b = 0$, then Z is said to be *locally synchronizable* [4], and making use of the Frobenius Theorem (see [23]), we deduce that it is foliated by a family of spacelike hypersurfaces $\mathcal{F} = \{\mathcal{F}_\lambda\}$ and each leaf $\mathcal{F}_\lambda$ of the induced foliation $\mathcal{F}$ represents the "space" for the family of observers in Z. It is well-known that being locally synchronizable is equivalent to assert that each $p \in M$ has a neighborhood where $dZ^b = f\,dt$, for certain smooth functions $f > 0$, t, and so, the hypersurfaces $\{t = \text{constant}\}$ locally coincide with the leaves of the foliation $\mathcal{F}$. Thus, any observer may be synchronized through the "compromise time" t, obtained rescaling its proper time. In the more restrictive case Z^b is closed, i.e., $dZ^b = 0$, it is said that the field of observer Z is *proper time locally synchronizable.*

The widely known Poincaré Lemma (see [5]) asserts that any closed 1-form is locally exact, which means that it is locally the differential of a smooth function. Therefore, locally, $dZ^b = dt$. Hence, observers are synchronized directly by its proper time (up to a constant).

3 Infinitesimal relative position vector fields in relativistic spacetimes

Let (M, g) be a relativistic spacetime, and Z a field of observers on M. Fix an observer $\gamma : I \longrightarrow M$ in Z such that $\gamma(0) = p$, and take an orthonormal basis $\{e_1, \ldots, e_n\}$ of the Euclidean vector subspace $\{\gamma'(0)\}^{\perp}$ of T_pM. Consider the Fermi-Walker parallel reference frame $\{E_1(t), \ldots, E_n(t)\}$ through γ, such that $E_i(0) = e_i$, $i = 1, ..., n$ and define a coordinate system on a tubular neighborhood of γ,

$$(t, x_1, \ldots, x_n) \mapsto \exp_{\gamma(t)}\big(\sum_{i=1}^{n} x_i\, E_i(t)\big).$$

This coordinate system arround γ, describes a new parametrization of the worldlines of the observers in Z close to γ and, as consequence, a new local timelike vector field $\overline{Z}$, obtained reparametring the local flux of Z, and such that $\overline{Z}_{\gamma(t)} = Z_{\gamma(t)}$. The local flux of

$\overline{Z}$, Ψ_s, does preserve the spacelike character of the basis $\{v_1, \ldots, v_n\}$, i.e., now we have that $d\Psi_t\big|_p(v_i) \in \{\gamma'(t)\}^\perp$ for all t and $i = 1, \ldots, n$. Let $\{v_1, \ldots, v_n\}$ be a basis of $\{\gamma'(0)\}^\perp$, therefore, we may construct the following (adapted to γ) coordinate system in a tubular neighborhood of γ, $\mathcal{V} \subseteq M$,

$$\phi : U \subseteq \mathbb{R} \longrightarrow \mathcal{V}, \qquad (t, x_1, \ldots, x_n) \mapsto \exp_{\gamma(t)}\big(\sum_{i=1}^{n} x_i \, V_i(t)\big),$$

where $\{V_i\}_{i=1}^n$ are the only $\overline{Z}$-Lie parallel vector fields along γ verifying $V_i(0) = v_i$, $i = 1, \ldots, n$.

Now, let $\mathcal{E}$ be a sufficiently small open set around of $\overrightarrow{0} \in \{\gamma'(0)\}^\perp$ and let $q \in \mathcal{S}_p \equiv \exp_p\big(\mathcal{E}\big)$ (i.e., the spacelike hypersurface $\{t = 0\}$ in the adapted tubular neighborhood, which γ perceives when its proper time is 0) be an event, and σ an observer in Z such that $\sigma(0) = q$. Hence, there exists a unique spacelike vector field along γ, $V(t)$ satisfying $\sigma(t) = \exp_{\gamma(t)}\big(V(t)\big)$. This vector field will be considered the *relative position vector field of the nearby observer σ with respect to γ*. Next we expose a formal version version of this notion.

Definition 2 *We define the* infinitesimal position vector field associated to v with respect to γ *as the only Ψ-invariant vector field along γ with $V(0) = v$,*

$$V(t) = d\Psi_t|_{\gamma(0)}(v).$$

Equivalently, given any spacelike vector field along γ, V, we say V is an *infinitesimal relative position vector field for γ* if it is invariant under Ψ, equivalently if it is Lie-parallel with respect to $\overline{Z}$.

Next, we proceed to characterize the infinitesimal position vector fields. Notice that, taking into account the previous construction, there exists necessarily, a positive smooth function on a tubular neighborhood of the observer γ, with $(h \circ \gamma)(t) = 1$, $\forall t \in I$, such that

$$\overline{Z} := h\, Z.$$

Given $v \in (\gamma'(0))^\perp$, a vector field $\overline{V}$ may be defined on a neighborhood of $\gamma(0)$ such that $\overline{V}_{\gamma(0)} = v$ and $[\overline{Z}, \overline{V}] = 0$. Hence, a direct computation shows that along γ

$$Z\big(g(\overline{V}, Z)\big) = \overline{V}(h) + g(\nabla_Z Z, \overline{V}).$$

Then, we may claim that

Lemma 3 *Let Z be a field of observers on the spacetime (M, g), $\gamma : I \longrightarrow M$ an observer in Z and h a smooth positive function, defined on a tubular neighborhood of γ . Then, the flux of $h\,Z$ defines infinitesimal relative position vector fields with respect to γ if and only if h satisfies,*

(a) $(h \circ \gamma)(t) = 1, \quad \forall t \in I.$

(b) $\nabla h\big|_{\gamma(t)} = -\nabla_{\gamma'(t)} Z\big|_{\gamma(t)}, \quad \forall t \in I.$

There is another procedure to obtain infinitesimal position vector fields. It may be described as follows.

Theorem 4 *Let Z be a field of observers on the spacetime (M, g), $\gamma : I \longrightarrow M$ an observer in Z and $v \in \big(\gamma'(0)\big)^{\perp}$. Let $h \in \mathcal{C}^{\infty}(M)$ satisfy hypothesis (a) and (b) of the previous Lemma. Consider the only Z-Lie parallel vector field $W(t)$ along γ, such that $W(0) = v$. Then, the orthogonal projection of W onto $\big(\gamma'(t)\big)^{\perp}$, $\forall t \in I$, is an infinitesimal relative position vector field for the observer γ*

Proof. Let $\overline{W}$ be a vector field defined on a neighborhood $U \subseteq M$ of γ extending to W. We may decompose $\overline{W} = \overline{V} + f\, Z$, with $f \in C^{\infty}(U)$ and $\overline{V}$ the $\overline{Z}$-Lie parallel vector field in a neighborhood of γ such that $\overline{V}(0) = v$. So

$$0 = [Z, \overline{W}] = [Z, \overline{V}] + Z(f)\, Z.$$

Thus, taking into account that $\overline{Z} = h\, Z$,

$$[\overline{Z}, \overline{V}] = h[Z, \overline{V}] - \overline{V}(h) Z = -\big(h\, Z(f) + \overline{V}(h)\big)\, Z.$$

Taking a function f satisfying $Z(f) = -\overline{V}\big(\log(h)\big)$, we conclude that $V(t) = \overline{V}\big|_{\gamma(t)}$ is an infinitesimal position vector field.

□

Taking into account the previous result, the Fermi-Walker derivative $\frac{\widehat{D}V}{dt}$ of an infinitesimal relative position vector field $V \in \mathfrak{X}(\gamma)$, represents the velocity of nearby neighboring observers in Z with respect to the observer γ. This Fermi-Walker derivative admits another geometrical interesting expression, as it follows from the next result.

Proposition 5 *Given a field of observers Z in the spacetime (M, g), and γ a observer in Z. For any infinitesimal relative position vector $V \in \mathfrak{X}(\gamma)$, the following identity holds*

$$\frac{\widehat{D}V}{dt} = -A'_Z(V) := \nabla_V Z. \tag{2}$$

Proof. By using that $\overline{Z} = h\, Z$ and $L_Z V = 0$, we get

$$g\big(\frac{\widehat{D}V}{dt}, U\big) = g\big(\nabla_Z V, U\big) = g\big(\nabla_Z \overline{V}, U\big) = g\big(\nabla_{\overline{V}} Z, U\big) = g\big(\nabla_V Z, U\big),$$

for any $U \in \mathfrak{X}(\gamma)$ orthogonal to $Z_{\gamma(t)}$. Where we have had into account that $h \circ \gamma \equiv 1$ and $\frac{\widehat{D}V}{dt} \perp Z$.

□

Next, we analyse the geometry of the linear operator $A'_Z : \Gamma(Z^{\perp}) \subset \Gamma(TM) \longrightarrow \Gamma(Z^{\perp})$. This linear operator may be decomposed in its symmetric $\widehat{S}$ and skew-symmetric $\widehat{\omega}$ parts,

$$-A'_Z = \widehat{S} + \widehat{\omega},$$

where $\widehat{S}$ is self-adjoint for g, and $\widehat{\omega}$ skew-adjoint. Denote by S and ω the corresponding fields of 2-covariant associated tensors,

$$S(V, W) = g(\widehat{S}(V), W) = \frac{1}{2}\big(g(\nabla_V Z, W) + g(\nabla_W Z, V)\big), \tag{3}$$

$$\omega(V,W) = g(\widehat{\omega}(V),W) = \frac{1}{2}\mathrm{Curl}(Z)(V,W), \tag{4}$$

where V,W are spacelike vector fields in $Z^{\perp}$, and ω is the vorticity or Coriolis tensor field defined in (**??**). The name "vorticity" means that, if the observers of Z represent the wordlines of the particles of a fluid and γ is the trajectory of one of them, $\omega(\gamma')$ gives certain measure about how γ see the others turn around. Notice that $\omega = \frac{1}{2}\mathrm{Curl}(Z)$ can also be expressed as $\omega = dZ^b$, being $Z^b = g(Z,\cdot)$ the 1-form metrically equivalent to Z.

On the other hand,the symmetric operator $\widehat{S}$ can be decomposed as

$$\widehat{S} = \frac{\mathrm{div}(Z)}{n} I + \Theta,$$

where I denotes the identity endomorphism of $\Gamma(Z^{\perp})$ and Θ is the traceless part of $\widehat{S}$ (the shear tensor) and $\theta(Z) = \mathrm{div}(Z)$. The term $\frac{\mathrm{div}(Z)}{n} I$ represents the expansion or contraction, i.e., fixed an observer in Z, it measures how nearby neighboring observers go away on average, while Θ measures the deviations of this average.

It seems natural to think that an observer γ in Z will affirm that his neighboring observers do not rotate if the Fermi-Walker derivative of each infinitesimal relative position vector field on γ is proportional to the proper infinitesimal relative position vector field. As it is immediate to show, this condition assures the irrotational character of Z. Nevertheless, If a field of observer Z is irrotational, it is not possible in general assures the condition on the Fermi-Walker derivative. Indeed,

Example 6 *The condition (6) can not be derived from the irrotational character of Z. For instance, let us consider the Lorentz-Minkowski spacetime $\mathbb{L}^3$ with its standard coordinates (u,x,y). Take the field of observers $Z = \sqrt{1+u^2}\,\partial_u + u\,\partial_x$. Any vector field orthogonal to Z takes the form*

$$V = \frac{uV_1}{\sqrt{1+u^2}}\partial_u + V_1\partial_x + V_2\partial_y,$$

for arbitrary smooth functions V_1, V_2. We can compute

$$\langle \nabla_V Z, W\rangle = \frac{u}{(1+u^2)^{3/2}} V_1 W_1,$$

for any V,W orthogonal to Z. From which we get that Z is irrotational.

Now, we fix $p \in \mathbb{L}^3$ and we denote by E_1 and E_2 the infinitesimal position vector fields along the integral curve of Z through p such that $E_1(p) = \frac{u}{\sqrt{1+u^2}}\partial_u(p) + \partial_x(p)$ and $E_2(p) = \partial_y(p)$. Then,

$$\frac{\widehat{D}E_1}{du}(p) = \frac{u}{(1+u^2)^{3/2}} E_1(p), \quad \textit{while} \quad \frac{\widehat{D}E_2}{du}(p) = 0.$$

4 Irrotational and spatially conformal Killing fields of observers in relativistic spacetimes

The notions of spatially conformal Killing field of observers and conformal Killing vector field have been reviewed in Section **??**. Anyway it is useful to note that a field of observers Z is spatially conformal Killing if and only if

$$g(\nabla_V Z, W) + g(\nabla_W Z, V) = 2\lambda\, g(V, W), \qquad \forall V, W \in \Gamma\big(Z^{\perp}\big),$$

i.e., $\widehat{S} = \lambda\, I$, with $\lambda = \frac{1}{n}\mathrm{div}(Z)$ (the shear tensor of Z is identically zero).
Moreover, Z satisfies the condition $L_Z g = 2\lambda\, g$ if and only if

$$g(\nabla_X Z, Y) + g(\nabla_X Z, Y) = 2\lambda\, g(X, Y), \qquad \forall X, Y \in \Gamma\big(TM\big). \tag{5}$$

On the other hand, a field of observers Z is said to be *irrotational* if $\widehat{\omega} = 0$, i.e.,

$$g(\nabla_V Z, W) = g(\nabla_W Z, V), \qquad \forall V, W \in \Gamma(Z^{\perp}).$$

From this observation, using (2) and the previous remark, we conclude that,

Proposition 7 *Let Z be a field of observers in a spacetime (M, g). Then, Z is irrotational and spatially conformal if and only if*

$$\frac{\widehat{D}V}{dt} = \frac{\mathrm{div}(Z)}{n}\, V, \tag{6}$$

for any infinitesimal position vector field $V \in \mathfrak{X}(\gamma)$, for any γ observer of Z.

Under the same hypothesis, Z will be rigid if and only if

$$\frac{\widehat{D}V}{dt} = 0, \tag{7}$$

for any infinitesimal position vector field $V \in \mathfrak{X}(\gamma)$.

The importance of conformaly stationary spacetimes has been described in the previous section. that if (M, g) is a conformaly stationary spacetime, whose timelike comformal Killing vector field X is irrootational, then it is not difficult to see, that the field of observer $\frac{X}{\|X\|}$ determined by X is irrotational and spatially conformal. So, the following question arises naturally: if Z is a field of observers in a spacetime (M, g), under which conditions on the geometry of M and Z does a function φ exist such that the vector field $\varphi\, Z$ is a conformal Killing vector field?

Suppose that Z satisfies condition (6). It is clear that $\varphi\, Z$ is irrotational and spatially conformal for any $\varphi > 0$, with conformal factor $\varphi\,\frac{\mathrm{div}(Z)}{n}$. Let γ be an observer of Z, and $v \in \big(\gamma'(t)\big)^{\perp}$. We have

$$g\big(\nabla_v(\varphi\, Z), \gamma'(t)\big) + g\big(\nabla_{\gamma'(t)}(\varphi\, Z), v\big) = v(\varphi) + \varphi\, g\big(\nabla_{\gamma'(t)} Z, v\big), \tag{8}$$

$$g\big(\nabla_{\gamma'(t)}(\varphi\, Z), \gamma'(t)\big) = \gamma'(t)(\varphi). \tag{9}$$

From (5), we deduce that φ must verify

$$v\big(\log(\varphi)\big) = -g\big(\nabla_{\gamma'(t)} Z, v\big), \qquad \text{and} \qquad Z\big(\log(\varphi)\big) = \frac{\mathrm{div}(Z)}{n},$$

for any $v \in (Z_{\gamma(t)})^{\perp}$. Both conditions may be summarize as

$$\nabla \log(\varphi) = -\nabla_Z Z - \frac{\mathrm{div}(Z)}{n}\, Z. \tag{10}$$

Hence, from a direct application of the converse to Poincar's Lemma, we get the following result.

Theorem 8 *Let Z be a field of observers in a spacetime (M,g) such that condition (6) holds for any observer γ of Z and for every infinitesimal position vector field $V \in \mathfrak{X}(\gamma)$.*

(a) If the following condition holds

$$\operatorname{Curl}\big(\nabla_Z Z + \frac{1}{n}\operatorname{div}(Z)\, Z\big) = 0, \tag{11}$$

then for each point $p \in M$ there exists a neighborhood U_p and a positive function $\varphi \in C^\infty(U_p)$ such that $\varphi\, Z$ is conformal Killing.

(b) If M is contractible to a point, then the function φ is globally defined.

(c) If there exists a positive function $\varphi \in C^\infty(M)$ such that $\varphi\, Z$ is conformal Killing, then (11) is satisfied.

In addition, note that if (11) holds, from (8) we have

$$\varphi\big(\gamma(t)\big) = \varphi\big(\gamma(0)\big) \exp\big(\frac{1}{n}\int_\gamma \operatorname{div}(Z)\big),$$

for each observer γ in Z. Thus, if we know the function φ on a hypersurface orthogonal to Z passing through a point p, we also know the function φ on a neighborhood of p.

Finally, we assume that $\frac{\widehat{D}V}{dt} = 0$ for any observer γ of Z and for all infinitesimal position vector field $V \in \mathfrak{X}(\gamma)$. We know that Z is irrotational and rigid, but, what must Z satisfy so that $\varphi\, Z$ is Killing for some positive function φ? The answer is contained in the following corollary of Th.8.

Theorem 9 *Let Z be a field of observers in a spacetime (M,g) such that $\frac{\widehat{D}V}{dt} = 0$ for any observer γ of Z and for every infinitesimal position vector field $V \in \mathfrak{X}(\gamma)$.*

(a) If the following condition holds

$$\operatorname{Curl}\big(\nabla_Z Z\big) = 0, \tag{12}$$

then for each point $p \in M$ there exists a neighborhood U_p and a positive function $\varphi \in C^\infty(U_p)$ such that φZ is Killing.

(b) If M is contractible to a point, then the function φ is globally defined.

(c) If there exists a positive function $\varphi \in C^\infty(M)$ such that $\varphi\, Z$ is conformal Killing, then (12) is satisfied.

Remark 10 *Given an observer γ of Z and an infinitesimal position vector field $V \in \mathfrak{X}(\gamma)$, a straightforward computation shows that condition (12) is satisfied if and only if $\nabla_Z Z$ is irrotational and $g(\nabla_Z Z, V)$ is constant along the worldline of γ.*

5 Characterizations of GRW spacetimes spacetimes

There is a particularly interesting class of spacetimes admitting an irrotational and spatially conformal field of observers, the Generalized Robertson-Walker (GRW) spacetimes. A GRW spacetime with base $(I, -dt^2)$, fiber (F, g_F) and warping function f is the product manifold $M = I \times F$ endowed with the Lorentzian metric

$$g^f = -dt^2 + f^2(t)g_F,$$

see [1]. GRW spacetimes are particular cases of warped products, which have been widely used in General Relativity (see Introduction). For any GRW spacetime, the vector field ∂_t is a field of observers which is geodesic (the observers in Z are free falling), irrotational, spatially conformal Killing and divergence non-spacelike depending, see [22]. Indeed, given a simply connected relativistic spacetimed (M, g) with a complete vector field Z satisfying the previously cited properties, then (M, g) is (globally) a GRW spacetime with $\partial_t = Z$, [22, Theorem 2.1]. If the global hypotheses (simply connection of M and completeness) are removed, then (M, g) is locally a GRW spacetime. From this results together with Proposition 7 we get the following theorem.

Theorem 11 *Let (M, g) be a simply connected relativistic spacetime with a complete vector field of observers Z and suppose that the observer in Z are in free falling, the gradient of* $\mathrm{div}(Z)$ *is pointwise parallel to Z and the Fermi-Walker derivative of any infinitesimal position vector field along any of its integral curves is proportional to the infinitesimal position vector field. Then (M, g) is (globally) a GRW spacetime with $\partial_t = Z$. If M is not asked to be simply connected and Z is not needed to be complete, then (M, g) is locally a GRW spacetime.*

This last result, constitutes a new characterization of the GRW spacetimes through behavior of the infinitesimal relative position vector field associates to certain class of observers in them. Others recent characterizations of this relevant family of spacetiemes can be found in [15], [9] and [17] and references thereim.

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