# The null space of fuzzy inclusion measures

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***Abstract*—Some formal relationships between the different axiomatic definitions of inclusion measure are analysed. In particular, the links between the different proposals about the null-space (the collection of pairs associated with a null degree of inclusion) are studied. Taking as starting point the well-known axiomatics of Kitainik and Sinha-Dougherty, we observe that other alternative proposals about the null-space are incompatible with both the null-space and the decomposition axioms of these authors. We also conclude that both the axiomatics of Kitainik and that of Sinha-Dougherty contain certain redundancies. Reduced equivalent lists of axioms are proposed.**

**Keywords: subsethood measure, fuzzy inclusion, Sinha-Dougherty axioms, Kitainik axioms keywords.**

## I. INTRODUCTION

Zadeh defined in his seminal paper from 1965 [12] the well known inclusion relation "$\subseteq$" between fuzzy sets defined as follows:

$$A \subseteq B \Leftrightarrow A(x) \leq B(x),\ \forall\, x.$$

It induced a partial order over the family of fuzzy subsets of a universe. Such a binary (0-1-valued) relation was not considered to match the essence of fuzzy set theory, and in order to overcome this issue, Bandler and Kohout [1] introduced for the first time the notion of *subsethood grade* in order to quantify (within a $[0,1]-$scale) to what extent a fuzzy set is included in another. Different axiomatic definitions of *subsethood* or *fuzzy inclusion measures* have been later on proposed successively by Kitainik [6], Sinha and Dougherty [9], Young [11], Fan el al. [4]. Some other authors as Bustince *et al.* [2], Vlachos *et al.* ( [10]) or Zhang [13] have provided additional discussions around those axiomatic definitions, or applied them in practical situations. A further axiomatic definition has been recently proposed by Santos *et al.* [8].

Most of those definitions coincide in requiring that the inclusion measure assigns the value $\sigma(A,B)=1$ to any pair of fuzzy sets satisfying the relation $A \subseteq B$. They also demand the mapping $\sigma$ to be decreasing in the first argument and increasing in the second, with respect to Zadeh's inclusion.

However, the different axiomatics disagree in the conditions required to a pair of fuzzy sets $(A,B)$ in order to satisfy the equality $\sigma(A,B)=0$.

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Furthermore, some axiomatics, such as those of Kitainik [6] and Sinha-Dougherty [9], include additional restrictions on the relationship between the degree of inclusion between the union of two sets and a third one, as well as the inclusion of a fuzzy set wrt to the intersection of two additional fuzzy sets. More specifically, these authors demand:

$$\sigma(A \cup B, C) = \min\{\sigma(A,C), \sigma(B,C)\} \tag{1}$$

$$\sigma(A, B \cap C) = \min\{\sigma(A,B), \sigma(A,C)\},\ \forall\, A,B,C. \tag{2}$$

The above constraints influence the functional expression of $\sigma$, and, as a consequence, they implicitly restrict the conditions under which a pair $(A,B)$ satisfies the equality $\sigma(A,B)=0$, as we will prove in this paper. In particular, we will show that Equations 1 and 2 are incompatible with some of the proposals about $\sigma(A,B)=0$ alternatively required by some authors such as Santos *et al.* [8], for example.

As a consequence of our study, we will see that there is a certain redundancy in the lists of axioms respectively proposed by Kitainik and Sinha-Dougherty, and we will show (equivalent) reduced versions of both definitions. As an intermediate step we will show some (alternative but not reduced) lists of axioms that are respectively equivalent to Kitainik and to Sinha-Dougherty proposals.

## II. PRELIMINARIES

Let us consider a finite universe $\Omega$ and let $\mathcal{F}(\Omega)$ denote the collection of fuzzy subsets of it. Sinha and Dougherty proposed the following list of axioms to be satisfied by a *fuzzy inclusion measure* $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0,1]$:

- SD1.- $\sigma(A,B)=1 \Leftrightarrow A \subseteq B$.
- SD2.- $\sigma(A,B)=0 \Leftrightarrow \exists\, x \in \Omega$ s.t. $A(x)=1, B(x)=0$.
- SD3.- $B \subseteq B' \Rightarrow \sigma(A,B) \leq \sigma(A,B')$.
- SD4.- $A \subseteq A' \Rightarrow \sigma(A,B) \geq \sigma(A',B)$.
- SD5.- $\sigma(A,B)=\sigma(S(A),S(B))$ for any $S : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ where $S(A)(x)=A(s(x)),\ \forall\, x \in \Omega$ and $s : \Omega \to \Omega$ is a one-to-one mapping (permutation).
- SD6.- $\sigma(A,B)=\sigma(B^c,A^c)$ where $\cdot^c$ denotes Zadeh's complement $(A^c(x)=1-A(x),\ \forall\, x \in \Omega.)$
- SD7.- $\sigma$ satisfies Eq. 1, i.e., $\sigma(A \cup B, C) = \min\{\sigma(A,C), \sigma(B,C)\},\ \forall\, A,B,C \in \mathcal{F}(\Omega)$.
- SD8.- $\sigma$ satisfies Eq. 2, i.e., $\sigma(A, B \cap C) = \min\{\sigma(A,B), \sigma(A,C)\},\ \forall\, A,B,C \in \mathcal{F}(\Omega)$.

Their original version included an additional axiom that was later proved to be equivalent to SD3. Kitainik independently studied this notion in [6] and [7]. He proposed the following alternative list of axioms:

- K1.- $\sigma(A,B) = \sigma(B^c, A^c)$ (SD6)
- K2.- $\sigma(A, B \cap C) = \min\{\sigma(A,B), \sigma(A,C)\},\ \forall\, A,B,C.$ (SD8)
- K3.- $\sigma(A,B) = \sigma(S(A), S(B))$ for any $S : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ where $S(A)(x) = A(s(x)),\ \forall\, x \in \Omega$ and $s : \Omega \to \Omega$ is a one-to-one mapping. (SD5)
- K4.- The restriction of $\sigma$ to the collection of pairs of crisp sets coincides with crisp set inclusion, i.e., for all $A, B \in \wp(\Omega)$:
  - K4a.- $\sigma(A,B) = 1 \Leftrightarrow A \subseteq B$ and
  - K4b.- $\sigma(A,B) = 0 \Leftrightarrow A \not\subseteq B$,

and proved that Properties SD3 and SD4 and SD7 can be derived from his collection of axioms, which are in turn implied by Sinha-Dougherty axioms. The interested reader can consult Cornelis *et al.* [3] for more detailed explanations. As a consequence of Kitainik result, SD3, SD4 and SD7 can be removed from Sinha & Dougherty list. Thus, the list of axioms proposed by Sinha and Dougherty is equivalent to the following reduced list:

- SD1.- $\sigma(A,B) = 1 \Leftrightarrow A \subseteq B$.
- SD2.- $\sigma(A,B) = 0 \Leftrightarrow \exists\, x \in \Omega$ s.t. $A(x) = 1, B(x) = 0$.
- SD5.- $\sigma(A,B) = \sigma(S(A), S(B))$ for any $S : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ where $S(A)(x) = A(s(x)),\ \forall\, x \in \Omega$ and $s : \Omega \to \Omega$ is a one-to-one mapping.
- SD6.- $\sigma(A,B) = \sigma(B^c, A^c)$.
- SD8.- $\sigma$ satisfies Eq. 2, i.e., $\sigma(A, B \cap C) = \min\{\sigma(A,B), \sigma(A,C)\},\ \forall\, A,B,C \in \mathcal{F}(\Omega)$.

Thus we can observe that both axiomatic definitions (the one proposed by Sinha & Dougherty, equivalent to the above reduced list, and the one proposed by Kitainik) essentially capture the same idea, except for the "boundary" conditions SD1 and SD2 (proposed by Sinha & Dougherty), which are stronger than condition K4 (proposed by Kitainik instead).

As a side remark, let us furthermore notice that the conjunction of SD6 (K1) plus SD8 (K2) is equivalent to the conjunction of SD6 (K1) plus SD7. Thus, Axiom SD8 (K2) can be alternatively replaced in both definitions (Sinha & Dougherty and Kitaink) by Axiom SD7. I.e. the following results hold:

*Corollary 1:* Consider a finite universe $\Omega$ and a mapping $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0,1]$.

- $\sigma$ satisfies Sinha & Dougherty's properties (SD1 to SD8) iff it satisfies SD1, SD2, SD5, SD6, SD7.
- $\sigma$ satisfies Kitainik's properties (K1 to K4) iff it satisfies K1, SD7, K3, K4.

Furthermore, the following additional properties concerning the boundary conditions can be deduced from Kitainik's list of axioms. The first one is stronger than K4a and the second one complements Property K4b. They are respectively weaker than SD1 and SD2:

*Theorem 1:* Let $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0,1]$ satisfy properties K1 to K4. Then the following properties hold:

- SD1$^-$.-
  - SD1$^-$a).- $\sigma(\emptyset, A) = 1,\ \forall\, A \in \mathcal{F}(\Omega)$
  - SD1$^-$b).- $\sigma(A, \Omega) = 1,\ \forall\, A \in \mathcal{F}(\Omega)$
- SD2$^-$.- $\exists\, x$ such that $A(x) = 1, B(x) = 0 \Rightarrow \sigma(A,B) = 0$.

**Proof:**

- SD1$^-$.- The first property (SD1$^-$) is straightforwardly derived from K4 and the monotonicity conditions SD3 and SD4. These two monotonicity conditions are in turn respectively derived from SD7 and SD8. Finally, as we have clarified in Corollary 1, SD7 is implied by the conjunction of properties SD6 (K1) and SD8 (K2).
- SD2$^-$.- Let us suppose that $\sigma$ satisfies Kitainik properties. Let us consider a pair of fuzzy subsets $A, B \in \mathcal{F}(\Omega)$ satisfying the following condition:

  $$\exists\, x \ \text{ such that } A(x) = 1, B(x) = 0.$$

  Let us now consider the crisp sets $\underline{A}$ and $\overline{B}$ respectively defined as follows:

  $$\underline{A}(x) = 1 \text{ if } \ A(x) = 1, \underline{A}(x) = 0 \text{ otherwise.}$$

  $$\overline{B}(x) = 0 \text{ if } \ B(x) = 0, \overline{B}(x) = 1 \text{ otherwise.}$$

  Under the above condition about $A$ and $B$, $\underline{A} \not\subseteq \overline{B}$, and therefore, according to Kitainik's restrictions $\sigma(\underline{A}, \overline{B}) = 0$. Furthermore, taking into account that $A \supseteq \underline{A}$ and $B \subseteq \overline{B}$, and according to the monotonicity of Kitainik's measures, we can deduce that $\sigma(A,B) \leq \sigma(\underline{A}, \overline{B}) = 0$. □

Some alternative axiomatic definitions were later proposed by Young [11], Fan *et al.* [4] (who proposed different versions). As we mentioned in our introduction, all those authors require some monotonicity properties such as SD3 and SD4 or weaker versions of them (as Young and Fan *et al.* do). Furthermore, all of them $\sigma(A,B) = 1 \Leftrightarrow A \subseteq B$ (although Kitainik restricts this condition to the case of pairs of crisp sets). We can nevertheless observe different proposals concerning necessary and sufficient conditions for $\sigma(A,B) = 0$. (In the next section, we will study those differences in detail).

Santos et al. [8] reviewed all those axiomatic definitions and proposed the following one:

*Definition 1:* $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0,1]$ is a subsethood measure if it satisfies the following properties:

- SA1.- $\sigma(A,B) = 1 \Leftrightarrow A \subseteq B$. (SD1)
- SA2.- $\sigma(A,B) = 0 \Leftrightarrow A = \Omega, B = \emptyset$.
- SA3.-
  - SA3a) $B \subseteq B' \Rightarrow \sigma(A,B) \leq \sigma(A', B)$. (SD3)
  - SA3b) $A \subseteq A' \Rightarrow \sigma(A,B) \geq \sigma(A', B)$. (SD4)

## III. The restriction of the inclusion measures to the family of crisp sets

As we have previously mentioned, the different axiomatic definitions of fuzzy inclusion disagree (among other things) on the conditions about the null space,

$$\{(A,B) \in \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \,:\, \sigma(A,B) = 0\}.$$

The different conditions correspond to basically two different viewpoints about the restriction of the inclusion measure to the family of pairs of crisp sets:

a) The restriction of $\sigma$ to the family of pairs of crisp sets takes values within the (binary) set $\{0,1\}$. In other words, given two crisp sets, we only distinguish the case in which one of them is included in the other ($\sigma(A,B)=1$), from the case where it is not ($\sigma(A,B)=0$).

b) The inclusion measure assigns intermediate values $0<\sigma(A,B)<1$ to some pairs of crisp sets.

Kitainik and Sinha-Dougherty definitions correspond to situation a), while Santos *et al.* matches situation b). In the three axiomatic definitions, the restriction of the kernel $\sigma^{-1}(\{1\})$ to the collection of pairs of crisp sets is:

$$\sigma^{-1}(\{1\})\cap\wp(\Omega)\times\wp(\Omega)=\\ \{(A,B)\in\wp(\Omega)\times\wp(\Omega)\,:\,A\subseteq B\},\qquad(3)$$

but the respective null-spaces do not coincide. In fact, according to Kitainik and Sinha-Doughery definitions, the corresponding restriction is:

$$\sigma_K^{-1}(\{0\})\cap\wp(\Omega)\times\wp(\Omega)=\\ \sigma_{SD}^{-1}(\{0\})\cap\wp(\Omega)\times\wp(\Omega)=\\ \{(A,B)\in\wp(\Omega)\times\wp(\Omega)\,:\,\exists x:A(x)=1,B(x)=0\}=\\ \wp(\Omega)\times\wp(\Omega)\setminus\sigma_K^{-1}(\{1\})$$

which strictly includes Santos *et al.* null space, as this last one is required to be the following singleton:

$$\sigma_{SA}^{-1}(\{0\})\cap\wp(\Omega)\times\wp(\Omega)=\sigma_{SA}^{-1}(\{0\})=\{(\Omega,\emptyset)\}.$$

Thus, under Santos *et al.* definition (and more concretely speaking, under conditions SA1 and SA2), all those pairs of crisp sets $(A,B)$ that do not either satisfy the condition $A\subseteq B$ or coincide with the pair $(\Omega,\emptyset)$ take intermediate inclusion values.

But Santos *et al.* boundary condition SA2 is not only incompatible with the corresponding boundary axioms respectively proposed by Sinha-Dougherty (SD2) and Kitainik (K4b), but also with axioms SD8 (K2) and SD7, as we will prove next:

*Proposition 1:* Consider a universe $\Omega$ with cardinality strictly greater than 1. Consider a mapping $\sigma:\mathcal{F}(\Omega)\times\mathcal{F}(\Omega)\to[0,1]$ satisfying SA2.

1) $\sigma$ does not satisfy SD7.
2) $\sigma$ does not satisfy SD8.

**Proof:** Let us prove the first result. According to the "if" part of condition SA2 $\sigma(\Omega,\emptyset)=0$. Furthermore, if $\sigma$ satisfies, in addition, Property SD7, then for any arbitrary $D\in\wp(\Omega)$, $\emptyset\subsetneq D\subsetneq\Omega$ $\sigma(D\cup D^c,\emptyset)=\min\{\sigma(D,\emptyset),\sigma(D^c,\emptyset)\}$. We therefore conclude that either $\sigma(D,\emptyset)=0$ or $\sigma(D^c,\emptyset)=0$. This implies, according to the "only if" part of SA2, that either $D=\Omega$ or $D=\emptyset$, which leads us to a contradiction.

The proof of the second part is analogous. $\square$

We could even consider an intermediate condition between the one proposed by Kitainik (K4b) and the one considered by Santos *et al.* (SD2), for the null-space, like the following one:

$$\sigma^{-1}(\{0\})\cap\wp(\Omega)\times\wp(\Omega)=\\ \{(A,B)\in\wp(\Omega)\times\wp(\Omega)\,:\,A\neq\emptyset,\,A\cap B=\emptyset\},\qquad(4)$$

but such a condition is also incompatible with Axioms SD7 and SD8, as we prove next:

*Proposition 2:* Consider a mapping $\sigma:\mathcal{F}(\Omega)\times\mathcal{F}(\Omega)\to[0,1]$ satisfying the following condition:

$$[\sigma(A,B)=0\Leftrightarrow A\neq\emptyset,\;A\cap B=\emptyset],\;\forall\,A,B\in\wp(X)$$

Then:

1) $\sigma$ does not satisfy SD7.
2) $\sigma$ does not satisfy SD8.

**Proof:**

1) Let us consider a non-empty proper subset of $\Omega$, $A\in\wp(\Omega)$, $\emptyset\subsetneq A\subsetneq\Omega$. According to the above property $\sigma(A,A^c)$ is null and therefore, the minimum $\min\{\sigma(A,A^c),\sigma(A^c,A^c)\}$ is also zero. However, according to the same property, $\sigma(\Omega,A^c)>0$ and therefore $\sigma$ fails to satisfy the equality $\sigma(A\cup A^c,A^c)=\min\{\sigma(A,A^c),\sigma(A^c,A^c)\}$.
2) The proof of the second part is quite similar. According to the above property $\sigma(\Omega,\emptyset)$ would be equal to zero, but given a proper nonempty subset $A\in\wp(\Omega)$, none of the inclusion values $\sigma(\Omega,A)$, $\sigma(\Omega,A^c)$ would be equal to zero. Therefore $\sigma(\Omega,\emptyset)=\sigma(\Omega,A\cap A^c)$ does not coincide with the minimum of both quantities, and therefore $\sigma$ fails to satisfy SD8. $\square$

In summary, as Propositions 1 and 2 show, conditions SD7 and SD8 (both of them included in Kitainik and Sinha-Dougherty definitions) imply some restrictions about the null-space of the fuzzy inclusion measure:

$$\sigma^{-1}(\{0\})=\{(A,B)\in\mathcal{F}(\Omega)\times\mathcal{F}(\Omega)\,:\,\sigma(A,B)=0\}.$$

The next section is devoted to a more profound study of those restrictions.

## IV. Conditions about the null space and the influence of other axioms

As we have illustrated in the previous section, the conditions about the null-space of fuzzy inclusion measures interact with the decomposition axioms proposed by Kitainik and Sinha-Dougherty (SD7 and SD8). In particular, we have proved that those decomposition properties are incompatible with some conditions about the null-space proposed in other axiomatic definitions like the one recently proposed by Santos *et al.* In this section, we will deepen into the study about the conditions about the null-space and other conditions included in Kitainik's list. We will conclude that part of the requirements about the null-space involved in Property K4b are implicitly derived from the rest of Kitainik's axioms. As a consequence of this, we will show that there is mild redundancy in Kitainik's list of axioms. We will deduce that Sinha-Dougherty list also contains the same redundancy.

### *A. Kitainik and Sinha & Dougherty properties and the functional expression of fuzzy inclusion measures*

Before studying the interaction between SD7 and SD8 and the conditions about the null space, we need to analyse their interaction with the following *decomposition properties*, that we will denote by D1 and D2:

- D1.- $\forall X \in \wp(\Omega), \forall A, B \in \mathcal{F}(\Omega), \sigma(A, B) = \min\{\sigma(A \cap X, B \cap X), \sigma(A \cap X^c, B \cap X^c)\}$.
- D2.- If $A, B, C \in \mathcal{F}(\Omega)$ with $A \cap C = \emptyset$ then $\sigma(A, B) = \sigma(A, B \cup C)$.

The following result shows some formal relations between our new decomposition properties and the conjunction of SD7, SD8 and K4a.

*Theorem 2:* Let us consider a mapping $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0, 1]$. The following implications hold:

(a) If $\sigma$ satisfies K4a, SD7 and SD8, then it also satisfies D2.
(b) If $\sigma$ satisfies SD7 and D2, then it also satisfies D1.
(c) If $\sigma$ satisfies SD7, SD8 and K4a, then it also satisfies D1.
(d) If $\sigma$ satisfies SD6 (K1), SD8 (K2) and K4a, then it satisfies D1.
(e) If $\sigma$ satisfies D1, SD4 and K4a, then it also satisfies SD7.

**Proof:**

(a) Suppose that $\sigma$ satisfies K4a, SD7 and SD8, and consider a triple of fuzzy sets $A, B, C \in \mathcal{F}(\Omega)$ with $A \cap C = \emptyset$. Under these conditions, there exists a crisp set $X \in \wp(\Omega)$ such that $A \subseteq X$ and $C \subseteq X^c$. By means of the distributive property, we can easily check that $B$ coincides with $(B \cup C) \cap (B \cup X)$. Thus:

$$\sigma(A, B) = \sigma(A, (B \cup C) \cap (B \cup X)),$$

which, according to SD8 coincides with $\min\{\sigma(A, B \cup C), \sigma(A, B \cup X\}$. Now by means of SD3 and SD4 (respectively derived from SD8 and SD7) we can state that $\sigma(X, X) \leq \sigma(A, B \cup X)$. Furthermore, according to K4a, $\sigma(X, X) = 1$ Therefore $\min\{\sigma(A, B \cup C), \sigma(A, B \cup X\} = \min\{\sigma(A, B \cup C), 1\} = \sigma(A, B \cup C)$ and thus $\sigma(A, B) = \sigma(A, B \cup C)$.

(b) Consider a pair of fuzzy sets $A, B \in \mathcal{F}(\Omega)$ and a crisp set $X \in \wp(\Omega)$. We can decompose $A$ as the union $A = (A \cap X) \cup (A \cap X^c)$, and then, according to SD7 we have:

$$\sigma(A, B) = \min\{\sigma(A \cap X, B), \sigma(A \cap X^c, B)\}.$$

Furthermore, according to D2, we can write

$$\sigma(A \cap X, B) = \sigma(A \cap X, B \cap X)$$

and

$$\sigma(A \cap X^c, B) = \sigma(A \cap X^c, B \cap X^c),$$

and thus

$$\sigma(A, B) = \min\{\sigma(A \cap X, B \cap X), \sigma(A \cap X^c, B \cap X^c)\}.$$

(c) This result is a direct consequence of the previous ones ((a) and (b)).

(d) This result is immediately derived from (c) and the fact that the conjunction of Properties SD6 and SD7 is equivalent to the conjunction of Properties SD6 and SD8.

(e) Consider three arbitrary fuzzy subsets $A, B, C \in \mathcal{F}(\Omega)$. Consider the crisp set

$$X = \{x \in \Omega \, : \, A(x) \geq C(x)\}.$$

According to D2 we can write $\sigma(A \cup C, B)$ coincides with the minimum of the quantities:

$$\{\sigma((A \cup C) \cap X, B \cap X)$$

and

$$\sigma((A \cup C) \cap X^c, B \cap X^c),$$

which, according to the construction of the crisp set $X$, respectively coincide with $\sigma(A \cap X, B \cap X)$ and $\sigma(C \cap X^c, B \cap X^c)$. Thus, we deduce the following equality:

$$\sigma(A \cup C, B) = \min\{\sigma(A \cap X, B \cap X), \sigma(C \cap X^c, B \cap X^c)\}.$$

Now, according to SD4, we have:

$$\sigma(A \cap X, B \cap X) \leq \sigma(C \cap X, B \cap X)$$

and

$$\sigma(C \cap X^c, B \cap X^c) \leq \sigma(A \cap X^c, B \cap X^c).$$

Thus, the above minimum can be alternatively expressed as the minimum of the four quantities $\sigma(A \cap X, B \cap X)$, $\sigma(A \cap X^c, B \cap X^c)$, $\sigma(C \cap X, B \cap X)$ and $\sigma(C \cap X^c, B \cap X^c)$. Now, again by means of D2,

$$\sigma(A, B) = \min\{\sigma(A \cap X, B \cap X), \sigma(A \cap X^c, B \cap X^c)\}$$

and

$$\sigma(C, B) = \min\{\sigma(C \cap X, B \cap X), \sigma(C \cap X^c, B \cap X^c)\}.$$

We derive the following equality:

$$\sigma(A \cup C, B) = \min\{\sigma(A, B), \sigma(C, B)\}. \quad \square$$

As a consequence, and taking into account the results proved by Kitainik and recalled in Section II, we can derive the following results establishing alternative lists of axioms respectively equivalent to Sinha & Dougherty's and to Kitainik's lists:

*Corollary 2:* Let us consider a mapping $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0, 1]$. The following results are equivalent:

- $\sigma$ satisfies the list of axioms proposed by Sinha & Dougherty (SD1 to SD8).
- $\sigma$ satisfies SD1, SD2, SD5, SD6 and SD7.
- $\sigma$ satisfies SD1, SD2, SD5, SD6 and SD8.
- $\sigma$ satisfies SD1, SD2, SD4, SD5, SD6 and D1.

*Corollary 3:* Let us consider a mapping $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0, 1]$. The following results are equivalent:

- $\sigma$ satisfies the list of axioms proposed by Kitainik (K1 to K4).
- $\sigma$ satisfies K1, SD7, K3 and K4.
- $\sigma$ satisfies K1, SD4, D1, K3 and K4.

The following result shows a sufficient condition for a mapping $\sigma$ to admit a decomposition in terms of the min-operator:

*Proposition 3:* Consider a finite universe $\Omega = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$. Suppose that a mapping $\sigma : \mathcal{F}(\Omega)\times\mathcal{F}(\Omega) \to [0,1]$ satisfies D1 and SD5 (K3). Then there exists some $I : [0,1] \times [0,1] \to [0,1]$ such that

$$\sigma(A,B) = \min_{x\in\Omega} I(A(x), B(x)), \forall\, A,B \in \mathcal{F}(\Omega).$$

**Proof:** Let us consider the family of nested (crisp) sets $X_i = \{x_{i+1}, \ldots, x_n\},\ i = 1, \ldots, n-1$. Every pair of sets $(\{x_i\}, X_i)$ forms a partition of the set $X_{i-1} = \{x_i, \ldots, x_n\}$. Let us now define the mapping $I : [0,1] \times [0,1] \to [0,1]$ as follows

$$I(a,b) = \sigma(a \cdot 1_{\{x_1\}}, b \cdot 1_{\{x_2\}}),\ \forall\,(a,b) \in [0,1] \times [0,1],$$

where $1_A : \Omega \to \{0,1\}$ denotes the indicator function of the crisp set $A$, for every $A \in \wp(\Omega)$. For an arbitrary pair of sets $A, B \in \mathcal{F}(\Omega)$, by means of successively applying the property D2 to each of those partitions, we get the following equality:

$$\sigma(A,B) = \min_{i=1}^{n} \sigma(A \cap \{x_i\}, B \cap \{x_i\}).$$

According to Property D5, $\sigma(A \cap \{x_i\}, B \cap \{x_i\})$ coincides with

$$I(A(x_i), B(x_i)) = \sigma(A(x_i) \cdot 1_{\{x_1\}}, B(x_i) \cdot 1_{\{x_1\}}),$$

for every $i = 1, \ldots, n$. Then $\sigma(A,B) = \min_{i=1,\ldots,n} I(A(x_i), B(x_i))$ and therefore $\sigma$ can be expressed as follows as a function of $I$:

$$\sigma(A,B) = \min_{x\in\Omega} I(A(x), B(x)),\ \forall\, A,B \in \mathcal{F}(\Omega). \quad \square$$

From now on, we will refer to $I$ as the *generating mapping*. Fodor and Yager had already proved that any inclusion measure fulfilling all the axioms proposed by Kitainik admits the previous functional expression. However, and as a consequence of the above results, we can deduce that condition K4b is not necessary in order to guarantee that a fuzzy inclusion measure can be expressed in terms of the "min" operator. More concretely:

*Corollary 4:* Consider a finite universe $\Omega = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$. Suppose that a mapping $\sigma : \mathcal{F}(\Omega)\times\mathcal{F}(\Omega) \to [0,1]$ satisfies K1(SD6), K2 (SD8), K3 (SD5) and K4a. Then there exists some $I : [0,1] \times [0,1] \to [0,1]$ such that

$$\sigma(A,B) = \min_{x\in\Omega} I(A(x), B(x)), \forall\, A,B \in \mathcal{F}(\Omega),$$

with $I$ decreasing in the first argument and increasing in the second one.

**Proof:** According to Theorem 2, part d), if $\sigma$ satisfies Properties K1, K2 and K4a then it satisfies D1. If it satisfies in addition K3 (SD5) then, according to Proposition 3, there exists a mapping $I : [0,1] \times [0,1] \to [0,1]$ such that it can be decomposed in terms of the min-operator as follows:

$$\sigma(A,B) = \min_{x\in\Omega} I(A(x), B(x)), \forall\, A,B \in \mathcal{F}(\Omega).$$

Now, according to the monotonicity conditions SD3 and SD4 (which are implied by the conjunction of K1 (SD6) and K2 (SD8)), we can straightforwardly derive that $I$ is decreasing wrt the first argument and increasing wrt the second one: given an arbitrary value $\alpha \in [0,1]$ let $\vec{\alpha}$ denote the fuzzy set with constant membership function $\vec{\alpha}(x) = \alpha,\ \forall\, x \in \Omega$. Let us now consider an arbitrary tuple of numbers $a, a', b, b' \in [0,1]$ satisfying the condition $a \le a'$ and $b \ge b'$. According to SD3 and SD4, we have:

$$\sigma(\vec{a}, \vec{b}) \ge \sigma(\vec{a}', \vec{b}').$$

But, according to the previous paragraph, $\sigma(\vec{a}, \vec{b}) = I(a,b)$ and $\sigma(\vec{a}', \vec{b}') = I(a', b')$ and therefore

$$I(a,b) \ge I(a', b'). \quad \square$$

The following result is easily derived from the above corollary:

*Corollary 5:* Consider a finite universe $\Omega = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$. Suppose that a mapping $\sigma : \mathcal{F}(\Omega)\times\mathcal{F}(\Omega) \to [0,1]$ satisfies K1 (SD6), K2 (SD8), K3 (SD5) and K4a. Consider $A, B \in \mathcal{F}(\Omega)$ such that $\exists\, x \in \Omega$ with $A(x) = 1$ and $B(x) = 0$. Then:

$$\sigma(A,B) = \sigma(\Omega, \emptyset) = \min_{(C,D)\in\mathcal{F}(\Omega)\times\mathcal{F}(\Omega)} \sigma(C,D).$$

In particular if $A, B \in \wp(\Omega)$ and $A \not\subseteq B$ then:

$$\sigma(A,B) = \sigma(\Omega, \emptyset).$$

As a consequence of the above results, we can prove some additional formal relations between Kitainik and the definition proposed by Young (and mentioned in the Introduction). Young suggests four properties to be satisfied by a fuzzy inclusion, that we will respectively denote Y1, Y2, Y3 and Y4. Properties Y1 and Y4 respectively coincide with SD1 and SD3 and Property Y3 is softer than SD4. Therefore, the three properties are implied by all the axiomatic definitions analysed in this paper (Kitainik, Sinha & Dougherty and Santos *et al.* definitions).

But there is a fourth property concerning the null space of the fuzzy inclusion that is clearly implied by Santos *et al.* definition. It reads as follows:

*Definition 2:* $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega)$ satisfies Property Y2 if:

- Y2a.- $\sigma(\Omega, \emptyset) = 0$.
- Y2b.- If $A(x) \ge 0.5,\ \forall\, x \in \Omega$, and $\sigma(A, A^c) = 0$ then $A = \Omega$.

The above property is compatible with Kitainik's boundary axiom K4b. In fact, if $A \in \wp(X)$ and $A(x) \ge 0.5,\ \forall\, x \in \Omega$ then necessarily coincides with $\Omega$ and therefore the consequent of Y2b is trivially satisfied $(\sigma(A, A^c) = \sigma(\Omega, \emptyset) = 0)$.

But, on the basis of the above results, we can prove that Young axiom Y2 is not compatible with the rest of the axioms proposed by Kitainik.

*Theorem 3:* Consider a finite universe $\Omega = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$. Suppose that a mapping $\sigma : \mathcal{F}(\Omega)\times\mathcal{F}(\Omega) \to [0,1]$ satisfies K1(SD6), K2 (SD8), K3 (SD5), K4a and Y2a. Then it does not satisfy Y2b.

**Proof:** According to Corollary 5 and assuming that $\sigma$ satisfies all the above properties, we can deduce that $\sigma(A,B) = 1,\ \forall\, A, B \in \mathcal{F}(\Omega)$ such that $\exists\, x \in \Omega$ with $A(x) = 1$ and

$B(x) = 0$. In particular, if we select an arbitrary $x_0 \in \Omega$ and define the fuzzy sets $A$ as follows:

$$A(x_0) = 1, \text{ and } A(x) = 0.5, \ \forall x \neq x_0,$$

then $\sigma(A, A^c) = 0$, which contradicts Property Y2b. □

### *B. Redundancy of Kitainik and Sinha & Dougherty lists of axioms*

As we have seen in the last result of the previous section, if a function fulfills the axioms K1, K2, K3 and K4a then it can be expressed as an aggregation of values in terms of the minimum operator. Now, if a function $\sigma$ admits this expression in terms of a function $I : [0,1] \times [0,1] \to [0,1]$, then the null-space of $\sigma$ can be expressed as follows:

$$\sigma^{-1}\{0\} = \{(A, B) : \exists x \in \Omega \text{ with } I(A(x), B(x)) = 0\}.$$

Based in turn on this result, we can see that the conjunction of the axioms K1, K2, K3 and K4a has some interaction with the axiom K4b. More concretely, we see that the K4b property can be replaced by another weaker property, resulting in an equivalent collection of axioms. Specifically:

*Theorem 4:* The following statements are equivalent:

(a) $\sigma$ satisfies Properties K1, K2, K3 and K4.
(b) $\sigma$ satisfies Properties K1, K2, K3, K4a and the following property:
   - K4b'.- $\min\{\sigma(A, B) : A, B \in \mathcal{F}(\Omega)\} = 0$.

**Proof:**

- (a) $\Rightarrow$ (b).- If $\sigma$ satisfies K4b, then $\sigma(\Omega, \emptyset) = 0$ and therefore $\min\{\sigma(A, B) : A, B \in \mathcal{F}(\Omega)\} = 0$. Thus the implication (a)$\Rightarrow$(b) is trivially satisfied.
- (b) $\Rightarrow$ (a).- Let us assume that $\sigma$ satisfies Properties K1, K2, K3, K4a and K4b'. It only remains to prove that it also satisfies Property K4b, i.e., that

$$\forall A, B \in \wp(X) \ [\sigma(A, B) = 0 \Leftrightarrow A \not\subseteq B].$$

   - Suppose that $A, B \in \wp(\Omega)$ and suppose that $\sigma(A, B) = 0$. Then, according to K4a, $A \not\subseteq B$ (otherwise we would have $\sigma(A, B) = 1$).
   - Suppose now that $A, B \in \wp(\Omega)$ and $A \not\subseteq B$. Therefore, there exists $x \in \Omega$ such that $A(x) = 1$ and $B(x) = 0$ and therefore according to Corollary 5, we have

$$\sigma(A, B) = \sigma(\Omega, \emptyset) = \min \sigma_{(C,D) \in \mathcal{F}(\Omega) \times \mathcal{F}(\Omega)}(C, D),$$

   which is equal to 0, according to Property K4b'. □

The above property ($\min \sigma = 0$) is clearly weaker than K4b, and therefore we conclude that the list of axioms of Kitainik contains a mild redundancy.

As a result of the same results, Sinha & Dougherty's list of axioms is also slightly redundant. Specifically, we can verify that, if we replace SD2 with the conjunction of the following properties (weaker than SD2):

- SD2'a.- $\min \sigma = 0$ and
- SD2'b.- $A(x) - B(x) < 1, \ \forall x \in \Omega \Rightarrow \sigma(A, B) > 0$,

we obtain a list of properties equivalent to the list proposed by Sinha & Dougherty, i.e.:

*Theorem 5:* The following statements are equivalent:

- (a).- $\sigma$ satisfies properties SD1, SD2, SD5, SD6, SD8.
- (b).- $\sigma$ satisfies properties SD1, SD5, SD6, SD8 and the following property:
   - SD2'a.- $\min \sigma = 0$
   - SD2'b.- $A(x) - B(x) < 1, \ \forall x \in \Omega \Rightarrow \sigma(A, B) > 0$

**Proof:** Properties SD2a' and SD2b' are clearly weaker than SD2, so we just need to prove the implication (b) $\Rightarrow$ (a). Let us suppose that $\sigma$ satisfies SD1, SD5, SD6, SD8 and SD2' and let us prove that the following equivalence holds:

$$\sigma(A, B) = 0 \Leftrightarrow \exists x \in \Omega : A(x) = 1, B(x) = 0.$$

- $\Rightarrow$).- Suppose that $\sigma(A, B) = 0$. Then, according to SD2'b we deduce that $A(x) - B(x) = 1$, i.e., $A(x) = 1$ and $B(x) = 0$, for some $x \in \Omega$.
- $\Leftarrow$).- Consider a pair of fuzzy sets $A, B \in \mathcal{F}(\Omega)$ satisfying the condition $[\exists x \in \Omega$ such that $A(x) = 1$ and $B(x) = 0]$. Thus, according to Corollary 5, we have that $\sigma(A, B) = \min_{(C,D) \in \mathcal{F}(\Omega) \times \mathcal{F}(\Omega)} \sigma(C, D)$. Now, according to Property SD2'a, we have that $\sigma(A, B) = 0$. □

The above property S2'b is implied by Sinha-Dougherty list of axioms, but not by Kitainik properties. It basically indicates that $\sigma^{-1}(\{0\})$ is the smallest possible non-empty set, according to the rest of the axioms.

As a conclusion of some of the results of this work, we deduce that the difference between Kitainik and Sinha & Dougherty falls on the following properties (satisfied by Sinha & Dougherty, but not Kitainik):

- $\sigma(A, B) = 1 \Rightarrow A \subseteq B$ and $[0 < A(x) \leq B(x) < 1 \forall x \in \Omega \Rightarrow \sigma(A, B) = 1]$.
- $A(x) - B(x) < 1, \ \forall x \in \Omega \Rightarrow \sigma(A, B) > 0$.

In this way, we deduce the following corollary:

*Corollary 6:* The following statements are equivalent:

- $\sigma$ satisfies properties SD1, SD2, SD5, SD6, SD8.
- $\sigma$ satisfies properties K1, K2, K3, K4a and:
   - $\sigma(A, B) = 1 \Rightarrow A \subseteq B$ and $[0 < A(x) \leq B(x) < 1 \forall x \in \Omega \Rightarrow \sigma(A, B) = 1]$.
   - $\min \sigma = 0$
   - $A(x) - B(x) < 1, \ \forall x \in \Omega \Rightarrow \sigma(A, B) > 0$.

The differences between the requirements of Sinha & Dougherty and Kitainik are reflected in the properties of the generating mapping $I : [0,1] \times [0,1] \to [0,1]$. The following result is easily deduced from the previous ones:

*Corollary 7:* Let $\Omega$ be a finite universe. Consider a mapping $\sigma : \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \to [0,1]$ satisfying properties K1, K2, K3 and K4a. Let $I_\sigma : [0,1] \times [0,1] \to [0,1]$ denote the (unique) generating mapping (the one satisfying the following equalities

$$\sigma(A, B) = \min_{x \in \Omega} I(A(x), B(x)), \ \forall A, B \in \mathcal{F}(\Omega).$$

Then $I$ satisfies the following properties:

- $I(0,0) = I(1,1) = 1$
- $I$ is decreasing on the first argument and increasing in the first one.

If, furthermore, $\sigma$ satisfies K4b (or equivalently, K4b') then $I^{-1}(\{0\}) \neq \emptyset$. If, on top of that, $\sigma$ satisfies SD1 and SD2 then, given a pair of numbers $a, b \in [0,1]$, we have:

- If $a \leq b$ then $I(a,b) = 1$.
- If $0 < a \leq b < 1$ then $I(a,b) = 1$.
- If $(a,b) \neq (1,0)$ then $I(a,b) > 0$.

## V. Concluding remarks

We have studied some formal relationships between different axioms of fuzzy inclusion. All the axiomatics coincide in assigning the maximum inclusion value ($\sigma(A,B) = 1$) in the case where $A \subseteq B$, but they differ mainly:

- In the axioms concerning the null-space and
- in the decomposition properties proposed by some authors such as Sinha & Dougherty and Kitainik.

In Sinha & Dougherty and Kitainik's axiomatics, the restriction of the inclusion measure to the family of pairs of crisp sets takes values inside the binary set $\{0,1\}$, since all those cases in which $A$ is not included in $B$ are associated with the minimum inclusion value (normally, 0). Thus, any fuzzy inclusion of that kind defines a "crisp" relationship between pairs of crisp sets. From the mathematical results proved in this manuscript, we can conclude that this property is already derived from other axioms. In particular, if the fuzzy inclusion satisfies the properties of decomposition and symmetry and, in addition, any pair of nested crisp sets $A \subseteq B$ reaches the maximum value for $\sigma$, then $\sigma(A,B)$ reaches the minimum value for all those pairs of crisp sets $(A,B)$ in which $A \not\subseteq B$. Thus, any other definition that allows some crisp pairs to be assigned an intermediate inclusion value between the minimum and the maximum will be incompatible with any of those axioms. These results have allowed us to find a certain redundancy in Sinha & Dougherty and Kitainik's lists of axioms.

In the near future, we intend to study the relationships between the properties of inclusion measures and other functional expressions not based on the minimum function. There has already been a precedent for this study in Santos et al. [8], where some relationships between the properties of alternative aggregation operators and the axioms of the inclusion measures are studied.

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