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STOCHASTIC INEQUALITIES BASED ON COBB–DOUGLAS UTILITY FUNCTIONS

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Abstract. Consumer theory studies how individuals make choices given the prices of goods, budget constraints and their preferences. The preferences of a consumer are represented by a utility measure. One of the most important examples of utility mappings is given by the Cobb-Douglas functions. Frequently the quantities of goods involved in the selection problem are random instead of deterministic. Motivated by the need to compare the preferences and investments of a consumer when the quantities of goods are random and the utility belongs to the Cobb-Douglas family, a new stochastic order is introduced. The order is analyzed in detail, providing characterizations, conditions which lead to the order and properties derived from the order. Special emphasis is placed on the antisymmetric property of the new ordering. The proposed stochastic order weakens the concave order.

1. Motivation of the analysis

In economics, a utility is a representation of preferences over some set of goods. A utility function can be defined as a mapping which specifies the satisfaction of a consumer for all combinations of goods involved in a problem. Roughly speaking, utility functions measure the degree of well-being such goods provide for consumers.

The microeconomic theory basic problem of consumer choice is the election between two goods given a budget constraint. Namely, let x_1 and x_2 stand for the units purchased of two goods by a consumer, let p_1 and p_2 be the unit prices of both goods respectively, and let m represent the income of the consumer, where prices and income are fixed strictly positive values. Assume that for a consumer the utility of the purchased goods is represented by a utility function $U(x_1,x_2)$, which follows the axioms of the revealed preferences as in [14]. Clearly the aim of a consumer is the maximization of his utility. That is, the optimization problem

$$\begin{aligned} \text{maximize} \quad & U(x_1,x_2) \\ \text{with the constraint} \quad & p_1x_1+p_2x_2 \leqslant m \end{aligned}$$

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since the amount spent on both goods cannot be greater than the income of the consumer. Obviously the utility function will be maximize when $p_1x_1 + p_2x_2 = m$ since that mapping is increasing in both arguments. As a consequence we can express the second good as a mapping of the first good, that is, $x_2 = -\frac{p_1}{p_2}x_1 + \frac{m}{p_2}$.

Therefore the above optimization problem can be rewritten as a problem in one dimension. Namely,

$$\begin{aligned} \text{maximize} \ \ & U(x_1, -\frac{p_1}{p_2}x_1 + \frac{m}{p_2}) \\ \text{with the constraint} \ \ & x_1 \in [0, \frac{m}{p_1}]. \end{aligned}$$

One of the most important families of utility (and production) functions is the so-called Cobb-Douglas family, which was introduced in [5]. These functions have been widely used in applied problems since they describe many economic problems and enjoy important properties such the constant elasticity. The readers are referred to [6] and [8] for the justification of the use of Cobb-Douglas utility functions in real-life problems, to [1] for the use in economic growth theory, and to [15] for a characterization of preferences represented by a Cobb-Douglas utility function.

Consider the family of Cobb-Douglas utility functions given by the mappings $U : \mathbb{R}_+^2 \to \mathbb{R}$, with $U(x_1,x_2) = kx_1^{\alpha}x_2^{1-\alpha}$ for any $(x_1,x_2) \in \mathbb{R}_+^2$, where $k > 0$ and $\alpha \in [0,1]$.

Assume that our optimization problem involves a Cobb-Douglas function. Let $a' = \frac{p_1}{p_2}$, $b = \frac{m}{p_2}$ and $a = \frac{b}{a'}$. The consumer choice problem becomes

$$\begin{aligned} \text{maximize} \ \ & ka'^{d}x_1^{\alpha}\,(-x_1 + a)^{1-\alpha} \\ \text{with the constraint} \ \ & x_1 \in [0, a]. \end{aligned}$$

In many applied problems the goods that a consumer purchases are better described by random variables than by constants since they are essentially random. This fact usually appears when some good depends on a random variable that represents a characteristic of that good. Clear examples arise when an investor buys all the harvest of a season, that depends on the humidity, the land, the rain, the temperature, etc., and so the final harvest is random instead of deterministic. The total amount of a good that an investor can purchase frequently depends on stocks, evolution of financial markets, climatology, political scene, foreign exchange rates, etc.

Under this framework, how could one compare investments? That is the main aim of this manuscript. Namely, we propose a mathematical method to compare investments in a two-good consumer choice problem with the above Cobb-Douglas utility functions, when the amounts of goods involved in the problem are subject to randomness.

The structure of the paper is as follows. In Section 2 we collect the concepts and results needed for our analysis. In Section 3 we introduce a new stochastic order to approach the problem described above. In Section 4 we develop some characterizations of the order, conditions which lead to the new stochastic order, and consequences of the order. Relevant properties of the order are studied in Section 5. To conclude we analyze the antisymmetric property of the new stochastic order in Section 6, providing a general family of distributions in which that property is satisfied.

2. Preliminaries

The mathematical concepts, notations and results needed for the analysis are included in this section.

Stochastic orders are pre-order relations on sets of probabilities. Basically, a stochastic order tries to rank probabilities in accordance with an appropriate criterion. In [11], [13] and [3], the reader can find a clear and comprehensive introduction to the theory of stochastic orderings.

Given a random variable X, F_X will denote its distribution function, $E(X)$ its expected value, P_X its induced probability, and if X is continuous, f_X will stand for a density mapping of X.

The integrated survival function of a random variable X with finite mean is the mapping $\pi_X : \mathbb{R} \to \mathbb{R}$, with $\pi_X(t) = E(X-t)_+$ for any $t \in \mathbb{R}$.

Let $\preceq$ denote a stochastic order on the set of probabilities on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, where $\mathscr{B}_{\mathbb{R}}$ denotes the usual Borel σ-algebra on $\mathbb{R}$. Let X and Y be random variables, $X \preceq Y$ will mean that $P_X \preceq P_Y$. Thus univariate stochastic orderings are sometimes introduced by means of random variables.

The following stochastic orderings will appear throughout the paper. Let X and Y be two random variables, then

i) X is said to be smaller than Y in the concave order if $E(f(X)) \leqslant E(f(Y))$ for all concave mappings $f : \mathbb{R} \to \mathbb{R}$ such that the above expectations exist. It will be denoted by $X \preceq_{cv} Y$,

ii) X is said to be smaller than Y in the convex order if $E(f(X)) \leqslant E(f(Y))$ for all convex mappings $f : \mathbb{R} \to \mathbb{R}$ such that the above expectations exist. It will be denoted by $X \preceq_{cx} Y$,

iii) X is said to be smaller than Y in the increasing convex order if $E(f(X)) \leqslant E(f(Y))$ for all increasing convex mappings $f : \mathbb{R} \to \mathbb{R}$ such that the above expectations exist. It will be denoted by $X \preceq_{icx} Y$.

A stochastic order $\preceq$ is said to be integral, when there exists a set $\mathscr{F}$ of real measurable mappings, such that two probabilities P_1 and P_2 satisfy

$$P_1 \preceq P_2 \quad \text{when} \quad \int f\,dP_1 \leqslant \int f\,dP_2$$

for any $f \in \mathscr{F}$ for which the above expectations exist. The set of mappings $\mathscr{F}$ is said to be a generator of the order. The reader is referred to [10] and Chapter 2 of [11] for a precise analysis of integral stochastic orders.

Let P be a probability on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, and $T : \mathbb{R} \to \mathbb{R}$ be a measurable mapping, $P \circ T^{-1}$ will denote the probability on $\mathscr{B}_{\mathbb{R}}$ given by $P \circ T^{-1}(B) = P(T^{-1}(B))$ for any $B \in \mathscr{B}_{\mathbb{R}}$.

Let $(\Omega, \mathscr{A})$ be a measurable space. Let $\mu, \nu : \mathscr{A} \to \mathbb{R}$ be σ-finite measures, $\mu \ll \nu$ will mean that μ is absolutely continuous with respect to ν, that is, $\mu(A) = 0$ for any $A \in \mathscr{A}$ such that $\nu(A) = 0$. In that case, $\frac{d\mu}{d\nu}$ will stand for a Radon-Nikodym derivative of μ with respect to ν (see for instance [12] or [2]).

We will denote by $\mathscr{L}^1(\mu)$ the set of mappings $\{f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \int_\Omega |f|\,d\mu < +\infty\}$.

The usual Borel measure on the real line will be denoted by θ.

If A is a subset of $\mathbb{R}$, I_A will stand for the indicator function of A.

The beta function will be denoted by B, that is, $B:(0,\infty)\times(0,\infty)\to\mathbb{R}$, with $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ for any $(x,y)\in(0,\infty)\times(0,\infty)$. On the other hand, Γ will stand for the gamma mapping given by $\Gamma:(0,+\infty)\to\mathbb{R}$, with $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ for any $x\in(0,+\infty)$.

When A belongs to $\mathscr{B}_{\mathbb{R}}$, $\mathscr{B}_A$ will stand for the inherited Borel σ-algebra on A.

For ease of reading of subsequent results, the following proposition on the differentiability under the integral sign is included here. It is an immediate consequence of Theorem 9.2 in [9].

PROPOSITION 1. *Let $(\Omega,\mathscr{A},\mu)$ be a measure space. Let $N\in\mathscr{A}$ be a set with $\mu(N)=0$. Let $I\subset\mathbb{R}$ be an open interval. Let $F:I\times\Omega\to\mathbb{R}$ be a function satisfying that*

i) for any $x\in\Omega\setminus N$, the mapping $F(\cdot,x)$ is differentiable on I,

ii) for any $t\in I$, the mapping $F(t,\cdot)$ is measurable,

iii) there is a mapping $g\in\mathscr{L}^1(\mu)$ with $|\frac{d}{dt}F(t,x)|\leqslant g(x)$ for each $x\in\Omega\setminus N$ and $t\in I$,

iv) there exists $t_0\in I$ such that $F(t_0,\cdot)\in\mathscr{L}^1(\mu)$.

Then $F(t,\cdot)\in\mathscr{L}^1(\mu)$ for any $t\in I$, and the mapping $f:I\to\mathbb{R}$ given by

$$f(t)=\int_\Omega F(t,\cdot)\,d\mu \text{ is differentiable on } I \text{ with } f'(t)=\int_\Omega \frac{d}{dt}F(t,\cdot)\,d\mu$$

for any $t\in I$.

3. The Cobb-Douglas stochastic order

A mathematical model to approach the problem described in Section 1 is proposed in this section. That model is based on a stochastic order.

In the first place we introduce the following families of probabilities and random variables.

Let $a>0$. Let us denote by $\mathbb{P}^a$ the set of probabilities associated with the measurable space $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$, such that $P([0,a])=1$.

Let $\mathbb{X}^a$ be the set of random variables whose induced probabilities belong to the class $\mathbb{P}^a$.

DEFINITION 1. Let $a>0$. The family of mappings

$$\mathscr{F}_a=\{\,f:[0,a]\to\mathbb{R} \text{ with } f(x)=kx^\alpha(-x+a)^{1-\alpha}\mid k>0,\ \alpha\in[0,1]\,\}$$

is said to be the Cobb-Douglas family.

The mappings of the family $\mathscr{F}_a$ are concave.

From now on and given $a > 0$, for any $\alpha \in [0,1]$ we will denote by f_α the function $f_\alpha : [0,a] \to \mathbb{R}$, with $f_\alpha(x) = x^\alpha(-x+a)^{1-\alpha}$ for any $x \in [0,a]$.

DEFINITION 2. Let $a > 0$ and $X, Y \in \mathbb{X}^a$. It will be said that X is less than Y in the Cobb-Douglas order if $E(f(X)) \leqslant E(f(Y))$ for any $f \in \mathscr{F}_a$. This relation will be denoted by $X \preceq_{CD} Y$.

Note that all mappings of $\mathscr{F}_a$ are bounded, thus the above expectations are finite. Therefore the relation $\preceq_{CD}$ is a pre-order.

Let us clarify the meaning of the order. Let X and Y stand for the random variables associated with the units of a good in our framework, for instance, the harvest of a cereal in two regions. Assume that an investor aims to buy one of them. How could he compare both investments? The relation $X \preceq_{CD} Y$ means that whatever Cobb-Douglas utility function of $\mathscr{F}_a$ we consider, the expected utility is greater, or at least not lower, when the number of units of the good is that associated with random variable Y. Thus in the above example the investor should acquire the harvest associated with random variable Y.

4. Necessary and/or sufficient conditions for the Cobb-Douglas order

In this section we develop some characterization results of the Cobb-Douglas order. These results connect the new order with a special family of beta distributions. Some inequalities for those beta distributions will be proved. Moreover, we obtain conditions implied by the new order, and conditions which lead to the new stochastic order.

In the first place we introduce a useful function for the comparison of random variables by means of the Cobb-Douglas order.

DEFINITION 3. Let $a > 0$ and $X \in \mathbb{X}^a$. The mapping $\phi_X : [0,1] \to \mathbb{R}$, with $\phi_X(\alpha) = \int_{[0,a]} f_\alpha dP_X$ for any $\alpha \in [0,1]$, will be said to be the discriminant Cobb-Douglas function of X.

Note that ϕ_X is well defined since f_α belongs to $\mathscr{L}^1(P_X)$ for any $\alpha \in [0,1]$. Observe that for any $X, Y \in \mathbb{X}^a$, $X \preceq_{CD} Y$ if and only if $\phi_X \leqslant \phi_Y$.

Characterizations of the order are proved below.

PROPOSITION 2. *Let $a > 0$ and $X, Y \in \mathbb{X}^a$. Then $X \preceq_{CD} Y$ if and only if*

$$E\big(f_{B(\alpha+1,2-\alpha)}(\frac{X}{a})\big) \leqslant E\big(f_{B(\alpha+1,2-\alpha)}(\frac{Y}{a})\big)$$

for any $\alpha \in [0,1]$, where $f_{B(\alpha+1,2-\alpha)}$ stands for the density mapping of a beta distribution with parameters $\alpha+1$ and $2-\alpha$.

Proof. We have that $X \preceq_{CD} Y$ if and only if $E(f(X)) \leqslant E(f(Y))$ for any $f \in \mathscr{F}_a$, that is, if and only if for any $\alpha \in [0,1]$

$$\int_{[0,a]} x^{\alpha}(-x+a)^{1-\alpha}\, dP_X \leqslant \int_{[0,a]} x^{\alpha}(-x+a)^{1-\alpha}\, dP_Y.$$

By a change of variable (see for instance [7]) given by the mapping $T : \mathbb{R} \to \mathbb{R}$, with $T(x) = x/a$, we have that

$$\int_{[0,a]} x^{\alpha}(-x+a)^{1-\alpha}\, dP_X = \int_{[0,1]} ax^{\alpha}(1-x)^{1-\alpha}\, dP_X \circ T^{-1}.$$

Thus $X \preceq_{CD} Y$ if and only if

$$\int_{[0,1]} ax^{\alpha}(1-x)^{1-\alpha}\, dP_{\frac{X}{a}} \leqslant \int_{[0,1]} ax^{\alpha}(1-x)^{1-\alpha}\, dP_{\frac{Y}{a}}$$

for any $\alpha \in [0,1]$. Observe that this is the same as

$$\int_{[0,1]} \frac{x^{\alpha}(1-x)^{1-\alpha}}{B(\alpha+1, 2-\alpha)}\, dP_{\frac{X}{a}} \leqslant \int_{[0,1]} \frac{x^{\alpha}(1-x)^{1-\alpha}}{B(\alpha+1, 2-\alpha)}\, dP_{\frac{Y}{a}}$$

for any $\alpha \in [0,1]$, which concludes the proof. $\square$

PROPOSITION 3. *Let $a > 0$ and $X, Y \in \mathbb{X}^a$ continuous random variables with densities f_X and f_Y respectively. We have that $X \preceq_{CD} Y$ if and only if $E(f_X(aZ_\alpha)) \leqslant E(f_Y(aZ_\alpha))$ for any $\alpha \in [0,1]$, where Z_α stands for a beta random variable with parameters $\alpha+1$ and $2-\alpha$.*

Proof. Since X and Y are continuous so are X/a and Y/a. Moreover $af_X(ax)$ and $af_Y(ax)$ are densities of X/a and Y/a respectively. Thus

$$E(f_{B(\alpha+1,2-\alpha)}(\frac{X}{a})) = \int_{[0,1]} \frac{x^{\alpha}(1-x)^{1-\alpha}}{B(\alpha+1, 2-\alpha)}\, af_X(ax)\, dx = aE(f_X(aZ_\alpha)),$$

which proves the proposition. $\square$

We state another characterization of the Cobb-Douglas order for continuous random variables. First, we state the following technical lemma.

LEMMA 1. *Let X be a continuous random variable. Then $P_X \circ F_X^{-1}$ is equal to θ on $\mathscr{B}_{[0,1]}$.*

Proof. Let $x \in [0,1]$, then $P_X \circ F_X^{-1}([0,x]) = P(F_X(X) \in [0,x]) = x = \theta([0,x])$ since $F_X(X)$ follows uniform distribution on the interval $[0,1]$.

Since the class $\{\,[0,x] \mid x \in [0,1]\,\}$ is a π-system which generates $\mathscr{B}_{[0,1]}$, we obtain the result (see for instance Theorem 3.3 in [4]). $\square$

PROPOSITION 4. *Let $a>0$ and $X,Y \in \mathbb{X}^a$ continuous random variables with densities f_X and f_Y respectively. Then $X \preceq_{CD} Y$ if and only if*

$$\int_{[0,a]} F_X(x)^\alpha (1-F_X(x))^{1-\alpha} f_X(aF_X(x)) f_X(x)\, d\theta$$
$$\leqslant \int_{[0,a]} F_Y(x)^\alpha (1-F_Y(x))^{1-\alpha} f_Y(aF_Y(x)) f_Y(x)\, d\theta$$

for any $\alpha \in [0,1]$.

Proof. Note that for any $\alpha \in [0,1]$ we have that

$$\int_{[0,a]} x^\alpha (a-x)^{1-\alpha}\, dP_X = \int_{[0,a]} x^\alpha (a-x)^{1-\alpha} f_X(x)\, d\theta$$
$$=a^2 \int_{[0,1]} x^\alpha (1-x)^{1-\alpha} f_X(ax)\, d\theta = a^2 \int_{[0,1]} x^\alpha (1-x)^{1-\alpha} f_X(ax)\, dP_X \circ F_X^{-1},$$

where the last inequality follows from Lemma 1. Observe that the last expression is the same as

$$a^2 \int_{[0,a]} F_X(x)^\alpha (1-F_X(x))^{1-\alpha} f_X(aF_X(x))\, dP_X$$
$$=a^2 \int_{[0,a]} F_X(x)^\alpha (1-F_X(x))^{1-\alpha} f_X(aF_X(x)) f_X(x)\, d\theta,$$

which concludes the proof. $\square$

By means of the above results we can derive some inequalities in relation to beta distributions.

Let $f: I \to \mathbb{R}$, with I an open interval of $\mathbb{R}$. The number of sign changes of f in I is defined by $S^-(f) = \sup_{\{x_1<\ldots<x_n, x_i\in I, n\in\mathbb{N}\}} S^-(f(x_1),\ldots,f(x_n))$, where $S^-(y_1,\ldots,y_n)$ is the number of sign changes of the sequence $y_1,\ldots,y_n$, where zero values are discarded.

PROPOSITION 5. *Let $a>0$ and $X,Y \in \mathbb{X}^a$ continuous random variables with the same mean. If $S^-(f_X - f_Y) = 2$ and the sequence of signs is $+,-,+$, then $E(f_X(aZ_\alpha)) \leqslant E(f_Y(aZ_\alpha))$ for any $\alpha \in [0,1]$.*

Proof. Theorem 3.A.44 in [13] reads that under the above conditions $Y \preceq_{cx} X$ holds. This is equivalent to $X \preceq_{cv} Y$. Note that all the mappings of $\mathscr{F}_a$ are concave, which implies that $X \preceq_{CD} Y$. Now the result is a consequence of Proposition 3. $\square$

We study some conditions implied by the Cobb-Douglas order and some which imply that order. For that purpose we prove the following lemma.

LEMMA 2. *Let $a>0$ and $X \in \mathbb{X}^a$ with $P(X\in(0,a))=1$. The discriminant Cobb-Douglas function ϕ_X is infinitely derivable on $(0,1)$, with $\phi_X^{(n}(\alpha) = \int_{(0,a)} x^\alpha (a-x)^{1-\alpha} \ln^n(\frac{x}{a-x})\, dP_X$ for any $\alpha \in (0,1)$ and any $n \in \mathbb{N}$.*

Proof. We prove the result by induction. Let us begin with the case $n = 1$.

Consider the probability space $((0,a), \mathscr{B}_{(0,a)}, P_X)$ and the open interval $I = (t_1, t_2)$, with $0 < t_1 < t_2 < 1$. Let $F : I \times (0,a) \to \mathbb{R}$, with $F(\alpha, x) = f_\alpha(x)$. Observe that $\phi_X(\alpha) = \int_{(0,a)} F(\alpha,x)\, dP_X$.

We will use Proposition 1 to obtain the first derivative of ϕ_X. Note that conditions $i), ii)$ and $iv)$ in that result are trivially satisfied.

Let us see condition $iii)$. Given any $\alpha \in (0,1)$, by means of L'Hôpital's rule we conclude that

$$\lim_{x \to 0^+} x^\alpha \ln x = 0 \quad \text{and} \quad \lim_{x \to a^-} (a-x)^{1-\alpha} \ln(a-x) = 0.$$

Therefore we have that $x^{t_1} \ln x$ and $(a-x)^{1-t_2} \ln(a-x)$ are bounded on $(0,a)$. So if $\alpha \in (t_1, t_2)$, it holds that $|x^\alpha \ln x| \leqslant \max\{|x^{t_1} \ln x|, |a \ln a|\} \leqslant K_1$ and $|(a-x)^{1-\alpha} \ln(a-x)| \leqslant \max\{|(a-x)^{1-t_2} \ln(a-x)|, |a \ln a|\} \leqslant K_2$ for some constants K_1 and K_2.

As a consequence, for any $\alpha \in (t_1, t_2)$

$$|\frac{\partial F}{\partial \alpha}(\alpha,x)| = |x^\alpha (a-x)^{1-\alpha} \ln(\frac{x}{a-x})| = |x^\alpha (a-x)^{1-\alpha} \ln x - x^\alpha (a-x)^{1-\alpha} \ln(a-x)|$$
$$\leqslant K_1 |(a-x)^{1-\alpha}| + K_2 |x^\alpha| \leqslant (K_1 + K_2) \max\{a, 1\}.$$

Thus condition $iii)$ in Proposition 1 is also satisfied. Therefore ϕ_X is derivable on (t_1, t_2) with $\phi_X'(\alpha) = \int_{(0,a)} x^\alpha (a-x)^{1-\alpha} \ln(\frac{x}{a-x})\, dP_X$. Since t_1 and t_2 are arbitrary values with $0 < t_1 < t_2 < 1$, we obtain the case $n = 1$.

Assume that the result is true for a natural number n, so $\phi_X^{(n}(\alpha) = \int_{(0,a)} x^\alpha (a-x)^{1-\alpha} \ln^n(\frac{x}{a-x})\, dP_X$. In order to apply Proposition 1 to obtain the differentiability of $\phi_X^{(n}$, and in relation to condition $iii)$, note that

$$x^\alpha (a-x)^{1-\alpha} \ln^{n+1}(\frac{x}{a-x}) = \left(x^{\frac{\alpha}{n+1}} (a-x)^{\frac{1-\alpha}{n+1}} \ln(\frac{x}{a-x})\right)^{n+1}.$$

By the previous inequalities we obtain that

$$|\frac{\partial^{n+1} F}{\partial \alpha^{n+1}}(\alpha,x)| \leqslant \left((K_1' + K_2') \max\{a,1\}\right)^{n+1}$$

for some constants K_1' and K_2'. Therefore condition $iii)$ is satisfied. Moreover, conditions $i)$ and $ii)$ of Proposition 1 are also held, and $iv)$ is a consequence of the above inequality. Thus we obtain that $\phi_X^{(n+1}(\alpha) = \int_{(0,a)} x^\alpha (a-x)^{1-\alpha} \ln^{n+1}(\frac{x}{a-x})\, dP_X$, which proves the lemma. □

PROPOSITION 6. *Let $a > 0$ and $X, Y \in \mathbb{X}^a$ with $P(X \in (0,a)) = P(Y \in (0,a)) = 1$, such that $X \preceq_{CD} Y$. Let $D = \{\alpha \in (0,1) \mid \phi_X(\alpha) = \phi_Y(\alpha)\}$. Then for every $d \in D$ we have that $\phi_X'(d) = \phi_Y'(d)$, and so $E(X^d (a-X)^{1-d} \ln(\frac{X}{a-X})) = E(Y^d (a-Y)^{1-d} \ln(\frac{Y}{a-Y}))$.*

Proof. We define the function $W:[0,1]\to\mathbb{R}$, with $W(\alpha)=\phi_Y(\alpha)-\phi_X(\alpha)$ for any $\alpha\in[0,1]$. By Lemma 2 that mapping is infinitely derivable on $(0,1)$, with $W'(\alpha)=E(Y^\alpha(a-Y)^{1-\alpha}\ln(\frac{Y}{a-Y}))-E(X^\alpha(a-X)^{1-\alpha}\ln(\frac{X}{a-X}))$. Since $X\preceq_{CD}Y$ we have that $W(\alpha)\geqslant 0$. Let $d\in D$, it holds that $W(d)=0$, so d is a local minimum. Therefore $W'(d)=\phi_Y'(d)-\phi_X'(d)=0$, which implies the result. □

A sufficient condition for the Cobb-Douglas order is analyzed in the following proposition.

PROPOSITION 7. *Let $a>0$ and $X,Y\in\mathbb{X}^a$ with $P(X\in(0,a))=P(Y\in(0,1))=1$ and $E(X)=E(Y)$. Let $D=\{\alpha\in(0,1)\mid\phi_X(\alpha)=\phi_Y(\alpha)\}$. If one of the following conditions is satisfied*

i) $D=\emptyset$ *and there exists* $d\in(0,1)$ *with* $\phi_X(d)<\phi_Y(d)$,

ii) $D\neq\emptyset$ *and for all* $d\in D$ *it holds that*

$$E\big(X^d(a-X)^{1-d}\ln(\frac{X}{a-X})\big)=E\big(Y^d(a-Y)^{1-d}\ln(\frac{Y}{a-Y})\big)$$

and

$$E\big(X^d(a-X)^{1-d}\ln^2(\frac{X}{a-X})\big)<E\big(Y^d(a-Y)^{1-d}\ln^2(\frac{Y}{a-Y})\big),$$

then we have that $X\preceq_{CD}Y$.

Proof. Consider the mapping $W:[0,1]\to\mathbb{R}$ with $W(\alpha)=\phi_Y(\alpha)-\phi_X(\alpha)$. Lemma 2 says that W is infinitely derivable on $(0,1)$, with $W^{(n}(\alpha)=E(Y^\alpha(a-Y)^{1-\alpha}\ln^n(\frac{Y}{a-Y}))$ $-E(X^\alpha(a-X)^{1-\alpha}\ln^n(\frac{X}{a-X}))$.

Let us study case *i*). Note that if $E(X)=E(Y)$ then $W(0)=W(1)=0$. Since $D=\emptyset$ and W is continuous on $(0,1)$, W is either positive or negative. We know that there exists $d\in(0,1)$ with $\phi_X(d)<\phi_Y(d)$, so $W>0$ on $(0,1)$. Thus for every $d\in(0,1)$ we have $\phi_X(d)<\phi_Y(d)$, therefore $X\preceq_{CD}Y$.

Let us analyze case *ii*). Suppose that the relation $X\preceq_{CD}Y$ is false. So there exists α with $W(\alpha)<0$. Since $E(X)=E(Y)$, such a value α should be in $(0,1)$. We know that $D\neq\emptyset$. Thus we can take $d_0\in D$ with $W(d_0)=0$. Suppose that $d_0>\alpha$, the case $d_0<\alpha$ is analogous. Let us define $I=\{d\in D\mid d>\alpha\}$. The set I is bounded and non-empty, so it has an infimum. The continuity of W on $(0,1)$ implies that such an infimum is a minimum on I. Let us denote it by $\underline{m}$. Using condition *ii*) we obtain that $\underline{m}$ satisfies that $W(\underline{m})=0$, $W'(\underline{m})=0$ and $W''(\underline{m})>0$.

Therefore $\underline{m}$ is also a local minimum of W on $(0,1)$. Thus there exists m' satisfying that $\alpha<m'<\underline{m}$ and $W(m')\geqslant W(\underline{m})=0$. We know that $W(\alpha)<0$, therefore there exists $m''\in[\alpha,m']$ with $W(m'')=0$. Thus $\underline{m}$ is not the minimum on I which is a contradiction. Therefore $X\preceq_{CD}Y$. □

The following result shows an approximation condition which leads to the Cobb-Douglas order.

Let $a>0$, $l_1,l_2,\ldots,l_n\in(0,a)$, with $0<l_1<l_2<\ldots<l_n<a$, and let $\alpha\in(0,1)$. Let $g^{\alpha,l_1,\ldots,l_n}:[0,a]\to\mathbb{R}$ be the mapping defined as $g^{\alpha,l_1,\ldots,l_n}(x)$

$$=\begin{cases} r[(0,0),(l_1,l_1^{\alpha}(-l_1+a)^{1-\alpha})](x) & \text{if } x\in[0,l_1],\\ r[(l_i,l_i^{\alpha}(-l_i+a)^{1-\alpha}),(l_{i+1},l_{i+1}^{\alpha}(-l_{i+1}+a)^{1-\alpha})](x) & \text{if } x\in(l_i,l_{i+1}],\\ & 1\leqslant i\leqslant n-1,\\ r[(l_n,l_n^{\alpha}(-l_n+a)^{1-\alpha}),(a,0)](x) & \text{if } x\in(l_n,a], \end{cases}$$

where $r[(x_1,y_1),(x_2,y_2)]$ stands for the equation of the line passing through the points (x_1,y_1) and (x_2,y_2).

PROPOSITION 8. *Let $a>0$ and let $X,Y\in\mathbb{X}^a$ with $E(X)=E(Y)$, such that $E\big(g^{\alpha,l_1,\ldots,l_n}(X)\big)\leqslant E\big(g^{\alpha,l_1,\ldots,l_n}(Y)\big)$ for any $\alpha\in(0,1)$, $n\in\mathbb{N}$ and $l_1,l_2,\ldots,l_n\in(0,a)$, with $0<l_1<l_2<\ldots<l_n<a$. Then $X\preceq_{CD}Y$.*

Proof. Consider the mappings f_α with $\alpha\in(0,1)$. Let D be a countable dense subset of $(0,a)$, say $D=\{a_1,a_2,\ldots\}$.

Let us consider the class of mappings $\{g^{\alpha,a_i}(x)\}_{a_i\in D}$.

Since f_α is concave, $g^{\alpha,a_i}(x)\leqslant f_\alpha(x)$ for any $x\in[0,a]$. Because of the density of D and the continuity of f_α, we obtain that for any $x\in[0,a]$, $f_\alpha(x)=\sup\{g^{\alpha,a_i}(x)\mid a_i\in(0,a), i\in\mathbb{N}\}$.

Let $f_{n,\alpha}:[0,a]\to\mathbb{R}$ given by $f_{n,\alpha}(x)=\max\{g^{\alpha,a_1}(x),\ldots,g^{\alpha,a_n}(x)\}$ for any $x\in[0,a]$.

The sequence $\{f_{n,\alpha}\}_n$ is increasing for any $\alpha\in(0,1)$. Moreover $f_{n,\alpha}(x)\leqslant f_\alpha(x)$ and $\lim_n f_{n,\alpha}(x)=f_\alpha(x)$ for any $x\in[0,a]$ and $\alpha\in(0,1)$.

By means of the Monotone Convergence Theorem we obtain that

$$\lim_{n\to\infty}\int_{[0,a]} f_{n,\alpha}\,dP_X=\int_{[0,a]} f_\alpha\,dP_X.$$

Let $a_{(1)},\ldots,a_{(n)}$ stand for the arrangement of $a_1,\ldots,a_n$ in increasing order. We should note that $f_{n,\alpha}(x)\leqslant g^{\alpha,a_{(1)},\ldots,a_{(n)}}(x)\leqslant f_\alpha(x)$ for any $x\in[0,a]$, $n\in\mathbb{N}$ and $\alpha\in(0,1)$. As a consequence $\lim_{n\to\infty}E(g^{\alpha,a_{(1)},\ldots,a_{(n)}}(X))=E(f_\alpha(X))$.

Thus we have seen that $E(f_\alpha(X))\leqslant E(f_\alpha(Y))$ for any $\alpha\in(0,1)$. The above inequality is also held when we consider $\alpha=0$ and $\alpha=1$ since X and Y have the same expected value. Therefore $X\preceq_{CD}Y$. □

5. Relevant properties of the order

Some properties of the Cobb-Douglas order are analyzed in this section.

The first results say that the Cobb-Douglas order implies the equality of expected values, and that order is preserved under weak convergence and mixtures.

PROPOSITION 9. *Let $a>0$ and $X,Y\in\mathbb{X}^a$ such that $X\preceq_{CD}Y$. Then $E(X)=E(Y)$.*

Proof. Take f_0 and f_1 in $\mathscr{F}_a$. Since $X \preceq_{CD} Y$ we obtain that $E(X) \leqslant E(Y)$ and $E(a-X) \leqslant E(a-Y)$, and so the result. $\square$

PROPOSITION 10. *The Cobb-Douglas order is closed with respect to weak convergence.*

Proof. Note that all the mappings of $\mathscr{F}_a$ are bounded and continuous. $\square$

PROPOSITION 11. *The Cobb-Douglas order is closed with respect to mixtures.*

Proof. The Cobb-Douglas order is integral which implies the result. $\square$

Using the budgetary constraint of our problem, we have written utility functions based on the first good since $x_2 = -x_1 + a$. If we refer to the second good, the utility function would be written as $k(-x_2+a)^{\alpha} x_2^{1-\alpha}$.

Now we prove that the Cobb-Douglas order is preserved if we consider the second good. From an economic point of view, the following result means that the consumer choice does not depend on the good studied by the order.

PROPOSITION 12. *Let $a > 0$ and $X, Y \in \mathbb{X}^a$. Then $-X+a, -Y+a \in \mathbb{X}^a$, and $X \preceq_{CD} Y$ if and only if $-X+a \preceq_{CD} -Y+a$.*

Proof. It is clear that $-X+a$ and $-Y+a$ belong to the set $\mathbb{X}^a$.

On the other hand, by means of Proposition 2, $-X+a \preceq_{CD} -Y+a$ if and only if

$$E\big(f_{B(\alpha+1,2-\alpha)}(\frac{-X+a}{a})\big) \leqslant E\big(f_{B(\alpha+1,2-\alpha)}(\frac{-Y+a}{a})\big)$$

for any $\alpha \in [0,1]$. We should observe that

$$f_{B(\alpha+1,2-\alpha)}(\frac{-X+a}{a}) = \frac{1}{B(\alpha+1,2-\alpha)}(1-\frac{X}{a})^{\alpha}(\frac{X}{a})^{1-\alpha} = f_{B(2-\alpha,\alpha+1)}(\frac{X}{a})$$

because of the symmetry of the beta mapping in its arguments. Therefore $-X+a \preceq_{CD} -Y+a$ if and only if

$$E\big(f_{B(2-\alpha,\alpha+1)}(\frac{X}{a})\big) \leqslant E\big(f_{B(2-\alpha,\alpha+1)}(\frac{Y}{a})\big)$$

for any $\alpha \in [0,1]$. Now observe that $\{(\alpha+1, 2-\alpha) \mid \alpha \in [0,1]\} = \{(2-\alpha, \alpha+1) \mid \alpha \in [0,1]\}$, which implies the result applying Proposition 2. $\square$

PROPOSITION 13. *Let $\lambda, a > 0$ and $X, Y \in \mathbb{X}^a$ such that $X \preceq_{CD} Y$. Then $\lambda X, \lambda Y \in \mathbb{X}^{\lambda a}$, and $X \preceq_{CD} Y$ if and only if $\lambda X \preceq_{CD} \lambda Y$.*

Proof. The first assertion is clear. In relation to the condition $\lambda X \preceq_{CD} \lambda Y$, by Proposition 2 that is held if and only if

$$E\big(f_{B(\alpha+1,2-\alpha)}(\frac{\lambda X}{\lambda a})\big) \leqslant E\big(f_{B(\alpha+1,2-\alpha)}(\frac{\lambda Y}{\lambda a})\big)$$

for any $\alpha \in [0,1]$, which is equivalent to $X \preceq_{CD} Y$. $\square$

The mappings of $\mathscr{F}_a$ are concave, does the Cobb-Douglas order weaken the concave order? As we prove below, the answer is affirmative. Of course we consider probabilities whose support is in the interval $[0,a]$, with $a>0$.

PROPOSITION 14. *The concave order implies the Cobb-Douglas stochastic order, but the converse is not true.*

Proof. It is clear that the concave order implies the Cobb-Douglas order since the mappings of $\mathscr{F}_a$ are concave for any $a>0$.

Let us see that the converse is false.

Take $a=1$ and let $X,Y \in \mathbb{X}^1$ discrete random variables whose supports are $S_X = \{0.25, 0.75\}$ and $S_Y = \{0.2, 0.575\}$ respectively, and with probability mass functions given by $P(X=0.25)=P(X=0.75)=0.5$, and $P(Y=0.2)=0.2$ and $P(Y=0.575)=0.8$.

In the first place we check that $X \preceq_{cv} Y$ is false. Observe that the relation $X \preceq_{cv} Y$ is the same as $Y \preceq_{cx} X$, and this implies $Y \preceq_{icx} X$.

Let us consider the integrated survival mappings π_X and π_Y of X and Y respectively. It can be seen that $\pi_Y(0.25) = 0.26 > 0.25 = \pi_X(0.25)$. Therefore the condition $\pi_Y(t) \leqslant \pi_X(t)$ for any $t \in \mathbb{R}$ is not held, condition which is equivalent to $Y \preceq_{icx} X$ (see for instance Theorem 1.5.13 in [11]). Thus $X \preceq_{cv} Y$ is false.

Consider the discriminant Cobb-Douglas functions ϕ_X and ϕ_Y. Note that by Lemma 2 these mappings are continuous on $(0,1)$. The right continuity at 0 and the left continuity at 1 are trivial in this case. Moreover $\phi_X(0) = \phi_X(1) = E(X) = 0.5 = E(Y) = \phi_Y(1) = \phi_Y(0)$.

The derivatives of ϕ_X and ϕ_Y are $\phi_X'(\alpha) = E(h_\alpha(X))$ and $\phi_Y'(\alpha) = E(h_\alpha(Y))$, with $h_\alpha(x) = x^\alpha(1-x)^{1-\alpha}\ln(\frac{x}{1-x})$ for any $\alpha \in (0,1)$ (see Lemma 2). Observe that those expected values exist since X and Y do not induce probability in the points 0 and 1.

By Lemma 2 we obtain that $\phi_X''(\alpha), \phi_Y''(\alpha) \geqslant 0$ for any $\alpha \in (0,1)$, and thus ϕ_X and ϕ_Y are convex.

Moreover, the right and left hand derivatives of ϕ_X and ϕ_Y at 0 and at 1 satisfy that $\phi_X^{'+}(0) = -0.2746 < -0.1190 = \phi_Y^{'+}(0)$ and $\phi_X^{'-}(1) = 0.2746 > 0.0835 = \phi_Y^{'-}(1)$.

Therefore there are neighborhoods of 0 and 1 (in the interval $[0,1]$) in which $\phi_X(\alpha) \leqslant \phi_Y(\alpha)$ for any α in both sets.

Since ϕ_Y is convex, the mapping ϕ_Y' is increasing and so $|\phi_Y'(\alpha)| < 0.12$ for any $\alpha \in (0,1)$.

By means of this inequality let us see that $\phi_Y(\alpha) > 0.47$ for any $\alpha \in [0,1]$. It holds that $\phi_Y(0.44) > \phi_Y(0.45)$ and $\phi_Y(0.45) < \phi_Y(0.47)$, those values greater than 0.47.

The convexity of ϕ_Y implies that its minimum is reached when $\alpha \in (0.44, 0.47)$. The Mean Value Theorem reads that if $\alpha \in (0.44, 0.47)$, we have that $\phi_Y(\alpha) = \phi_Y(0.44) + \phi_Y'(\beta_\alpha)(\alpha - 0.44)$ for some $\beta_\alpha \in (0.44, 0.47)$, and therefore $\phi_Y(\alpha) \geqslant \phi_Y(0.44) - 0.12(0.47 - 0.44) = 0.47171 > 0.47$, as we have affirmed before.

Now observe that $\phi_X(0.13) = \phi_X(0.87) < 0.47$. So the convexity of ϕ_X implies that $\phi_X(\alpha) \leqslant \phi_Y(\alpha)$ for any $\alpha \in [0.13, 0.87]$.

To conclude, note that for any $\alpha \in (0,1)$ we have $\phi_X'(0.13) = -0.1987 < -0.12 \leqslant \phi_Y'(\alpha)$ and $\phi_X'(0.87) = 0.1987 > 0.12 \geqslant \phi_Y'(\alpha)$.

Using the convexity of ϕ_X, for any $\alpha_1, \alpha_2 \in [0, 0.13)$ it holds that $\phi_X'(\alpha_1) < \phi_Y'(\alpha_2)$. In the same way, for any $\alpha_1, \alpha_2 \in (0.87, 1]$, $\phi_X'(\alpha_1) > \phi_Y'(\alpha_2)$.

The Mean Value Theorem implies that for any $\alpha \in (0, 0.13)$ we have that $\phi_X(\alpha) = \phi_X(0) + \phi_X'(\gamma_\alpha)\alpha$ for some $\gamma_\alpha \in (0, \alpha)$, and $\phi_Y(\alpha) = \phi_Y(0) + \phi_Y'(\zeta_\alpha)\alpha$ for some $\zeta_\alpha \in (0, \alpha)$. Since $\phi_X'(\gamma_\alpha) < \phi_Y'(\zeta_\alpha)$, we obtain that $\phi_X(\alpha) < \phi_Y(\alpha)$ for any $\alpha \in (0, 0.13)$.

In the same way it is possible to prove that $\phi_X(\alpha) < \phi_Y(\alpha)$ for any $\alpha \in (0.87, 1)$.

Thus we have obtained that $\phi_X \leqslant \phi_Y$, that is, $X \preceq_{CD} Y$. Therefore the Cobb-Douglas order weakens the concave order. □

PROPOSITION 15. *Let $a > 0$. Let $\mathscr{F}_a^{cv} = \{f : [0,a] \to \mathbb{R} \mid f \text{ is concave}\}$. Let $\widetilde{\mathscr{F}_a}$ be the convex cone spanned by $\mathscr{F}_a$. Then $\widetilde{\mathscr{F}_a}$ is not dense in $\mathscr{F}_a^{cv}$ when we consider the topology of uniform convergence.*

Proof. Let us suppose that $\widetilde{\mathscr{F}_a}$ is dense in $\mathscr{F}_a^{cv}$ when we consider the topology of uniform convergence. By Theorem 2.3.5 a) of [11], the class $\widetilde{\mathscr{F}_a}$ is a generator of the Cobb-Douglas order.

Let $X, Y \in \mathbb{X}^a$ such that $X \preceq_{CD} Y$ and $f \in \mathscr{F}_a^{cv}$. Therefore there exists a sequence $\{f_m\}_m \subset \widetilde{\mathscr{F}_a}$ such that $\lim_m f_m = f$ uniformly.

It holds that

$$\int_{[0,a]} f_m \, dP_X \leqslant \int_{[0,a]} f_m \, dP_Y$$

for any $m \in \mathbb{N}$ since $\widetilde{\mathscr{F}_a}$ is a generator of $\preceq_{CD}$. The continuity of f on $[0,a]$ and the uniform convergence of $\{f_m\}_m$ to f, imply that the sequence $\{f_m\}_m$ is uniformly bounded. Thus by the Dominated Convergence Theorem

$$\int_{[0,a]} f \, dP_X \leqslant \int_{[0,a]} f \, dP_Y,$$

hence $X \preceq_{cv} Y$, which is a contradiction with Proposition 14. Therefore $\widetilde{\mathscr{F}_a}$ is not dense in $\mathscr{F}_a^{cv}$ for the uniform convergence. □

To conclude this section we include the following example in relation to beta distributions.

EXAMPLE 1. Let $a = 1$. Let X_n be a beta random variable with parameters n and n for any $n \in \mathbb{N}$. Then $X_n \preceq_{CD} X_{n+1}$ for any $n \in \mathbb{N}$.

Consider Proposition 2. We have that

$$E\left(f_{B(\alpha+1,2-\alpha)}(X_n)\right)=\frac{B(n+\alpha,n-\alpha+1)}{B(\alpha+1,2-\alpha)B(n,n)} \quad \text{and}$$

$$E\left(f_{B(\alpha+1,2-\alpha)}(X_{n+1})\right)=\frac{B(n+\alpha+1,n-\alpha+2)}{B(\alpha+1,2-\alpha)B(n+1,n+1)}.$$

By the properties of B and Γ functions, $E\left(f_{B(\alpha+1,2-\alpha)}(X_n)\right)\leqslant E\left(f_{B(\alpha+1,2-\alpha)}(X_{n+1})\right)$ if and only if $n(n+1)\leqslant(n+\alpha)(n+1-\alpha)$, equivalently $0\leqslant\alpha(1-\alpha)$, which holds since $\alpha\in[0,1]$. Thus $X_n\preceq_{CD}X_{n+1}$ for any $n\in\mathbb{N}$.

6. On the antisymmetric property

In this section we approach the question of the antisymmetric property of the Cobb-Douglas stochastic order.

Let $a>0$ and $X,Y\in\mathbb{X}^a$, we denote by $X\sim_{CD}Y$ when $X\preceq_{CD}Y$ and $Y\preceq_{CD}X$.

Let $X\in\mathbb{X}^a$. Assume that $P(X\in(0,a))>0$. Define $P':\mathscr{B}_{(0,a)}\to\mathbb{R}$ with

$$P'(B)=\frac{P(X\in B)}{P(X\in(0,a))}$$

for any $B\in\mathscr{B}_{(0,a)}$. Clearly P' is a probability.

Let $\widehat{X}$ be a random variable such that $P_{\widehat{X}}=P'$. Obviously $\widehat{X}\in\mathbb{X}^a$.

When we write $\widehat{X}$ we will assume the existence of the probability $P_{\widehat{X}}$, that is, $P(X\in(0,a))>0$.

First of all, we prove that the antisymmetric property depends essentially on the behaviour of the random variables in the open interval $(0,a)$.

PROPOSITION 16. *Let $a>0$ and $X,Y\in\mathbb{X}^a$ such that $\widehat{X}$ and $\widehat{Y}$ exist. The following conditions are equivalent*

i) $X\sim_{CD}Y$

ii) $P(X=0)=P(Y=0)$, $P(X=a)=P(Y=a)$ *and* $\widehat{X}\sim_{CD}\widehat{Y}$.

Proof. In the first place note that for any $\alpha\in[0,1]$ and $f_\alpha\in\mathscr{F}_a$,

$$\begin{aligned}E(f_\alpha(X))&=\int_{[0,a]}f_\alpha(x)\,dP_X=f_\alpha(0)P(X=0)+f_\alpha(a)P(X=a)+\int_{(0,a)}f_\alpha(x)\,dP_X\\&=f_\alpha(0)P(X=0)+f_\alpha(a)P(X=a)+P(X\in(0,a))\int_{(0,a)}f_\alpha(x)\,dP_{\widehat{X}}\\&=f_\alpha(0)P(X=0)+f_\alpha(a)P(X=a)+P(X\in(0,a))\int_{[0,a]}f_\alpha(x)\,dP_{\widehat{X}}\\&=f_\alpha(0)P(X=0)+f_\alpha(a)P(X=a)+P(X\in(0,a))E(f_\alpha(\widehat{X})).\end{aligned}$$

The same formula holds for random variable Y.

Now it is clear that condition $ii)$ implies $i)$.

Let us see that $i)$ leads to $ii)$.

Take $\alpha \in (0,1)$ and $f_\alpha \in \mathscr{F}_a$. Observe that $f_\alpha(0) = f_\alpha(a) = 0$. Therefore $E(f_\alpha(X)) = P(X \in (0,a))E(f_\alpha(\widehat{X}))$. Since $X \sim_{CD} Y$ we have that

$$P(X \in (0,a)) \lim_{\alpha \to 1^-} E(f_\alpha(\widehat{X})) = P(Y \in (0,a)) \lim_{\alpha \to 1^-} E(f_\alpha(\widehat{Y}))$$

and

$$P(X \in (0,a)) \lim_{\alpha \to 0^+} E(f_\alpha(\widehat{X})) = P(Y \in (0,a)) \lim_{\alpha \to 0^+} E(f_\alpha(\widehat{Y})).$$

Now note that the functions $\{f_\alpha\}_{\alpha \in [0,1]}$ are uniformly bounded and

$$\lim_{\alpha \to 1^-} f_\alpha = f_1 \ a.s.\ [P_{\widetilde{X}}] \ \text{ and } \ \lim_{\alpha \to 0^+} f_\alpha = f_0 \ a.s.\ [P_{\widetilde{X}}].$$

The same convergence results are obtained when we consider $P_{\widetilde{Y}}$.

As a consequence of the Dominated Convergence Theorem we conclude

$$\lim_{\alpha \to 1^-} E(f_\alpha(\widehat{X})) = E(f_1(\widehat{X})) \ \text{ and } \ \lim_{\alpha \to 0^+} E(f_\alpha(\widehat{X})) = E(f_0(\widehat{X})).$$

Take $\alpha = 1$. We have that

$$\begin{aligned} E(f_1(X)) &= P(X = a)f_1(a) + P(X \in (0,a))E(f_1(\widehat{X})) = E(f_1(Y)) \\ &= P(Y = a)f_1(a) + P(Y \in (0,a))E(f_1(\widehat{Y})). \end{aligned}$$

Therefore $P(Y = a)f_1(a) = P(X = a)f_1(a)$ and so $P(Y = a) = P(X = a)$.

Reasoning in the same way with $\alpha = 0$, we deduce that $P(X = 0) = P(Y = 0)$. As a consequence we obtain that $P(X \in (0,a)) = P(Y \in (0,a))$.

Now the result follows from the fact that

$$E(f_\alpha(X)) = f_\alpha(0)P(X = 0) + f_\alpha(a)P(X = a) + P(X \in (0,a))E(f_\alpha(\widehat{X})),$$

which leads to $\widehat{X} \sim_{CD} \widehat{Y}$. □

Note that if $a > 0$, $X, Y \in \mathbb{X}^a$ and $\widehat{X}$ and $\widehat{Y}$ do not exist, $X \sim_{CD} Y$ if and only if $P(X = 0) = P(Y = 0)$ and $P(X = a) = P(Y = a)$, because $P(X \in (0,a)) = 0$ implies that $\int_{(0,a)} f_\alpha(x)\, dP_X = 0$ for any $\alpha \in [0,1]$.

Moreover, it is easy to prove that if $a > 0$, $X, Y \in \mathbb{X}^a$ and $X \sim_{CD} Y$, then $\widehat{X}$ and $\widehat{Y}$ do exist or, $\widehat{X}$ and $\widehat{Y}$ do not exist, other cases are not possible.

Now we will focus our attention on finding a subfamily of $\mathbb{X}^a$ in which the order satisfies the antisymmetric property.

DEFINITION 4. Let $a > 0$. Let $\mathbb{X}^a_{\ln}$ be the set of random variables given by

$$\mathbb{X}^a_{\ln} = \{X \in \mathbb{X}^a \mid E\big(\big|(-X + a)\ln^n(\frac{X}{-X + a})\big|\big) < +\infty \text{ for any } n \in \mathbb{N}\}.$$

Note that if $X \in \mathbb{X}_{\ln}^{a}$, then $P(X=0)=P(X=a)=0$.

We prove that the Cobb-Douglas order satisfies the antisymmetric property on $\mathbb{X}_{\ln}^{a}$, where equality is understood in distribution.

PROPOSITION 17. *Let $a>0$ and $X,Y \in \mathbb{X}_{\ln}^{a}$. If $X \sim_{CD} Y$, then $F_X = F_Y$.*

Proof. Let $f:[0,a] \to \mathbb{R}$, with $f(x)=a-x$. Let $M=\int_{[0,a]} f\,dP_X$. Note that $M>0$ since $X \in \mathbb{X}_{\ln}^{a}$.

By Proposition 9 we know that the condition $X \preceq_{CD} Y$ implies that $E(X)=E(Y)$, and so $M=a-E(X)=a-E(Y)$.

We define the measure $P:\mathscr{B}_{[0,a]} \to \mathbb{R}$, with

$$P(B)=\int_B f\frac{1}{M}\,dP_X$$

for any $B \in \mathscr{B}_{[0,a]}$.

It is clear that P is a probability. Moreover P is absolutely continuous with respect to P_X ($P \ll P_X$), and $f\frac{1}{M}$ is a Radon-Nikodym derivative of P with respect to P_X ($\frac{dP}{dP_X}=f\frac{1}{M}$). Thus, for any measurable mapping $g:[0,a] \to \mathbb{R}$

$$\int_{(0,a)} g\,dP = \int_{(0,a)} g\frac{f}{M}\,dP_X.$$

Let X' be a random variable such that $P_{X'}=P$. Let $\alpha \in [0,1]$ and $f_\alpha \in \mathscr{F}_a$. It holds that

$$\begin{aligned}\int_{(0,a)} x^\alpha(-x+a)^{-\alpha}\,dP_{X'} &= \int_{(0,a)} x^\alpha(-x+a)^{-\alpha}\frac{-x+a}{M}\,dP_X\\ &= \frac{1}{M}\int_{(0,a)} x^\alpha(-x+a)^{1-\alpha}\,dP_X = \frac{1}{M}E(f_\alpha(X)).\end{aligned}$$

Consider now the mappings $g:(-\infty,\infty) \to \mathbb{R}$ with $g(x)=e^{\alpha x}$, and $T:(0,a) \to (-\infty,\infty)$ given by $T(x)=\ln(\frac{x}{-x+a})$. By a change of variable (see for instance [7]),

$$\int_{(0,a)} g\circ T\,dP_{X'} = \int_{(-\infty,\infty)} g\,dP_{X'}\circ T^{-1},$$

and so

$$\frac{1}{M}E(f_\alpha(X)) = \int_{(0,a)}\left(\frac{x}{-x+a}\right)^\alpha dP_{X'} = \int_{(-\infty,\infty)} e^{\alpha x}\,dP_{X'}\circ T^{-1}.$$

Observe that this is true for any $\alpha \in [0,1]$.

Proceeding in the same way with random variable Y,

$$\frac{1}{M}E(f_\alpha(Y)) = \int_{(0,a)}\left(\frac{x}{-x+a}\right)^\alpha dP_{Y'} = \int_{(-\infty,\infty)} e^{\alpha x}\,dP_{Y'}\circ T^{-1},$$

where $P_{Y'} \ll P_Y$ and $\frac{dP_{Y'}}{dP_Y}=\frac{f}{M}$.

Thus, if $X \preceq_{CD} Y$ and $Y \preceq_{CD} X$, then $E(f_\alpha(X)) = E(f_\alpha(Y))$, and so

$$\int_{(-\infty,\infty)} e^{\alpha x}\, dP_{X'} \circ T^{-1} = \int_{(-\infty,\infty)} e^{\alpha x}\, dP_{Y'} \circ T^{-1}$$

for any $\alpha \in [0,1]$.

Note that for any $B \in \mathscr{B}_{\mathbb{R}}$,

$$P_{X'} \circ T^{-1}(B) = P_{\ln(\frac{X'}{-X'+a})}(B) \quad \text{and} \quad P_{Y'} \circ T^{-1}(B) = P_{\ln(\frac{Y'}{-Y'+a})}(B).$$

Let $\widetilde{X} = \ln(\frac{X'}{-X'+a})$ and $\widetilde{Y} = \ln(\frac{Y'}{-Y'+a})$.

Observe that for any $n \in \mathbb{N}$ we have that

$$\begin{aligned} E(\widetilde{X}^n) &= \int_{\mathbb{R}} x^n\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^n\, dP_{X'} \circ T^{-1} = \int_{(0,a)} \ln^n\big(\frac{x}{-x+a}\big)\, dP_{X'} \\ &= \int_{(0,a)} \frac{-x+a}{M} \ln^n\big(\frac{x}{-x+a}\big)\, dP_X = \frac{1}{M} E\big((-X+a)\ln^n\big(\frac{X}{-X+a}\big)\big). \end{aligned}$$

This value is in $\mathbb{R}$ since $X \in \mathbb{X}^a_{\ln}$. Clearly the same result is obtained when we consider random variables $\widetilde{Y}$ and Y.

Therefore $E(\widetilde{X}^n), E(\widetilde{Y}^n) \in \mathbb{R}$ for any $n \in \mathbb{N}$. Moreover, we have seen that

$$\int_{\mathbb{R}} e^{\alpha x}\, dP_{\widetilde{X}} = \int_{\mathbb{R}} e^{\alpha x}\, dP_{\widetilde{Y}} \in \mathbb{R}$$

for any $\alpha \in [0,1]$. Thus the generating functions of $\widetilde{X}$ and $\widetilde{Y}$ exist and are equal in $[0,1]$. Note that this interval does not contain a neighborhood of 0, and as a consequence the equality in distribution of $\widetilde{X}$ and $\widetilde{Y}$ cannot be guaranteed with the above formula.

We will prove that

$$\frac{d}{d\alpha^n} \int_{\mathbb{R}} e^{\alpha x}\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^n e^{\alpha x}\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^n e^{\alpha x}\, dP_{\widetilde{Y}} = \frac{d}{d\alpha^n} \int_{\mathbb{R}} e^{\alpha x}\, dP_{\widetilde{Y}}$$

for any $n \in \mathbb{N}$ and $\alpha \in (0,1)$. As a consequence we will be able to derive that $E(\widetilde{X}^n) = E(\widetilde{Y}^n)$, which implies that $\widetilde{X}$ and $\widetilde{Y}$ are equal in distribution. For this purpose we will use Proposition 1.

We will argue by induction. Let us consider the case $n = 1$. Take the probability space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, P_{\widetilde{X}})$ and the open interval $I = (0,z) \subset \mathbb{R}$, with $z < 1$. We define the mapping $F : I \times \mathbb{R} \to \mathbb{R}$, with $F(\alpha,x) = e^{\alpha x}$ for any $(\alpha,x) \in I \times \mathbb{R}$. In the first place we have that $\frac{d}{d\alpha}F(\alpha,x) = xe^{\alpha x}$ for any $(\alpha,x) \in I \times \mathbb{R}$. It holds that

$$\left|\frac{d}{d\alpha}F(\alpha,x)\right| = |xe^{\alpha x}| \leqslant |x| I_{(-\infty,0]}(x) + \frac{1}{z'} e^{(z'+z)x} I_{(0,\infty)}(x)$$

with $z' > 0$ and $z' + z < 1$.

Let $\omega:\mathbb{R}\to\mathbb{R}$ be the mapping on the right-hand side of the above inequality. Let us see that $\omega\in\mathscr{L}^1(P_{\widetilde{X}})$. On the one hand

$$\int_{(-\infty,0]}|x|\,dP_{\widetilde{X}}=\int_{(-\infty,0]}|x|\,dP_{T(X')}=\int_{(-\infty,0]}|x|\,dP_X\circ T^{-1}=\int_{(0,\frac{a}{2}]}|\ln(\frac{x}{a-x})|\,dP_{X'}$$
$$\leqslant\int_{(0,a)}\frac{1}{M}(a-x)|\ln(\frac{x}{a-x})|\,dP_X=E(|(a-X)\ln(\frac{X}{a-X})|)\frac{1}{M}<+\infty.$$

On the other hand

$$\int_{(0,+\infty)}\frac{1}{z'}e^{(z+z')x}\,dP_{\widetilde{X}}=\int_{(\frac{a}{2},a)}\frac{1}{z'}(\frac{x}{a-x})^{(z+z')}\,dP_{X'}$$
$$\leqslant\int_{(0,a)}\frac{1}{z'}(\frac{x}{a-x})^{(z+z')}\frac{1}{M}(a-x)\,dP_X<+\infty$$

since $z+z'<1$.

As a consequence, condition $iii)$ in Proposition 1 holds. Moreover, conditions $i)$, $ii)$ and $iv)$ are easy to check. Applying that proposition we conclude that

$$\frac{d}{d\alpha}\int_{\mathbb{R}}F(\alpha,x)\,dP_{\widetilde{X}}=\int_{\mathbb{R}}xe^{\alpha x}\,dP_{\widetilde{X}}$$

for any $\alpha\in(0,z)$. Since $z<1$ is arbitrary, the result is also true when $\alpha\in(0,1)$. The same result holds when we consider random variable $\widetilde{Y}$.

Since we have that

$$\int_{\mathbb{R}}e^{\alpha x}\,dP_{\widetilde{X}}=\int_{\mathbb{R}}e^{\alpha x}\,dP_{\widetilde{Y}}$$

for any $\alpha\in[0,1]$, we deduce that

$$\int_{\mathbb{R}}xe^{\alpha x}\,dP_{\widetilde{X}}=\int_{\mathbb{R}}xe^{\alpha x}\,dP_{\widetilde{Y}}$$

for any $\alpha\in(0,1)$.

By means of the Dominated Convergence Theorem, taking limit as α tends to 0 from the right, the mapping ω being a bound of $xe^{\alpha x}$ of class $\mathscr{L}^1(P_{\widetilde{X}})$ and $\mathscr{L}^1(P_{\widetilde{Y}})$, we deduce that

$$\int_{\mathbb{R}}x\,dP_{\widetilde{X}}=\int_{\mathbb{R}}x\,dP_{\widetilde{Y}},$$

that is, $E(\widetilde{X})=E(\widetilde{Y})$. So the result is true when $n=1$.

Let us suppose that the result is true for some natural number n. Let us see that it holds for the value $n+1$.

Consider the probability space $(\mathbb{R},\mathscr{B}_{\mathbb{R}},P_{\widetilde{X}})$ and the interval $I=(0,z)$, with $z<1$. We define the mapping $F:I\times\mathbb{R}\to\mathbb{R}$, with $F(\alpha,x)=x^ne^{\alpha x}$ for any $(\alpha,x)\in I\times\mathbb{R}$. We have that $\frac{d}{d\alpha}F(\alpha,x)=x^{n+1}e^{\alpha x}$ for any $(\alpha,x)\in I\times\mathbb{R}$. On the other hand

$$|\frac{d}{d\alpha}F(\alpha,x)|=|x^{n+1}e^{\alpha x}|\leqslant|x^{n+1}|I_{(-\infty,0]}(x)+K_{n+1}e^{(z'+z)x}I_{(0,\infty)}(x)$$

where $z' > 0$ and $z' + z < 1$, and K_{n+1} is any constant satisfying that $x^{n+1} \leqslant K_{n+1} e^{z' x}$.

Let $\widetilde{\omega} : \mathbb{R} \to \mathbb{R}$ be the mapping on the right-hand side of the above inequality. In a similar way to the case $n = 1$, it can be seen that $\widetilde{\omega} \in \mathscr{L}^1(P_{\widetilde{X}})$. That is, we have condition $iii)$ in Proposition 1. Conditions $i)$, $ii)$ and $iv)$ are easy to analyze.

By means of Proposition 1 we obtain that

$$\frac{d}{d\alpha} \int_{\mathbb{R}} F(\alpha, x)\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^{n+1} e^{\alpha x}\, dP_{\widetilde{X}}$$

for any $\alpha \in (0, z)$. Since $z < 1$ is arbitrary, we have the same result for any $\alpha \in (0, 1)$. Obviously we obtain the same equality with random variable $\widetilde{Y}$.

By hypothesis we have that

$$\int_{\mathbb{R}} x^n e^{\alpha x}\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^n e^{\alpha x}\, dP_{\widetilde{Y}}$$

for any $\alpha \in (0, 1)$, and so

$$\int_{\mathbb{R}} x^{n+1} e^{\alpha x}\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^{n+1} e^{\alpha x}\, dP_{\widetilde{Y}}.$$

Reasoning in the same way as $n = 1$, the Dominated Convergence Theorem implies that

$$E(\widetilde{X}^{n+1}) = \int_{\mathbb{R}} x^{n+1}\, dP_{\widetilde{X}} = \int_{\mathbb{R}} x^{n+1}\, dP_{\widetilde{Y}} = E(\widetilde{Y}^{n+1}).$$

Thus we conclude that $E(\widetilde{X}^n) = E(\widetilde{Y}^n)$ for any $n \in \mathbb{N}$. As a consequence $\widetilde{X}$ and $\widetilde{Y}$ have the same distribution.

The injectivity of the logarithm implies that $X'/(X' - a)$ and $Y'/(Y' - a)$ have the same distribution. Applying the same argument, we have that X' and Y' are equal in distribution.

Now note that $P_{X'} \ll P_X$ and f/M is a Radon-Nikodym derivative of $P_{X'}$ with respect to P_X. Thus for any $b \in (0, a)$ it holds that

$$\int_{(0,a)} \frac{M}{-x + a} I_{(0,b]}(x)\, dP_{X'} = \int_{(0,a)} I_{(0,b]}(x)\, dP_X = F_X(b),$$

and the same formula holds for the random variables Y and Y'.

So we obtain that $F_X(b) = F_Y(b)$ for any $b \in (0, a)$. The right continuity of distribution functions implies that $F_X(0) = F_Y(0)$. Moreover $F_X(a) = F_Y(a) = 1$ since $X, Y \in \mathbb{X}^a$.

Therefore we have seen that $F_X = F_Y$, which concludes the proof. $\square$

COROLLARY 1. *Let $a > 0$ and $X, Y \in \mathbb{X}^a$ such that $X \sim_{CD} Y$. If any of the following conditions is satisfied*

$i)$ *for any $n \in \mathbb{N}$*

$$E\left(\left|X \ln^n\left(\frac{a - X}{X}\right)\right|\right) < +\infty \quad \text{and} \quad E\left(\left|Y \ln^n\left(\frac{a - Y}{Y}\right)\right|\right) < +\infty,$$

ii) *there exists* $\varepsilon \in (0,a)$ *with* $P(X \in (0,a-\varepsilon]) = P(Y \in (0,a-\varepsilon]) = 1$,

iii) *there exists* $\varepsilon \in (0,a)$ *such that* $P(X \in [\varepsilon,a)) = P(Y \in [\varepsilon,a)) = 1$,

then we have that $F_X = F_Y$.

Proof. In relation to $i)$, observe that such a condition implies that $a-X, a-Y \in \mathbb{X}^a_{\ln}$. Moreover $X \sim_{CD} Y$ is equivalent to $a-X \sim_{CD} a-Y$ as Proposition 12 reads. Applying Proposition 17 we deduce that $F_{a-X} = F_{a-Y}$ and then $F_X = F_Y$.

With respect to $ii)$, observe that such a condition implies those in $i)$. Note that

$$E\left(\left|X \ln^n\left(\frac{a-X}{X}\right)\right|\right) = E\left(\left|\left(X^{\frac{1}{n}} \ln(a-X) - X^{\frac{1}{n}} \ln X\right)^n\right|\right).$$

It is clear that $P(X \in (0,a-\varepsilon]) = 1$ implies that $X^{\frac{1}{n}} \ln(a-X)$ is a bounded random variable. Moreover, by means of L'Hôpital's rule it is possible to prove that the mapping $x \to x^{\frac{1}{n}} \ln x$ is bounded when $x \in (0,a-\varepsilon]$. As a consequence, the above expectation is finite and the result follows from $i)$.

To conclude with $iii)$, note that if $P(X \in [\varepsilon,a)) = 1$ then $P(a-X \in (0,a-\varepsilon]) = 1$. Now the result follows from Proposition 12 and $ii)$ in this corollary. $\square$

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