Harmonizing two approaches to fuzzy random variables

Miriam Alonso de la Fuente
Departamento de Estadística e I.O. y D.M.
Universidad de Oviedo
E-33071 Oviedo, Spain
e-mail: uo233280@uniovi.es

Pedro Terán
Escuela Politécnica de Ingeniería
Departamento de Estadística e I.O. y D.M.
Universidad de Oviedo
E-33071 Gijón, Spain
e-mail: teranpedro@uniovi.es

Abstract

We prove a measurability result which implies that the measurable events concerning the values of a fuzzy random variable, in two related mathematical approaches wherein the codomains of the variables are different spaces, are the same (provided both approaches apply). Further results on the perfectness of probability distributions of fuzzy random variables are presented.

Keywords: Borel measurability; Fuzzy random variable; Perfect distribution; Probability distribution.

1 Introduction

Fuzzy random variables are a well established mathematical model for the simultaneous handling of probabilistic and fuzzy uncertainty. Since fuzziness may appear in almost all aspects of a decision problem, they are not only a tool for statistical data analysis but have also been used in decision making. A few examples are Bayesian decision [7, 22], finance problems [29, 24, 30], multiobjective decision [27, 28], multicriteria decision [10], group decision [25].

A fuzzy random variable is, intuitively, just a random variable whose values are fuzzy sets instead of numbers. Those values can be given different interpretations [17, 2, 6], most commonly either as fuzzy perceptions and linguistic descriptions of an underlying precise random variable, or as intrinsically fuzzy data. However, at the mathematical level, fuzzy random variables are significantly harder to handle than ordinary random variables and present specific challenges related to the topology, arithmetics, and order of the fuzzy sets which are allowed as values.

This paper revolves around one such challenge. While random variables take on values in the real line, there are several spaces of fuzzy sets which can serve as the codomain of a fuzzy random variable. Traditionally, for statistical purposes it has been assumed that those fuzzy sets have bounded support [20, 17], and the metrics d_∞ and d_p [3, Chapter 7] had been used. However, Krätschmer [15, 16] and then Trutschnig *et al.* [26, 9] started an elegant approach restricted to convex fuzzy sets but in which some fuzzy sets with unbounded support (and bounded α-cuts for $\alpha > 0$) are allowed. That space is then endowed with a metric which allows one to embed it as a closed cone into an L^p-space (typically $p = 2$ since then the final space is a Hilbert space) while being topologically equivalent to d_p in the bounded case.

These approaches are not intrinsically conflictive. The fact that d_p is not complete for fuzzy sets with bounded support means that, if one performed a similar embedding and used known techniques of statistics in Hilbert spaces (if $p = 2$) to, e.g., find an estimator, we would risk the situation that this estimator failed to correspond to any fuzzy set in the original space (this is a nice example that topological properties which might appear to be far removed from practice have a definite impact on whether a method will work or not). However, to use this approach one also has to accept some limitations. First, only convex fuzzy sets can be used. Second, while some fuzzy sets with unbounded support are allowed as values, others are not, for no good reason other than mathematical convenience. Moreover, it is not possible to work with the metric d_∞, the strongest in the literature, since it takes infinite values, leading one to settling for weaker consistency and convergence results.

The lack of a single mathematical approach encompassing their advantages makes it important to ensure that both approaches can work in harmony. By this we mean that, if a fuzzy random variable from one framework lives in the space used by the other, we should be able to pass on to that framework and use its results without problems.

Let $\mathcal{F}_c(\mathbb{R})$ denote the space of all fuzzy numbers with bounded support (see next section for further details). The question is to model fuzzy random variables as random elements of $\mathcal{F}_c(\mathbb{R})$ or of a larger space when the fuzzy values we are working with are actually elements of $\mathcal{F}_c(\mathbb{R})$ (which is the case in nearly all applications). These will be called 'the smaller framework' and 'the larger framework' in the sequel. The ideal situation would be for this mathematical choice to have no working consequence whatsoever.

But an essential question is whether $\mathcal{F}_c(\mathbb{R})$ is a measurable set in the larger space (i.e., an element of its σ-algebra). If that failed, unexpected problems would arise. For example, if $\mathcal{X}$ is a fuzzy random variable with distribution $P_\mathcal{X}$ and values in $\mathcal{F}_c(\mathbb{R})$, then $P_\mathcal{X}(\mathcal{F}_c(\mathbb{R})) = 1$ would be a true formula in the smaller framework whereas the quantity $P_\mathcal{X}(\mathcal{F}_c(\mathbb{R}))$ would be undefined in the larger framework.

More generally, the distributions of a fuzzy random variable in both frameworks (the probability measures induced in the two spaces of possible values) would use σ-algebras with the one for the smaller framework not being contained in the one for the larger framework: the sets of values of which we can speak and calculate probabilities would not be mutually consistent. Another consequence: if $\mathcal{X}$ and $\mathcal{Y}$ are independent in the smaller framework, by definition

$$P_{(\mathcal{X},\mathcal{Y})}(A \times B) = P_\mathcal{X}(A) \cdot P_\mathcal{Y}(B)$$

for certain sets of values A, B whereas those probabilities might be undefined in the larger framework as soon as A, B are measurable in $\mathcal{F}_c(\mathbb{R})$ but not in the larger space (due to the non-measurability of $\mathcal{F}_c(\mathbb{R})$ itself).

The aim of this paper is to prove that, fortunately, these problems actually do not happen as indeed $\mathcal{F}_c(\mathbb{R})$ is measurable in the L^p-type spaces used in the larger framework. As a subproduct, we also show that all distributions of fuzzy random variables are perfect, a property introduced by Gnedenko and Kolmogorov which avoids some non-intuitive features of arbitrary probability measures.

The structure of the paper is as follows. Next section presents the preliminary notions and results, Section 3 proves the measurability result, and Section 4 establishes the results about distributions of fuzzy random variables. The paper concludes with some final remarks in Section 5.

2 Preliminaries

2.1 Fuzzy sets

Recall a *fuzzy subset* of $\mathbb{R}$ is a function $\widetilde{U} : \mathbb{R} \to [0,1]$. Its α-cuts are the sets

$$\widetilde{U}_\alpha = \{x \in \mathbb{R} \mid \widetilde{U}(x) \geq \alpha\}, \quad \alpha \in (0,1],$$

with the notation $\widetilde{U}_0$ representing the closure of its support.

The set $\mathcal{F}_c(\mathbb{R})$ is formed by all fuzzy subsets of $\mathbb{R}$ such that $\widetilde{U}_\alpha$ is a non-empty compact interval for all $\alpha \in [0,1]$. The elements of $\mathcal{F}_c(\mathbb{R})$ are characterized by the properties of normalization, (fuzzy) convexity and upper semicontinuity.

The metric d_∞ in $\mathcal{F}_c(\mathbb{R})$ is defined by

$$d_\infty(\widetilde{U}, \widetilde{V}) = \sup_{\alpha \in (0,1]} d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha),$$

where d_H is the *Hausdorff metric* between non-empty compact sets, which in the case of intervals can be written as

$$d_H(K, L) = \max\{|\min K - \min L|, |\max K - \max L|\}.$$

The *norm* is then the distance to the zero set,

$$\|K\| = d_H(K, \{0\}).$$

For each $p \in [1, \infty)$, the metric d_p in $\mathcal{F}_c(\mathbb{R})$ is defined by

$$d_p(\widetilde{U}, \widetilde{V}) = (\int_0^1 d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha)^p d\alpha)^{1/p}.$$

We define now a metric space $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ containing $(\mathcal{F}_c(\mathbb{R}), d_p)$. Set

$$\widehat{\mathcal{F}}_{c,p}(\mathbb{R}) = \left\{ \widetilde{U} : \mathbb{R} \longrightarrow [0,1] : \forall \alpha \in (0,1] \ \widetilde{U}_\alpha \in \mathcal{K}_c(\mathbb{R}), \|\widetilde{U}\|_p < \infty \right\},$$

where

$$\|\widetilde{U}\|_p = \left(\int_0^1 \|\widetilde{U}_\alpha\|^p d\alpha \right)^{1/p}.$$

In $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ the definition of the d_p-metric still makes sense and, in a natural way, we still use the same notation for it.

2.2 Metric spaces

A *Polish space* is a topological space whose topology is generated by a complete separable metric. The *Borel σ-algebra* $\mathcal{B}_{\mathbb{E}}$ of a metric space $\mathbb{E}$ is the smallest σ-algebra which contains the open sets. In particular, the closed sets and the *G_δ-sets* (those which can be written as the intersection of countably many open sets) are in $\mathcal{B}_{\mathbb{E}}$.

In the case of $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$, for simplicity the Borel σ-algebra will be denoted by $\mathcal{B}_{d_p}$.

The space $\mathcal{F}_c(\mathbb{R})$ is the one used in the 'smaller framework', while some $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ (typically $p = 1, 2$) is used in the 'larger framework'. The reader may note, by checking the original sources, that the latter uses similar metrics but not identical to d_p. It is known though that those metrics are equivalent [15, 26], whence they generate the same Borel σ-algebra as d_p. Since our proofs involve working with d_p, we will not discuss those metrics further here.

The restriction of a function $f : \mathbb{E} \to \mathbb{R}$ to a subset $A \subseteq \mathbb{E}$ will be denoted by $f|_A$.

2.3 Fuzzy random variables

The notation for a probability space will be $(\Omega, \mathcal{A}, P)$, where Ω is the sample space, $\mathcal{A}$ the σ-algebra of events, and P the probability measure.

A mapping $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ is called a *fuzzy random variable* if each α-set mapping $\mathcal{X}_\alpha : \omega \in \Omega \mapsto (\mathcal{X}(\omega))_\alpha$ is a random compact interval, equivalently if both $\min \mathcal{X}_\alpha$ and $\max \mathcal{X}_\alpha$ are random variables.

A mapping $\mathcal{X} : \Omega \to \widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ is called a *fuzzy random variable* if it is measurable with respect to the σ-algebras $\mathcal{A}$ and $\mathcal{B}_{d_p}$.

That leaves us with two definitions of the same concept, one from each framework. But, as shown by Krätschmer, they are equivalent when both apply.

It is important, however, to underline that the codomain of $\mathcal{X}$ indicates whether $\mathcal{X}$ is being thought of as a fuzzy random variable in one framework or the other.

3 Measurability of $\mathcal{F}_c(\mathbb{R})$

In this section, we establish the measurability result which is the basis for the subsequent results in the paper. Our starting point is a continuous extension theorem from [23, Proposition 2.2.3, p. 54].

Lemma 3.1. *Let $\mathbb{E}$ be a metric space and $\mathbb{F}$ a complete metric space. Let $A \subseteq \mathbb{E}$. If $f : A \longrightarrow \mathbb{F}$ is continuous, then it has a continuous extension $\bar{f} : B \longrightarrow \mathbb{F}$ to a G_δ-set B which contains A.*

That lemma allows us to prove the following support result which has independent interest.

Proposition 3.2. *Let $\mathbb{E}$ be a metric space endowed with its Borel σ-algebra and $f : \mathbb{E} \longrightarrow \mathbb{R}$. If there exists an increasing sequence $\{A_n\}_n$ of (possibly non-measurable) subsets which covers $\mathbb{E}$, and if $f|_{A_n}$ is continuous for each $n \in \mathbb{N}$, then f is a random variable.*

Proof. Set $h_n = f|_{A_n}$. For each $n \in \mathbb{N}$, let B_n be the G_δ-set provided by Lemma 3.1 such that $A_n \subseteq B_n$, $B_n \in \mathcal{B}_{\mathbb{E}}$, and $\bar{h}_n : B_n \longrightarrow \mathbb{R}$ is a continuous extension of h_n from A_n to B_n. Set

$$g_n(x) = \begin{cases} \bar{h}_n(x) & \text{if } x \in B_n, \\ 0 & \text{if } x \notin B_n. \end{cases}$$

Let us show each g_n is measurable, namely $g_n^{-1}((-\infty, a)) \in \mathcal{B}_{\mathbb{E}}$ for each $a \in \mathbb{R}$.

Assume for now $a < 0$. Then

$$\begin{aligned} g_n^{-1}((-\infty, a)) &= (B_n \cap g_n^{-1}((-\infty, a))) \cup (B_n^c \cap g_n^{-1}((-\infty, a))) \\ &= \bar{h}_n^{-1}((-\infty, a)) \cup \emptyset = \bar{h}_n^{-1}((-\infty, a)) \in \mathcal{B}_{\mathbb{E}} \end{aligned}$$

Else, if $a \geq 0$,

$$g_n^{-1}((-\infty, a)) = (B_n \cap g_n^{-1}((-\infty, a))) \cup (B_n^c \cap g_n^{-1}((-\infty, a)))$$
$$= \bar{h}_n^{-1}((-\infty, a)) \cup (B_n^c \cap g_n^{-1}(\{0\})) = \bar{h}_n^{-1}((-\infty, a)) \cup B_n^c \in \mathcal{B}_{\mathbb{E}}$$

For each $x \in \mathbb{E}$, there exists $n \in \mathbb{N}$ such that $x \in A_n \subseteq \mathbb{E}$. Therefore, $g_m(x) = f(x)$ for all $m \geq n$, whence f is a random variable because it is the pointwise limit of the measurable functions g_n. □

The following left-continuity properties are well known.

Lemma 3.3. *Let $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ and $\alpha \in (0, 1]$. If $\alpha_n \nearrow \alpha$, then*

1. $d_H(\widetilde{U}_{\alpha_n}, \widetilde{V}_{\alpha_n}) \to d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha)$.
2. $\|\widetilde{U}_{\alpha_n}\| \to \|\widetilde{U}_\alpha\|$.

We will also use the fact that the d_p-metrics are increasing in p.

Lemma 3.4. *Let $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$. If $1 \leq p \leq q < \infty$, then $d_p(\widetilde{U}, \widetilde{V}) \leq d_q(\widetilde{U}, \widetilde{V})$.*

Proof. Define the function

$$\begin{array}{rcl} f : (0,1] & \to & \mathbb{R} \\ \alpha & \mapsto & d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha) \end{array}$$

By Lemma 3.3, f is left-continuous and thus measurable. Since

$$d_p(\widetilde{U}, \widetilde{V}) = \left(\int_{[0,1]} d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha)^p d\alpha \right)^{1/p} = \left(\int_{[0,1]} f^p d\alpha \right)^{1/p} = \|f\|_p,$$

applying Minkowski's inequality we have $\|f\|_p \leq \|f\|_q$, that is, $d_p(\widetilde{U}, \widetilde{V}) \leq d_q(\widetilde{U}, \widetilde{V})$. □

The main properties of the metric spaces $(\widehat{\mathcal{F}}_{c,p}(\mathbb{R}), d_p)$ were established by Krätschmer [14, Corollary 3.3].

Lemma 3.5. *Let $p \in [1, \infty)$. Then $(\widehat{\mathcal{F}}_{c,p}(\mathbb{R}), d_p)$ is a complete, separable metric space and a completion of $(\mathcal{F}_c(\mathbb{R}), d_p)$ (namely, $\mathcal{F}_c(\mathbb{R})$ is dense in the complete space $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$).*

The following real functions play an important role in the proof of our main result. Let ϕ_α and $\phi_{\alpha,k}$ be defined as follows.

- For each $\alpha \in (0,1]$,

$$\begin{aligned} \phi_\alpha : \widehat{\mathcal{F}}_{c,p}(\mathbb{R}) &\to \mathbb{R} \\ \widetilde{U} &\mapsto \|\widetilde{U}_\alpha\| \end{aligned}$$

- For each $k > \frac{1}{\alpha}$,

$$\begin{aligned} \phi_{\alpha,k} : \widehat{\mathcal{F}}_{c,p}(\mathbb{R}) &\to \mathbb{R} \\ \widetilde{U} &\mapsto \left(k \cdot \int_{[\alpha-\frac{1}{k},\alpha]} \|\widetilde{U}_\beta\|^p d\beta \right)^{1/p} \end{aligned}$$

In the sequel, when the values of α and k are fixed by the context, we will denote $I = [\alpha - \frac{1}{k}, \alpha]$.

We will now prove the measurability of the $\phi_{\alpha,k}$.

Theorem 3.6. *The mapping $\phi_{\alpha,k}$ is a random variable for each $\alpha \in (0,1]$ and $k > \frac{1}{\alpha}$.*

Proof. Fix $\alpha \in (0,1]$, and $k > \frac{1}{\alpha}$. Set

$$A_n = \{\widetilde{U} \in \widehat{\mathcal{F}}_{c,p}(\mathbb{R}) : \|\widetilde{U}_{\alpha-\frac{1}{k}}\| \leq n\}.$$

Since $\alpha - \frac{1}{k} > 0$, the set $\widetilde{U}_{\alpha-\frac{1}{k}}$ is compact for all $\widetilde{U} \in \widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ and then $\|\widetilde{U}_{\alpha-\frac{1}{k}}\| < \infty$. Therefore, $\{A_n\}_n$ is an increasing sequence of sets with

$$\widehat{\mathcal{F}}_{c,p}(\mathbb{R}) = \bigcup_{n\in\mathbb{N}} A_n.$$

By Proposition 3.2, to complete the proof it is enough to show that the restrictions $\phi_{\alpha,k}|_{A_n}$ are continuous for all $n \in \mathbb{N}$. We will indeed prove that its power $\phi^p_{\alpha,k}$ is Lipschitzian in A_n, whence, by taking the pth root, the continuity of $\phi_{\alpha,k}|_{A_n}$ follows.

For any fixed $n \in \mathbb{N}$, let $\widetilde{U}, \widetilde{V} \in A_n$. Then

$$\|\phi^p_{\alpha,k}(\widetilde{U}) - \phi^p_{\alpha,k}(\widetilde{V})\| = \left| k \cdot \int_I \|\widetilde{U}_\beta\|^p d\beta - k \cdot \int_I \|\widetilde{V}_\beta\|^p d\beta \right|$$

$$\leq k \cdot \int_I \left| \|\widetilde{U}_\beta\|^p - \|\widetilde{V}_\beta\|^p \right| d\beta. \tag{1}$$

Since $\beta \in I$, we have $\widetilde{U}_\beta \subseteq \widetilde{U}_{\alpha-\frac{1}{k}}$, therefore

$$0 \leq \|\widetilde{U}_\beta\| \leq \|\widetilde{U}_{\alpha-\frac{1}{k}}\| \leq n.$$

Analogously, $\|\widetilde{V}_\beta\| \in [0, n]$ as well. Denote by f_p the function $f_p : x \mapsto x^p$; since it is continuously derivable, it is Lipschitzian on $[0, n]$ with the Lipschitz constant

$$\sup_{x\in[0,n]} f_p'(x) = \sup_{x\in[0,n]} px^{p-1} = pn^{p-1}$$

because $p \geq 1$.

Using the triangle inequality for d_H,

$$\left| \|\widetilde{U}_\beta\|^p - \|\widetilde{V}_\beta\|^p \right| \leq pn^{p-1} \left| \|\widetilde{U}_\beta\| - \|\widetilde{V}_\beta\| \right|$$

$$= pn^{p-1} |d_H(\widetilde{U}_\beta, \{0\}) - d_H(\widetilde{V}_\beta, \{0\})| \leq pn^{p-1} d_H(\widetilde{U}_\beta, \widetilde{V}_\beta)$$

Plugging that into (1),

$$\|\phi^p_{\alpha,k}(\widetilde{U}) - \phi^p_{\alpha,k}(\widetilde{V})\| \leq kpn^{p-1} \int_I d_H(\widetilde{U}_\beta, \widetilde{V}_\beta) d\beta$$

$$\leq kpn^{n-1} \int_{(0,1]} d_H(\widetilde{U}_\beta, \widetilde{V}_\beta) d\beta = kpn^{p-1} d_1(\widetilde{U}, \widetilde{V}).$$

By Lemma 3.4 we know $d_1 \leq d_p$, whence

$$|\phi^p_{\alpha,k}(\widetilde{U}) - \phi^p_{\alpha,k}(\widetilde{V})| \leq kpn^{p-1} d_p(\widetilde{U}, \widetilde{V}).$$

Indeed $\phi^p_{\alpha,k}$ is a Lipschitz function on A_n (with constant kpn^{p-1}), and the proof is complete. □

We will show now that ϕ_α is a random variable.

Proposition 3.7. *Let $p \in [1, \infty)$. For each $\widetilde{U} \in \widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ and $\alpha \in (0, 1]$,*

$$\phi_\alpha(\widetilde{U}) = \|\widetilde{U}_\alpha\| = \lim_{k\to\infty} \phi_{\alpha,k}(\widetilde{U}).$$

Proof. Let $\widetilde{U} \in \widehat{\mathcal{F}}_{c,p}(\mathbb{R})$. Since

$$\int_I d\beta = \ell(I) = \frac{1}{k}$$

we have

$$\|\widetilde{U}_\alpha\|^p = \|\widetilde{U}_\alpha\|^p \cdot k \cdot \int_I d\beta = k \cdot \int_I \|\widetilde{U}_\alpha\|^p d\beta.$$

Then

$$\left|\phi^p_{\alpha,k}(\widetilde{U}) - \|\widetilde{U}_\alpha\|^p\right| = \left|k \cdot \int_I \|\widetilde{U}_\beta\|^p d\beta - k \cdot \int_I \|\widetilde{U}_\alpha\|^p d\beta\right|$$

$$= \left|k \cdot \int_I (\|\widetilde{U}_\beta\|^p - \|\widetilde{U}_\alpha\|^p) d\beta\right| \leq k \cdot \int_I \left|\|\widetilde{U}_\beta\|^p - \|\widetilde{U}_\alpha\|^p\right| d\beta.$$

$$|\|\widetilde{U}_\beta\|^p - \|\widetilde{U}_\alpha\|^p| = \|\widetilde{U}_\beta\|^p - \|\widetilde{U}_\alpha\|^p \leq \|\widetilde{U}_{\alpha-\frac{1}{k}}\|^p - \|\widetilde{U}_\alpha\|^p$$

for all $\beta \in I = [\alpha - \frac{1}{k}, \alpha]$. Therefore

$$\left|\phi^p_{\alpha,k}(\widetilde{U}) - \|\widetilde{U}_\alpha\|^p\right| \leq k \cdot \int_I (\|\widetilde{U}_{\alpha-\frac{1}{k}}\|^p - \|\widetilde{U}_\alpha\|^p) d\beta = k \cdot \frac{1}{k} \cdot (\|\widetilde{U}_{\alpha-\frac{1}{k}}\|^p - \|\widetilde{U}_\alpha\|^p)$$

By Lemma 3.3, $\|\widetilde{U}_{\alpha-\frac{1}{k}}\| \longrightarrow \|\widetilde{U}_\alpha\|$, whence

$$\left|\phi^p_{\alpha,k}(\widetilde{U}) - \|\widetilde{U}_\alpha\|^p\right| \longrightarrow 0$$

and then

$$\phi^p_{\alpha,k}(\widetilde{U}) \to \|\widetilde{U}_\alpha\|.$$

□

Corollary 3.8. *Let $\alpha \in (0, 1]$. The mapping $\phi_\alpha : (\widehat{\mathcal{F}}_{c,p}(\mathbb{R}), \mathcal{B}_{d_p}) \longrightarrow \mathbb{R}$ is a random variable.*

Proof. It is so because it is the pointwise limit of the random variables $\phi_{\alpha,k}$. □

Now we will use the measurability of the ϕ_α to achieve our main result.

Theorem 3.9. *For each $p \in [1, \infty)$, the subspace $\mathcal{F}_c(\mathbb{R})$ is a Borel measurable set in $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$.*

Proof. First let us show

$$\|\widetilde{U}_0\| = \sup_n \|\widetilde{U}_{1/n}\|.$$

Recall

$$\|\widetilde{U}_\alpha\| = \max\{|\inf \widetilde{U}_\alpha|, |\sup \widetilde{U}_\alpha|\}.$$

Since $\widetilde{U}_0$ is the closure of $\bigcup_{n=1}^{\infty} \widetilde{U}_{1/n}$, the quantity $|\inf \widetilde{U}_{1/n}|$ decreases to $|\inf \widetilde{U}_0|$ as well as $|\sup \widetilde{U}_{1/n}|$ increases to $|\sup \widetilde{U}_0|$. Thus $\|\widetilde{U}_{1/n}\| \longrightarrow \|\widetilde{U}_0\|$ and, being a monotone sequence, $\|\widetilde{U}_0\| = \sup_n \|\widetilde{U}_{1/n}\|$.

Now we will show

$$\mathcal{F}_c(\mathbb{R}) = \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \phi_{1/n}^{-1}(-\infty, m]. \tag{2}$$

($\subseteq$) Let $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$, $||\widetilde{U}_0|| < \infty$. Let $m \in \mathbb{N}$ be such that $m \geq ||\widetilde{U}_0||$. Then

$$\|\widetilde{U}_0\| = \sup_n \|\widetilde{U}_{1/n}\| \Longrightarrow \|\widetilde{U}_{1/n}\| \leq m \ \forall n \in \mathbb{N} \Longrightarrow \phi_{1/n}(\widetilde{U}) \leq m \ \ \forall n \in \mathbb{N}$$

$$\Longrightarrow \widetilde{U} \in \phi_{1/n}^{-1}((-\infty, m]) \ \forall n \in \mathbb{N} \Longrightarrow \widetilde{U} \in \bigcap_{n\in\mathbb{N}} \phi_{1/n}^{-1}((-\infty, m]).$$

($\supseteq$) Let $\widetilde{U} \in \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \phi_{1/n}^{-1}(-\infty, m]$. We know $\widetilde{U} \in \widehat{\mathcal{F}}_{c,p}(\mathbb{R})$, there exists $m \in \mathbb{N}$ such that $\phi_{1/n}(\widetilde{U}) \leq m$ for all $n \in \mathbb{N}$, yielding $||\widetilde{U}_{1/n}|| \leq m$ for all $n \in \mathbb{N}$. Then

$$\sup_n \|\widetilde{U}_{1/n}\| \leq m \Rightarrow \|\widetilde{U}_0\| \leq m \Rightarrow \|\widetilde{U}_0\| < \infty \Rightarrow \widetilde{U} \in \mathcal{F}_c(\mathbb{R}).$$

By Corollary 3.8, ϕ_α is a random variable whence, for each $n, m \in \mathbb{N}$,

$$\phi_{1/n}^{-1}((-\infty, m]) \in \mathcal{B}_{d_p}.$$

By (2) and the properties of a σ-algebra, $\mathcal{F}_c(\mathbb{R}) \in \mathcal{B}_{d_p}$ and the proof is complete. □

4 Distributions of fuzzy random variables with values in $\mathcal{F}_c(\mathbb{R})$ and $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$

In this section we use the measurability result Theorem 3.9 to study the distributions of fuzzy random variables.

The following support result is from [19, Theorem 1.9].

Lemma 4.1. *Let $\mathbb{E}$ be a metric space and let $A \subseteq \mathbb{E}$. If A is a Borel set, then*

$$\mathcal{B}_A = \{B \in \mathcal{B}_{\mathbb{E}} \mid B \subseteq A\}.$$

Recall the definition of a fuzzy random variable $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ as a mapping such that each $\mathcal{X}_\alpha$ is measurable. This is the same as to require $\mathcal{X}$ to be measurable with respect to the σ-algebras $\mathcal{A}$ and σ_L, the smallest σ-algebra which makes the mappings $L_\alpha : \widetilde{U} \in \mathcal{F}_c(\mathbb{R}) \mapsto \widetilde{U}_\alpha$ measurable.

Accordingly, the distribution of a fuzzy random variable $\mathcal{X}$ in the 'smaller framework' is a probability measure on the σ-algebra σ_L of subsets of $\mathcal{F}_c(\mathbb{R})$ whereas its distribution in the 'larger framework' is a probability measure on σ_{d_p}, a family of subsets of $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$.

Theorem 4.2. *The σ-algebras associated to the distributions of $\mathcal{F}_c(\mathbb{R})$-valued and $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$-valued fuzzy random variables satisfy the following.*

(a) If $A \in \sigma_L$ then $A \in \sigma_{d_p}$.

(b) If $A \in \sigma_{d_p}$ and $A \subset \mathcal{F}_c(\mathbb{R})$, then $A \in \sigma_L$.

Proof. By part (ii)$\Leftrightarrow$(iv) in [13, Theorem 6.6], σ_L is the Borel σ-algebra of $\mathcal{F}_c(\mathbb{R})$ endowed with the d_p metric. By Lemma 4.1, it is formed by the elements of $\mathcal{B}_{d_p}$ which are contained in $\mathcal{F}_c(\mathbb{R})$. That proves part (a) as well as part (b). □

Theorem 4.2 means that, whenever $\mathcal{X}$ is a fuzzy random variable with values in $\mathcal{F}_c(\mathbb{R})$, it can be handled in both frameworks and the events $A \subset \mathcal{F}_c(\mathbb{R})$ of possible values for which the expression $P_{\mathcal{X}}(A)$ makes sense are exactly the same.

In the remainder of this section, we show that all distributions of fuzzy random variables, with values in either $\mathcal{F}_c(\mathbb{R})$ or $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$, are perfect.

Kolmogorov's approach to probability, now standard, is based on measure theory and thus the sample space is allowed to be an arbitrary set. Former approaches, like e.g. Paul Lévy's, constructed probability distributions as limits of discrete (finite) situations and thus would not allow for arbitrary sample spaces (one may remark that still today explicit constructions from finite situations are favoured by some, e.g. [11]).

However, Kolmogorov's generality gave rise in the 1940s to a number of striking or pathological examples. One such situation is the following, observed by Doob [4] and Jessen [12].

Proposition 4.3. (Doob–Jessen) *There exists a probability space* $(\Omega, \mathcal{A}, P)$, *independent random variables* $\xi, \eta : \Omega \longrightarrow \mathbb{R}$ *and sets* $A, B \subseteq \mathbb{R}$ *such that*

(a) $\{\xi \in A\} \in \mathcal{A}$, $\{\eta \in B\} \in \mathcal{A}$.

(b) $P(\xi \in A, \eta \in B) \neq P(\xi \in A) \cdot P(\eta \in B)$.

At first sight this seems self-contradictory; the explanation is that such A, B are not Borel sets whereas distributions of random variables are defined in the σ-algebra of Borel sets. Thus ξ, η can satisfy the formal definition of independence while they are intuitively not independent since

$$P(\xi \in A \mid \eta \in B) \neq P(\xi \in A)$$

for certain sets A, B.

To overcome this and other anti-intuitive situations (see, e.g., [1]), the notions of a perfect probability measure and a perfect measurable space were introduced by Gnedenko and Kolmogorov [8]. A probability measure P in a measurable space $(\Omega, \mathcal{A})$ is called *perfect* if, for every $A \subseteq \mathbb{R}$ and every random variable $\xi : \Omega \longrightarrow \mathbb{R}$ such that $\{\xi \in A\} \in \mathcal{A}$, there exist Borel sets $B_1, B_2 \subseteq \mathbb{R}$ such that

- $B_1 \subseteq A \subseteq B_2$.
- $P(\xi \in B_2 \setminus B_1) = 0$.

Thus a perfect probability measure is a probability measure which leaves no room for a situation like Proposition 4.3 to happen since all the information about the values of ξ in the experiment is contained in the Borel sets: if A is non-Borel then, for appropriate Borel sets $B_1 \subseteq A \subseteq B_2$, we have

$$P(\xi \in A) = P(\xi \in B_1) = P(\xi \in B_2).$$

In its turn, a measurable space is called *perfect* if all probability measures which can be defined on it are perfect.

Let us show that all distributions of fuzzy random variables with values in $\mathcal{F}_c(\mathbb{R})$ or $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ are perfect.

Lemma 4.4. *Every Polish space, endowed with its Borel σ-algebra, is perfect.*

Proof. This follows from [21, Facts C1 and P2, pp. 770–771]. Namely, every probability measure in a Polish space is compact in the sense of measure theory, and every compact probability measure is perfect. $\square$

It is immediate that $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ is a perfect space, since it is Polish because the metric d_p is separable and complete.

Corollary 4.5. *Let $p \in [1, \infty)$. Let $\mathcal{X} : \Omega \longrightarrow \widehat{\mathcal{F}}_{c,p}(\mathbb{R})$ be a fuzzy random variable. Then its distribution $P_{\mathcal{X}}$ is perfect.*

To show that $\mathcal{F}_c(\mathbb{R})$ is perfect too we need another known result [5, Exercise 342X.(n).(i), p. 181].

Lemma 4.6. *Every measurable subspace of a perfect measurable space is perfect.*

Therefore distributions in $\mathcal{F}_c(\mathbb{R})$ are perfect.

Proposition 4.7. *Let $\mathcal{X} : \Omega \longrightarrow \mathcal{F}_c(\mathbb{R})$ be a fuzzy random variable. Then its distribution $P_{\mathcal{X}}$ is perfect.*

Proof. By Theorem 3.9, $\mathcal{F}_c(\mathbb{R})$ is a measurable subspace of the perfect measurable space $\widehat{\mathcal{F}}_{c,p}(\mathbb{R})$. $\square$

5 Concluding remarks

The informal terms 'smaller framework' and 'larger framework' only make sense in the context of variables with convex values, since the latter stops to apply in the case of non-convex values.

We have dealt with fuzzy random variables with convex values in $\mathcal{F}_c(\mathbb{R})$. It should be emphasized that a similar concern with the relationship between $\mathcal{F}_c(\mathbb{R})$ and its superspace $\mathcal{F}(\mathbb{R})$ formed by withdrawing the requirement of convexity is not necessary, as it is not hard to show that $\mathcal{F}_c(\mathbb{R})$ is measurable in $\mathcal{F}(\mathbb{R})$.

Indeed, the space of all compact convex subsets is closed in the space of all compact subsets (e.g. [18, Theorem 1.1.2]), whence each set

$$\{\widetilde{U} \in \mathcal{F}(\mathbb{R}) \mid \widetilde{U}_\alpha \text{ is convex}\}$$

is measurable. Writing $\mathcal{F}_c(\mathbb{R})$ as the countable intersection

$$\mathcal{F}_c(\mathbb{R}) = \bigcap_{\alpha\in(0,1]\cap\mathbb{Q}} \{\widetilde{U} \in \mathcal{F}(\mathbb{R}) \mid \widetilde{U}_\alpha \text{ is convex}\},$$

we see that it is indeed measurable.

Acknowledgements. Research in this paper was partially funded by Spain's *Ministerio de Economía y Competitividad* (MTM2015–63971–P) and Asturias' *Consejería de Empleo, Industria y Turismo* (GRUPIN-IDI2018-000132). Moreover, it was partially carried out while the second author enjoyed a *Beca de colaboración* from Spain's *Secretaría de Estado de Educación, Formación Profesional y Universidades.*

References

[1] D. Blackwell (1956). On a class of probability spaces. Proc. Third Berkeley Symp. Math. Statist. Probab. (J. Neyman, ed.), Vol. 2, 1–6. Univ. California Press, Berkeley.

[2] I. Couso, D. Dubois, L. Sánchez (2014). Random sets and random fuzzy sets as ill-perceived random variables. Springer, Cham.

[3] P. Diamond, P. Kloeden (1994). Metric spaces of fuzzy sets. World Scientific, Singapore.

[4] J. L. Doob (1948). On a problem of Marczewski. Colloq. Math. 1, 216–217.

[5] D. H. Fremlin (2002). Measure theory, Vol. 3. Torres Fremlin, Colchester.

[6] M. A. Gil, A. Colubi, P. Terán (2014). Random fuzzy sets: why, when, how. Bol. Estad. Invest. Oper. 30, 5–29.

[7] M. A. Gil, M. López-Díaz (1996). Fundamentals and Bayesian analyses of decision problems with fuzzy-valued utilities. Int. J. Approx. Reasoning 15, 203–224.

[8] B. V. Gnedenko, A. N. Kolmogorov (1954). Limit distributions for sums of independent random variables. Addison-Wesley, Reading.

[9] G. González-Rodríguez, A. Colubi, M. A. Gil (2012). Fuzzy data treated as functional data: A one-way ANOVA test approach. Comput. Statist. Data Anal. 56, 943–955.

[10] M. Inuiguchi, K. Kato, H. Katagiri (2016). Fuzzy multi-criteria optimization: possibilistic and fuzzy/stochastic approaches. In: Multiple criteria decision analysis (S. Greco, M. Ehrgott, J. R. Figueira, eds.), 2nd edition, 851–902. Springer, New York.

[11] E. T. Jaynes (2003). Probability theory: the logic of science. Cambridge Univ. Press, Cambridge.

[12] B. Jessen (1948). On two notions of independent functions. Colloq. Math. 1, 214–215.

[13] V. Krätschmer (2001) A unified approach to fuzzy random variables. Fuzzy Sets and Systems 123, 1–9.

[14] V. Krätschmer (2002). Some complete metrics on spaces of fuzzy subsets. Fuzzy Sets and Systems 130, 357–365.

[15] V. Krätschmer (2004). Probability theory in fuzzy sample spaces. Metrika 60, 167–189.

[16] V. Krätschmer (2006). Integrals of random fuzzy sets. Test 15, 433–469.

[17] R. Kruse, K. D. Meyer (1987). Statistics with vague data. Reidel, Dordrecht.

[18] S. Li, Y. Ogura, V. Kreinovich (2002). Limit theorems and applications of set-valued and fuzzy set-valued random variables. Kluwer, Dordrecht.

[19] K. R. Parthasarathy (1967). Probability measures on metric spaces. Academic Press, New York.

[20] M. L. Puri, D. A. Ralescu (1986). Fuzzy random variables. J. Math. Anal. Appl. 114, 409–422.

[21] D. Ramachandran (2002). Perfect measures and related topics. In: Handbook of measure theory (E. Pap, ed.), Chapter 18, 765–786. Elsevier, Amsterdam.

[22] L. J. Rodríguez-Muñiz, M. López-Díaz (2008). A new framework for the Bayesian analysis of single-stage decision problems with imprecise utilities. Fuzzy Sets and Systems 159, 3271–3280.

[23] S. M. Srivastava (1998). A course on Borel sets. Springer, New York.

[24] P. Terán (2006). A note on Yoshida's optimal stopping model for option pricing. Eur. J. Oper. Res. 170, 672–676.

[25] P. Terán, J. M. Moreno-Jiménez (2008). Group decision making with soft AHP based on the random set view of fuzzy sets. In: Soft methods for handling variability and imprecision (D. Dubois, M. A. Lubiano, H. Prade, M. Á. Gil, P. Grzegorzewski, O. Hryniewicz, eds.), 427–434. Springer, Berlin.

[26] W. Trutschnig, G. González-Rodríguez, A. Colubi, M. A. Gil (2009). A new family of metrics for compact, convex (fuzzy) sets based on a generalized concept of mid and spread. Inform. Sci. 179, 3964–3972.

[27] J. Xu, Z. Zeng (2014). Fuzzy-like multiple objective multistage decision making. Springer, Cham.

[28] H. Yano (2017). Interactive multiobjective decision making under uncertainty. CRC, Boca Raton.

[29] Y. Yoshida (2003). The valuation of European options in uncertain environment. Eur. J. Oper. Res. 145, 221–229.

[30] Y. Yoshida, M. Yasuda, J. Nakagami, M. Kurano (2006). A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty. Fuzzy Sets and Systems 157, 2614–2626.