Mixed discontinuous Galerkin approximation of the elasticity eigenproblem

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Abstract

We introduce a discontinuous Galerkin method for the mixed formulation of the elasticity eigenproblem with reduced symmetry. The analysis of the resulting discrete eigenproblem does not fit in the standard spectral approximation framework since the underlying source operator is not compact and the scheme is nonconforming. We show that the proposed scheme provides a correct approximation of the spectrum and prove asymptotic error estimates for the eigenvalues and the eigenfunctions. Finally, we provide several numerical tests to illustrate the performance of the method and confirm the theoretical results.

Keywords: Mixed elasticity equations, spectral problems, finite elements, discontinuous Galerkin methods, error estimates.

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1 Introduction

We present a discontinuous Galerkin (DG) approximation of the linearized vibrations of an elastic structure. In many applications, the displacement field is not necessarily the

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variable of primary interest. We consider here the dual-mixed formulation of the elasticity eigenproblem because it delivers a direct finite element approximation of the Cauchy stress tensor and it permits to deal safely with nearly incompressible materials.

A mixed finite element approximation of the eigenvalue elasticity problem with reduced symmetry has been analyzed in [19]. It consists in a formulation that only maintains the stress tensor as primary unknown, besides the rotation whose role is the weak imposition of the symmetry restriction. It is shown that a discretization based on the lowest order Arnorld-Falk-Winther element provides a correct spectral approximation and quasi optimal asymptotic error estimates for the eigenvalues and the eigenfunctions.

The ability of DG methods handle efficiently *hp*-adaptive strategies make them suitable for the numerical simulation of physical systems related to elastodynamics. Our aim here is to introduce an interior penalty discontinuous Galerkin version for the H(div)-conforming finite element space employed in [19]. The k^{th}-order of this method amounts to approximate the Cauchy stress tensor and the rotation by discontinuous finite element spaces of degree k and $k-1$ respectively. We point out that an H(curl)-based interior penalty discontinuous Galerkin method has also been introduced in [9] for the Maxwell eigensystem. The DG approximation we are considering here may be regarded as its counterpart in the H(div)–setting. As in [9], our analysis requires conforming meshes, but the DG method still permits one to employ different polynomial element orders in the same triangulation. A further advantage of this DG scheme is that it allows to implement high-order elements in a mixed formulation by using standard shape functions. Let us remark that the DG method has also been analyzed in [1] for the Laplace operator.

It is well known that the underlying source operator corresponding to mixed formulations is generally not compact. In our case, this operator admits a non physical zero eigenvalue whose eigenspace is infinite dimensional. It is then essential to use a scheme that is safe from the pollution that may appear in the form of spurious eigenvalues interspersed among the physically relevant ones. It turns out (cf. [3, 6]) that, for mixed eigenvalue problems, the conditions guarantying the convergence of the source problem does not necessarily provide a correct spectral approximation (as it happens for compact operators [2]).

It has been shown in [9] that DG methods can also benefit from the general theory developed in [11, 12] to deal with the spectral numerical analysis of non-compact operators. We follow here the same strategy, combined with techniques from [19, 18], to prove that our numerical scheme is spurious free. We also establish asymptotic error estimates for the eigenvalues and eigenfunctions. We treat with special care the analysis of the limit problem obtained when the Lamé coefficient tends to infinity.

We end this section with some of the notations that we will use below. Given any Hilbert space V, let V^n and $V^{n\times n}$ denote, respectively, the space of vectors and tensors of order n ($n = 2, 3$) with entries in V. In particular, $\boldsymbol{I}$ is the identity matrix of $\mathbb{R}^{n\times n}$ and $\mathbf{0}$ denotes a generic null vector or tensor. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{n\times n}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^n \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^n \tau_{ij}\sigma_{ij}$.

Let Ω be a polyhedral Lipschitz bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$. For

$s \geq 0$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathrm{H}^s(\Omega)^n$ or $\mathrm{H}^s(\Omega)^{n\times n}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define for $s \geq 0$ the Hilbert space $\mathrm{H}^s(\mathbf{div}, \Omega) := \{\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^{n\times n} : \ \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^n\}$, whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}^s(\mathbf{div},\Omega)} := \|\boldsymbol{\tau}\|^2_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{s,\Omega}$ and denote $\mathrm{H}(\mathbf{div}, \Omega) := \mathrm{H}^0(\mathbf{div}, \Omega)$.

Henceforth, we denote by C generic constants independent of the discretization parameter, which may take different values at different places.

2 The model problem

In this section, we recall the mixed variational formulation of the elasticity eigenvalue problem analyzed in [19]. Moreover, we summarize some results from this reference.

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be an open bounded Lipschitz polygon/polyhedron representing an elastic body. We denote by $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$ and assume that $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\mathrm{int}(\Gamma_D) \cap \mathrm{int}(\Gamma_N) = \emptyset$. The solid is supposed to be isotropic and linearly elastic with mass density ρ and Lamé constants μ and λ. We assume that the structure is fixed at $\Gamma_D \neq \emptyset$ and free of stress on Γ_N. We can combine the constitutive law

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega,$$

and the equilibrium equation

$$\omega^2 \boldsymbol{u} = \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma} \qquad \text{in } \Omega, \tag{1}$$

to eliminate either the displacement field $\boldsymbol{u}$ or the Cauchy stress tensor $\boldsymbol{\sigma}$ from the global spectral formulation of the elasticity problem. Here, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}]$ is the linearized strain tensor, and $\mathcal{C} : \ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the Hooke operator, which is given in terms of the Lamé coefficients λ and μ by

$$\mathcal{C}\boldsymbol{\tau} := \lambda\,(\mathrm{tr}\,\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{R}^{n\times n}.$$

Opting for the elimination of the displacement $\boldsymbol{u}$ and maintaining the stress tensor $\boldsymbol{\sigma}$ as a main variable leads to the following dual mixed formulation of the elasticity eigenproblem: Find $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{n\times n}$ symmetric, $\boldsymbol{r} : \Omega \to \mathbb{R}^{n\times n}$ skew symmetric and the corresponding natural frequencies $\omega \in \mathbb{R}$ such that,

$$\begin{aligned} -\nabla\left(\rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}\right) &= \omega^2\left(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}\right) && \text{in } \Omega,\\ \mathbf{div}\,\boldsymbol{\sigma} &= \mathbf{0} && \text{on } \Gamma_D,\\ \boldsymbol{\sigma}\boldsymbol{n} &= \mathbf{0} && \text{on } \Gamma_N. \end{aligned} \tag{2}$$

We notice that the additional variable $\boldsymbol{r} := \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ is the rotation. It acts as a Lagrange multiplier for the symmetry restriction. We also point out that the displacement can be recovered and also post-processed at the discrete level by using identity (1).

Taking into account that the Neumann boundary condition becomes essential in the mixed formulation, we consider the closed subspace $\boldsymbol{\mathcal{W}}$ of $\mathrm{H}(\mathbf{div}, \Omega)$ given by

$$\boldsymbol{\mathcal{W}} := \left\{\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div}, \Omega) : \quad \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_N\right\}.$$

The rotation $\boldsymbol{r}$ will be sought in the space

$$\boldsymbol{\mathcal{Q}} := \{\boldsymbol{s} \in \mathrm{L}^2(\Omega)^{n\times n} : \ \boldsymbol{s}^{\mathtt{t}} = -\boldsymbol{s}\}.$$

We denote the Hilbertian product norm on $\mathrm{H}(\mathbf{div}, \Omega) \times \mathrm{L}^2(\Omega)^{n\times n}$ by

$$\|(\boldsymbol{\tau}, \boldsymbol{s})\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{s}\|^2_{0,\Omega}.$$

Now, in order to write the variational formulation of the spectral problem, we introduce the following symmetric bilinear forms in $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$:

$$B\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{r} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{s} : \boldsymbol{\sigma},$$

$$A\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) := \int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} + B\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big).$$

The variational formulation of the eigenvalue problem (2) is given as follows in terms of $\kappa := 1 + \omega^2$ (see [19] for more details): Find $\kappa \in \mathbb{R}$ and $\mathbf{0} \neq (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ such that

$$A\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) = \kappa\, B\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}. \tag{3}$$

We observe that the definition of bilinear form $A(\cdot,\cdot)$ includes bilinear form $B(\cdot,\cdot)$. This has been done in order to build an inf-sup stable bilinear form on the left-hand side of the spectral problem, which will allow us to define a solution operator (cf. (8)). The above standard procedure is called a shift argument and we notice that the original eigenvalues have been shifted to κ.

We notice that the bilinear form

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{C},\mathbf{div}} := \int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau}$$

also defines an inner product on $\boldsymbol{\mathcal{W}}$. Moreover, the following well-known result establishes that the norm induced by $(\cdot,\cdot)_{\mathcal{C},\mathbf{div}}$ is equivalent to $\|\cdot\|_{\mathrm{H}(\mathbf{div},\Omega)}$ uniformly in the Lamé coefficient λ.

Proposition 2.1. *There exist constants $c_2 \geq c_1 > 0$ independent of λ such that*

$$c_1\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)} \leq \|\boldsymbol{\tau}\|_{\mathcal{C},\mathbf{div}} \leq c_2\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

where $\|\boldsymbol{\tau}\|_{\mathcal{C},\mathbf{div}} := \sqrt{(\boldsymbol{\tau}, \boldsymbol{\tau})_{\mathcal{C},\mathbf{div}}}$.

Proof. The bound from above follows immediately from the fact that

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} = \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \frac{1}{n(n\lambda + 2\mu)}\int_\Omega (\operatorname{tr}\boldsymbol{\sigma})(\operatorname{tr}\boldsymbol{\tau}) \tag{4}$$

is bounded by a constant independent of λ. The left inequality may be found, for example, in [19, Lemma 2.1]. □

As a consequence of Proposition 2.1, there exists a constant $M > 0$ independent of λ such that

$$\left| A\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) \right| \leq M \, \|(\boldsymbol{\sigma}, \boldsymbol{r})\| \|(\boldsymbol{\tau}, \boldsymbol{s})\| \quad \forall (\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}. \tag{5}$$

The following result establishes an inf-sup condition for the bilinear form $A(\cdot, \cdot)$ uniformly in the Lamé coefficient λ.

Proposition 2.2. *There exists a constant $\alpha > 0$, depending on ρ, μ and Ω (but not on λ), such that*

$$\sup_{(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}} \frac{A\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big)}{\|(\boldsymbol{\tau}, \boldsymbol{s})\|} \geq \alpha \|(\boldsymbol{\sigma}, \boldsymbol{r})\| \quad \forall (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}. \tag{6}$$

Proof. It follows from Proposition 2.1 that

$$A\Big((\boldsymbol{\tau}, \mathbf{0}), (\boldsymbol{\tau}, \mathbf{0})\Big) = (\boldsymbol{\tau}, \boldsymbol{\tau})_{\mathcal{C}, \mathbf{div}} \geq C_1^2 \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div}, \Omega)} \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

with $C_1 > 0$ independent of λ. On the other hand, there exists a constant $\beta > 0$ depending only on Ω (see, for instance, [5]) such that

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\int_\Omega \boldsymbol{s} : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div}, \Omega)}} \geq \beta \|\boldsymbol{s}\|_{0,\Omega} \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{7}$$

Consequently, the Babuška-Brezzi theory (see [16]) shows that, for any bounded linear form $L \in \mathcal{L}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$, the problem: find $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ such that

$$A\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) = L\big(\boldsymbol{\tau}, \boldsymbol{s}\big) \qquad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$$

is well-posed, which proves (6). □

We deduce from Proposition 2.2 and from the symmetry of $A(\cdot, \cdot)$ that the solution operator $\boldsymbol{T} : [\mathrm{L}^2(\Omega)^{n \times n}]^2 \to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ defined for any $(\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n \times n}]^2$, by

$$A\Big(\boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) = B\Big((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \tag{8}$$

is well-defined and symmetric with respect to $A(\cdot, \cdot)$. Moreover, there exists a constant $C > 0$ independent of λ such that

$$\|\boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g})\| \leq C \|(\boldsymbol{f}, \boldsymbol{g})\|_{0,\Omega} \quad \forall (\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n \times n}]^2. \tag{9}$$

It is clear that $(\kappa, (\boldsymbol{\sigma}, \boldsymbol{r}))$ is a solution of (3) if and only if $\big(\eta = \frac{1}{\kappa}, (\boldsymbol{\sigma}, \boldsymbol{r})\big)$ is an eigenpair for $\boldsymbol{T}$. Let

$$\boldsymbol{\mathcal{K}} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} : \ \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0} \ \text{ in } \Omega\}. \tag{10}$$

From the definition of $\boldsymbol{T}$, it is clear that $\boldsymbol{T}|_{\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}} : \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ reduces to the identity. Thus, $\eta = 1$ is an eigenvalue of $\boldsymbol{T}$ with eigenspace $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$. We introduce the orthogonal subspace to $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ in $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ with respect to the bilinear form B,

$$[\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}]^{\perp_B} := \Big\{ (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} : \ B\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) = 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \Big\}.$$

Lemma 2.1. *The subspace* $[\mathcal{K}\times\mathcal{Q}]^{\perp_B}$ *is invariant for* $\boldsymbol{T}$, *i.e.,*

$$\boldsymbol{T}([\mathcal{K}\times\mathcal{Q}]^{\perp_B})\subset[\mathcal{K}\times\mathcal{Q}]^{\perp_B}.$$

Moreover, we have the direct and stable decomposition

$$\mathcal{W}\times\mathcal{Q}=[\mathcal{K}\times\mathcal{Q}]\oplus[\mathcal{K}\times\mathcal{Q}]^{\perp_B}. \tag{11}$$

Proof. See Lemma 3.3 and Lemma 3.4 of [19]. □

To the best of the authors' knowledge no intrinsic characterization of $[\mathcal{K}\times\mathcal{Q}]^{\perp_B}$ is known (see [19] for some further details). Nevertheless, we deduce from Lemma 2.1 that there exists a unique projection $\boldsymbol{P}:\mathcal{W}\times\mathcal{Q}\to\mathcal{W}\times\mathcal{Q}$ with range $[\mathcal{K}\times\mathcal{Q}]^{\perp_B}$ and kernel $\mathcal{K}\times\mathcal{Q}$ associated to the splitting (11). This is all what is needed for the ongoing analysis.

Let us consider the elasticity problem posed in Ω with a volume load in $\mathrm{L}^2(\Omega)^n$ and with homogeneous Dirichlet and Neumann boundary conditions on Γ_D and Γ_N, respectively: Given $\boldsymbol{f}\in\mathrm{L}^2(\Omega)^n$, let $(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\sigma}})\in\mathrm{H}^1(\Omega)^n\times\mathrm{H}(\mathbf{div},\Omega)$ be such that

$$\begin{aligned}
-\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}&=\boldsymbol{f} && \text{in }\Omega,\\
\widetilde{\boldsymbol{\sigma}}&=\mathcal{C}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) && \text{in }\Omega,\\
\widetilde{\boldsymbol{\sigma}}\boldsymbol{n}&=\mathbf{0} && \text{on }\Gamma_N,\\
\widetilde{\boldsymbol{u}}&=\mathbf{0} && \text{on }\Gamma_D.
\end{aligned}$$

This problem has a unique solution and, according to [10, 17], there exists $\widehat{s}\in(0,1]$ and $\widehat{C}>0$ that depend on Ω, λ and μ such that $\widetilde{\boldsymbol{u}}\in\mathrm{H}^{1+s}(\Omega)^n$ and

$$\|\widetilde{\boldsymbol{u}}\|_{\mathrm{H}^{1+s}(\Omega)^n}\leq\widehat{C}\|\boldsymbol{f}\|_{0,\Omega}\quad\forall s\in(0,\widehat{s}). \tag{12}$$

We point out that, in principle, the exponent $\widehat{s}$ and the constant $\widehat{C}$ in (12) depend on the Lamé coefficient λ. However, we know that (12) also holds true when $\lambda=+\infty$ (see the Appendix). Hence, it is natural to expect (12) to be satisfied uniformly in λ. To the best of authors' knowledge, such a result is not available in the literature. For this reason, from now on we make the following assumption.

Assumption 2.1. *Constants* $\widehat{s}$ *and* $\widehat{C}$ *in* (12) *are independent of* λ

Now, we are in a position to show that $\boldsymbol{P}$ and $\boldsymbol{T}\circ\boldsymbol{P}$ are regularizing operators.

Lemma 2.2. *For all* $s\in(0,\widehat{s})$, $\boldsymbol{P}(\mathcal{W}\times\mathcal{Q})\subset\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n}$ *and* $\boldsymbol{T}(\boldsymbol{P}(\mathcal{W}\times\mathcal{Q}))\subset\{(\boldsymbol{\tau},\boldsymbol{s})\in\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n}:\ \mathbf{div}\,\boldsymbol{\tau}\in\mathrm{H}^{1+s}(\Omega)^n\}$. *Moreover, there exists a constant* $C>0$, *independent of* λ, *such that*

$$\|\boldsymbol{P}(\boldsymbol{\tau},\boldsymbol{s})\|_{\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n}}\leq C\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}\quad\forall(\boldsymbol{\tau},\boldsymbol{s})\in\mathcal{W}\times\mathcal{Q} \tag{13}$$

and, if $(\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}):=\boldsymbol{T}\circ\boldsymbol{P}(\boldsymbol{\tau},\boldsymbol{s})$, *then*

$$\|(\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}})\|_{\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n}}+\|\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^{1+s}(\Omega)^n}\leq C\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}\quad\forall(\boldsymbol{\tau},\boldsymbol{s})\in\mathcal{W}\times\mathcal{Q}. \tag{14}$$

Proof. Estimate (13) is proved in [19, Lemma 3.2] and (14) follows as a consequence of (13) by an argument essentially identical to that of [19, Proposition 3.5]. □

The next result gives the spectral characterization for the solution operator $\boldsymbol{T}$.

Proposition 2.3. *The spectrum* $\mathrm{sp}(\boldsymbol{T})$ *of* $\boldsymbol{T}$ *decomposes as follows*

$$\mathrm{sp}(\boldsymbol{T}) = \{0, 1\} \cup \{\eta_k\}_{k\in\mathbb{N}}$$

where $\{\eta_k\}_k \subset (0, 1)$ *is a real sequence of finite-multiplicity eigenvalues of* $\boldsymbol{T}$ *which converges to 0. The ascent of each of these eigenvalues is* 1 *and the corresponding eigenfunctions lie in* $\boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$*. Moreover,* $\eta = 1$ *is an infinite-multiplicity eigenvalue of* $\boldsymbol{T}$ *with associated eigenspace* $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ *and* $\eta = 0$ *is not an eigenvalue.*

Proof. See [19, Theorem 3.7]. □

We end this section by providing a bound of the resolvent $\big(z\boldsymbol{I} - \boldsymbol{T}\big)^{-1}$.

Proposition 2.4. *If* $z \notin \mathrm{sp}(\boldsymbol{T})$*, there exists a constant* $C > 0$ *independent of* λ *and* z *such that*

$$\|\big(z\boldsymbol{I} - \boldsymbol{T}\big)(\boldsymbol{\sigma}, \boldsymbol{r})\| \geq C \ \mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big) \,\|(\boldsymbol{\sigma}, \boldsymbol{r})\| \quad \forall (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}},$$

where $\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)$ *represents the distance between* z *and the spectrum of* $\boldsymbol{T}$ *in the complex plane, which in principle depends on* λ*.*

Proof. See Proposition 2.4 in [18]. □

3 A discontinuous Galerkin discretization

We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega}$ into triangles/tetrahedra K of diameter h_K. The parameter $h := \max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. Hereafter, given an integer $m \geq 0$ and a domain $D \subset \mathbb{R}^n$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D.

We say that a closed subset $F \subset \overline{\Omega}$ is an interior edge/face if F has a positive $(n-1)$-dimensional measure and if there are distinct elements K and K' such that $F = \bar{K} \cap \bar{K}'$. A closed subset $F \subset \overline{\Omega}$ is a boundary edge/face if there exists $K \in \mathcal{T}_h$ such that F is an edge/face of K and $F = \bar{K} \cap \partial\Omega$. We consider the set $\mathcal{F}_h^0$ of interior edges/faces and the set $\mathcal{F}_h^\partial$ of boundary edges/faces. We assume that the boundary mesh $\mathcal{F}_h^\partial$ is compatible with the partition $\partial\Omega = \Gamma_D \cup \Gamma_N$, i.e.,

$$\bigcup_{F\in\mathcal{F}_h^D} F = \Gamma_D \qquad \text{and} \qquad \bigcup_{F\in\mathcal{F}_h^N} F = \Gamma_N,$$

where $\mathcal{F}_h^D := \{F \in \mathcal{F}_h^\partial; \quad F \subset \Gamma_D\}$ and $\mathcal{F}_h^N := \{F \in \mathcal{F}_h^\partial; \quad F \subset \Gamma_N\}$. We denote

$$\mathcal{F}_h := \mathcal{F}_h^0 \cup \mathcal{F}_h^\partial \qquad \text{and} \qquad \mathcal{F}_h^* := \mathcal{F}_h^0 \cup \mathcal{F}_h^N,$$

and for any element $K \in \mathcal{T}_h$, we introduce the set

$$\mathcal{F}(K) := \{F \in \mathcal{F}_h; \quad F \subset \partial K\}$$

of edges/faces composing the boundary of K. The space of piecewise polynomial functions of degree at most m relatively to $\mathcal{T}_h$ is denoted by

$$\mathcal{P}_m(\mathcal{T}_h) := \{v \in L^2(\Omega); \quad v|_K \in \mathcal{P}_m(K), \quad \forall K \in \mathcal{T}_h\}.$$

For any $k \geq 1$, we consider the finite element spaces

$$\boldsymbol{\mathcal{W}}_h := \mathcal{P}_k(\mathcal{T}_h)^{n\times n}, \qquad \boldsymbol{\mathcal{W}}_h^c := \boldsymbol{\mathcal{W}}_h \cap \boldsymbol{\mathcal{W}} \qquad \text{and} \qquad \boldsymbol{\mathcal{Q}}_h := \mathcal{P}_{k-1}(\mathcal{T}_h)^{n\times n} \cap \boldsymbol{\mathcal{Q}}.$$

The discrete space $\boldsymbol{\mathcal{W}}_h^c$ corresponds to the well-known Brezzi-Douglas-Marini (BDM) mixed finite element (see [7]) and will be useful in the forthcoming analysis. Let us now recall some well-known properties of $\boldsymbol{\mathcal{W}}_h^c$. For $t > 1/2$, the tensorial version of the BDM-interpolation operator $\Pi_h : \mathrm{H}^t(\Omega)^{n\times n} \to \boldsymbol{\mathcal{W}}_h^c$, satisfies the following classical error estimate, see [4, Proposition 2.5.4],

$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^{\min\{t,k+1\}}\|\boldsymbol{\tau}\|_{t,\Omega} \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^t(\Omega)^{n\times n}, \quad t > 1/2. \tag{15}$$

For less regular tensorial fields we also have the following error estimate

$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^t(\|\boldsymbol{\tau}\|_{t,\Omega} + \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}) \quad \forall \boldsymbol{\tau} \in \mathrm{H}^t(\Omega)^{n\times n} \cap \mathrm{H}(\mathbf{div},\Omega), \quad t \in (0, 1/2]. \tag{16}$$

Moreover, thanks to the commutativity property, if $\mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^t(\Omega)^n$, then

$$\|\mathbf{div}(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\|_{0,\Omega} = \|\mathbf{div}\,\boldsymbol{\tau} - \mathcal{R}_h\,\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^{\min\{t,k\}}\|\mathbf{div}\,\boldsymbol{\tau}\|_{t,\Omega}, \tag{17}$$

where $\mathcal{R}_h$ is the $\mathrm{L}^2(\Omega)^n$-orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{T}_h)^n$. Finally, we denote by $\mathcal{S}_h : \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $\mathrm{L}^2(\Omega)^{n\times n}$-norm. It is well-known that, for any $t > 0$, we have

$$\|\boldsymbol{s} - \mathcal{S}_h\boldsymbol{s}\|_{0,\Omega} \leq Ch^{\min\{t,k\}}\|\boldsymbol{s}\|_{t,\Omega} \qquad \forall \boldsymbol{s} \in \mathrm{H}^t(\Omega)^{n\times n} \cap \boldsymbol{\mathcal{Q}}. \tag{18}$$

For the analysis we need to decompose adequately the space $\boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$. We consider,

$$\boldsymbol{\mathcal{K}}_h = \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h^c; \quad \mathbf{div}\,\boldsymbol{\tau} = 0\} \subset \boldsymbol{\mathcal{K}}.$$

Lemma 3.1. *There exists a projection $\boldsymbol{P}_h : \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h \to \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$ with kernel $\boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h$ such that for all $s \in (0, \widehat{s})$, there exists a constant C independent of h and λ such that*

$$\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \leq C\,h^s\|\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} \quad \forall(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h.$$

Proof. The proof is similar to that of estimate (ii) of Lemma 4.2 from [19] □

For any $t \geq 0$, we consider the broken Sobolev space

$$\mathrm{H}^t(\mathcal{T}_h) := \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^n; \quad \boldsymbol{v}|_K \in \mathrm{H}^t(K)^n \quad \forall K \in \mathcal{T}_h\}.$$

For each $\boldsymbol{v} := \{\boldsymbol{v}_K\} \in \mathrm{H}^t(\mathcal{T}_h)^n$ and $\boldsymbol{\tau} := \{\boldsymbol{\tau}_K\} \in \mathrm{H}^t(\mathcal{T}_h)^{n\times n}$ the components $\boldsymbol{v}_K$ and $\boldsymbol{\tau}_K$ represent the restrictions $\boldsymbol{v}|_K$ and $\boldsymbol{\tau}|_K$. When no confusion arises, the restrictions of these functions will be written without any subscript. We will also need the space given on the skeletons of the triangulations $\mathcal{T}_h$ by

$$\mathrm{L}^2(\mathcal{F}_h) := \prod_{F\in\mathcal{F}_h} \mathrm{L}^2(F).$$

Similarly, the components χ_F of $\chi := \{\chi_F\} \in \mathrm{L}^2(\mathcal{F}_h)$ coincide with the restrictions $\chi|_F$ and we denote

$$\int_{\mathcal{F}_h} \chi := \sum_{F\in\mathcal{F}_h} \int_F \chi_F \quad \text{and} \quad \|\chi\|^2_{0,\mathcal{F}_h} := \int_{\mathcal{F}_h} \chi^2, \qquad \forall \chi \in \mathrm{L}^2(\mathcal{F}_h).$$

Analogously, $\|\chi\|^2_{0,\mathcal{F}_h^*} := \sum_{F\in\mathcal{F}_h^*} \int_F \chi_F^2$ for all $\chi \in \mathrm{L}^2(\mathcal{F}_h^*) := \prod_{F\in\mathcal{F}_h^*} \mathrm{L}^2(F)$.

From now on, $h_\mathcal{F} \in \mathrm{L}^2(\mathcal{F}_h)$ is the piecewise constant function defined by $h_\mathcal{F}|_F := h_F$ for all $F \in \mathcal{F}_h$ with h_F denoting the diameter of edge/face F.

Given a vector valued function $\boldsymbol{v} \in \mathrm{H}^t(\mathcal{T}_h)^n$ with $t > 1/2$, we define averages $\{\boldsymbol{v}\} \in \mathrm{L}^2(\mathcal{F}_h)^n$ and jumps $[\![\boldsymbol{v}]\!] \in \mathrm{L}^2(\mathcal{F}_h)$ by

$$\{\boldsymbol{v}\}_F := (\boldsymbol{v}_K + \boldsymbol{v}_{K'})/2 \quad \text{and} \quad [\![\boldsymbol{v}]\!]_F := \boldsymbol{v}_K \cdot \boldsymbol{n}_K + \boldsymbol{v}_{K'} \cdot \boldsymbol{n}_{K'} \quad \forall F \in \mathcal{F}(K) \cap \mathcal{F}(K'),$$

where $\boldsymbol{n}_K$ is the outward unit normal vector to ∂K. On the boundary of Ω we use the following conventions for averages and jumps:

$$\{\boldsymbol{v}\}_F := \boldsymbol{v}_K \quad \text{and} \quad [\![\boldsymbol{v}]\!]_F := \boldsymbol{v}_K \cdot \boldsymbol{n} \quad \forall F \in \mathcal{F}(K) \cap \partial\Omega.$$

Similarly, for matrix valued functions $\boldsymbol{\tau} \in \mathrm{H}^t(\mathcal{T}_h)^{n\times n}$, we define $\{\boldsymbol{\tau}\} \in \mathrm{L}^2(\mathcal{F}_h)^{n\times n}$ and $[\![\boldsymbol{\tau}]\!] \in \mathrm{L}^2(\mathcal{F}_h)^n$ by

$$\{\boldsymbol{\tau}\}_F := (\boldsymbol{\tau}_K + \boldsymbol{\tau}_{K'})/2 \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_F := \boldsymbol{\tau}_K \boldsymbol{n}_K + \boldsymbol{\tau}_{K'} \boldsymbol{n}_{K'} \quad \forall F \in \mathcal{F}(K) \cap \mathcal{F}(K')$$

and on the boundary of Ω we set

$$\{\boldsymbol{\tau}\}_F := \boldsymbol{\tau}_K \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_F := \boldsymbol{\tau}_K \boldsymbol{n} \quad \forall F \in \mathcal{F}(K) \cap \partial\Omega.$$

Given $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h$ we define $\mathbf{div}_h\, \boldsymbol{\tau} \in \mathrm{L}^2(\Omega)^n$ by $\mathbf{div}_h\, \boldsymbol{\tau}|_K = \mathbf{div}(\boldsymbol{\tau}|_K)$ for all $K \in \mathcal{T}_h$ and endow $\boldsymbol{\mathcal{W}}(h) := \boldsymbol{\mathcal{W}} + \boldsymbol{\mathcal{W}}_h$ with the seminorm

$$|\boldsymbol{\tau}|^2_{\boldsymbol{\mathcal{W}}(h)} := \|\mathbf{div}_h\, \boldsymbol{\tau}\|^2_{0,\Omega} + \|h_\mathcal{F}^{-1/2} [\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{F}_h^*}.$$

Let us remark that this seminorm is actually well defined for any function $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} + \boldsymbol{\mathcal{W}}_h$. Indeed, although in principle the jump $[\![\boldsymbol{\tau}]\!]_F$ appearing in this seminorm is defined for

$\boldsymbol{\tau} \in \mathrm{H}^t(\mathcal{T}_h)^{n\times n}$ with $t > 1/2$, this definition remains valid for any $\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div};\Omega)$ since, in such a case, $[\![\boldsymbol{\tau}]\!]_F$ vanishes (see [9], for a similar analysis in $\mathrm{H}(\mathbf{curl};\Omega)$).

Then, for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}(h)$ we define the norm

$$\|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{W}}(h)} := |\boldsymbol{\tau}|^2_{\boldsymbol{\mathcal{W}}(h)} + \|\boldsymbol{\tau}\|^2_{0,\Omega}.$$

For the sake of simplicity, we will also use the notation

$$\|(\boldsymbol{\tau},\boldsymbol{s})\|^2_{DG} := \|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{W}}(h)} + \|\boldsymbol{s}\|^2_{0,\Omega}$$

which is the norm in $\boldsymbol{\mathcal{W}}(h) \times \boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$.

The following result will be used in the sequel to ultimately derive a method free of spurious modes. Since, according to Proposition 2.3, the spectrum of $\boldsymbol{T}$ lies in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, we restrict our attention to this subset of the complex plane.

Lemma 3.2. *There exists a constant $C > 0$ independent of h and λ such that for all $z \in \mathbb{D} \setminus \mathrm{sp}(\boldsymbol{T})$ with $|z| \leq 1$, there holds*

$$\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \geq C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|\, \|(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \quad \forall (\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{W}}(h) \times \boldsymbol{\mathcal{Q}}.$$

Proof. We introduce

$$(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) := \boldsymbol{T}(\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$$

and notice that

$$(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) = \boldsymbol{T}(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s}).$$

By virtue of Proposition 2.4 and the boundedness of $\boldsymbol{T} : [\mathrm{L}^2(\Omega)^{n\times n}]^2 \to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ we have that

$$\begin{aligned} C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)\|(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)\| &\leq \|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)\| \leq \|\boldsymbol{T}(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\| \\ &\leq \|\boldsymbol{T}\|\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_0 \leq \|\boldsymbol{T}\|\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}. \end{aligned}$$

Finally, by the triangle inequality,

$$\begin{aligned} \|(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} &\leq |z|^{-1}\|(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)\| + |z|^{-1}\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \\ &\leq |z|^{-1}\left(1 + \frac{\|\boldsymbol{T}\|}{C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)}\right)\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \\ &\leq |z|^{-1}\left(\frac{C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big) + \|\boldsymbol{T}\|}{C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)}\right)\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}. \end{aligned}$$

Hence,

$$C|z|\left(\frac{\operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)}{\|\boldsymbol{T}\| + \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)}\right)\|(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \leq \|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}.$$

Since $\operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big) \leq |z| \leq 1$ and $\|\boldsymbol{T}\| \leq C'$ (with C' independent of λ), we derive from the above estimate that

$$\frac{C|z|}{1 + C'} \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)\|(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \leq \|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG},$$

and the result follows. □

Remark 3.1. *If E is a compact subset of $\mathbb{D} \setminus \mathrm{sp}(\boldsymbol{T})$, we deduce from Lemma 3.2 that there exists a constant $C > 0$ independent of h and λ such that, for all $z \in E$,*

$$\|\big(z\boldsymbol{I} - \boldsymbol{T}\big)^{-1}\|_{\mathcal{L}(\boldsymbol{\mathcal{W}}(h)\times\boldsymbol{\mathcal{Q}},\boldsymbol{\mathcal{W}}(h)\times\boldsymbol{\mathcal{Q}})} \leq \frac{C}{\mathrm{dist}\,\big(E, \mathrm{sp}(\boldsymbol{T})\big)|z|}.$$

Let us now introduce the discrete counterpart of (3). Given a parameter $\mathtt{a}_S > 0$, we introduce the symmetric bilinear form

$$\begin{aligned} A_h\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) &:= \int_\Omega \rho^{-1}\,\mathbf{div}_h\,\boldsymbol{\sigma}\cdot\mathbf{div}_h\,\boldsymbol{\tau} + B\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) \\ &+ \int_{\mathcal{F}_h^*} \mathtt{a}_S h_{\mathcal{F}}^{-1}\,[\![\boldsymbol{\sigma}]\!]\cdot[\![\boldsymbol{\tau}]\!] - \int_{\mathcal{F}_h^*}\big(\{\rho^{-1}\,\mathbf{div}_h\,\boldsymbol{\sigma}\}\cdot[\![\boldsymbol{\tau}]\!] + \{\rho^{-1}\,\mathbf{div}_h\,\boldsymbol{\tau}\}\cdot[\![\boldsymbol{\sigma}]\!]\big) \end{aligned} \tag{19}$$

and consider the DG method: Find $\kappa_h \in \mathbb{R}$ and $0 \neq (\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ such that

$$A_h\Big((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) = \kappa_h B\Big((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) \qquad \forall(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h. \tag{20}$$

We notice that, as it is usually the case for DG methods, the essential boundary condition is directly incorporated within the scheme.

A straightforward application of the Cauchy-Schwarz inequality shows that, for all $(\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}}(h) \times \boldsymbol{\mathcal{Q}}$ such that $\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^t(\Omega)^n$ with $t > 1/2$, there exists a constant $M^* > 0$ independent of h and λ such that

$$\Big|A_h\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big)\Big| \leq M^*\|(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG}^*\,\|(\boldsymbol{\tau}, \boldsymbol{s})\|_{DG}^*, \tag{21}$$

where

$$\|(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG}^* := \Big(\|(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG}^2 + \|h_{\mathcal{F}}^{1/2}\{\mathbf{div}\,\boldsymbol{\sigma}\}\|_{0,\mathcal{F}_h^*}^2\Big)^{1/2}.$$

Moreover, we deduce from the discrete trace inequality (see [13])

$$\|h_{\mathcal{F}}^{1/2}\{v\}\|_{0,\mathcal{F}_h} \leq C\|v\|_{0,\Omega} \quad \forall v \in \mathcal{P}_k(\mathcal{T}_h), \tag{22}$$

that for all $(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$,

$$\Big|A_h\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big)\Big| \leq M_{DG}\|(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG}^*\,\|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG}, \tag{23}$$

with $M_{DG} > 0$ is independent of h and λ.

4 The DG-discrete source operator

The following discrete projection operator from the DG-space $\boldsymbol{\mathcal{W}}_h$ onto the $\mathrm{H}(\mathbf{div}, \Omega)$-conforming mixed finite element space $\boldsymbol{\mathcal{W}}^c$ will be used in the forthcoming analysis to deduce the stability of the DG source problem by taking advantage of the inf-sup condition (27).

Proposition 4.1. *There exists a projection $\mathcal{I}_h : \boldsymbol{\mathcal{W}}_h \to \boldsymbol{\mathcal{W}}_h^c$ such that the norm equivalence*

$$\underline{C}\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{W}}(h)} \leq \Big(\|\mathcal{I}_h\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} + \|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{F}_h^*}\Big)^{1/2} \leq \bar{C}\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{W}}(h)} \tag{24}$$

holds true on $\boldsymbol{\mathcal{W}}_h$ with constants $\underline{C} > 0$ and $\bar{C} > 0$ independent of h. Moreover, we have that

$$\|\mathbf{div}_h(\boldsymbol{\tau} - \mathcal{I}_h\boldsymbol{\tau})\|^2_{0,\Omega} + \sum_{K\in\mathcal{T}_h} h_K^{-2}\|\boldsymbol{\tau} - \mathcal{I}_h\boldsymbol{\tau}\|^2_{0,K} \leq C_0\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{F}_h^*}, \tag{25}$$

with $C_0 > 0$ independent of h.

Proof. See [18, Proposition 5.2]. □

We can prove, with the aid of this result, that the bilinear form A_h, which according to its definition (19) depends on $\mathtt{a}_S$, satisfies the following inf-sup condition.

Proposition 4.2. *Let A_h be defined as in* (19)*. Then, there exists a positive parameter $\mathtt{a}_S^*$ such that, for all $\mathtt{a}_S \geq \mathtt{a}_S^*$,*

$$\sup_{(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h} \frac{A_h\Big((\boldsymbol{\sigma}_h,\boldsymbol{r}_h),(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\Big)}{\|(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\|_{DG}} \geq \alpha_{DG}\|(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\|_{DG} \quad \forall(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h \tag{26}$$

with $\alpha_{DG} > 0$ independent of h and λ.

Proof. It is shown in [18, Proposition 3.1] that there exists a constant $\alpha_A^c > 0$ independent of h and λ such that

$$\sup_{(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h^c\times\boldsymbol{\mathcal{Q}}_h} \frac{A\Big((\boldsymbol{\sigma}_h,\boldsymbol{r}_h),(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\Big)}{\|(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\|} \geq \alpha_A^c\|(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\| \quad \forall(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\in\boldsymbol{\mathcal{W}}_h^c\times\boldsymbol{\mathcal{Q}}_h. \tag{27}$$

It follows that there exists an operator $\Theta_h : \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h \to \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$ satisfying

$$A\Big((\boldsymbol{\sigma}_h,\boldsymbol{r}_h),\Theta_h(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\Big) = \alpha_A^c\|(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\|^2 \quad \text{and} \quad \|\Theta_h(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\| \leq \|(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\| \tag{28}$$

for all $(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\in\boldsymbol{\mathcal{W}}_h^c\times\boldsymbol{\mathcal{Q}}_h$.

Given $(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h$, the decomposition $\boldsymbol{\tau}_h = \boldsymbol{\tau}_h^c + \tilde{\boldsymbol{\tau}}_h$, with $\boldsymbol{\tau}_h^c := \mathcal{I}_h\boldsymbol{\tau}_h$ and $\tilde{\boldsymbol{\tau}}_h := \boldsymbol{\tau}_h - \mathcal{I}_h\boldsymbol{\tau}_h$, and (28) yield

$$\begin{aligned} A_h\Big((\boldsymbol{\tau}_h,\boldsymbol{s}_h),\Theta_h(\boldsymbol{\tau}_h^c,\boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h,\mathbf{0})\Big) &= \alpha_A^c\|(\boldsymbol{\tau}_h^c,\boldsymbol{s}_h)\|^2 + \\ A_h\Big((\boldsymbol{\tau}_h^c,\boldsymbol{s}_h),(\tilde{\boldsymbol{\tau}}_h,\mathbf{0})\Big) &+ A_h\Big((\tilde{\boldsymbol{\tau}}_h,\mathbf{0}),\Theta_h(\boldsymbol{\tau}_h^c,\boldsymbol{s}_h)\Big) + A_h\Big((\tilde{\boldsymbol{\tau}}_h,\mathbf{0}),(\tilde{\boldsymbol{\tau}}_h,\mathbf{0})\Big). \end{aligned} \tag{29}$$

By the Cauchy-Schwarz inequality,

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) = \rho^{-1}\|\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\|^2_{0,\Omega} + \mathtt{a}_S\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{F}_h^*} + \int_\Omega \mathcal{C}^{-1}\tilde{\boldsymbol{\tau}}_h : \tilde{\boldsymbol{\tau}}_h$$
$$- 2\int_{\mathcal{F}_h^*}\{\rho^{-1}\,\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\}\cdot[\![\tilde{\boldsymbol{\tau}}_h]\!] \;\geq\; \mathtt{a}_S\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{F}_h^*}$$
$$- 2\rho^{-1}\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\}\|_{0,\mathcal{F}_h^*}\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}$$

and we deduce from (22) and (25) that

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq (\mathtt{a}_S - C_1)\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{F}_h^*},$$

with a constant C_1 independent of h and λ.

We proceed similarly for the terms in the right-hand side of (29). Indeed, it is straightforward that

$$A_h\Big((\boldsymbol{\tau}_h^c, \boldsymbol{s}_h), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq -\rho^{-1}\|\mathbf{div}\, \boldsymbol{\tau}_h^c\|_{0,\Omega}\|\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} - C_2\|\tilde{\boldsymbol{\tau}}_h\|_{0,\Omega}(\|\boldsymbol{\tau}_h^c\|_{0,\Omega} + \|\boldsymbol{s}_h\|_{0,\Omega}) -$$
$$\rho^{-1}\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}\, \boldsymbol{\tau}_h^c\}\|_{0,\mathcal{F}_h^*}\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*},$$

and using again (22) and (25) we obtain

$$A_h\Big((\boldsymbol{\tau}_h^c, \boldsymbol{s}_h), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq -C_3\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\| \geq$$
$$-\frac{\alpha_A^c}{4}\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 - C_4\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{F}_h^*}$$

with $C_4 > 0$ independent of h and λ. Similar estimates lead to

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\Big) \geq -C_5\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}\|\Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\| \geq$$
$$- C_5\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|,$$

where the last inequality follows from (28). We conclude that there exists $C_6 > 0$ independent of h and λ such that

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\Big) \geq -\frac{\alpha_D^c}{4}\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 - C_6\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{F}_h^*}.$$

We then have shown that,

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq \frac{\alpha_A^c}{2}\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 + \big(\mathtt{a}_S - C_7\big)\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{F}_h},$$

with $C_7 := C_1 + C_4 + C_6$. Consequently, if $\mathtt{a}_S > \mathtt{a}_S^* := C_7 + \frac{\alpha_A^c}{2}$,

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq \frac{\alpha_A^c}{2}\Big(\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 + \|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{F}_h^*}\Big),$$

and thanks to (24), we conclude that there exists $\alpha_{DG} > 0$ such that,

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq \alpha_{DG} \|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG}\Big(\|\Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\|_{DG}\Big),$$

which gives (26). □

In the sequel, we assume that the stabilization parameter is large enough (namely $\mathtt{a}_S \geq \mathtt{a}_S^*$) so that the inf-sup condition (26) is guaranteed. The first consequence of this inf-sup condition is that the discrete solution operator $\boldsymbol{T}_h : \mathrm{L}^2(\Omega)^{n\times n} \times \mathrm{L}^2(\Omega)^{n\times n} \to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ characterized, for any $(\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2$, by

$$A_h\Big(\boldsymbol{T}_h(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) = B\Big((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \tag{30}$$

is well-defined, symmetric with respect to $A_h(\cdot,\cdot)$ and there exists a constant $C > 0$ independent of λ and h such that

$$\|\boldsymbol{T}_h(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq C \|(\boldsymbol{f}, \boldsymbol{g})\|_{0,\Omega} \quad \forall (\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2. \tag{31}$$

We observe that if $(\kappa_h, (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)) \in \mathbb{R} \times \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}$ is a solution of problem (20) if and only if $(\mu_h, (\boldsymbol{\sigma}_h, \boldsymbol{r}_h))$, with $\mu_h = 1/(1+\kappa_h)$ is an eigenpair of $\boldsymbol{T}_h$, i.e.

$$\boldsymbol{T}_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) = \frac{1}{1+\kappa_h}(\boldsymbol{\sigma}_h, \boldsymbol{r}_h).$$

The following result establishes the convergence properties (Céa estimate) for the solutions opetators $\boldsymbol{T}$ and $\boldsymbol{T}_h$. We recall that $\widehat{s}$ is the Sobolev exponent for which (12) holds true.

Theorem 4.1. *Let* $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ *and* $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}) := \boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g})$. *Then,*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq \left(1 + \frac{M_{DG}}{\alpha_{DG}}\right) \inf_{(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h} \|(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}) - (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG}^*, \tag{32}$$

with M_{DG} *and* α_{DG} *as in* (23) *and* (26), *respectively. Moreover, for all* $s \in (0, \widehat{s})$, *the error estimate*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq C\, h^s \Big(\|\tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^s(\Omega)^{n\times n}} + \|\tilde{\boldsymbol{r}}\|_{\mathrm{H}^s(\Omega)^{n\times n}} + \|\mathbf{div}\, \tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^{1+s}(\Omega)^n}\Big), \tag{33}$$

holds true with a constant $C > 0$ *independent of* h *and* λ.

Proof. We first notice that the DG approximation (30) is consistent with regards to its continuous counterpart (8) in the sense that

$$A_h\Big((\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) = 0 \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h. \tag{34}$$

Indeed, by definition,

$$A_h\Big((\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}),(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\Big) = \int_\Omega \rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\cdot\mathbf{div}_h\,\boldsymbol{\tau}_h + B\Big((\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}),(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\Big) - \int_{\mathcal{F}_h^*}\{\rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\}\cdot[\![\boldsymbol{\tau}_h]\!]. \quad (35)$$

Note that the average in the last term above is well defined since, according to Lemma 2.2, $\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\in \mathrm{H}^{1+s}(\Omega)^n$.

It is straightforward to deduce from (8)

$$\nabla\left(\rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\right) = \mathcal{C}^{-1}(\tilde{\boldsymbol{\sigma}}-\boldsymbol{f}) + \tilde{\boldsymbol{r}} - \boldsymbol{g} \quad \text{and} \quad (\tilde{\boldsymbol{\sigma}}-\tilde{\boldsymbol{\sigma}}^{\mathtt{t}})/2 = (\boldsymbol{f}-\boldsymbol{f}^{\mathtt{t}})/2. \quad (36)$$

Moreover, an integration by parts yields

$$\begin{aligned}\int_\Omega \rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\cdot\mathbf{div}_h\,\boldsymbol{\tau}_h &= -\sum_{K\in\mathcal{T}_h}\int_K \nabla(\rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}) : \boldsymbol{\tau}_h + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\tau}_h\boldsymbol{n}_K \\ &= -\sum_{K\in\mathcal{T}_h}\int_K \nabla(\rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}) : \boldsymbol{\tau}_h + \int_{\mathcal{F}_h^*}\{\rho^{-1}\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\}\cdot[\![\boldsymbol{\tau}_h]\!].\end{aligned}$$

Substituting back the last identity and (36) into (35) we obtain

$$A_h\Big((\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}),(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\Big) = B\Big((\boldsymbol{f},\boldsymbol{g}),(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\Big) \quad \forall(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h$$

and (34) follows.

The Céa estimate (32) follows now in the usual way by taking advantage of (34), the inf-sup condition (26), estimate (23), and the triangle inequality.

Now, in order to obtain (33), from (32) we have that

$$\|(\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{f},\boldsymbol{g})\|_{DG} \leq \left(1+\frac{M_{DG}}{\alpha_{DG}}\right)\|(\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}) - (\Pi_h\tilde{\boldsymbol{\sigma}},\mathcal{S}_h\tilde{\boldsymbol{r}})\|_{DG}^*. \quad (37)$$

Using the interpolation error estimates (15), (17), (18) and the additional regularity in Lemma 2.2, we immediately obtain for all $s\in(0,\widehat{s})$,

$$\begin{aligned}\|(\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}) - (\Pi_h\tilde{\boldsymbol{\sigma}},\mathcal{S}_h\tilde{\boldsymbol{r}})\|_{DG} &= \|(\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{r}}) - (\Pi_h\tilde{\boldsymbol{\sigma}},\mathcal{S}_h\tilde{\boldsymbol{r}})\| \\ &\leq C_0\,h^s\left(\|\tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^s(\Omega)^{n\times n}} + \|\tilde{\boldsymbol{r}}\|_{\mathrm{H}^s(\Omega)^{n\times n}} + \|\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^{1+s}(\Omega)^n}\right). \quad (38)\end{aligned}$$

Moreover, we notice that

$$\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}(\tilde{\boldsymbol{\sigma}}-\Pi_h\tilde{\boldsymbol{\sigma}})\}\|_{0,\mathcal{F}_h^*}^2 \leq \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(K)} h_F\|\mathbf{div}(\tilde{\boldsymbol{\sigma}}-\Pi_h\tilde{\boldsymbol{\sigma}})\|_{0,F}^2.$$

Under the regularity hypotheses on $\tilde{\boldsymbol{\sigma}}$, the commuting diagram property satisfied by Π_h, the trace theorem and standard scaling arguments give

$$h_F \|\mathbf{div}(\tilde{\boldsymbol{\sigma}} - \Pi_h \tilde{\boldsymbol{\sigma}})\|_{0,F}^2 = h_F \|\mathbf{div}\, \tilde{\boldsymbol{\sigma}} - \mathcal{R}_K \,\mathbf{div}\, \tilde{\boldsymbol{\sigma}}\|_{0,F}^2 \leq C_2 h_K^{2+2s} \|\mathbf{div}\, \tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^{1+s}(K)^n}^2$$

for all $F \in \mathcal{F}(K)$, where the $\mathrm{L}^2(K)$-orthogonal projection $\mathcal{R}_K := \mathcal{R}_h|_K$ onto $\mathcal{P}_{k-1}(K)$ is applied componentwise. It follows that

$$\|h_{\mathcal{F}}^{1/2} \{\mathbf{div}(\tilde{\boldsymbol{\sigma}} - \Pi_h \tilde{\boldsymbol{\sigma}})\}\|_{0,\mathcal{F}_h^*} \leq C_3 h_K^{1+s} \left(\sum_{K \in \mathcal{T}_h} \|\mathbf{div}\, \tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^{1+s}(K)^n}^2 \right)^{1/2} \leq C_3 h_K^{1+s} \|\mathbf{div}\, \tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^{1+s}(\Omega)^n}. \tag{39}$$

Combining (39) and (38) with (37) proves the asymptotic error estimate (33). □

We end this section by providing technical results that will be used to establish the spectral approximation properties of the proposed DG method.

Corollary 4.1. *For all $s \in (0, \widehat{s})$, there exists a constant $C > 0$ independent of h and λ, such that for all $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG} \leq C\, h^s \,\|\mathbf{div}\, \boldsymbol{\sigma}\|_{0,\Omega}.$$

Proof. The result is a consequence of Theorem 4.1 and Lemma 2.2. □

Lemma 4.1. *For all $s \in (0, \widehat{s})$, there exists a constant $C > 0$ independent of h and λ such that*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG} \leq C\, h^s \,\|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG} \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h.$$

Proof. For any $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h$ we consider the splitting $\boldsymbol{\tau}_h = \boldsymbol{\tau}_h^c + \tilde{\boldsymbol{\tau}}_h$ with $\boldsymbol{\tau}_h^c := \mathcal{I}_h \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h^c$. We have that

$$\begin{aligned}(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h) &= (\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0}) + (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) \\ &= (\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0}) + (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h),\end{aligned}$$

where the last identity is due to the fact that $(\boldsymbol{I} - \boldsymbol{P}_h)(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h$ and $\boldsymbol{T} - \boldsymbol{T}_h$ vanishes identically on this subspace. It follows that

$$(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h) = (\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0}) + (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h),$$

and the triangle inequality together with (9) and (31) yield

$$\begin{aligned}\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG} &\leq \|(\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\|_{DG} + \|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG} \\ &+ \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG} \leq \Big(\|\boldsymbol{T}\|_{\mathcal{L}([\mathrm{L}^2(\Omega)^{n\times n}]^2, \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})} + \|\boldsymbol{T}_h\|_{\mathcal{L}([\mathrm{L}^2(\Omega)^{n\times n}]^2, \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h)} \Big) \\ &\Big(\|\tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\| \Big) + \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG}.\end{aligned}$$

Using (25), Lemma 3.1 and Corollary 4.1, we have that

$$\|\tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} \leq Ch\|\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{W}}(h)},$$

$$\|(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\| \leq Ch^s\|\mathbf{div}\,\boldsymbol{\tau}_h^c\|_{0,\Omega} \leq Ch^s\|\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{W}}(h)}$$

and

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG} \leq Ch^s\|\mathbf{div}\,\boldsymbol{\tau}_h^c\|_{0,\Omega} \leq Ch^s\|\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{W}}(h)},$$

respectively, which gives the result. □

5 Spectral correctness of the DG method

The convergence analysis follows the same steps introduced in [11, 12], we only need to adapt it to the DG context (cf. also [9]).

For the sake of brevity, we will denote in this section $\mathbb{X} := \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, $\mathbb{X}_h := \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ and $\mathbb{X}(h) := \boldsymbol{\mathcal{W}}(h) \times \boldsymbol{\mathcal{Q}}$. Moreover, when no confusion can arise, we will use indistinctly $\boldsymbol{x}$, $\boldsymbol{y}$, etc. to denote elements in $\mathbb{X}$ and, analogously, $\boldsymbol{x}_h$, $\boldsymbol{y}_h$, etc. for those in $\mathbb{X}_h$. Finally, we will use $\|\cdot\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}$ to denote the norm of an operator restricted to the discrete subspace $\mathbb{X}_h$; namely, if $\boldsymbol{S} : \mathbb{X}(h) \to \mathbb{X}(h)$, then

$$\|\boldsymbol{S}\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))} := \sup_{\mathbf{0}\neq\boldsymbol{x}_h\in\mathbb{X}_h} \frac{\|\boldsymbol{S}\boldsymbol{x}_h\|_{DG}}{\|\boldsymbol{x}_h\|_{DG}}. \tag{40}$$

The following result will be used to establish that the proposed DG scheme does not introduce spurious eigenvalues.

Lemma 5.1. *If $z \in \mathbb{D} \setminus \mathrm{sp}(\boldsymbol{T})$, there exists $h_0 > 0$ such that if $h \leq h_0$,*

$$\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h\|_{DG} \geq C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|\,\|\boldsymbol{x}_h\|_{DG} \quad \forall \boldsymbol{x}_h \in \mathbb{X}_h.$$

with $C > 0$ independent of h and λ.

Proof. It follows from

$$(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h = (z\boldsymbol{I} - \boldsymbol{T})\boldsymbol{x}_h + (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{x}_h$$

and Lemma 3.2 that

$$\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h\|_{DG} \geq \Big(C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z| - \|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\Big)\|\boldsymbol{x}_h\|_{DG}$$

and the result follows from Lemma 4.1. □

Analogously to the continuous case, we prove that the discrete resolvent associated to the discrete operator $\boldsymbol{T}_h$ is bounded.

Lemma 5.2. *If $z \in \mathbb{D} \setminus \mathrm{sp}(\boldsymbol{T})$, there exists $h_0 > 0$ such that if $h \leq h_0$,*

$$\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG} \geq C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|^2 \,\|\boldsymbol{x}\|_{DG} \quad \forall \boldsymbol{x} \in \mathbb{X}(h),$$

with $C > 0$ independent of h and λ.

Proof. Given $\boldsymbol{x} \in \mathbb{X}(h)$ we let

$$\boldsymbol{x}_h^* = \boldsymbol{T}_h \boldsymbol{x} \in \mathbb{X}_h.$$

We deduce from the identity

$$(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h^* = \boldsymbol{T}_h(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}$$

and from Lemma 5.1 that

$$C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z| \|\boldsymbol{x}_h^*\|_{DG} \leq \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h^*\|_{DG} \leq \|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)} \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG}.$$

This and the triangle inequality leads to

$$\begin{aligned}
\|\boldsymbol{x}\|_{DG} &\leq |z|^{-1} \|\boldsymbol{x}_h^*\|_{DG} + |z|^{-1} \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG} \\
&\leq |z|^{-1} \left(1 + \frac{\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}}{C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|}\right) \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG}. \\
&\leq |z|^{-1} \left(\frac{C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z| + \|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}}{C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|}\right) \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG}.
\end{aligned}$$

Hence,

$$C|z| \left(\frac{C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|}{\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)} + C \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|}\right) \|\boldsymbol{x}\|_{DG} \leq \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG}.$$

Now, using that $\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big) \leq |z| \leq 1$ and $\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)} \leq C'$ (with C' independent of λ), from the estimate above we derive

$$C|z|^2 \,\mathrm{dist}\,\big(z, \mathrm{sp}(\boldsymbol{T})\big) \|\boldsymbol{x}\|_{DG} \leq \|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau}, \boldsymbol{s})\|_{DG},$$

and the result follows. □

Remark 5.1. *If E is a compact subset of $\mathbb{D} \setminus \mathrm{sp}(\boldsymbol{T})$ and h is small enough, we deduce from Lemma 5.2 that $(z\boldsymbol{I} - \boldsymbol{T}_h) : \mathbb{X}(h) \to \mathbb{X}(h)$ is invertible for all $z \in E$. Hence, $E \subset \mathbb{D} \setminus \mathrm{sp}(\boldsymbol{T}_h)$. Consequently, for h small enough, the numerical method does not introduce spurious eigenvalues. Moreover, we have that there exists a constant $C > 0$ independent of h and λ such that, for all $z \in E$,*

$$\|\big(z\boldsymbol{I} - \boldsymbol{T}_h\big)^{-1}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))} \leq \frac{C}{\mathrm{dist}(E, \mathrm{sp}(\boldsymbol{T}))|z|^2}.$$

For $\boldsymbol{x} \in \mathbb{X}(h)$ and $\mathbb{E}$ and $\mathbb{F}$ closed subspaces of $\mathbb{X}(h)$, we set $\delta(\boldsymbol{x}, \mathbb{E}) := \inf_{\boldsymbol{y}\in\mathbb{E}} \|\boldsymbol{x}-\boldsymbol{y}\|_{DG}$, $\delta(\mathbb{E}, \mathbb{F}) := \sup_{\boldsymbol{y}\in\mathbb{E}:\, \|\boldsymbol{y}\|=1} \delta(\boldsymbol{y}, \mathbb{F})$, and $\widehat{\delta}(\mathbb{E}, \mathbb{F}) := \max\{\delta(\mathbb{E}, \mathbb{F}), \delta(\mathbb{F}, \mathbb{E})\}$, the latter being the so called *gap* between subspaces $\mathbb{E}$ and $\mathbb{F}$.

Given an isolated eigenvalue $\kappa \neq 1$ of $\boldsymbol{T}$, we define

$$\mathtt{d}_\kappa := \frac{1}{2} \operatorname{dist}\big(\kappa, \operatorname{sp}(\boldsymbol{T}) \setminus \{\kappa\}\big).$$

It follows that the closed disk $D_\kappa := \{z \in \mathbb{C} : \quad |z-\kappa| \leq \mathtt{d}_\kappa\}$ of the complex plane, with center κ and boundary γ is such that $D_\kappa \cap \operatorname{sp}(\boldsymbol{T}) = \{\kappa\}$. We deduce from Remark 3.1 that the operator $\boldsymbol{\mathcal{E}} := \frac{1}{2\pi i}\int_\gamma (z\boldsymbol{I}-\boldsymbol{T})^{-1}\, dz : \mathbb{X}(h) \longrightarrow \mathbb{X}(h)$ is well-defined and bounded uniformly in h. Moreover, $\boldsymbol{\mathcal{E}}|_{\mathbb{X}}$ is a spectral projection in $\mathbb{X}$ onto the (finite dimensional) eigenspace $\boldsymbol{\mathcal{E}}(\mathbb{X})$ corresponding to the eigenvalue κ of $\boldsymbol{T}$. In fact,

$$\boldsymbol{\mathcal{E}}(\mathbb{X}(h)) = \boldsymbol{\mathcal{E}}(\mathbb{X}). \tag{41}$$

To prove this, let $\kappa^* \in D_\kappa$ be an eigenvalue of $\boldsymbol{T} : \mathbb{X}(h) \to \mathbb{X}(h)$ and $\boldsymbol{x}^* \in \mathbb{X}(h)$ be the corresponding eigenfunction. Since $\kappa^* \neq 0$ and $\boldsymbol{T}(\mathbb{X}(h)) \subset \mathbb{X}$, we actually have that $\boldsymbol{x}^* \in \mathbb{X}$. Then, necessarily, $\kappa^* = \kappa$ and taking into account that $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is the eigenspace associated with κ we deduce (41).

Similarly, we deduce from Remark 5.1 that, for h small enough, the operator $\boldsymbol{\mathcal{E}}_h := \frac{1}{2\pi i}\int_\gamma (z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\, dz : \mathbb{X}(h) \longrightarrow \mathbb{X}(h)$ is also well-defined and bounded uniformly in h. Moreover, $\boldsymbol{\mathcal{E}}_h|_{\mathbb{X}_h}$ is a projector in $\mathbb{X}_h$ onto the eigenspace $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ corresponding to the eigenvalues of $\boldsymbol{T}_h : \mathbb{X}_h \to \mathbb{X}_h$ contained in γ. The same arguments as above show that we also have,

$$\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)) = \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h).$$

Our aim now is to compare $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ to $\boldsymbol{\mathcal{E}}(\mathbb{X})$ in terms of the gap $\widehat{\delta}$.

Lemma 5.3. *There exists $C > 0$, independent of h and λ, such that*

$$\|\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))} \leq \frac{C}{\mathtt{d}_\kappa} \|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))}. \tag{42}$$

Proof. We deduce from the identity

$$(z\boldsymbol{I}-\boldsymbol{T})^{-1} - (z\boldsymbol{I}-\boldsymbol{T}_h)^{-1} = (z\boldsymbol{I}-\boldsymbol{T})^{-1}(\boldsymbol{T}-\boldsymbol{T}_h)(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}$$

that, for any $\boldsymbol{x}_h \in \mathbb{X}_h$,

$$\begin{aligned}
\|(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h)\boldsymbol{x}_h\|_{DG} &\leq \frac{1}{2\pi}\int_\gamma \|[(z\boldsymbol{I}-\boldsymbol{T})^{-1} - (z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}]\boldsymbol{x}_h\|_{DG} |dz| \\
&= \frac{1}{2\pi}\int_\gamma \|[(z\boldsymbol{I}-\boldsymbol{T})^{-1}(\boldsymbol{T}-\boldsymbol{T}_h)(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}]\boldsymbol{x}_h\|_{DG} |dz| \\
&\leq \frac{1}{2\pi}\int_\gamma \|(z\boldsymbol{I}-\boldsymbol{T})^{-1}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))} \|\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))} \|(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}_h)} \|\boldsymbol{x}_h\|_{DG} |dz|
\end{aligned}$$

and the result follows from Lemmas 3.2 and 5.2, the definition (40) and the fact that for all $z \in \gamma$, $|z| \geq \kappa - \mathtt{d}_\kappa \geq \frac{1}{2}\kappa$. ☐

The following theorem will be used to establish the approximation properties of the eigenfunctions of problem (3) by means of those of problem (20).

Theorem 5.1. *There exists a constant $C > 0$ independent of h and λ such that*

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) \leq C \left(\frac{\|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))}}{\mathtt{d}_\kappa} + \delta(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) \right).$$

Proof. As $\boldsymbol{\mathcal{E}}_h$ is a projector, for h sufficiently small, we have that $\boldsymbol{\mathcal{E}}_h \boldsymbol{x}_h = \boldsymbol{x}_h$ for all $\boldsymbol{x}_h \in \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$. It follows from (41) that $\boldsymbol{\mathcal{E}} \boldsymbol{x}_h \in \boldsymbol{\mathcal{E}}(\mathbb{X})$, which leads to

$$\delta(\boldsymbol{x}_h, \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq \|\boldsymbol{\mathcal{E}}_h \boldsymbol{x}_h - \boldsymbol{\mathcal{E}} \boldsymbol{x}_h\|_{DG} \leq \|\boldsymbol{\mathcal{E}}_h - \boldsymbol{\mathcal{E}}\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))} \|\boldsymbol{x}_h\|_{DG}$$

for all $\boldsymbol{x}_h \in \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$. We deduce from (42) that

$$\delta(\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h), \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq \frac{C}{\mathtt{d}_\kappa} \|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))}. \tag{43}$$

On the other hand, as $\boldsymbol{\mathcal{E}} \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}(\mathbb{X})$, for h small enough and $\boldsymbol{y}_h \in \mathbb{X}_h$,

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{\mathcal{E}}_h \boldsymbol{y}_h\|_{DG} &\leq \|\boldsymbol{\mathcal{E}}(\boldsymbol{x} - \boldsymbol{y}_h)\|_{DG} + \|(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h)\boldsymbol{y}_h\|_{DG} \leq \\ &\quad \|\boldsymbol{\mathcal{E}}\|_{\mathcal{L}(\mathbb{X}(h), \mathbb{X}(h))} \|(\boldsymbol{x} - \boldsymbol{y}_h)\|_{DG} + \|(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h)\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))} \|\boldsymbol{y}_h\|_{DG} \\ &\leq \big(\|\boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}(h), \mathbb{X}(h))} + 2\|\boldsymbol{\mathcal{E}}\|_{\mathcal{L}(\mathbb{X}(h), \mathbb{X}(h))}\big) \|\boldsymbol{x} - \boldsymbol{y}_h\|_{DG} + \|\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))} \|\boldsymbol{x}\|_{DG}. \end{aligned}$$

Consequently,

$$\delta(\boldsymbol{x}, \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) \leq C(\delta(\boldsymbol{x}, \mathbb{X}_h) + \|\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))})$$

for all $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}(\mathbb{X})$ such that $\|\boldsymbol{x}\|_{DG} = 1$ and using that the eigenspace $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is finite dimensional we deduce that

$$\delta(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) \leq C(\delta(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) + \|\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))})$$

and the result follows from the last estimate and (43). □

We end this section with a result that establishes the convergence properties of the eigenvalues and eigenfunctions.

Theorem 5.2. *Let $\kappa \neq 1$ be an eigenvalue of $\boldsymbol{T}$ of algebraic multiplicity m and let D_κ be a closed disk in the complex plane centered at κ with boundary γ such that $D_\kappa \cap \mathrm{sp}(\boldsymbol{T}) = \{\kappa\}$. Let $\kappa_{1,h}, \ldots, \kappa_{m(h),h}$ be the eigenvalues of $\boldsymbol{T}_h : \mathbb{X}_h \to \mathbb{X}_h$ lying in D_κ and repeated according to their algebraic multiplicity. Then, we have that $m(h) = m$ for h sufficiently small and*

$$\lim_{h \to 0} \max_{1 \leq i \leq m} |\kappa - \kappa_{i,h}| = 0.$$

Moreover, if $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is the eigenspace corresponding to κ and $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ is the $\boldsymbol{T}_h$-invariant subspace of $\mathbb{X}_h$ spanned by the eigenspaces corresponding to $\{\kappa_{i,h},\ i = 1, \ldots, m\}$ then

$$\lim_{h \to 0} \widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) = 0.$$

Proof. To prove the result, we will use Theorem 5.1. First, we deduce from Lemma 4.1 that

$$\lim_{h\to 0}\|\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}=0.$$

Moreover, since $\boldsymbol{\mathcal{E}}(\mathbb{X})\subset \boldsymbol{T}\circ\boldsymbol{P}(\boldsymbol{\sigma},\boldsymbol{r})\subset\{(\boldsymbol{\tau},\boldsymbol{s})\in[\mathrm{H}^s(\Omega)^{n\times n}]^2 : \mathbf{div}\,\boldsymbol{\tau}\in \mathrm{H}^{1+s}(\Omega)^n\}$ for all $s\in(0,\widehat{s})$, it follows from (33) that

$$\lim_{h\to 0}\delta(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h)=0.$$

Hence, by virtue of Theorem 5.1, we have that

$$\lim_{h\to 0}\widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}),\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h))=0,$$

and, as a consequence, $\boldsymbol{\mathcal{E}}(\mathbb{X})$ and $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ have the same dimension provided h is sufficiently small. Finally, being κ an isolated eigenvalue and the radius of the circle γ arbitrary, we deduce that

$$\lim_{h\to 0}\max_{1\le i\le m}|\kappa-\kappa_{i,h}|=0.$$

□

6 Asymptotic error estimates

Along this section we fix a particular eigenvalue $\kappa\neq 1$ of $\boldsymbol{T}$. We wish to obtain error estimates for the eigenfunctions and the eigenvalues in terms of the quantity

$$\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h):=\sup_{\boldsymbol{x}\in\boldsymbol{\mathcal{E}}(\mathbb{X}),\|\boldsymbol{x}\|=1}\ \inf_{\boldsymbol{x}_h\in\mathbb{X}_h}\|\boldsymbol{x}-\boldsymbol{x}_h\|^*_{DG}.$$

Theorem 6.1. *For h small enough, there exists a constant C independent of h such that*

$$\widehat{\delta}\big(\boldsymbol{\mathcal{E}}(\mathbb{X}),\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)\big)\le\frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h).\tag{44}$$

Proof. As $\boldsymbol{\mathcal{E}}(\mathbb{X}(h))=\boldsymbol{\mathcal{E}}(\mathbb{X})$ and $\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))=\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$, it is equivalent to show that

$$\widehat{\delta}\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)),\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big)\le\frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h).$$

We consider here again the disk D_κ centered at κ with radius $\mathtt{d}_\kappa$ and boundary γ. We first notice that for all $z\in\gamma$

$$(z\boldsymbol{I}-\boldsymbol{T})^{-1}-(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}=(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}(\boldsymbol{T}-\boldsymbol{T}_h)(z\boldsymbol{I}-\boldsymbol{T})^{-1},$$

which, by virtue of Remarks 3.1 and 5.1, implies

$$\begin{aligned}\|(\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\| &\leq \frac{1}{2\pi}\int_\gamma \|(z\boldsymbol{I}-\boldsymbol{T})^{-1}-(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\||dz|\\ &= \frac{1}{2\pi}\int_\gamma \|(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}(\boldsymbol{T}-\boldsymbol{T}_h)(z\boldsymbol{I}-\boldsymbol{T})^{-1}|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\||dz|\\ &\leq \frac{1}{2\pi}\int_\gamma \|(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))}\|(\boldsymbol{T}-\boldsymbol{T}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}\|(z\boldsymbol{I}-\boldsymbol{T})^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}|dz|\\ &\leq \frac{C}{\mathtt{d}_\kappa}\|(\boldsymbol{T}-\boldsymbol{T}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))} \quad (45)\end{aligned}$$

Now, on the one hand, it is clear that

$$\delta\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)),\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big)\leq \|(\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}.$$

On the other hand, (45), the Céa estimate given by (32) and the fact that $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is finite dimensional yield

$$\|(\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}\leq \frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h), \tag{46}$$

which proves that

$$\delta\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)),\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big)\leq \frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h). \tag{47}$$

Consequently, as $\boldsymbol{\mathcal{E}}(\mathbb{X})\subset \boldsymbol{T}\circ\boldsymbol{P}(\boldsymbol{\sigma},\boldsymbol{r})\subset\{(\boldsymbol{\tau},\boldsymbol{s})\in[\mathrm{H}^s(\Omega)^{n\times n}]^2:\ \mathbf{div}\,\boldsymbol{\tau}\in\mathrm{H}^{1+s}(\Omega)^n\}$ for all $s\in(0,\widehat{s})$, we have that

$$\lim_{h\to 0}\delta\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)),\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big)=0. \tag{48}$$

It is shown in [12] that (48) implies that, for h small enough, $\Lambda_h:=\boldsymbol{\mathcal{E}}_h|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}:\boldsymbol{\mathcal{E}}(\mathbb{X})\to\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))$ is bijective and Λ_h^{-1} exists and is uniformly bounded with respect to h. Furthermore, it holds that,

$$\sup_{\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)),\|\boldsymbol{x}_h\|_{DG}=1}\|\Lambda_h^{-1}\boldsymbol{x}-\boldsymbol{x}\|_{DG}\leq 2\sup_{\boldsymbol{y}\in\boldsymbol{\mathcal{E}}(\mathbb{X}(h)),\|\boldsymbol{y}\|_{DG}=1}\|\Lambda_h\boldsymbol{y}-\boldsymbol{y}\|_{DG}.$$

Hence,

$$\delta\Big(\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)),\boldsymbol{\mathcal{E}}(\mathbb{X}(h))\Big)\leq \sup_{\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)),\|\boldsymbol{x}_h\|_{DG}=1}\|\boldsymbol{x}_h-\Lambda_h^{-1}\boldsymbol{x}\|_{DG}\leq 2\sup_{\boldsymbol{y}\in\boldsymbol{\mathcal{E}}(\mathbb{X}),\|\boldsymbol{y}\|_{DG}=1}\|\boldsymbol{\mathcal{E}}\boldsymbol{y}-\boldsymbol{\mathcal{E}}_h\boldsymbol{y}\|_{DG},$$

and (46) shows that we also have $\delta(\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)),\boldsymbol{\mathcal{E}}(\mathbb{X}(h)))\leq\frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h)$, and the result follows from this last estimate and (47). □

Finally, we prove the following rates of convergence for the eigenfunctions and eigenvalues.

Theorem 6.2. *Let $r > 0$ be such that $\boldsymbol{\mathcal{E}}(\mathbb{X}) \subset \{(\boldsymbol{\tau}, \boldsymbol{s}) \in [\mathrm{H}^r(\Omega)^{n\times n}]^2 : \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^{1+r}(\Omega)^n\}$ with*

$$\|\boldsymbol{\tau}\|_{\mathrm{H}^r(\Omega)^{n\times n}} + \|\boldsymbol{s}\|_{\mathrm{H}^r(\Omega)^{n\times n}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\mathrm{H}^{1+r}(\Omega)^n} \leq C\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \quad \forall(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{E}}(\mathbb{X}).$$

Then, there exists $C > 0$ independent of h and λ such that, for h small enough,

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) \leq \frac{C}{\mathsf{d}_\kappa} h^{\min\{r,k\}}. \tag{49}$$

Moreover, there exists $C' > 0$ independent of h such that

$$\max_{1\leq i\leq m} |\kappa - \kappa_{i,h}| \leq \frac{C'}{\mathsf{d}_\kappa} h^{2\min\{r,k\}}. \tag{50}$$

Proof. Proceeding as in the proof of (33), with s substituted by r, but considering the additional regularity of the eigenfunctions, we obtain

$$\frac{C}{\mathsf{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) \leq \frac{C}{\mathsf{d}_\kappa} h^{\min\{r,k\}} \left(\|\boldsymbol{\sigma}\|_{\mathrm{H}^r(\Omega,\Omega)^{n\times n}} + \|\boldsymbol{r}\|_{\mathrm{H}^r(\Omega)^{n\times n}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\mathrm{H}^{1+r}(\Omega)^n}\right), \tag{51}$$

which by virtue of (44) yields (49).

Let $\kappa_{1,h}, \cdots, \kappa_{m,h}$ be the eigenvalues of $\boldsymbol{T}_h : \mathbb{X}_h \to \mathbb{X}_h$ lying in D_κ and repeated according to their algebraic multiplicity. We denote by $\boldsymbol{x}_{i,h}$ an eigenfunction corresponding to $\kappa_{i,h}$ satisfying $\|\boldsymbol{x}_{i,h}\|_{DG} = 1$. We know from Theorem 6.1 that, if h is sufficiently small, then

$$\delta(\boldsymbol{x}_{i,h}, \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq \frac{C}{\mathsf{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h).$$

Then, there exists an eigenfunction $\boldsymbol{x} := (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{E}}(\mathbb{X})$ satisfying

$$\|\boldsymbol{x}_{i,h} - \boldsymbol{x}\|_{DG} \leq C\delta(\boldsymbol{x}_{i,h}, \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq C\widehat{\delta}(\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h), \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq \frac{C}{\mathsf{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) \to 0 \quad \text{as } h \to 0, \tag{52}$$

which proves that, for h small enough, $\|\boldsymbol{x}\|_{DG}$ is bounded from below and above by a constant independent of h. Proceeding as in the proof of the consistency property in Theorem 4.1 we readily obtain that

$$A_h(\boldsymbol{x}, \boldsymbol{y}_h) = \kappa B(\boldsymbol{x}, \boldsymbol{y}_h) \tag{53}$$

for all $\boldsymbol{y}_h := (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \mathbb{X}_h$. With the aid of (53), it is easy to show that the identity

$$A_h(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h}) - \kappa B(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h}) = (\kappa_{i,h} - \kappa)\, B(\boldsymbol{x}_{i,h}, \boldsymbol{x}_{i,h})$$

holds true. Now, according to Lemma 3.6 of [19], for any $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}(\mathbb{X})$, $\boldsymbol{x} \neq 0$, it holds that $B(\boldsymbol{x}, \boldsymbol{x}) > 0$. Thus, since $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is finite-dimensional, there exists $c > 0$, independent of h, such that $B(\boldsymbol{x}, \boldsymbol{x}) \geq c\|\boldsymbol{x}\|_{DG}$ $\forall \boldsymbol{x} \in \boldsymbol{\mathcal{E}}(\mathbb{X})$. This proves that $B(\boldsymbol{x}_{ih}, \boldsymbol{x}_{ih}) \geq \frac{c}{2}$ for h sufficiently small. We obtain from (21) that

$$\frac{c}{2}|\kappa_{i,h} - \kappa| \leq |A_h(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h})| + |\kappa||B(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h})| \leq C(\|\boldsymbol{x} - \boldsymbol{x}_{i,h}\|_{DG}^*)^2. \tag{54}$$

Since $\boldsymbol{x} := (\boldsymbol{\sigma}, \boldsymbol{r})$ and $\boldsymbol{x}_{i,h} := (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)$, and by definition of $\|\cdot\|^*_{DG}$ we have

$$\|\boldsymbol{x}-\boldsymbol{x}_{i,h}\|^*_{DG} := \|(\boldsymbol{\sigma}, \boldsymbol{r})-(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|^*_{DG} = \|(\boldsymbol{\sigma}, \boldsymbol{r})-(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG}+\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*}.$$

Now, we bound separately the two terms on the right hand side of the last identity. For the first term, from (52) and (51), we immediately obtain that

$$\begin{aligned}\|(\boldsymbol{\sigma}, \boldsymbol{r}) - (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG} &\leq \frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h)\\ &\leq \frac{C}{\mathtt{d}_\kappa} h^{\min\{r,k\}}\left(\|\boldsymbol{\sigma}\|_{\mathrm{H}^r(\Omega,\Omega)^{n\times n}} + \|\boldsymbol{r}\|_{\mathrm{H}^r(\Omega)^{n\times n}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\mathrm{H}^{1+r}(\Omega)^n}\right).\end{aligned} \tag{55}$$

On the other hand,

$$\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*} \leq \|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\}\|_{\mathcal{F}_h^*}+\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*} \tag{56}$$

and proceeding as to derive (39) in the proof of Theorem 4.1, we have that

$$\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\}\|_{\mathcal{F}_h^*} \leq C h^{\min\{1+r,k\}}\|\mathbf{div}\,\boldsymbol{\sigma}\|_{\mathrm{H}^{1+r}(\Omega)^n,\Omega}. \tag{57}$$

Finally, using (22), (17) and (55) yield

$$\begin{aligned}\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}_h(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*} &\leq C\|\,\mathbf{div}_h(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}\\ &\leq C\big(\|\,\mathbf{div}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma})\|_{0,\Omega}+\|\,\mathbf{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}\big)\\ &\leq C\big(\|\,\mathbf{div}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma})\|_{0,\Omega}+\|(\boldsymbol{\sigma}, \boldsymbol{r}) - (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG}\big)\\ &\leq \frac{C}{\mathtt{d}_\kappa} h^r\left(\|\boldsymbol{\sigma}\|_{\mathrm{H}^r(\Omega,\Omega)^{n\times n}} + \|\boldsymbol{r}\|_{\mathrm{H}^r(\Omega)^{n\times n}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\mathrm{H}^{1+r}(\Omega)^n}\right).\end{aligned} \tag{58}$$

Combining (54), (56)-(58) and (55), we obtain (50). □

Remark 6.1. *In the proof provided above for the error estimate* (50) *the constant* C' *is not independent of* λ. *Indeed, according to the proof of Lemma 3.6 from [19], we have that*

$$B((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\sigma}, \boldsymbol{r})) = \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \min\left\{\frac{n}{n\lambda+2\mu}, \frac{1}{2\mu}\right\}\|\boldsymbol{\sigma}\|^2_{0,\Omega} \geq 0.$$

Therefore, the constant c in the proof above tends to zero when λ *goes to infinity. However, the numerical experiments presented below suggest that* (50) *holds true uniformly in* λ.

Remark 6.2. *We notice that there is in* (49) *and* (50) *a hidden reliance on* λ *through the constant* $\mathtt{d}_\kappa := \frac{1}{2}\operatorname{dist}\big(\kappa, \operatorname{sp}(\boldsymbol{T}) \setminus \{\kappa\}\big)$ *because* $\operatorname{sp}(\boldsymbol{T})$ *depends on* λ. *The constant* $\mathtt{d}_\kappa$ *measures the deterioration of the error estimates given in Theorem 6.2 when the eigenvalue* κ *is too close to the accumulation point 0.*

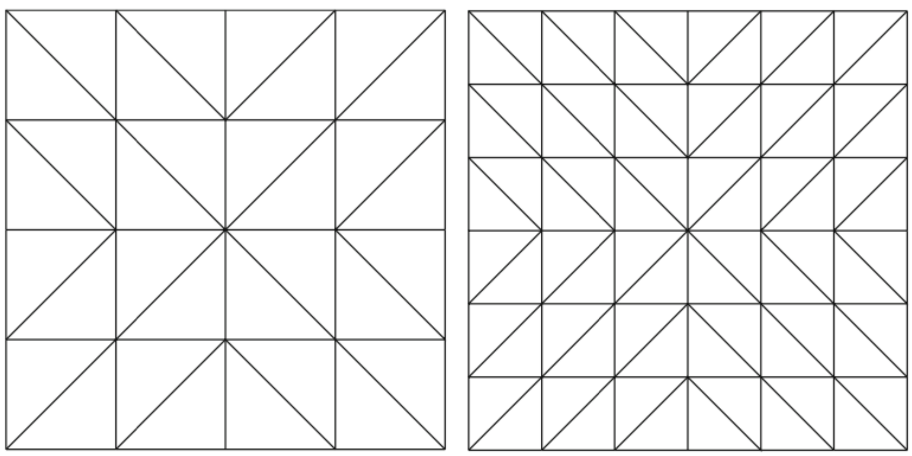
Remark 6.3. *We point out that, thanks to Lemma 2.2, we always have that* $\boldsymbol{\mathcal{E}}(\mathbb{X}) \subset \{(\boldsymbol{\tau}, \boldsymbol{r}) \in [\mathrm{H}^s(\Omega)^{n\times n}]^2 : \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^{1+s}(\Omega)^n\}$ *for all* $s \in (0, \widehat{s})$. *Consequently, the error estimates given in Theorem 6.2 will always hold true for any* $r \in (0, \widehat{s})$. *However, it may happen that some eigenspaces satisfy the regularity assumption of the theorem with* $r \geq \widehat{s}$ *(see, for instance, the last tests in the following section). In such a case, estimates* (49) *and* (50) *holds true even though* $r \geq \widehat{s}$.

7 Numerical results

We present a series of numerical experiments to solve the elasticity eigenproblem in mixed form with the discontinuous Galerkin scheme (20). All the numerical results have been obtained by using the FEniCS Problem Solving Environment [15]. For simplicity we consider a two-dimensional model problem. We choose $\Omega = (0,1) \times (0,1)$, fixed at its bottom ($\Gamma_D$) and free at the rest of the boundary (Γ_N). The material constants have been chosen $\rho = 1$ and Young modulus $E = 1$. We will let the Poisson ratio ν take different values in $(0, 1/2]$. We recall that the Lamé coefficients are related to E and ν by

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu := \frac{E}{2(1+\nu)}.$$

The limit problem corresponding to $\lambda = \infty$ is obtained by taking $\nu = 1/2$. In all our experiments we used uniform meshes with the symmetry pattern shown in Figure 1. The refinement parameter N represents the number of elements on each edge.



$N = 2$ $N = 6$

Figure 1: Uniform meshes

In the first test we are concerned with the determination of a reliable stabilization parameter $\mathtt{a}_S$. We point out that, the *hp*-DGFEM analysis given in [14] for the Maxwell source problem shows that the stabilization parameter should be sufficiently big and proportional to k^2. Moreover, the numerical tests presented in [8] confirm this fact for DG formulations of the Maxwell eigenproblem. We conjecture here the same behaviour in the H(div)-setting and take $\mathtt{a}_S = \mathtt{a}k^2$. We know that the spectral correctness of the method can only be guaranteed if $\mathtt{a}$ is sufficiently large (Proposition 4.2) and if the meshsize h is sufficiently small (cf. Remark 5.1). In a first stage, we fix the refinement level to $N = 8$ and report in Tables 1, 2 and 3 the 10 smallest vibration frequencies $\omega_{hi} := \sqrt{\kappa_{hi} - 1}$ computed for different values of $\mathtt{a} := 1/2, 1, 2, 4, 8$. The polynomial degrees are given by $k = 3, 4, 5$, respectively. The boxed numbers are spurious eigenvalues. We observe that they emerge at random positions when we vary $\mathtt{a}$ and k and they disappear completely when $\mathtt{a}$ is sufficiently large. Actually, $a \geq 4$ ensures the stability of the method for $k = 3$, 4, 5. This seems to confirm our supposition on the influence of the approximation order k on the penalty parameter.

$\mathtt{a} = 1/2$	$\mathtt{a} = 1$	$\mathtt{a} = 2$	$\mathtt{a} = 4$	$\mathtt{a} = 8$
0.6804473	0.6804477	0.6804462	0.6804472	0.6804472
1.6988806	1.6879653	1.6988720	1.6988796	1.6988800
1.8222055	1.6916177	1.8222037	1.8222050	1.8222051
2.9476928	1.6989049	2.9476908	2.9476927	2.9476933
3.0174155	1.8222057	3.0173631	3.0174089	3.0174112
3.4432205	2.9476990	3.4432037	3.4432155	3.4432167
4.1416494	3.0174223	4.1417279	4.1417682	4.1417745
4.6308330	3.4432178	4.6307482	4.6308440	4.6308541
4.6871661	3.6678401	4.7616006	4.7616214	4.7616310
4.7614527	3.6812705	4.7878815	4.7880137	4.7880286

Table 1: Computed lowest vibration frequencies for $k = 3$, $\mathtt{a}_S = \mathtt{a} \cdot k^2$ and $\nu = 0.35$.

$\mathtt{a} = 1/2$	$\mathtt{a} = 1$	$\mathtt{a} = 2$	$\mathtt{a} = 4$	$\mathtt{a} = 8$
0.6805737	0.6805734	0.6805736	0.6805737	0.6805737
1.6990333	1.6990335	1.6990325	1.6990330	1.6990331
1.8222096	1.8222095	1.8222093	1.8222096	1.8222096
2.9476922	2.9476921	2.9476918	2.9476922	2.9476922
3.0176437	3.0176229	3.0176395	3.0176427	3.0176430
3.4432474	3.4432468	3.4432456	3.4432472	3.4432472
4.1417705	4.1417583	4.1417494	4.1417709	4.1417710
4.6309442	4.4935293	4.5057494	4.6309431	4.6309435
4.7615804	4.6309218	4.6310303	4.7615811	4.7615813
4.7882418	4.7615802	4.6536939	4.7882397	4.7882404

Table 2: Computed lowest vibration frequencies for $k = 4$, $\mathtt{a}_S = \mathtt{a} \cdot k^2$ and $\nu = 0.35$.

$\mathtt{a} = 1/2$	$\mathtt{a} = 1$	$\mathtt{a} = 2$	$\mathtt{a} = 4$	$\mathtt{a} = 8$
0.6806522	0.6806522	0.6806522	0.6806522	0.6806522
1.6991251	1.6991236	1.6991253	1.6991254	1.6991254
1.8222137	1.8222137	1.8222137	1.8222137	1.8222137
1.8708971	2.6889275	2.9476935	2.9476935	2.9476935
1.9500361	2.7330944	3.0177841	3.0177845	3.0177845
2.9476935	2.9476937	3.4432655	3.4432656	3.4432656
3.0177850	3.0177865	4.1417851	4.1417852	4.1417852
3.4432657	3.4432669	4.6310192	4.6310196	4.6310197
4.1417854	4.1417852	4.7615802	4.7615803	4.7615803
4.6310206	4.6310213	4.7883870	4.7883879	4.7883881

Table 3: Computed lowest vibration frequencies for $k = 5$, $\mathtt{a}_S = \mathtt{a} \cdot k^2$ and $\nu = 0.35$.

The subsequent numerical tests are aimed to determine the convergence rate of the scheme. With the boundary conditions considered in our model problem, it turns out that (cf. [19] and the references therein) the regularity exponents $\widehat{s}$ defined in Lemma 2.2 are given by Table 4 for different values of the Poisson ratio ν.

ν	$\widehat{s}$
0.35	0.6797
0.49	0.5999
0.5	0.5946

Table 4: Sobolev regularity exponents.

We present in Tables 5, 6 and 7 (corresponding to the polynomial degrees $k = 2, 3, 4$, respectively) the first two vibration frequencies computed on a series of nested meshes for a range of Poisson ratios given by $\nu = 0.35, 0.49, 0.5$. We also report in these tables an estimate of the order of convergence α, and in the last column, more accurate values (ω_{extr}) of the vibration frequencies, extrapolated from the computed ones by means of a least-squares fitting of the model

$$\omega_{hi} \approx \omega_i + C_i h^{\alpha_i}.$$

This fitting has been done for each vibration mode separately. The fitted parameters ω_i and α_i are the extrapolated vibration frequency and the estimated order of convergence, respectively.

Comparing with the exponents given in Table 4, we observe that our method provides a double order of convergence for the vibration frequencies. Namely, in all cases we have

$\alpha \simeq 2\min\{r,k\} \approx 2\widehat{s}$, which corresponds to the the worst possible order of convergence for this problem. The eigenfunctions corresponding to higher natural frequencies are oscillating but they can be more regular (see Remark 6.3), which justifies the use of high polynomial orders of approximation. Finally, we point out that the method is clearly locking-free.

ν	$N=16$	$N=32$	$N=48$	$N=64$	α	ω_{extr}
0.35	0.6806068	0.6807467	0.6807850	0.6808020	1.34	0.6808381
	1.6990672	1.6992327	1.6992773	1.6992969	1.37	1.6993373
0.49	0.6987402	0.6991833	0.6993160	0.6993779	1.19	0.6995295
	1.8359946	1.8366760	1.8368781	1.8369722	1.20	1.8372009
0.5	0.7007298	0.7012091	0.7013534	0.7014210	1.18	0.7015881
	1.8472390	1.8479824	1.8482043	1.8483081	1.19	1.8485623

Table 5: Computed lowest vibration frequencies for $k=2$, $\mathtt{a}=250$ and convergence order

ν	$N=16$	$N=32$	$N=48$	$N=64$	α	ω_{extr}
0.35	0.6806839	0.6807775	0.6808029	0.6808142	1.35	0.6808379
	1.6991607	1.6992690	1.6992981	1.6993109	1.37	1.6993373
0.49	0.6989872	0.6992929	0.6993836	0.6994258	1.20	0.6995284
	1.8363810	1.8368436	1.8369810	1.8370450	1.20	1.8372002
0.5	0.7009977	0.7013286	0.7014275	0.7014736	1.19	0.7015868
	1.8476611	1.8481669	1.8483181	1.8483888	1.19	1.8485618

Table 6: Computed lowest vibration frequencies for $k=3$, $\mathtt{a}=250$ and convergence order

ν	$N=16$	$N=32$	$N=48$	$N=64$	α	ω_{extr}
0.35	0.6807342	0.6807973	0.6808144	0.6808219	1.36	0.6808376
	1.6992195	1.6992917	1.6993112	1.6993198	1.36	1.6993377
0.49	0.6991499	0.6993638	0.6994272	0.6994567	1.20	0.6995284
	1.8366280	1.8369510	1.8370470	1.8370917	1.20	1.8372000
0.5	0.7011738	0.7014060	0.7014751	0.7015075	1.19	0.7015869
	1.8479310	1.8482851	1.8483911	1.8484407	1.19	1.8485618

Table 7: Computed lowest vibration frequencies for $k=4$, $\mathtt{a}=250$ and convergence order

In the following test, we apply the method to a problem with smooth eigenfunctions, so that, the rate of convergence becomes $\alpha \simeq 2\min\{r,k\} = 2k$. With this end, we consider homogeneous Dirichlet condition on the whole boundary. We report in Table 8 the three lowest vibration frequencies computed considering different polynomial degrees $k = 1, 2, 3$ on uniform meshes (as in the previous tests) and $\nu = 0.5$ (for other values of ν the results are similar). The table includes order of convergence α, as well as more accurate values ω_{extr} of the vibration frequencies extrapolated by means of a least-squares fitting.

k	$N = 16$	$N = 32$	$N = 48$	$N = 64$	α	ω_{extr}
1	4.2141726	4.1863705	4.1812241	4.1794230	2.00	4.1771053
	5.6282584	5.5631132	5.5510951	5.5468923	2.01	5.5415667
	5.6282584	5.5631132	5.5510951	5.5468923	2.01	5.5415667
2	4.1772498	4.1771168	4.1771096	4.1771084	3.98	4.1771078
	5.5420361	5.5415262	5.5414986	5.5414939	3.98	5.5414917
	5.5420361	5.5415262	5.5414986	5.5414939	3.98	5.5414917
3	4.1771082	4.1771079	4.1771078	4.1771078	5.89	4.1771078
	5.5414937	5.5414918	5.5414917	5.5414917	5.94	5.5414917
	5.5414937	5.5414918	5.5414917	5.5414917	5.94	5.5414917

Table 8: Computed lowest vibration frequencies for $k = 1, 2, 3$ and $\mathtt{a} = 250$ on uniform structured meshes.

In this case, it can be clearly seen, that when using degree k, the order of convergence is $2k$ as the theory predicts.

Secondly we solve the problem with smooth eigenfunctions considering in this case non uniform meshes. We built the meshes with Fenics command "generate-mesh" with different levels of the refinement parameter N (which, roughly speaking, is proportional to h^{-1}). We report in Table 9 the three lowest vibration frequencies computed considering different polynomial degrees $k = 1, 2, 3$ and $\nu = 0.5$ (for other values of ν the results are similar). The table includes order of convergence α, as well as more accurate values ω_{extr} of the vibration frequencies extrapolated by means of a least-squares fitting. Once more, an order $2k$ can be clearly seen.

k	$N=16$	$N=32$	$N=48$	$N=64$	α	ω_{extr}
1	4.2065330	4.1842721	4.1803804	4.1789412	2.06	4.1772865
	5.6098823	5.5580421	5.5490942	5.5457585	2.08	5.5420534
	5.6113067	5.5584393	5.5492467	5.5457964	2.07	5.5419677
2	4.1771973	4.1771133	4.1771090	4.1771082	4.06	4.1771079
	5.5418368	5.5415130	5.5414961	5.5414931	4.03	5.5414919
	5.5418519	5.5415136	5.5414962	5.5414931	4.05	5.5414919
3	4.1771080	4.1771079	4.1771078	4.1771078	5.93	4.1771078
	5.5414927	5.5414918	5.5414917	5.5414917	6.08	5.5414917
	5.5414928	5.5414918	5.5414917	5.5414917	6.13	5.5414917

Table 9: Computed lowest vibration frequencies for $k = 1, 2, 3$ and $\mathtt{a} = 250$ on non-structured meshes.

8 Conclusions

We have introduced and analyzed a DG method for the elasticity eigensystem based on a mixed variational formulation that is written in terms of the Cauchy stress tensor and the rotation. We have proved that, if the penalty parameter is large enough, the numerical scheme provides a correct spectral approximation and the error estimates are of optimal order for the eigenfunctions and eigenvalues. We have reported several numerical experiments that validate our theoretical results. Our numerical tests have also confirmed the stability of our scheme in the nearly incompressible case and even in the limit case $\lambda = \infty$.

9 Appendix. The limit problem

As was shown in the previous section, the proposed method works fine also for the limit problem ($\lambda = +\infty$), namely, for perfectly incompressible elasticity. In this appendix, we will establish a spectral characterization in this case. Also, we will prove that the eigenvalues of the nearly incompressible elasticity problem converge to those of the incompressible elasticity problem as $\lambda \to \infty$.

In the limit case $\lambda = +\infty$, the bilinear forms A and B change in their definitions, since the term where λ appears in (4) vanishes. Therefore, the limit eigenvalue problem reads as follows: Find $\kappa \in \mathbb{R}$ and $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ such that

$$A_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) = \kappa B_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) \qquad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \tag{59}$$

with

$$B_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) := \frac{1}{2\mu} \int_\Omega \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \boldsymbol{r} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{s} : \boldsymbol{\sigma}$$

and

$$A_\infty((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})) := \int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + B_\infty((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s}))$$

for all $(\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$.

It is easy to check that A_∞ is a bounded bilinear form. Moreover, the arguments used in the proofs of Propositions 2.1 and 2.2 hold true for $\lambda = +\infty$, so that A_∞ satisfies the following inf-sup condition:

$$\sup_{(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}} \frac{A_\infty((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s}))}{\|(\boldsymbol{\tau},\boldsymbol{s})\|} \geq \alpha\|(\boldsymbol{\sigma},\boldsymbol{r})\| \qquad \forall(\boldsymbol{\sigma},\boldsymbol{r}) \in \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}.$$

In consequence, we are in a position to introduce a solution operator for the limit eigenvalue problem. Let $\boldsymbol{T}_\infty : [\mathrm{L}^2(\Omega)^{n\times n}]^2 \to \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ be defined for any $(\boldsymbol{f},\boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2$ by

$$A_\infty(\boldsymbol{T}_\infty(\boldsymbol{f},\boldsymbol{g}),(\boldsymbol{\tau},\boldsymbol{s})) = B_\infty((\boldsymbol{f},\boldsymbol{g}),(\boldsymbol{\tau},\boldsymbol{s})) \qquad \forall(\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}.$$

It is easy to check that μ is a non-zero eigenvalue of $\boldsymbol{T}_\infty$ with eigenfunction $(\boldsymbol{\sigma}_\infty,\boldsymbol{r}_\infty)$ if and only of $\kappa = 1/\mu$ is a non-vanishing eigenvalue of problem (59) with the same eigenfunction.

Our first goal is to prove that the operators $\boldsymbol{T}$ defined by (8) converges to $\boldsymbol{T}_\infty$ as λ goes to infinity. To recall that $\boldsymbol{T}$ actually depends on λ, in what follows we will denote it by $\boldsymbol{T}_\lambda$.

Before proving the convergence of $\boldsymbol{T}_\lambda$ to $\boldsymbol{T}_\infty$, we will characterize the spectrum of $\boldsymbol{T}_\infty$. Let $\boldsymbol{\mathcal{K}}$ be defined as in (10) and

$$[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_{B_\infty}} := \{(\boldsymbol{\sigma},\boldsymbol{r}) \in \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}} :\ B_\infty((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})) = 0 \quad \forall(\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}\}\,.$$

We observe that $\boldsymbol{T}_\infty|_{\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}} : \boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$ reduces to the identity, so that $\mu = 1$ is an eigenvalue of $\boldsymbol{T}_\infty$. Moreover, its associated eigenspace is precisely $\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$.

Let us introduce the following operator which will play a role similar to that of $\boldsymbol{P}$ in the limit problem:

$$\begin{aligned} \boldsymbol{P}_\infty : \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}} &\to \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}, \\ (\boldsymbol{\sigma},\boldsymbol{r}) &\mapsto \boldsymbol{P}_\infty\boldsymbol{\sigma} := (\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{r}}). \end{aligned}$$

where $(\widetilde{\boldsymbol{\sigma}},(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{r}})) \in \boldsymbol{\mathcal{W}}\times[\mathrm{L}^2(\Omega)^n\times\boldsymbol{\mathcal{Q}}]$ is the solution of the following problem:

$$\frac{1}{2\mu}\int_\Omega \widetilde{\boldsymbol{\sigma}}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \widetilde{\boldsymbol{u}}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \widetilde{\boldsymbol{r}} = 0 \qquad \forall\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{60}$$

$$\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} + \int_\Omega \widetilde{\boldsymbol{\sigma}} : \boldsymbol{s} = \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} \qquad \forall(\boldsymbol{v},\boldsymbol{s}) \in \mathrm{L}^2(\Omega)^n\times\boldsymbol{\mathcal{Q}}. \tag{61}$$

The previous problem is well posed, since the ellipticity of $\int_\Omega \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}}$ in the corresponding kernel is established in Lemma 2.3 of [20] and the following inf-sup condition holds true (see [5]):

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}} \frac{\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{s}:\boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \geq \beta(\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{s}\|_{0,\Omega}) \qquad \forall(\boldsymbol{v},\boldsymbol{s})\in \mathrm{L}^2(\Omega)^n\times\boldsymbol{\mathcal{Q}}.$$

We observe that problem (60)–(61) is a dual mixed formulation with weakly imposed symmetry of the following incompressible elasticity problem with volumetric force density $-\,\mathbf{div}\,\boldsymbol{\sigma}$

$$-\,\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} = -\,\mathbf{div}\,\boldsymbol{\sigma} \qquad \text{in}\,\Omega, \tag{62}$$

$$\frac{1}{2\mu}\widetilde{\boldsymbol{\sigma}}^{\mathtt{D}} = \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) \qquad \text{in}\,\Omega, \tag{63}$$

$$\widetilde{\boldsymbol{\sigma}}\boldsymbol{n} = \mathbf{0} \qquad \text{on}\,\Gamma_N, \tag{64}$$

$$\widetilde{\boldsymbol{u}} = \mathbf{0} \qquad \text{on}\,\Gamma_D. \tag{65}$$

It is easy to check that $(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{u}})\in \mathrm{H}(\mathbf{div},\Omega)\times\mathrm{H}^1(\Omega)^n$ satisfies (62)–(65) if and only if $(\widetilde{\boldsymbol{\sigma}},(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{r}}))\in\boldsymbol{\mathcal{W}}\times[\mathrm{L}^2(\Omega)^n\times\boldsymbol{\mathcal{Q}}]$ is the solution of (60)–(61) with $\widetilde{\boldsymbol{r}} = \frac{1}{2}[\nabla\widetilde{\boldsymbol{u}} - (\nabla\widetilde{\boldsymbol{u}})^{\mathtt{t}}]$.

Now, by resorting to the relation between the incompressible elasticity and the Stokes problem, we conclude that there exists $\widehat{s}_\infty\in(0,1)$ depending only on Ω and μ (see for instance [16]) such that, for all $s\in(0,\widehat{s}_\infty)$ the solution $\widetilde{\boldsymbol{u}}$ of (62)–(65) belongs to $\mathrm{H}^{1+s}(\Omega)^n$ and the following estimate hold true

$$\|\widetilde{\boldsymbol{u}}\|_{1+s,\Omega} \leq C\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega},$$

with a constant C independent of $\boldsymbol{\sigma}$.

The following lemma is a consequence of this regularity result.

Lemma 9.1. *For all $s\in(0,\widehat{s})$ and $(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$, if $(\widetilde{\boldsymbol{\sigma}},(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{r}}))$ is the solution of* (60)–(61)*, then $\widetilde{\boldsymbol{\sigma}}\in\mathrm{H}^s(\Omega)^{n\times n}$, $\widetilde{\boldsymbol{u}}\in\mathrm{H}^{1+s}(\Omega)^{n\times n}$, $\widetilde{\boldsymbol{r}}\in\mathrm{H}^s(\Omega)^{n\times n}$ and*

$$\|\widetilde{\boldsymbol{\sigma}}\|_{s,\Omega} + \|\widetilde{\boldsymbol{u}}\|_{1+s,\Omega} + \|\widetilde{\boldsymbol{r}}\|_{s,\Omega} \leq C\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega},$$

with a constant C independent of $\boldsymbol{\sigma}$. Consequently, $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})\subset\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n}$.

We observe that $\boldsymbol{P}_\infty$ is idempotent and that $\ker(\boldsymbol{P}_\infty) = \boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$. Moreover, being $\boldsymbol{P}_\infty$ a projector, the orthogonal decomposition $\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}} = (\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}})\oplus\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$ holds true. On the other hand, $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$ is an invariant space of $\boldsymbol{T}_\infty$ (see Proposition A.1 in [19]).

The following is the key point for the spectral characterization of $\boldsymbol{T}_\infty$.

Lemma 9.2. *For all $s\in(0,\widehat{s})$*

$$\boldsymbol{T}_\infty(\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}))\subset\{(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)\in\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n} :\, \mathbf{div}\,\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega)^n\}, \tag{66}$$

and there exists $C>0$ such that for all $(\boldsymbol{f},\boldsymbol{g})\in\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$, if $(\boldsymbol{\sigma}^*,\boldsymbol{r}^*) = \boldsymbol{T}_\infty(\boldsymbol{f},\boldsymbol{g})$, then

$$\|\boldsymbol{\sigma}^*\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}^*\|_{1,\Omega} + \|\boldsymbol{r}^*\|_{s,\Omega} \leq C\|(\boldsymbol{f},\boldsymbol{g})\|. \tag{67}$$

Moreover, $\boldsymbol{T}_\infty|_{\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})} : \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})\to\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$ is a compact operator.

Proof. Let $(\boldsymbol{f},\boldsymbol{g}) \in \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ and $(\boldsymbol{\sigma}^*,\boldsymbol{r}^*) = \boldsymbol{T}_\infty(\boldsymbol{f},\boldsymbol{g})$. Hence, we have

$$\int_\Omega \rho^{-1}\operatorname{\mathbf{div}}\boldsymbol{\sigma}^* \cdot \operatorname{\mathbf{div}}\boldsymbol{\tau} + \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{*\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \boldsymbol{r}^* : \boldsymbol{\tau} = \frac{1}{2\mu}\int_\Omega \boldsymbol{f}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \boldsymbol{g} : \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$
$$\int_\Omega \boldsymbol{\sigma}^* : \boldsymbol{s} = \int_\Omega \boldsymbol{f} : \boldsymbol{s} \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}.$$

Then, testing the first equation of the system above with $\boldsymbol{\tau} \in \mathcal{D}(\Omega)^{n\times n} \subset \boldsymbol{\mathcal{W}}$, we have that

$$-\rho^{-1}\nabla(\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*) + \frac{1}{2\mu}\boldsymbol{\sigma}^{*\mathtt{D}} + \boldsymbol{r}^* = \frac{1}{2\mu}\boldsymbol{f}^{\mathtt{D}} + \boldsymbol{g}.$$

Hence, since ρ and μ are constants, we conclude that $\operatorname{\mathbf{div}}\boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega)^n$.

Since $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ is invariant with respect to $\boldsymbol{T}_\infty$, applying Lemma 9.1 we obtain directly (66). On the other hand, (67) is a consequence of Lemma 9.1. Finally, the compactness of $\boldsymbol{T}_\infty|_{\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})}$ is a consequence of the following compact embedding

$$\{(\boldsymbol{\sigma}^*,\boldsymbol{r}^*) \in \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n} : \operatorname{\mathbf{div}}\boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega)^n\} \hookrightarrow \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}},$$

which allow us to conclude the proof. □

Now we are in position to establish a spectral characterization for $\boldsymbol{T}_\infty$.

Theorem 9.1. *The spectrum of* $\boldsymbol{T}_\infty$ *decomposes as follows:* $\mathrm{sp}(\boldsymbol{T}_\infty) = \{0,1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, *where:*

(i) $\mu = 1$ *is an infinite-multiplicity eigenvalue of* $\boldsymbol{T}_\infty$ *and its associated eigenspace is* $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$.

(ii) $\mu = 0$ *is an eigenvalue of* $\boldsymbol{T}_\infty$ *and its associated eigenspace is* $\mathcal{Z} \times \boldsymbol{\mathcal{Q}}$, *where*

$$\mathcal{Z} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} : \ \boldsymbol{\tau}^{\mathtt{D}} = 0\} = \{q\boldsymbol{I} : \ q \in \mathrm{H}^1(\Omega) \ \textit{and} \ q = 0 \quad \textit{on} \ \Gamma_N\}.$$

(iii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0,1)$ *is a sequence of nondefective finite-multiplicity eigenvalues of* $\boldsymbol{T}_\infty$ *which converge to zero and the corresponding eigenspaces lie in* $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$.

Proof. It is enough to follow the steps of Theorem 3.5 from [20]. □

Now we are in position to establish the following convergence result.

Lemma 9.3. *There exists a constant* $C > 0$ *such that*

$$\|(\boldsymbol{T}_\lambda - \boldsymbol{T}_\infty)((\boldsymbol{f},\boldsymbol{g}))\| \le \frac{C}{\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega} \qquad \forall (\boldsymbol{f},\boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2.$$

Proof. Let $(\boldsymbol{f},\boldsymbol{g})\in[\mathrm{L}^2(\Omega)^{n\times n}]^2$ and let $(\boldsymbol{\sigma}_\lambda,\boldsymbol{r}_\lambda):=\boldsymbol{T}_\lambda(\boldsymbol{f},\boldsymbol{g})$ and $(\boldsymbol{\sigma}_\infty,\boldsymbol{r}_\infty):=\boldsymbol{T}_\infty(\boldsymbol{f},\boldsymbol{g})$. Then, from (8) and the definition of $\mathcal{C}$ we have

$$\int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}_\lambda\cdot\mathbf{div}\,\boldsymbol{\tau}+\frac{1}{2\mu}\int_\Omega\boldsymbol{\sigma}_\lambda^{\mathtt{D}}:\boldsymbol{\tau}^{\mathtt{D}}+\frac{1}{n(n\lambda+2\mu)}\int_\Omega\operatorname{tr}(\boldsymbol{\sigma}_\lambda)\operatorname{tr}(\boldsymbol{\tau})+\int_\Omega\boldsymbol{r}_\lambda:\boldsymbol{\tau}$$
$$=\frac{1}{2\mu}\int_\Omega\boldsymbol{f}^{\mathtt{D}}:\boldsymbol{\tau}^{\mathtt{D}}+\frac{1}{n(n\lambda+2\mu)}\int_\Omega\operatorname{tr}(\boldsymbol{f})\operatorname{tr}(\boldsymbol{\tau})+\int_\Omega\boldsymbol{g}:\boldsymbol{\tau},$$
$$\int_\Omega\boldsymbol{\sigma}_\lambda:\boldsymbol{s}=\int_\Omega\boldsymbol{f}:\boldsymbol{s}.$$

Whereas

$$\int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}_\infty\cdot\mathbf{div}\,\boldsymbol{\tau}+\frac{1}{2\mu}\int_\Omega\boldsymbol{\sigma}_\infty^{\mathtt{D}}:\boldsymbol{\tau}^{\mathtt{D}}+\int_\Omega\boldsymbol{r}_\infty:\boldsymbol{\tau}=\frac{1}{2\mu}\int_\Omega\boldsymbol{f}^{\mathtt{D}}:\boldsymbol{\tau}^{\mathtt{D}}+\int_\Omega\boldsymbol{g}:\boldsymbol{\tau}\quad\forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}},$$
$$\int_\Omega\boldsymbol{\sigma}_\infty:\boldsymbol{s}=\int_\Omega\boldsymbol{f}:\boldsymbol{s}\qquad\forall\boldsymbol{s}\in\boldsymbol{\mathcal{Q}}.$$

Subtracting the above equations we have

$$\int_\Omega \rho^{-1}\,\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\cdot\mathbf{div}\,\boldsymbol{\tau}+\frac{1}{2\mu}\int_\Omega(\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}):\boldsymbol{\tau}^{\mathtt{D}}$$
$$+\int_\Omega(\boldsymbol{r}_\lambda-\boldsymbol{r}_\infty):\boldsymbol{\tau}=\frac{1}{n(n\lambda+2\mu)}\int_\Omega\operatorname{tr}(\boldsymbol{f}-\boldsymbol{\sigma}_\lambda)\operatorname{tr}(\boldsymbol{\tau})\qquad\forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}},\tag{68}$$
$$\int_\Omega(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty):\boldsymbol{s}=0\qquad\forall\boldsymbol{s}\in\boldsymbol{\mathcal{Q}}.\tag{69}$$

Testing this equation with $\boldsymbol{\tau}:=\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty$ and $\boldsymbol{s}:=\boldsymbol{r}_\lambda-\boldsymbol{r}_\infty$ we have

$$\begin{aligned}\rho^{-1}\|\,\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2+\frac{1}{2\mu}\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2=&\frac{1}{n(n\lambda+2\mu)}\int_\Omega(\operatorname{tr}(\boldsymbol{f})-\operatorname{tr}(\boldsymbol{\sigma}_\lambda))\operatorname{tr}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\\ \leq&\frac{1}{n(n\lambda+2\mu)}\int_\Omega\|\operatorname{tr}(\boldsymbol{f})-\operatorname{tr}(\boldsymbol{\sigma}_\lambda)\|_{0,\Omega}\,\|\operatorname{tr}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}\\ \leq&\frac{1}{n\lambda+2\mu}(\|\boldsymbol{f}\|_{0,\Omega}+\|\boldsymbol{\sigma}_\lambda\|_{0,\Omega})\|\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty\|_{0,\Omega}\\ \leq&\frac{C}{n\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega}\|\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty\|_{0,\Omega},\end{aligned}$$

where we have used (9) to bound $\|\boldsymbol{\sigma}_\lambda\|_{0,\Omega}$. Moreover

$$\underbrace{\min\left\{\rho^{-1},\frac{1}{2\mu}\right\}}_{C_{\rho,\mu}}\left(\|\,\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2+\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2\right)\leq\frac{C}{n\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega}\|\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty\|_{0,\Omega}.$$

We observe that $(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\in\boldsymbol{\mathcal{W}}$ is symmetric due to equation (69). Then, we resort to the following estimate (see [4] for instance)

$$C\|\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty\|_{0,\Omega}^2\leq\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2+\|\,\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2$$

with $C>0$ to deduce that

$$C\|\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty\|_{\mathrm{H}(\mathbf{div},\Omega)}\leq(\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2+\|\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2)^{1/2}.$$

Hence

$$\begin{aligned}\|\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2&+\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2\\&\leq\frac{C_{\rho,\mu}}{n\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega}(\|\boldsymbol{\sigma}^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2+\|\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2)^{1/2}\end{aligned}\tag{70}$$

and, finally,

$$\|\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty\|_{\mathrm{H}(\mathbf{div},\Omega)}\leq\frac{C}{\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega},\tag{71}$$

with C a positive constant depending on ρ, μ and n.

On the other hand, taking into account the inf-sup condition (7), (68), Cauchy-Schwarz inequality, (70) and (71), we have

$$\begin{aligned}&\beta\|\boldsymbol{r}_\lambda-\boldsymbol{r}_\infty\|_{0,\Omega}\\&\leq\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}}\frac{\frac{1}{n(n\lambda+2\mu)}\int_\Omega\operatorname{tr}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\operatorname{tr}(\boldsymbol{\tau})-\int_\Omega\rho^{-1}\,\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\cdot\mathbf{div}\,\boldsymbol{\tau}-\frac{1}{2\mu}\int_\Omega(\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}):\boldsymbol{\tau}^{\mathtt{D}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}\\&\leq\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}}\frac{\frac{C}{n\lambda+2\mu}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega}\|\boldsymbol{\tau}\|_{0,\Omega}+\rho^{-1}\|\mathbf{div}(\boldsymbol{\sigma}_\lambda-\boldsymbol{\sigma}_\infty)\|_{0,\Omega}\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}+\frac{1}{2\mu}\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}}-\boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}\|\boldsymbol{\tau}^{\mathtt{D}}\|_{0,\Omega}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}\\&\leq\frac{C}{\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega}.\end{aligned}\tag{72}$$

Hence, the proof follows by combining (71) and (72).

□

Now we are in a position to establish the following result.

Theorem 9.2. *Let $\mu_\infty>0$ be an eigenvalue of $\boldsymbol{T}_\infty$ of multiplicity m. Let D be any disc of the complex plane centered at μ_∞ and containing no other element of the spectrum of $\boldsymbol{T}_\infty$. Then, for λ large enough, D contains exactly m eigenvalues of $\boldsymbol{T}_\lambda$ (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu_\infty>0$ of $\boldsymbol{T}_\infty$ is a limit of eigenvalues μ of $\boldsymbol{T}_\lambda$, as λ goes to infinity.*

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