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# Beyond the near-horizon limit: stringy corrections to heterotic black holes

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Abstract: We study the first-order in $\alpha'$ corrections to 4-charge black holes (with the Reissner-Nordström black hole as a particular example) beyond the near-horizon limit in the Heterotic Superstring effective action framework. The higher-curvature terms behave as delocalized sources in the equations of motion and in the Bianchi identity of the 3-form. For some charges, this introduces a shift between their values measured at the horizon and asymptotically. Some of these corrections and their associated charge shifts, but not all of them, can be canceled using appropriate SU(2) instantons for the heterotic gauge fields. The entropy, computed using Wald's formula, is in agreement with the result obtained via microstate counting when the delocalized sources are properly taken into account.

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## Contents



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## 1 Introduction

In our recent work ref. [1] we have constructed a large family of solutions of the Heterotic Superstring effective action to first order in $\alpha'$. Generically, these solutions describe well-known systems consisting of intersections of fundamental strings (F1) with momentum flowing along them (W), solitonic 5-branes (NS or S5) and Kaluza-Klein monopoles (KK). The 5-dimensional, extremal, 3-charge black holes studied in refs. [2, 3] are simple members of this family with no KK monopoles and they describe the first-order in $\alpha'$ corrections of the heterotic version of the Strominger-Vafa black hole [4]. Our main task in this paper will be to study the case with KK monopoles. The corresponding solutions are 4-dimensional,

extremal, 4-charge black holes which will contain the first-order in $\alpha'$ corrections to the heterotic version of the black holes whose microscopic entropy was computed and compared with the supergravity result in refs. [5–7].[1]

The agreement between the Bekenstein-Hawking (BH) entropy of 4- and 5-dimensional black holes and the degeneracy of string microstates in the backgrounds mentioned above, initially obtained at the $\alpha' \to 0$ level in regimes in which the $\alpha'$ corrections can be safely ignored, is one of the triumphs of String Theory. These results have been extended in several directions to include rotation [11], non-trivial topology of the horizon (black rings) [12] etc.

A very important question to study is whether this agreement between the values of the BH entropy calculated by macroscopic and microscopic methods still holds when $\alpha'$ corrections (genuinely stringy effects associated to the finite string size $\ell_S = \sqrt{\alpha'}$) are taken into account.

In the calculation of the BH entropy by microstate counting the AdS/CFT correspondence tools have proven extraordinarily useful, shedding results that account for all the contributions in the $\alpha'$ perturbative expansion in the large charge regime.

The near-horizon geometry of all the black hole solutions we consider is $\mathrm{AdS}_3 \times \mathrm{S}^3/\mathbb{Z}_n \times \mathrm{T}^4$. The $\mathrm{AdS}_3$ and $\mathrm{S}^3$ factors are standard in the three-charge family of extremal black holes, and the quotient of the sphere by $\mathbb{Z}_n$ is related to the presence of a charge-$n$ KK monopole. Heterotic String Theory on this background was studied in ref. [13], identifying the central charges of the dual CFT. Then, applying the Cardy formula one obtains the following expression for the entropy

$$\mathbb{S}_{\mathrm{CFT}} = 2\pi\sqrt{N_{F1} N_W \, (k+2)}\,, \tag{1.1}$$

where $N_{F1}$ is the number of fundamental strings present in the background, $N_W$ is the number of units of momentum flowing along them and $k$ is the total level of affine algebra $\widehat{\mathrm{SL}(2)}$ in the right-moving sector. This number, minus two units, was identified in ref. [13] with the product of the KK monopole charge and S5 charge: $k = n\mathcal{Q}_{S5} + 2$. As we will see along this paper and in the discussion section, distinguishing correctly between charges (total charges, evaluated at spatial infinity) and numbers of stringy objects (KK monopoles and S5-branes in this case) is essential in order to compare this microscopic result with the macroscopic one.

The macroscopic ("supergravity") calculation of the $\alpha'$ corrections to the BH entropy faces a number of difficulties:

1. Finding the $\alpha'$-corrected solutions is a very complicated task, owing to the higher-order in curvature terms present in the equations of motion and the complicated interactions between them. In the Heterotic Superstring effective action there is an infinite series of terms related to the supersymmetrization of the Chern-simons terms present in the NS 3-form that can be introduced in an interative way following refs. [14, 15] but there are further terms of higher order in the curvature that seem to be unrelated to them first appearing at $\mathcal{O}(\alpha'^3)$ [16].

[1] See also refs. [8, 9] and references therein. The $\alpha' \to 0$ limits of these solutions are well known and were first obtained in ref. [10] directly in the heterotic version.



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2. The BH entropy of the $\alpha'$-corrected solutions is no longer simply given by the area of the horizon, and one needs to use the Wald formula [17, 18].

In order to circumvent these difficulties one may try to work with the near-horizon region of the solution only, assuming that it will have the same geometry after the $\alpha'$ corrections are taken into acount. The entropy function formalism developed by Sen [9, 19] provides an elegant and powerful strategy to find the near-horizon solutions of extremal black holes and to compute their entropy, making a comparison with the microscopic result eq. (1.1) possible. This approach has important drawbacks, though: it is not guaranteed that a solution interpolating between the near-horizon geometry and Minkowski spacetime describing an asymptotically-flat black-hole spacetime exists and, if it does, it does not give any information on how the non-linear interactions introduced by the higher-order corrections affect the physical properties of the solution, such as the values of the conserved charges.

The family of solutions constructed in ref. [1] makes unnecessary the restriction to the near-horizon limit, because they can describe the complete black hole spacetime to first order in $\alpha'$, as shown for the 5-dimensional case in ref. [3]. This allows us to take into account the non-linear interactions and compute explicitly the asymptotic charges. In the S5 and W cases, these will receive contributions from localized sources signaling the presence of the corresponding fundamental objects in the String Theory background, and contributions from the non-linear interactions that arise at first order in $\alpha'$. Being able to make this distinction is essential in order to write the BH entropy in terms of the same variables used in the counting of string microstates.

The non-linear contributions to the total S5 charge are analogous to those of SU(2) instantons over the KK monopole[2] with the wrong sign and can be exactly cancelled through the introduction of heterotic SU(2) gauge fields with the same instanton configuration.[3] It can, then, be argued that certain components of the fields associated to the S5 charge will not receive any further $\alpha'$ corrections.

The non-linear contributions to the momentum (W), though, are of a more mysterious nature. They take the form of the contribution of a generalization of the electric sector of the dyon of ref. [22] with the standard sign and cannot be cancelled using the same Green-Schwarz-type mechanism. However, it can still be argued that further $\alpha'$ corrections connected to the supersymmetrization of the Chern-Simons terms will vanish or will be arbitrarily small.

Having the 4-dimensional, extremal, 4-charge black holes with their first-order in $\alpha'$ corrections under control just leaves us with the calculation of the entropy using Wald's formula. This calculation can be conveniently performed directly in 10 dimensions using

[2]When the charge of the KK monopole is larger than one, the Yang-Mills instantons are somewhat exotic because the KK monopole contains a conical singularity. However, this singularity is resolved in the full supergravity metric by a conformal factor and the corresponding instantons are regular in 10-dimensional spacetime as well. Nevertheless, this deficit in the angle is reflected in terms of a fractional (although discrete) instanton number.

[3]These Yang-Mills fields coincide with those of some of the non-Abelian supergravity solutions we have studied in refs. [2, 20–24].



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the same trick we used in the 5-dimensional case ref. [3] and the result, to first order in $\alpha'$, is found to be

$$\mathbb{S}_{\text{Wald}} = 2\pi\sqrt{N_{F1}N_W n N_{S5}}\left(1+\frac{1}{nN_{S5}}+\cdots\right). \tag{1.2}$$

Comparing this macroscopic result with the expansion in the large charge limit of the result coming from the counting of string microstates eq. (1.1) we see that they agree with each other upon the identification $k = nN_{S5}$, where $N_{S5}$ is the number of S5-branes. This is the main result of this article.

This article is organized as follows: the Heterotic Superstring effective action at first order in $\alpha'$ is described in section 2. In section 3 we review the generic structure of the 10-dimensional fields for the family of solutions studied and discuss some aspects of their lower-dimensional descendants. In section 4 we present the construction of two families of spherically symmetric selfdual instantons over KK monopoles, which can be used as gauge fields in the solution. Additional details about these constructions are contained in appendix B. Our most important results are described in section 5: the description of the 4-charge corrected black hole and the identification of the parameters in terms of fundamental objects in the String Theory background. In section 6 we briefly describe some physical properties of our solutions from an effective 4-dimensional perspective. We particularize the discussion for the special example of an extremal Reissner-Nordström solution and comment on the non-perturbative nature of the small black holes. In section 7 we compute in detail the Wald entropy. Finally, we discuss our results and compare them with the previous literature in section 8.

## 2 The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$

In this section we review the Heterotic Superstring effective action [15] to $\mathcal{O}(\alpha')$, as already appearing in [1]. We start by defining the zeroth-order 3-form field strength of the Kalb-Ramond 2-form $B$:

$$H^{(0)} \equiv dB\,, \tag{2.1}$$

and constructing with it the zeroth-order torsionful spin connections

$$\Omega^{(0)\,a}_{(\pm)\,b} = \omega^a{}_b \pm \frac{1}{2}H^{(0)a}_{\mu}{}_b dx^\mu\,, \tag{2.2}$$

where $\omega^a{}_b$ is the Levi-Civita spin connection 1-form.[4] With them we define the zeroth-order Lorentz curvature 2-form and Chern-Simons 3-forms

$$R^{(0)\,a}_{(\pm)\,b} = d\Omega^{(0)\,a}_{(\pm)\,b} - \Omega^{(0)\,a}_{(\pm)\,c}\wedge\Omega^{(0)\,c}_{(\pm)\,b}\,, \tag{2.3}$$

$$\omega^{\text{L}\,(0)}_{(\pm)} = d\Omega^{(0)\,a}_{(\pm)\,b}\wedge\Omega^{(0)\,b}_{(\pm)\,a} - \frac{2}{3}\Omega^{(0)\,a}_{(\pm)\,b}\wedge\Omega^{(0)\,b}_{(\pm)\,c}\wedge\Omega^{(0)\,c}_{(\pm)\,a}\,. \tag{2.4}$$

Next, we introduce the gauge fields. We will only activate a SU(2) × SU(2) subgroup of the full gauge group of the Heterotic Theory and we will denote by $A^{A_{1,2}}$ ($A_{1,2} = 1,2,3$)

[4] We follow the conventions of ref. [25] for the spin connection and the curvature.



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the components, with the understanding that repeated gauge indices imply summation over both SU(2) factors. The gauge field strength and the Chern-Simons 3-form of each SU(2) factor are defined by

$$F^A = dA^A + \frac{1}{2}\epsilon^{ABC} A^B \wedge A^C \,, \tag{2.5}$$

$$\omega^{\rm YM} = dA^A \wedge A^A + \frac{1}{3}\epsilon^{ABC} A^A \wedge A^B \wedge A^C \,. \tag{2.6}$$

Then, we are ready to define recursively

$$\begin{aligned}
H^{(1)} &= dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L\,(0)}_{(-)}\right) \,, \\
\Omega^{(1)\ a}_{(\pm)\ b} &= \omega^a{}_b \pm \frac{1}{2} H^{(1)a}_{\mu\ b} dx^\mu \,, \\
R^{(1)\ a}_{(\pm)\ b} &= d\Omega^{(1)\ a}_{(\pm)\ b} - \Omega^{(1)\ a}_{(\pm)\ c} \wedge \Omega^{(1)\ c}_{(\pm)\ b} \,, \\
\omega^{\rm L\,(1)}_{(\pm)} &= d\Omega^{(1)\ a}_{(\pm)\ b} \wedge \Omega^{(1)\ b}_{(\pm)\ a} - \frac{2}{3}\Omega^{(1)\ a}_{(\pm)\ b} \wedge \Omega^{(1)\ b}_{(\pm)\ c} \wedge \Omega^{(1)\ c}_{(\pm)\ a} \,. \\
H^{(2)} &= dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L\,(1)}_{(-)}\right) \,,
\end{aligned} \tag{2.7}$$

and so on.

In practice only $\Omega^{(0)}_{(\pm)}, R^{(0)}_{(\pm)}, \omega^{\rm L\,(0)}_{(\pm)}, H^{(1)}$ will occur to the order we want to work at, but, often, it is more convenient to work with the higher-order objects ignoring the terms of higher order in $\alpha'$ when necessary. Thus we will suppress the $(n)$ upper indices from now on.

Finally, we define three "$T$-tensors" associated to the $\alpha'$ corrections

$$\begin{aligned}
T^{(4)} &\equiv \frac{3\alpha'}{4}\left[F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\right] \,, \\
T^{(2)}{}_{\mu\nu} &\equiv \frac{\alpha'}{4}\left[F^A{}_{\mu\rho} F^A{}_\nu{}^\rho + R_{(-)\,\mu\rho}{}^a{}_b R_{(-)\,\nu}{}^{\rho\,b}{}_a\right] \,, \\
T^{(0)} &\equiv T^{(2)\,\mu}{}_\mu \,.
\end{aligned} \tag{2.8}$$

In terms of all these objects, the Heterotic Superstring effective action in the string frame and to first-order in $\alpha'$ can be written as

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left\{R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!} H^2 - \frac{1}{2} T^{(0)}\right\} \,, \tag{2.9}$$

where $G_N^{(10)}$ is the 10-dimensional Newton constant, $\phi$ is the dilaton field and the vacuum expectation value of $e^\phi$ is the Heterotic Superstring coupling constant $g_s$. $R$ is the Ricci scalar of the string-frame metric $g_{\mu\nu}$.

The derivation of the complete equations of motion is quite a complicated challenge. Following ref. [26], we separate the variations with respect to each field into those corresponding to occurrences via $\Omega_{(-)}{}^a{}_b$, that we will call *implicit*, and the rest, that we will



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call *explicit*:

$$\begin{aligned}\delta S &= \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta S}{\delta A^{A_i}{}_\mu}\delta A^{A_i}{}_\mu + \frac{\delta S}{\delta\phi}\delta\phi \\ &= \frac{\delta S}{\delta g_{\mu\nu}}\bigg|_{\text{exp.}}\delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}}\bigg|_{\text{exp.}}\delta B_{\mu\nu} + \frac{\delta S}{\delta A^{A_i}{}_\mu}\bigg|_{\text{exp.}}\delta A^{A_i}{}_\mu + \frac{\delta S}{\delta\phi}\delta\phi \\ &\quad + \frac{\delta S}{\delta\Omega_{(-)}{}^a{}_b}\left(\frac{\delta\Omega_{(-)}{}^a{}_b}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta\Omega_{(-)}{}^a{}_b}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta\Omega_{(-)}{}^a{}_b}{\delta A^{A_i}{}_\mu}\delta A^{A_i}{}_\mu\right) . \end{aligned} \tag{2.10}$$

We can then apply a lemma proven in ref. [15]: $\delta S/\delta\Omega_{(-)}{}^a{}_b$ is proportional to $\alpha'$ and to the zeroth-order equations of motion of $g_{\mu\nu}, B_{\mu\nu}$ and $\phi$ plus terms of higher order in $\alpha'$.

The upshot is that, if we consider field configurations which solve the zeroth-order equations of motion[5] up to terms of order $\alpha'$, the contributions to the equations of motion associated to the implicit variations are at least of second order in $\alpha'$ and we can safely ignore them here.

If we restrict ourselves to this kind of field configurations, the equations of motion reduce to

$$R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - T^{(2)}{}_{\mu\nu} = 0\,, \tag{2.11}$$

$$(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{4\cdot 3!}H^2 + \frac{1}{8}T^{(0)} = 0\,, \tag{2.12}$$

$$d\left(e^{-2\phi}\star H\right) = 0\,, \tag{2.13}$$

$$\alpha' e^{2\phi}\mathfrak{D}_{(+)}\left(e^{-2\phi}\star F^{A_i}\right) = 0\,, \tag{2.14}$$

where $\mathfrak{D}_{(+)}$ stands for the exterior derivative covariant with respect to each SU(2) subgroup and with respect to the torsionful connection $\Omega_{(+)}$: suppressing the subindices $1, 2$ that distinguish the two subgroups, it takes the explicit form

$$e^{2\phi}d\left(e^{-2\phi}\star F^A\right) + \epsilon^{ABC}A^B\wedge\star F^C + \star H\wedge F^A = 0\,. \tag{2.15}$$

If the ansatz is given in terms of the 3-form field strength, we also need to solve the Bianchi identity

$$dH - \frac{1}{3}T^{(4)} = 0\,, \tag{2.16}$$

as well.

## 3 The 10-dimensional solutions and their $d = 5, 4$ descendents

The 5- and 4-dimensional black holes we are interested in belong to the class of $\alpha'$-corrected solutions constructed in ref. [1]. These preserve 1/4 of the 16 supersymmetries of the

[5] These can be obtained from eqs. (2.11)–(2.14) by setting $\alpha' = 0$. This eliminates the Yang-Mills fields, the $T$-tensors and the Chern-Simons terms in $H$.

heterotic theory and are completely determined by

1. A choice of 4-dimensional hyperKähler metric

$$d\sigma^2 = h_{\underline{mn}} dx^m dx^n \,, \quad m, n = \sharp, 1, 2, 3 \,, \tag{3.1}$$

with self-dual Riemann curvature 2-form with respect to some standard orientation.[6] If we are interested in 5-dimensional solutions, which are obtained by compactification on $\mathrm{T}^5$, we can use any non-compact 4-dimensional hyperKähler space. If we are interested in 4-dimensional supersymmetric solutions, though, the hyperKähler space must admit an additional isometry and we will take it to have a Gibbons-Hawking metric of the form[7]

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta + \chi)^2 + \mathcal{H} dx^x dx^x \,, \quad d\mathcal{H} = \star_{(3)} d\chi \,, \tag{3.3}$$

where $\star_{(3)}$ denotes the Hodge dual in $\mathbb{E}^3$.

In the Gibbons-Hawking case (and perhaps in more general hyperKähler spaces) one can write the Lorentz Chern-Simons 3-form of this 4-dimensional space as

$$\omega^{\mathrm{LHK}} = \star_{(4)} dW \,, \tag{3.4}$$

$$R^{mn} \wedge R^{nm} = d\omega^{\mathrm{LHK}} = d \star_{(4)} dW = -\nabla^2_{(4)} W |v| d^4x \,, \tag{3.5}$$

where $|v| d^4x$ is the volume 4-form and $W$ is some function defined on the hyperKähler space and where the subscript (4) indicates that the operator that carries it is defined in the 4-dimensional hyperKähler space. For Gibbons-Hawking spaces, which is the only class for which we have tested this property [1], we get

$$W = (\partial \log \mathcal{H})^2 \,. \tag{3.6}$$

2. Two SU(2) gauge fields $A^{A_{1,2}}$ defined on the hyperKähler manifold whose field strength 2-forms $F^{A_{1,2}}$ are self-dual there with respect to the same standard orientation (i.e. they are instanton fields)

$$F^{A_{1,2}} = + \star_{(4)} F^{A_{1,2}} \,. \tag{3.7}$$

These are directly the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ heterotic gauge fields of the solution. The most general solution to this equation in an arbitrary hyperKähler space does not seem to be available in the literature and we will consider some particular constructions. All of them, though, have the important property

$$\omega_i^{\mathrm{YM}} = - \star_{(4)} dV_i \,, \tag{3.8}$$

$$F^{A_i} \wedge F^{A_i} = d\omega_i^{\mathrm{YM}} = -d \star_{(4)} dV_i = \nabla^2_{(4)} V_i |v| d^4x \,, \tag{3.9}$$

for some functions $V_i$ defined on the hyperKähler space. These functions are computed for several instantons in appendix B.

[6] We use $\varepsilon^{\sharp 123} = +1$ in an appropriate Vierbein basis $v^m$

$$h_{\underline{mn}} = v^p{}_{\underline{m}} v^p{}_{\underline{n}} \,. \tag{3.2}$$

[7] Here $\eta = x^\sharp$ and we are using the 3-dimensional, curved, indices $x, y, z = 1, 2, 3$ which should not be mistaken with coordinates.



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3. Three functions $\mathcal{Z}_{0,+,-}$ defined on the hyperKähler space which are explicitly given by

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - \frac{\alpha'}{2}\left(\frac{\partial_m \mathcal{Z}_+^{(0)} \partial_m \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)} \mathcal{Z}_-^{(0)}}\right) + \mathcal{O}(\alpha'^2)\,, \tag{3.10}$$

$$\mathcal{Z}_- = \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, \tag{3.11}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} - \frac{\alpha'}{4}\left[V_1 + V_2 - \left(\partial \log \mathcal{Z}_0^{(0)}\right)^2 - W\right] + \mathcal{O}(\alpha'^2)\,, \tag{3.12}$$

where all the functions with a $^{(0)}$ superscript are harmonic in the hyperKähler space. Notice that the internal product implied in some of the terms in (3.10) and (3.12) is performed using the hyperKähler metric.

Using these building blocks, the remaining fields take the form

$$ds^2 = \frac{2}{\mathcal{Z}_-} du\left[dv - \frac{1}{2}\mathcal{Z}_+ du\right] - \mathcal{Z}_0 d\sigma^2 - dy^i dy^i\,, \quad i,j = 1,2,3,4\,, \tag{3.13}$$

$$H = d\mathcal{Z}_-^{-1} \wedge du \wedge dv + \star_{(4)} d\mathcal{Z}_0\,, \tag{3.14}$$

$$e^{-2\phi} = e^{-2\phi_\infty}\frac{\mathcal{Z}_-}{\mathcal{Z}_0}\,, \tag{3.15}$$

where $e^{\phi_\infty} = g_s$. The Kalb-Ramond 2-form $B$ satisfies

$$dB = d\mathcal{Z}_-^{-1} \wedge du \wedge dv + \star_{(4)} d\mathcal{Z}_0^{(0)}\,. \tag{3.16}$$

### 3.1 The 5-dimensional solutions

For our purposes, we only need to know the metric and the two scalar fields of the 5-dimensional solution (the 5-dimensional dilaton field $\phi$ and the Kaluza-Klein scalar that measures the radius of the $6 \to 5$ compactification, $k$). These are obtained from the 10-dimensional metric and dilaton with the same relations used in absence of $\alpha'$ corrections, [2], and read

$$\begin{aligned} ds^2 &= f^2 dt^2 - f^{-1} d\sigma^2\,, \\ e^{2\phi} &= e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \qquad k = k_\infty \frac{\mathcal{Z}_+^{1/2}}{\mathcal{Z}_0^{1/4}\mathcal{Z}_-^{1/4}}\,, \end{aligned} \tag{3.17}$$

where $d\sigma^2$ is the 4-dimensional hyperKähler metric in eq. (3.1), $\phi_\infty$ and $k_\infty$ are the asymptotic values of $\phi$ and $k$, and the metric function $f$ is given by

$$f^{-3} = \mathcal{Z}_0\, \mathcal{Z}_+\, \mathcal{Z}_-\,. \tag{3.18}$$

The functions $\mathcal{Z}_{0,+,-}$ are given in eqs. (3.10)–(3.12).

We should also mention that the SU(2) instanton fields have exactly the same expression as in 10 dimensions.



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### 3.2 The 4-dimensional solutions

If the hyperKähler metric $d\sigma^2$ in eq. (3.17) is a Gibbons-Hawking space eq. (3.3) and the other fields of the 5-dimensional solution do not depend on the isometric coordinate $\eta$, we can dimensionally reduce all fields along that coordinate, obtaining the metric and scalar fields. However, before doing so, it is convenient to rescale the coordinate $\eta = R\Psi/2$, where the dimensionless coordinate $\Psi \in [0, 4\pi)$ and the length of the circle is $2\pi R$. The, the Kaluza-Klein scalar of the $5 \to 4$ compactification, that we will denote by $\ell$, has the asymptotic value $\ell_\infty = \frac{1}{2}R/\ell_s$. Taking into account these points we get

$$\begin{aligned} ds^2_{(4)} &= e^{2U} dt^2 - e^{-2U} d\vec{x}^{\,2}\,, \\ e^{2\phi} &= e^{2\phi_\infty} \frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \qquad k = k_\infty \frac{\mathcal{Z}_+^{1/2}}{\mathcal{Z}_0^{1/4}\mathcal{Z}_-^{1/4}}\,, \qquad \ell = \ell_\infty \frac{\mathcal{Z}_0^{1/6}\,\mathcal{Z}_+^{1/6}\,\mathcal{Z}_-^{1/6}}{\mathcal{H}^{1/2}}\,, \end{aligned} \tag{3.19}$$

where the metric function $e^{-2U}$ is given by

$$e^{-2U} = \sqrt{\mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\mathcal{H}}\,. \tag{3.20}$$

The compactification of the Heterotic Superstring we are considering here gives an extension of the STU model of $\mathcal{N} = 2, d = 4$ Supergravity (plus the $\alpha'$ corrections related to $\Omega_{(-)}$).[8] The three scalars above are the imaginary parts of the three complex scalars of that model.

As for the non-Abelian gauge fields, their reduction follows Kronheimer's prescription [27], slightly modified by the introduction of the parameter $R$. It gives rise to adjoint Higgs fields $\Phi^{A_i}$ and gauge fields $\breve{A}^{A_i}$ in $\mathbb{E}^3$ related to the components of the gauge field in the Gibbons-Hawking space by

$$\Phi^{A_i} = -\mathcal{H} A^{A_i}_{\sharp}/(R/2) \qquad \breve{A}^{A_i}_{\underline{x}} = A^{A_i}_{\underline{x}} - \chi_{\underline{x}} A^{A_i}_{\sharp}\,. \tag{3.21}$$

The self-duality of the field strength in the hyperKähler space implies the that $\Phi^{A_i}$ and $\breve{A}^{A_i}$ are related by the Bogomol'nyi equation in $\mathbb{E}^3$ [28]

$$\breve{\mathcal{D}}\Phi^{A_i} = \star_{(3)} \breve{F}^{A_i}\,. \tag{3.22}$$

As shown in [21], the gauge fields constructed in this way enjoy the 3-dimensional version of the "Laplacian property":

$$\omega_i^{\rm YM} = \star_{(4)} d\left(\Phi^{A_i}\Phi^{A_i}/\mathcal{H}\right)\,, \tag{3.23}$$

$$F^{A_i} \wedge F^{A_i} = d\omega_i^{\rm YM} = \nabla^2_{(4)}\left(\Phi^{A_i}\Phi^{A_i}/\mathcal{H}\right)|v| d^4x\,. \tag{3.24}$$

This result is reviewed in appendix A.

In the current setup, in order to get a 4-dimensional solution we only need to choose a set of four harmonic functions $\mathcal{Z}^{(0)}_{+,-,0}, \mathcal{H}$ and a solution of the Bogomol'nyi equations

[8] The Yang-Mills fields of the Heterotic Superstring appear as non-Abelian vector multiplets of the 4-dimensional, gauged, supergravity.



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in $\mathbb{E}^3$ (3.22). If the solution is, at zeroth-order in $\alpha'$, a single (spherically-symmetric), asymptotically-flat, regular, extremal black hole, the functions $\mathcal{Z}^{(0)}_{+,-,0}$ are of the form

$$\mathcal{Z}^{(0)}_{+,-,0} = 1 + \frac{q_{+,-,0}}{r}\,, \tag{3.25}$$

and we are bound to choose[9]

$$\mathcal{H} = 1 + \frac{q}{r}\,, \tag{3.26}$$

where $r = |\vec{x}|$ and $q$ plays the rôle of a U(1) magnetic charge in $d = 4$. The corresponding Gibbons-Hawking space is a Kaluza-Klein (KK) monopole (also known as Euclidean Taub-NUT space) which, for large values of $r$, asymptotes to $\mathbb{E}^3 \times \mathrm{S}^1$. For this reason, the KK monopole is not used to construct asymptotically-flat 5-dimensional solutions and $\mathcal{H} = 1/r$, which corresponds to $\mathbb{E}^4_{-\{0\}}$, is used instead in that case. Since the $q/r$ term dominates in the $r \to 0$ limit, the 4-dimensional solutions have a 5-dimensional core even though asymptotically the have only four non-compact dimensions.

As for the solutions of eq. (3.22) (by definition, BPS magnetic monopoles, among which the 't Hooft-Polyakov magnetic monopole [29, 30] in the Prasad-Sommerfield limit [31] is a particular example), we must look for spherically-symmetric field configurations. Fortunately, all of them where found by Protogenov in ref. [32]. Independently of their singular character in $\mathbb{E}^3$, all of them have been used to construct regular, extremal, spherically-symmetric black holes in $d = 4$ dimensions and the 't Hooft-Polyakov BPS monopole has been used to construct globally regular solutions [20, 33–36].[10]

Using these BPS monopole solutions and Kronheimer's prescription we get selfdual instanton fields in 5 and 10 dimensions and we want them to be regular. For a given BPS monopole, this property depends critically on the choice of $\mathcal{H}$ and, more precisely, on its singularities and the behavior of the adjoint Higgs field $\Phi^A$ at those singularities. In ref. [27] Kronheimer gave the conditions under which the contributions to the instanton number (action) in a neighborhood of the singularities of $\mathcal{H}$ are finite. It is easy to see that all the monopoles found by Protogenov satisfy them if $\mathcal{H}$ is also spherically-symmetric, that is, if $\mathcal{H} = a + b/r$ for two positive constants $ab \neq 0$.

Observe that, even if Kronheimer's conditions are satisfied at $r = 0$, the instanton number density $F^{A_i} \wedge F^{A_i}$ may not fall off fast enough for large values of $r$ to give a finite integral. For $\mathcal{H} = 1/r$, the choice that leads to asymptotically-flat 5-dimensional solutions, this problem was shown in ref. [37] to arise for all of Protogenov's solutions except for the 1-parameter family of "colored" monopoles discussed in refs. [20, 36], which turn out to correspond to the BPST instanton [38]. These instantons were used in ref. [21] as part of the solution-generating technique found in ref. [39] to construct black holes with non-Abelian hair and, in ref. [23] to construct a globally regular instanton solution which corresponds to the compactification of the 10-dimensional heterotic "gauge 5-brane" of ref. [40].

[9] Multicenter solutions demand multicenter $\mathcal{Z}^{(0)}_{+,-,0}$s and multicenter $\mathcal{H}$s, but this case will be studied elsewhere.

[10] $\mathbb{E}^3$ is just an auxiliary space. The complete physical solution can be regular even if one uses a monopole solution which is singular in that auxiliary space. Typically, the singularity of the monopole in $\mathbb{E}^3$ is resolved by the extremal horizon.



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For $\mathcal{H} = 1 + q/r$, all the monopoles found by Protogenov give instantons over the KK monopole with good asymptotic behavior. We present them in detail in the next section. Then, in the following sections we study the 4- 5- and 10-dimensional solutions they give rise to.

## 4 Ingredients of the 4-dimensional black hole solution

### 4.1 KK monopole of arbitrary charge

We consider the Gibbons-Hawking metric

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta + \chi)^2 + \mathcal{H} dx^i dx^i\,, \quad \text{where } d\mathcal{H} = \star_{(3)} d\chi\,. \tag{4.1}$$

Here $\eta$ is a compact coordinate of periodicity $\eta \sim \eta + 2\pi R$. We choose the following harmonic function $\mathcal{H}$

$$\mathcal{H} = 1 + \frac{q}{r}\,. \tag{4.2}$$

It is convenient to use spherical coordinates $\theta, \phi$ defined by

$$\frac{x^1}{r} = \sin\theta\cos\phi\,, \quad \frac{x^2}{r} = -\sin\theta\sin\phi\,, \quad \frac{x^3}{r} = -\cos\theta\,, \tag{4.3}$$

so that, locally, the 1-form $\chi$ reads

$$\chi = q\cos\theta d\phi\,. \tag{4.4}$$

In these coordinates, the metric of the KK monopole takes the form

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta + q\cos\theta d\phi)^2 + \mathcal{H}\left(dr^2 + r^2 d\Omega^2_{(2)}\right)\,, \tag{4.5}$$

where

$$d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\phi^2\,, \tag{4.6}$$

is the metric of the round $S^2$ of unit radius. However, a global description of the solution requires two patches. The 1-form $\chi = q\cos\theta d\phi$ contains a Dirac-Misner string at the poles $\theta = 0, \pi$, or equivalently, in the line $x^1 = x^2 = 0$. This can be easily checked by computing the norm of $\chi$:

$$|\cos\theta d\phi|^2 = q^2 \frac{\cot^2\theta}{\mathcal{H} r^2}\,, \tag{4.7}$$

which is divergent at those points. In order to fix this singularity, we work with two different patches:

$$\chi^{(+)} = q\,(\cos\theta - 1)\,d\phi\,, \quad \chi^{(-)} = q\,(\cos\theta + 1)\,d\phi\,. \tag{4.8}$$

In this way, $\chi^{(+)}$ is regular everywhere except at $\theta = \pi$, and $\chi^{(-)}$ is regular everywhere except at $\theta = 0$. We also have to use different coordinates $\eta^{(+)}$ and $\eta^{(-)}$ in every patch, but in the intersection we must have

$$d\eta^{(+)} + \chi^{(+)} = d\eta^{(-)} + \chi^{(-)} \Rightarrow d\left(\eta^{(+)} - \eta^{(-)}\right) = 2q d\phi\,. \tag{4.9}$$



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Hence, we conclude that

$$\eta^{(+)} - \eta^{(-)} = 2q\phi\,. \tag{4.10}$$

Since $\phi$ has period $2\pi$ and both $\eta^{(\pm)}$ have period $2\pi R$ by definition, this relation can only hold if $q$ satisfies the quantization condition

$$q = \frac{nR}{2}\,, \quad n = 1, 2, \ldots \tag{4.11}$$

The reason is that $\eta \sim \eta + 2\pi R$ trivially implies that $\eta \sim \eta + 2\pi nR$, and thus the Dirac-Misner string is avoided for all $n = 1, 2 \ldots$ Let us then introduce the angular coordinate

$$\Psi = \frac{2\eta}{R} \Rightarrow \Psi \sim \Psi + 4\pi\,. \tag{4.12}$$

Taking into account the quantization of the charge $q$, we can write then the metric (locally) as

$$d\sigma^2 = \mathcal{H}^{-1}\frac{R^2}{4}(d\Psi + n\cos\theta d\phi)^2 + \mathcal{H}\left(dr^2 + r^2 d\Omega^2_{(2)}\right)\,. \tag{4.13}$$

It is important for future purposes to understand the $r \to 0$ and $r \to \infty$ limits of this space.

- In the $r \to 0$ limit we must use that $\mathcal{H} \sim \frac{nR}{2r}$, and after performing the change of coordinates

$$r = \frac{\rho^2}{2nR}\,, \tag{4.14}$$

we obtain

$$d\sigma^2(r \to 0) \sim d\rho^2 + \frac{\rho^2}{4}\left[\left(\frac{d\Psi}{n} + \cos\theta d\phi\right)^2 + d\theta^2 + \sin^2\theta d\phi^2\right]\,. \tag{4.15}$$

When $n = 1$, we recognize the factor that $\rho^2$ multiplies as the metric of the round $S^3$. However, for $n > 1$ the cyclic coordinate $\Psi$ does not cover the full sphere, but only a $1/n$ part of it. This corresponds to the metric of a lens space $S^3/\mathbb{Z}_n$, and hence the full space near $r = 0$ is the orbifold $\mathbb{E}^4/\mathbb{Z}_n$. Although lens spaces are regular, the full Gibbons-Hawking metric contains a conical singularity at $r = 0$, because at this point the periodicity of $\Psi$ is not "the right one" for $n > 1$. Nevertheless, it is important to notice that, when one takes into account the full 10-dimensional metric with non-vanishing $q_0$ there is no conical singularity, see eqs. (3.13) and (3.25).

- In the asymptotic limit $r \to \infty$ we have $\mathcal{H} \to 1$ and the metric becomes the direct product $S^1 \times \mathbb{E}^3$:

$$d\sigma^2(r \to \infty) = d\eta^2 + dx^i dx^i\,. \tag{4.16}$$

This is better seen by using Cartesian coordinates $x^i$ and the two patches introduced previously. In that case, the 1-forms $\chi^{(\pm)}$ read

$$\chi^{(\pm)} = q\frac{x^1 dx^2 - x^2 dx^1}{r(x^3 \pm r)}\,, \quad \text{where } r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\,. \tag{4.17}$$



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We use $\chi^{(+)}$ in the upper space $x^3 \geq 0$ and $\chi^{(-)}$ in the lower one $x^3 \leq 0$. In this way, it is explicit that $\chi^{(\pm)}$ are regular in their respective regions, and we observe that $\lim_{r\to\infty} \chi^{(\pm)} = 0$, where the limit is again taken in the respective region. Hence, the metric (4.1) becomes (4.16).

### 4.2 SU(2) instantons over KK monopoles

The BPS monopoles and the corresponding instantons that we consider can be written in terms of the functions $f(r), h(r)$ by

$$\Phi^A = -\frac{x^A}{r}(rf)\,, \qquad \breve{A}^A{}_{\underline{x}} = -\epsilon^A{}_{xz}\frac{x^z}{r}(rh)\,, \tag{4.18}$$

$$A^A = \mathcal{H}^{-1}\frac{x^A}{r}\,(rf)\,\frac{R}{2}(d\Psi + n\cos\theta d\phi) - \epsilon^A{}_{xz}r^2 h\frac{x^z}{r}d\left(\frac{x^x}{r}\right)\,. \tag{4.19}$$

There are two independent families of solutions. One of them corresponds to the *colored monopole*, which depends on a parameter $\lambda$:

$$f_\lambda(r) = h_\lambda(r) = -\frac{1}{gr^2}\frac{1}{1+\lambda^2 r}\,. \tag{4.20}$$

The other family depends on two parameters $(\mu, s)$, and it is given by

$$rf_{\mu,s}(r) = -\frac{1}{gr}\left[1 - \mu r\coth(\mu r + s)\right]\,, \quad rh_{\mu,s}(r) = -\frac{1}{gr}\left[1 - \frac{\mu r}{\sinh(\mu r + s)}\right]\,. \tag{4.21}$$

The parameter $\mu$ can always be taken to be positive, while $s$, which in this context is known as *Protogenov hair parameter*, can take any real value. However, we will only consider $s \geq 0$ to avoid singularities.

The $s = 0$ member of this family is the 't Hooft-Polyakov magnetic monopole [29, 30] in the BPS limit [31] with mass parameter $\mu$ and it is the only regular monopole in this family. On the other hand, in the $s \to \infty$ limit we get

$$rf_{\mu,\infty}(r) = \frac{\mu}{g} - \frac{1}{gr}\,, \quad rh_{\mu,\infty}(r) = -\frac{1}{gr}\,, \tag{4.22}$$

which is a $\mu$-dependent generalization of the Wu-Yang SU(2) monopole [41].

We are going to characterize the instantons over a KK monopole that all these monopoles give rise to. Observe, first of all, that the norm of the Higgs field is given by

$$|\Phi| = (\Phi^A\Phi^A)^{1/2} = |rf|\,, \tag{4.23}$$

and that Kronheimer's conditions are satisfied:

$$\lim_{r\to 0} r|\Phi_\lambda| = \frac{1}{g}\,, \qquad \lim_{r\to 0} d(r|\Phi_\lambda|) = -\frac{\lambda^2}{g}dr \tag{4.24}$$

for the colored monopole, and

$$\lim_{r\to 0} r|\Phi_{\mu,s}| = \begin{cases} \dfrac{1}{g}\,, & s \neq 0\,, \\ 0\,, & s = 0\,, \end{cases} \qquad \lim_{r\to 0} d(r|\Phi_{\mu,s}|) = \begin{cases} -\dfrac{\mu\coth s}{g}dr\,, & s \neq 0\,, \\ 0\,, & s = 0 \end{cases} \tag{4.25}$$

for the $(\mu, s)$ family.



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Except for the $(\mu, s = 0)$ case, the vector fields $A^A$ in eq. (4.19) are singular at $r = 0$ in the Gibbons-Hawking space (4.1). The fact that Kronheimer's conditions are satisfied ensures that these singularities can be removed by performing an appropriate gauge transformation (in fact, the field strength $F^A$ is regular). It is not too difficult to show that this is the case when $n = 1$, for all the instantons constructed from the above monopoles. However, for $n > 1$, the presence of a conical singularity in the GH space at $r = 0$ makes the problem more complicated, and we have not found the gauge in which the vectors become regular when defined over that space. Nevertheless, we stress that the physical fields are not defined over the GH space, but over the physical manifold whose metric is given by eq. (3.13). As explained in the previous section, in the full spacetime the conical singularity at $r = 0$ is blown away by the conformal factor in front of the hyperKähler metric. This mechanism also resolves the original divergences of these vectors, rendering the instanton fields regular.

The regularity conditions on the SU(2) gauge fields have relevant physical implications. For example, in the $(\mu, s)$ family, the parameter $\mu$ is related to the charge of the instanton. However, from the String Theory point of view, the charge cannot depend on an arbitrary continuous parameter. As we explain in appendix B, the correct construction of this family of solutions implies that the parameter $\mu$ should actually be quantized [42], according to eq. (B.32), which we write here for convenience

$$\mu = \frac{2m}{Rn}, \quad m = 0, 1, 2, \dots \tag{4.26}$$

In the case of the $\lambda$-family of instanton solutions built from the colored monopoles, the charge is independent of any continuous parameter, as we show below. Moreover, as we will see in the following section, the instantons of the $\lambda$-family are the most interesting ones from the point of view of $\alpha'$ corrections.

The "charge" of these instantons is just their instanton number, which is given by

$$\mathfrak{n} = -\frac{g^2}{8\pi^2}\int \mathrm{Tr}\left[\hat{F}\wedge\hat{F}\right] = \frac{g^2}{16\pi^2}\int F^A\wedge F^A\,, \tag{4.27}$$

where the integral is taken over the GH space.[11] This integral takes a simple form once we take into account that $F^A \wedge F^A$ is the Laplacian of some function $V$ (see appendix A):

$$F^A\wedge F^A = -d\star_{(4)}\, dV\,. \tag{4.28}$$

Using that relation, for the cases that we consider $V = V(r)$, we can write

$$d\star_{(4)}\, dV = \star_{(4)}\frac{1}{|v|}\partial_r\left(\mathcal{H}^{-1}|v|\partial_r V\right)\,, \tag{4.29}$$

and integration yields

$$\mathfrak{n} = -\frac{g^2}{16\pi^2}\int \frac{R}{2} d\Psi\wedge d\theta\wedge d\phi\sin\theta r^2\partial_r V\Big|_{r=\infty}^{r=0} = -\frac{g^2 R}{2} r^2\partial_r V\Big|_{r=\infty}^{r=0}\,. \tag{4.30}$$

[11] More precisely, it is defined over the 4-dimensional space conformal to the GH space, but the integral is invariant under Weyl transformations.



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For our instantons the function $V$ is given by (see appendix A)

$$V = \frac{\Phi^A \Phi^A}{\mathcal{H}} = \frac{r^2 f(r)^2}{\mathcal{H}}\,, \tag{4.31}$$

and we obtain

$$\mathfrak{n}_\lambda = \frac{1}{n}\,, \qquad \mathfrak{n}_{\mu,s} = \frac{(m+1)^2 - \delta_{s,0}}{n}\,. \tag{4.32}$$

As we see, the instanton number is quantized although its value is not necessarily an integer. This is related to the presence of the lens space $S^3/\mathbb{Z}_n$ and it shows that these instantons, when $n > 1$, are somewhat exotic in a mathematical sense. Other solutions with rational but discrete instanton number are known in the literature, see, for instance, refs. [42, 43].

## 5 Explicit 4-dimensional black hole solutions and their charges

Having described all the basic building blocks necessary to construct a solution, we just need to specify our choices for them.

We choose single-pole, spherically symmetric, harmonic functions $\mathcal{Z}_{0,+,-}$ of the form eqs. (3.25) and the GH space of the charge-$q$ KK monopole described in the last section with harmonic function $\mathcal{H}$ given by eq. (3.26). In addition, we are going to include an arbitrary number $N_\lambda$ and $N_{\mu,s}$ of instantons of the $\lambda$ and $(\mu, s)$ families, respectively, which we have described in section 4.2. Introducing this input in eqs. (3.10)–(3.12) and using the relations eqs. (B.33)–(B.35) for the contribution from the instantons $V_1, V_2$, we obtain the $\alpha'$-corrected functions

$$\mathcal{Z}_+ = 1 + \frac{q_+}{r} - \frac{\alpha'}{2}\frac{q_+ q_-}{r(r+q)(r+q_0)(r+q_-)} + \mathcal{O}(\alpha'^2)\,, \tag{5.1}$$

$$\mathcal{Z}_- = 1 + \frac{q_-}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.2}$$

$$\mathcal{Z}_0 = 1 + \frac{q_0}{r} \tag{5.3}$$

$$+ \frac{\alpha'}{4r(r+q)}\left\{\frac{q_0^2}{(r+q_0)^2} + \frac{q^2}{(r+q)^2} - \sum_{i=1}^{N_\lambda}\frac{1}{(1+\lambda_i^2 r)^2} - \sum_{i=1}^{N_{\mu,s}}\left[1 - \mu_i r \coth\left(\mu_i r + s_i\right)\right]^2\right\}, \tag{5.4}$$

$$\mathcal{H} = 1 + \frac{q}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.5}$$

where we recall that the charge $q$ is quantized according to eq. (4.11). Note, however, that these functions are not univocally determined: we are free to add an arbitrary $\mathcal{O}(\alpha')$ harmonic function to each of them, and the resulting field configuration is still a solution of the equations of motion at first order in $\alpha'$. We use this freedom to impose that the $1/r$ pole of the $\mathcal{Z}$ functions is not changed by the $\alpha'$ corrections and to ensure that $\mathcal{Z} \to 1$ at infinity. For the functions above this amounts to the changes

$$\mathcal{Z}_+ \to \mathcal{Z}_+ + \frac{\alpha'}{2rq}\frac{q_+}{q_0}\,, \qquad \mathcal{Z}_0 \to \mathcal{Z}_0 - \frac{\alpha'}{4rq}\left[2 - N_\lambda - N_{\mu,s} + \sum_{i=1}^{N_{\mu,s}}\left(\delta_{s,0} + rq\mu_i^2\right)\right]. \tag{5.6}$$



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There are two reasons why we must eliminate the poles from the $\alpha'$ corrections:

1. The $\alpha'$ corrections are associated to the curvatures of the gauge instantons and torsionful spin connection, which are regular. Thus, they should be regular as well. The poles are spurious and their presence is solely due to the fact that we are using a singular gauge to write the different connections.

2. We want to associate the residues of the poles with the sources of the solution, and these should not be modified by the $\alpha'$ corrections.

From now on we focus the discussion on the gauge fields on the $\lambda$-instantons, which will shortly be proven as the most interesting family. Then, for simplicity reasons, we will set $N_{\mu,s} = 0$. Taking into account all these points, the functions that determine the solution read

$$\mathcal{Z}_+ = 1 + \frac{q_+}{r} + \frac{\alpha' q_+}{2qq_0} \frac{r^2 + r(q_0 + q_- + q) + qq_0 + qq_- + q_0 q_-}{(r+q)(r+q_0)(r+q_-)} + \mathcal{O}(\alpha'^2)\,, \tag{5.7}$$

$$\mathcal{Z}_- = 1 + \frac{q_-}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.8}$$

$$\mathcal{Z}_0 = 1 + \frac{q_0}{r} + \alpha' \left\{ - F(r;q_0) - F(r;q) + \sum_{i=1}^{N_\lambda} F(r;\lambda_i^{-2}) \right\} + \mathcal{O}(\alpha'^2)\,, \tag{5.9}$$

$$\mathcal{H} = 1 + \frac{q}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.10}$$

where we have introduced the function

$$F(r;k) := \frac{(r+q)(r+2k)+k^2}{4q(r+q)(r+k)^2}\,. \tag{5.11}$$

Expressed in this way, it is obvious that we can eliminate all the $\alpha'$ corrections to $\mathcal{Z}_0$ if we use $N_\lambda = 2$ instantons of sizes $\lambda_1^{-2} = q_0$, $\lambda_2^{-2} = q$ ($N_{\mu,s} = 0$). We will come back to this point later.

In the configuration at hand, $q_0$ is related to the number of solitonic (or Neveu-Schwarz, NS) 5-branes (S5), $q_-$ is related to the winding number of a string wrapped along the $u$ direction (F1) and $q_+$ represents the momentum of a wave (W) along that direction. We also have a KK monopole of charge $n$ ($q = nR/2$) and a number $N_\lambda$ of gauge 5-branes, sourced by the SU(2) instantons.

The easiest way to determine the number of stringy objects is by looking at the near-horizon geometry $r \to 0$. In that case, we introduce the coordinate $\rho$ in (4.14) such that $d\sigma^2$ becomes explicitly $\mathbb{E}^4/\mathbb{Z}_n$. The full 10-dimensional geometry is regular and corresponds to the spacetime geometry $\mathrm{AdS}_3 \times \mathrm{S}^3/\mathbb{Z}_n \times \mathrm{T}^4$. The $\mathcal{Z}$ functions behave in that limit as

$$\mathcal{Z}_0 \sim \frac{2nRq_0}{\rho^2}\,, \quad \mathcal{Z}_- \sim \frac{2nRq_-}{\rho^2}\,, \quad \mathcal{Z}_+ \sim \frac{2nRq_+}{\rho^2}\,. \tag{5.12}$$

This near-horizon geometry is the same as the one of the 5-dimensional black holes considered in refs. [2, 3] up to a $\mathbb{Z}_n$ quotient, and therefore we can apply the results there in order to obtain the number of stringy objects from the coefficients of $1/\rho^2$. There is one



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difference though: here we only have $1/n$-th of the sphere, and this means that the field produced by one of these objects is $n$ times larger than in the case in which we have the full sphere. Taking into account this effect, we expect the result to be

$$q_0 = \frac{\ell_s^2}{2R} N_{S5}\,, \qquad q_- = \frac{\ell_s^2 g_s^2}{2R} N_{F1}\,, \qquad q_+ = \frac{g_s^2 \ell_s^4}{2R R_u^2} N_W\,. \tag{5.13}$$

Indeed, one can check that the values of $q_0$ and $q_-$ above agree with the ones computed by using the relations[12]

$$g_s^2 N_{S5} T_{S5} = \frac{g_s^2}{16\pi G_N^{(10)}} \left\{ \int_{\mathrm{S}\times\mathrm{S}^2_\infty} \star e^{2\phi} \tilde{H} - \frac{\alpha'}{4} \int \left( F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a \right) \right\}\,, \tag{5.15}$$

$$T_{F1} N_{F1} = \frac{g_s^2}{16\pi G_N^{(10)}} \int_{\mathrm{S}\times\mathrm{S}^2_\infty\times\mathrm{T}^4} \star e^{-2\phi} H\,, \tag{5.16}$$

obtained by coupling the 10-dimensional Heterotic Superstring effective action to the world-volume effective actions of $N_{S5}$ solitonic 5-branes and $N_{F1}$ fundamental strings [2].

The interpretation of $q_+$ is, however, less transparent. This is related to the fact that the parameters $q_0$, $q_-$ and $q_+$ represent localized sources of solitonic 5-brane, string and momentum charge respectively. But due to the effect of the $\alpha'$ corrections, those do not necessarily amount for the corresponding total charges measured at infinity. For instance, when $r \to \infty$, the functions $\mathcal{Z}_{+,0}$ take the form

$$\mathcal{Z}_+ = 1 + \frac{1}{r}\left( q_+ + \frac{\alpha' q_+}{2 q q_0} \right) + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{5.17}$$

$$\mathcal{Z}_0 = 1 + \frac{1}{r}\left[ q_0 + \frac{\alpha'}{2R}\left( -\frac{2}{n} + N_\lambda \mathfrak{n}_\lambda \right) \right] + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{5.18}$$

where $\mathfrak{n}_\lambda = 1/n$ is the instanton number, as given in eq. (4.32). Therefore, there are additional $\mathcal{O}(\alpha')$ contributions to the $S5$ and $W$ charges at infinity, while the $F1$ charge does not receive any contributions at $\mathcal{O}(\alpha')$.

Let us first consider the corrections/contributions to the S5 charge through the functions $\mathcal{Z}_0$. Some of these are immediately identified as produced by gauge 5-branes, which play the rôle of non-localized S5 sources. They are responsible for the appearance of the instanton number in the above expressions and are well understood.

The rest of the $\alpha'$ corrections in $\mathcal{Z}_0$ are associated to the $\Omega_{(-)}$ connection. As we already remarked some lines above, their net effect is the same as that of $N_\lambda = 2$ gauge 5-branes but with a negative sign. This means that we can eliminate all the $\alpha'$ corrections in $\mathcal{Z}_0$ simply by taking $N_\lambda = 2$. This choice is that of the *symmetric* 5-brane of ref. [44] adapted to include a KK monopole of charge $n$.

The case of S5 charge is paradigmatic, because we can explicitly identify and compute the different types of sources; the pole in $\mathcal{Z}_0$ at $r = 0$ gives the number of solitonic 5-branes,

[12] Here the tensions read

$$T_{S5} = \frac{1}{(2\pi\ell_s)^5 \ell_s g_s^2}\,, \qquad T_{F1} = \frac{1}{2\pi\alpha'}\,, \tag{5.14}$$

and the NSNS 7-form field strength $\tilde{H}$ is defined as $\tilde{H} = \star e^{-2\phi} H$.



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while the remaining regular functions account for delocalized sources of non-Abelian nature coming from $\alpha'$-corrections. Furthermore, these corrections can be completely cancelled by an adequate choice of Yang-Mills fields eliminating any possible ambiguity in the identification of the number of S5-branes that source the solution.

Now we turn back our attention to the W charge and the interpretation of $q_+$. At this stage and after the preceding discussion, the direct identification of $q_+$ in terms of $N_W$ units of momentum charge carried by the string proposed in eq. (5.13) is rather natural; the pole in $\mathcal{Z}_+$ should correspond to a localized source of momentum, while the remaining regular contribution to the function represents delocalized sources of momentum arising from higher-order interactions. The additional contributions to $N_W$ at infinity coming from the $\alpha'$ corrections have positive sign. We do not know how to cancel them by introducing some delocalized gauge brane. The contribution of a SU(2) dyonic gauge field, like that of ref. [22], would only add up to it. This is an $\alpha'$ correction that we have to live with and which has important physical consequences.

Performing a T-duality transformation along the $u$ coordinate using the $\alpha'$-corrected rules [45] amounts to the following changes in the functions [1, 3]

$$\begin{aligned}
\mathcal{Z}_- \to \mathcal{Z}'_- &= 1 + \frac{1}{r}\left(q_+ + \frac{\alpha' q_+}{2qq_0}\right) , \\
\mathcal{Z}_+ \to \mathcal{Z}'_+ &= 1 + \frac{1}{r}\left(q_- - \frac{\alpha' q_-}{2qq_0}\right) + \frac{\alpha' q_-}{2qq_0}\frac{r^2 + r(q_0+q_++q) + qq_0 + qq_+ + q_0q_+}{(r+q)(r+q_0)(r+q_+)} .
\end{aligned} \tag{5.19}$$

As we see, T-duality interchanges the total (asymptotic) string and momentum charges.[13] However, since the total momentum charge receives contributions from delocalized sources while the total string charge does not, the microscopic T-duality rules interchanging $N_W$ and $N_{F1}$ have to be modified as

$$q'_- = \left(q_+ + \frac{\alpha' q_+}{2qq_0}\right) \qquad \longrightarrow \qquad N'_{F1} = N_W\left(1 + \frac{2}{nN_{S5}}\right) , \tag{5.20}$$

$$q'_+ = \left(q_- - \frac{\alpha' q_-}{2qq_0}\right) \qquad \longrightarrow \qquad N'_W = N_{F1}\left(1 - \frac{2}{nN_{S5}}\right) . \tag{5.21}$$

We emphasize that the identification of the parameters of the solution in terms of fundamental String Theory objects and their relation with the asymptotic charges is affected by the global aspects of the solution when $\alpha'$ corrections are present. Therefore, the use of merely near-horizon geometries for this purpose, which, as we have discussed in the Introduction, has been a common strategy in the literature, can introduce errors in the determination of the sources. We shall come back to this issue in the discussion section.

To close this section, we note that when one or both of $q, q_0$ vanish, a qualitatively different family of solutions, known as "small" black holes, is obtained. In order to avoid difficulties in the limits $q, q_0 \to 0$ in $\mathcal{Z}_0$ given by eq. (5.9) it is convenient to work with the symmetric solution with $N_\lambda = 2$ gauge 5-branes that kills the $\alpha'$ corrections. On the

[13] On the other hand, the near-horizon geometry and entropy are preserved at $\mathcal{O}(\alpha')$. Moreover, temperature and entropy of the BTZ black hole in a simplified model remain invariant under the $\alpha'$-corrected T-duality transformation, [46].



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other hand, this limit is singular in the expression of $\mathcal{Z}_+$ given in eq. (5.7), but the reason is that the subtraction made in eq. (5.6) is not pertinent in this case. Hence, we should use eq. (5.1). For example, in the $q_0 \to 0$ limit, we get

$$\mathcal{Z}_+ = 1 + \frac{q_+}{r} - \frac{\alpha' q_+ q_-}{2r^2(r+q)(r+q_-)} + \mathcal{O}(\alpha'^2)\,, \tag{5.22}$$

$$\mathcal{Z}_- = 1 + \frac{q_-}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.23}$$

$$\mathcal{Z}_0 = 1\,, \tag{5.24}$$

$$\mathcal{H} = 1 + \frac{q}{r} + \mathcal{O}(\alpha'^2)\,. \tag{5.25}$$

An analogous solution is obtained if instead we set $q = 0$, $q_0 \neq 0$, just by interchanging $\mathcal{H}$ with $\mathcal{Z}_0$ and $q_0$ with $q$ in $\mathcal{Z}_+$. These solutions are characterized by the anomalous degree of divergence of $\mathcal{Z}_+$ at $r \to 0$, which now is $\sim 1/r^2$, instead of the usual $\sim 1/r$. If both $q$ and $q_0$ vanish, the divergence is $\sim 1/r^3$. As we discuss in the next section, this behavior potentially could stretch the horizon of these small black holes to yield a non-vanishing area. However, we will see in section 6.2 that these solutions are non-perturbative and therefore the functions given in eqs. (5.22)–(5.25) cannot give a good description of the small black holes near the horizon.

## 6 $\alpha'$ corrections to $d = 4$ black holes

From the solutions constructed in the previous section and the compactification given by eq. (3.19), we obtain a family of 4-dimensional black holes with $\alpha'$ corrections. Let us determine the main properties of these solutions.

### 6.1 4-charge black holes

Let us first consider the case in which all charges are non-vanishing, $qq_0q_+q_- \neq 0$. We write here the metric of these black holes for convenience,

$$ds^2 = e^{2U}dt^2 - e^{-2U}d\vec{x}^{\,2}\,, \quad \text{where} \quad e^{-2U} = \sqrt{\mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\mathcal{H}}\,. \tag{6.1}$$

First, defining the mass $M$ from the asymptotic behavior

$$e^{-2U} = 1 + \frac{2G_N^{(4)}M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{6.2}$$

and using eqs. (5.3)–(5.5), we obtain

$$M = \frac{1}{4G_N^{(4)}}\left[q_0 + \frac{\alpha'(N_\lambda - 2)}{4q} + q_- + q_+\left(1 + \frac{\alpha'}{2qq_0}\right) + q\right]\,. \tag{6.3}$$

This expression takes a more meaningful form in terms of the stringy objects that form the black hole. The charges are characterized by the integer numbers $N_{S5}$, $N_{F1}$, $N_W$



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and $n$ according to eqs. (5.13) and (4.11). On the other hand, the 4-dimensional Newton's constant is obtained from the 10-dimensional one as

$$G_N^{(4)} = \frac{G_N^{(10)}}{(2\pi R)(2\pi R_u)(2\pi \ell_s)^4} = \frac{g_s^2 \ell_s^4}{8 R R_u}\,, \tag{6.4}$$

where we used the value of the 10-dimensional Newton's constant $G_N^{(10)} = 8\pi^6 \ell_s^8 g_s^2$. Plugging this into eq. (6.3), we get

$$M = \frac{R_u}{g_s^2 \ell_s^2}\left(N_{S5} + \frac{N_\lambda - 2}{n}\right) + \frac{R_u}{\ell_s^2} N_{F1} + \frac{N_W}{R_u}\left(1 + \frac{2}{n N_{S5}}\right) + n \frac{R^2 R_u}{g_s^2 \ell_s^4}\,. \tag{6.5}$$

We see that every SU(2) instanton contributes to the mass as one S5-brane times the instanton number.[14] If we do not include SU(2) fields, the $\alpha'$ corrections induce a negative mass term that looks as that of certain kind of anti S5-brane. Hence, in order to avoid negative mass contributions the most natural choice corresponds to $N_\lambda = 2$, and in particular to the choice of two instantons that cancel all the $\alpha'$ corrections in the function sourced by the 5-branes $\mathcal{Z}_0$. Then, the S5-branes are symmetric.

An intriguing observation in the symmetric case is the following. As we see, the mass is the sum of four terms, that actually correspond to the asymptotic charges as should happen in a BPS state. Let us write it as $M = Q_1 + Q_2 + Q_2 + Q_4$. Then, if we consider instead the quantity

$$16\pi G_N^{(4)} \sqrt{Q_1 Q_2 Q_3 Q_4} = 2\pi \sqrt{N_{F1} N_W (n N_{S5} + 2)}\,, \tag{6.6}$$

we are going to see that it coincides with the exact result for the entropy computed counting string microstates. At the very least, this relation suggests that in the symmetric case we do not expect to have further corrections to the mass, and hence the resulting eq. (6.5) with $N_\lambda = 2$ would be exact. Going further, this also supports the conjecture, based on the form of the $T$-tensors that account for all the $\alpha'$ corrections associated to the supersymmetrization of the Chern-Simons terms, that the symmetric solution could be actually exact at all orders in $\alpha'$.[15]

On the other hand, the near-horizon geometry $r \to 0$ is that of $\mathrm{AdS}_2 \times \mathrm{S}^2$ and it does not receive any explicit $\alpha'$ correction:

$$ds^2_{\rm nh} = (q_0 q_+ q_- q)^{-1/2} r^2 dt^2 + (q_0 q_+ q_- q)^{1/2} \left(\frac{dr^2}{r^2} + d\Omega^2_{(2)}\right)\,, \tag{6.7}$$

and the area of the horizon area is

$$A = 4\pi \sqrt{q_0 q_+ q_- q}\,. \tag{6.8}$$

[14] In our previous works (refs. [1–3, 23, 47]) we have used, following ref. [40], a (wrong) normalization that differs from the one used in this paper by a factor of 8 for the $\alpha'$ corrections in the action, so in that references every instanton contributes as 8 S5 branes. Here we have eliminated this factor, in agreement with ref. [48]. We thank Prof. M.J. Duff for sharing this information with us.

[15] The reasons that support this conjecture are the exactly the same that support it in the 5-dimensional case treated in ref. [3]. Unfortunately, the $\alpha'$ corrections which are unrelated to the supersymmetrization of the Chern-Simons terms are not well known and there is no much that can be said about them that supports or contradicts the conjecture.



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However, due to the terms of hiher order in the curvature present in the Heterotic Superstring effective action, the entropy is not simply given by $A/(4G_N^{(4)})$. We will compute its value in the next section by applying Wald's formula [17, 18].

**Bonus example: corrections to Reissner-Nordstrom black hole.** The Reissner-Nordström black hole corresponds to the zeroth-order in $\alpha'$ solution with $q_+ = q_- = q_0 = q$. This choice of charges gives constant scalars $e^{\phi} = e^{\phi_\infty}, k = k_\infty$ and $\ell = \ell_\infty$ at this order. However, taking into account the constituents of the black hole, we can only take those charges equal at given points in moduli space

$$g_s = e^{\phi_\infty} = \sqrt{\frac{N_{S5}}{N_{F1}}}\,, \qquad R_z/\ell_s = k_\infty = \sqrt{\frac{N_W}{N_{F1}}}\,, \qquad R/\ell_s = 2\ell_\infty = \sqrt{\frac{N_{S5}}{n}}\,, \tag{6.9}$$

which fixes the asymptotic values of the scalars to their attractor values.

Applying the general result to this particular case is straightforward. Taking the symmetric case, we find that the corrected metric function $e^{-2U}$ and the scalars take the form

$$e^{-2U} = \left(1+\frac{q}{r}\right)^2 + \frac{\alpha'}{4q}\frac{[r^2+3rq+3q^2]}{r(r+q)^2} + \mathcal{O}(\alpha'^2)\,, \tag{6.10}$$

$$e^{2\phi} = e^{2\phi_\infty}\,, \tag{6.11}$$

$$k = k_\infty + \frac{\alpha' k_\infty}{4q}\frac{r[r^2+3rq+3q^2]}{(r+q)^4} + \mathcal{O}(\alpha'^2)\,, \tag{6.12}$$

$$\ell = \ell_\infty + \frac{\alpha' \ell_\infty}{12q}\frac{r[r^2+3rq+3q^2]}{(r+q)^4} + \mathcal{O}(\alpha'^2)\,, \tag{6.13}$$

with

$$q = \frac{\ell_s}{2}\sqrt{nN_{S5}}\,. \tag{6.14}$$

We also have to take into account that the 4-dimensional Newton constant given in eq. (6.4) now has the value

$$G_N^{(4)} = \frac{\ell_s^2}{8}\sqrt{\frac{nN_{S5}}{N_W N_{F1}}}\,. \tag{6.15}$$

Then, if we do not want $G_N^{(4)}$ to change with $q$, we must set $N_W N_{F1} = \aleph^2 N_{S5} n$ for some positive dimensionless constant $\aleph$ so that

$$G_N^{(4)} = \frac{\ell_s^2}{8\aleph}\,. \tag{6.16}$$

Staying at weak coupling (so the loop corrections can be safely ignored) and away of the self-dual radii at which new massless degrees of freedom arise, demands the following hierarchy

$$N_W > N_{F1} > N_{S5} > n\,, \quad \Rightarrow \quad \aleph \gg 1\,. \tag{6.17}$$

At this point in moduli space, the mass of the black hole will be given by

$$M = \frac{4}{\ell_s}\sqrt{N_W N_{F1}}\left(1+\frac{1}{2nN_{S5}}\right)\,, \quad \text{or} \quad 2G_N^{(4)}M = \ell_s\left(\sqrt{nN_{S5}}+\frac{1}{2\sqrt{nN_{S5}}}\right)\,, \tag{6.18}$$



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while the area of the horizon takes the value

$$A = 4\pi q^2 = \pi \ell_s^2 n N_{S5}\,, \tag{6.19}$$

and the leading contribution to the entropy will be

$$\frac{A}{4G_N^{(4)}} = 2\pi\sqrt{n N_{S5} N_W N_{F1}} = 2\pi \aleph n N_{S5}\,. \tag{6.20}$$

### 6.2 Small black holes

It has been known for some time that, in the String Theory context and at lowest order in $\alpha'$, four charges are needed in order to obtain a 4-dimensional extremal black hole with a regular horizon. When one or more of the charges vanish, the horizon (still located at $r = 0$) has zero area and becomes singular. Usually, the small black holes considered in the literature only contain $q_-$ and $q_+$ charges (corresponding to strings and waves). There is a curvature singularity at $r = 0$ and the scalars behave there as

$$e^{2\phi} \to 0\,, \quad k \sim \frac{1}{r^{1/4}}\,, \quad \ell \sim \frac{1}{r^{1/3}}\,. \tag{6.21}$$

Although at zeroth-order in $\alpha'$ the area and entropy vanish, the exact entropy of such black holes computed by counting string microstates is finite (see e.g. ref. [9]). Hence, from the Supergravity perspective, it was expected that the singularity at the would-be horizon of these solutions could be fixed somehow. Since the string coupling vanishes at $r = 0$, quantum corrections cannot be of help here, but $\alpha'$ corrections can be relevant because the curvature is very large (divergent) there.

Let us consider the $\alpha'$-corrected solution in the case $q_0 = 0$, allowing $q$ to be arbitrary, so that the solution still contains 3 independent charges. This solution is determined by the functions in eqs. (5.22)–(5.25). Let us recall that we are choosing a symmetric KK monopole in this case, so that $\mathcal{Z}_0 = 1$. This solution is qualitatively different from the 4-charge black hole. For example, the limit $q_0 \to 0$ in the mass formula eq. (6.3) would be divergent, but we find instead

$$M = \frac{1}{4G_N^{(4)}}\left[q_- + q_+ + q\right]\,. \tag{6.22}$$

On the other hand, in the limit $r \to 0$ the functions $\mathcal{H}$ and $\mathcal{Z}_-$ diverge with the usual $1/r$ behavior, but the $\alpha'$ corrections produce another divergence in $\mathcal{Z}_+$ that dominates in this limit:

$$\mathcal{Z}_+ \sim -\frac{\alpha' q_+}{2 q r^2}\,. \tag{6.23}$$

Thus, assuming that $q_+ q_- < 0$, the metric becomes again $\mathrm{AdS}_2 \times \mathrm{S}^2$ and the area of the horizon is

$$A = 4\pi\sqrt{-\alpha' q_+ q_-/2}\,. \tag{6.24}$$

Interestingly, only two charges contribute to this area even if the solution contains three of them. It is immediate to check that the previous formula also holds if we set $q = 0$



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(because then $\mathcal{Z}_+ \sim -\frac{\alpha' q_+}{2r^3}$) or if we set instead $q_0 \neq 0$, $q = 0$. Summarizing all the cases, if $q_0 q = 0$ it seems that the higher curvature corrections pump the area of the horizon up from zero to the value in eq. (6.24).

However, it is possible to see that this analysis is not rigorous enough. Our solution generating technique, summarized in section 3, assumes that the solution admits a perturbative expansion. In particular, to obtain the corrections to the function $\mathcal{Z}_+$ as given in eq. (3.10) we implicitly assume that the dominant contribution to $\partial_n \mathcal{Z}_+$ is given by $\partial_n \mathcal{Z}_+^{(0)}$ everywhere, allowing a perturbative construction. This is true in the 4-charge family of solutions, but when $q_0 q = 0$ we immediately see from eq. (6.23) that $\partial_n \mathcal{Z}_+^{(0)}$ is subdominant in the near-horizon region. This is a contradiction which signals that our method to build solutions is not valid in this case and that small black holes are non perturbative, so the solution eqs. (5.22)–(5.25) should not be trusted at this stage. They will be studied in more detail in a forthcoming paper ref. [49].

Let us close this section by mentioning that $\alpha'$ corrections also introduce very curious properties in 4-charge solutions when two of the charges are negative. At zeroth order in $\alpha'$ such solutions represent naked singularities, but it was shown in ref. [50] that the first-order corrections turn them into globally regular black holes whose horizon does not contain any singularity. These solutions suffer from the same issue as the one mentioned for small black holes, hence a thorough analysis of the higher-order corrections is necessary to determine if these appealing regular black holes can be trusted.

## 7 Black hole entropy

In this section we will compute the Wald entropy of the black holes we have obtained, up to terms of $\mathcal{O}(\alpha')$. Following refs. [17, 18], the Wald entropy formula for a $D+1$-dimensional theory is

$$\mathbb{S} = -2\pi \int_\Sigma d^{D-1}x \sqrt{|h|} \mathcal{E}_R^{abcd} \epsilon_{ab} \epsilon_{cd} \,, \tag{7.1}$$

where $\Sigma$ is a cross section of the horizon, $h$ is the determinant of the metric induced on $\Sigma$, $\epsilon_{ab}$ is the binormal to $\Sigma$ with normalization $\epsilon_{ab}\epsilon^{ab} = -2$, and $\mathcal{E}_R^{abcd}$ is the equation of motion one would obtain for the Riemann tensor $R_{abcd}$ treating it as an independent field of the theory,

$$\mathcal{E}_R^{abcd} = \frac{1}{\sqrt{|g|}} \frac{\delta S_{(D+1)}}{\delta R_{abcd}} \,, \tag{7.2}$$

where $S_{(D+1)}$ is the action of the theory.

Then it seems that, in order to compute the entropy of the lower dimensional solutions one needs to know the $\alpha'$ corrections to the Lagrangians of the dimensionally reduced theories. The determination of such correction terms starting from the 10-dimensional theory would require a long and intricate calculation, but luckily it turns out to be unnecessary since, as we will now show following the discussion in ref. [3], the entropy of the 4- and 5-dimensional solutions can be re-expressed entirely in terms of integrals of 10-dimensional quantities.



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It is convenient to work in frames adapted to the dimensional reduction. To this end, we start by rewriting the 10- and 5-dimensional line elements eqs. (3.13) and (3.17) in the following form:

$$ds^2_{(10)} = e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3} ds^2_{(5)} - (k/k_\infty)^2\left(du - \frac{dt}{\mathcal{Z}_+}\right)^2\right] - dy^i dy^i\,, \tag{7.3}$$

$$ds^2_{(5)} = R/(2\ell)\, ds^2_{(4)} - \ell^2\,(d\Psi + \chi)^2\,, \tag{7.4}$$

where the 4-dimensional line element $ds^2_{(4)}$, the dilaton $\phi$ and the Kaluza-Klein scalars $k$ and $\ell$ are given by (3.19), which we report here for convenience:

$$\begin{aligned} ds^2_{(4)} &= e^{2U}dt^2 - e^{-2U}d\vec{x}^{\,2}\,,\\ e^{2\phi} &= e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \qquad k = k_\infty \frac{\mathcal{Z}_+^{1/2}}{\mathcal{Z}_0^{1/4}\mathcal{Z}_-^{1/4}}\,, \qquad \ell = \ell_\infty \frac{\mathcal{Z}_0^{1/6}\,\mathcal{Z}_+^{1/6}\,\mathcal{Z}_-^{1/6}}{\mathcal{H}^{1/2}}\,, \end{aligned} \tag{7.5}$$

with $e^{\phi_\infty} = g_s$ and

$$e^{-2U} = \sqrt{\mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\mathcal{H}}\,. \tag{7.6}$$

We are also going to need the metric function of the 5-dimensional black holes $f$, which is given in eq. (3.18) and which we also quote here for convenience:

$$f^{-3} = \mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\,. \tag{7.7}$$

We define the following Vielbein bases in 4, 5 and 10 dimensions, respectively:

$$v^0 = e^U dt\,, \qquad v^1 = e^{-U}dr\,, \qquad v^2 = e^{-U} r d\theta\,, \qquad v^3 = e^{-U} r\sin\theta d\phi\,. \tag{7.8}$$

$$V^{0,\ldots,3} = (R/(2\ell))^{1/2}\, v^{0,\ldots,3}\,, \qquad V^4 = \ell\,(d\Psi + \cos\theta d\phi)\,, \tag{7.9}$$

$$W^{0,\ldots,4} = e^{\frac{\phi-\phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}} V^{0,\ldots,4}\,, \quad W^5 = e^{\frac{\phi-\phi_\infty}{2}}\frac{k}{k_\infty}\left(du - \frac{dt}{\mathcal{Z}_+}\right)\,, \; W^{6,\ldots,9} = dy^{6,\ldots,9}\,, \tag{7.10}$$

where $(r,\theta,\phi)$ are spherical coordinates on $\mathbb{E}^3$.

As we can see, the Vielbein 1-form bases we have introduced in 4,5 and 10 dimensions are related by multiplication by a common factor, and consequently the components of the Riemann tensors in these frames are related by

$$R_{(10)abcd} = e^{-(\phi-\phi_\infty)}(\frac{k}{k_\infty})^{2/3} R_{(5)abcd} + \ldots\,, \tag{7.11}$$

for $a,b,c,d = 0,\ldots,4$ and

$$R_{(5)abcd} = \frac{2\ell}{R}\, R_{(4)abcd} + \ldots \quad \Rightarrow \quad R_{(10)abcd} = \frac{2\ell}{R}\, e^{-(\phi-\phi_\infty)}(\frac{k}{k_\infty})^{2/3} R_{(4)abcd} + \ldots\,, \tag{7.12}$$

for $a,b,c,d = 0,\ldots,3$, so that

$$\frac{\delta S_{(10)}}{\delta R_{(5)abcd}} = e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\int \frac{\delta S_{(10)}}{\delta R_{(10)abcd}}\,, \tag{7.13}$$



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and

$$\frac{\delta S_{(10)}}{\delta R_{(4)abcd}} = \frac{2\ell}{R}\, e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3} \int \frac{\delta S_{(10)}}{\delta R_{(10)abcd}}\,, \tag{7.14}$$

where the integrations are on the appropriate compact coordinates. The action is, of course, the same in any dimension, and, therefore,

$$\frac{\delta S_{(5)}}{\delta R_{(5)abcd}} = \frac{\delta S_{(10)}}{\delta R_{(5)abcd}}\,, \qquad \frac{\delta S_{(4)}}{\delta R_{(4)abcd}} = \frac{\delta S_{(10)}}{\delta R_{(4)abcd}}\,. \tag{7.15}$$

In the solutions we are interested in, the horizon $\Sigma$ is located at $r = 0$, where the timelike Killing vector becomes null. Then we have

$$|g_{(5)}| = f\mathcal{H}|h_{(5)}|\,, \qquad |g_{(4)}| = |h_{(4)}|\,, \tag{7.16}$$

on $\Sigma$, and the components of the binormal $\epsilon_{ab}$ when expressed in the frames defined above are the same in any dimension, $\epsilon_{01} = 1$.

All this allows us to write the Wald entropy for the 4 and 5-dimensional solutions in 10-dimensional language, the explicit expressions being

$$\mathbb{S}_{(5)} = -2\pi \int_{\Sigma_3} d^3x \sqrt{\left|\frac{h_{(5)}}{g_{(5)}}\right|}\, \frac{\delta S_{(5)}}{\delta R_{(5)abcd}} \epsilon_{ab}\epsilon_{cd} \tag{7.17}$$

$$= -2\pi \int_{\Sigma_3\times S^1\times \mathrm{T}^4} d^8x (f\mathcal{H})^{-1/2} e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3} \frac{\delta S_{(10)}}{\delta R_{(10)abcd}} \epsilon_{ab}\epsilon_{cd}\,, \tag{7.18}$$

and

$$\mathbb{S}_{(4)} = -2\pi \int_{\Sigma_2} d^2x \sqrt{\left|\frac{h_{(4)}}{g_{(4)}}\right|}\, \frac{\delta S_{(4)}}{\delta R_{(4)abcd}} \epsilon_{ab}\epsilon_{cd} \tag{7.19}$$

$$= -2\pi \int_{\Sigma_2\times S^1\times S^1\times \mathrm{T}^4} d^8x \frac{2\ell}{R}\, e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3} \frac{\delta S_{(10)}}{\delta R_{(10)abcd}} \epsilon_{ab}\epsilon_{cd}\,. \tag{7.20}$$

Observe that, taking into account the expression for $\ell$ in eq. (7.5), if $\Sigma_3 = \Sigma_2 \times \mathrm{S}^1$, then $\mathbb{S}_{(5)} = \mathbb{S}_{(4)}$.

The action $S_{(10)}$ is given in eq. (2.9). At leading order it depends on the Riemann tensor only through the Einstein-Hilbert term, while at first order in $\alpha'$ there are additional contributions from terms depending on the curvature of the torsionful spin connection $\Omega_{(-)}$, denoted by $R_{(-)}$. Since we are only interested in the first order corrections and $R_{(-)}$ already appears at first order in the Lagrangian, it is enough to consider the leading order dependence of $R_{(-)}$ on the Riemann tensor,

$$R_{(-)abcd} = R_{(10)abcd} + \dots\,. \tag{7.21}$$



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Since at this order no derivatives of the Riemann tensor appear in the Lagrangian, one has on $\Sigma$

$$\begin{aligned}\frac{1}{\sqrt{|g_{(10)}|}}\frac{\delta S_{(10)}}{\delta R_{(10)abcd}}\epsilon_{ab}\epsilon_{cd} &= \frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_N^{(10)}}\frac{\partial}{\partial R_{(10)abcd}}\left(R+\frac{1}{2\cdot 3!}H^2-\frac{\alpha'}{8}R_{(-)}{}^a{}_b R_{(-)}{}^b{}_a\right)\epsilon_{ab}\epsilon_{cd}\\ &= \frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_N^{(10)}}\left[\eta^{ac}\eta^{bd}+\frac{H^{efg}}{3!}\frac{\delta H_{efg}}{\delta R_{(10)abcd}}\right]\epsilon_{ab}\epsilon_{cd}\\ &= \frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_N^{(10)}}\left[\eta^{ac}\eta^{bd}-\frac{\alpha'}{8}H^{abg}\Omega_{(-)g}{}^{cd}\right]\epsilon_{ab}\epsilon_{cd}\\ &= -\frac{e^{-2(\phi-\phi_\infty)}}{8\pi G_N^{(10)}}\left[1+\frac{\alpha'}{4}H^{01g}\Omega_{(-)g}{}^{01}\right]. \end{aligned} \tag{7.22}$$

The term quadratic in $R_{(-)}$ does not contribute because $R_{(-)}$ vanishes on $\Sigma$.[16] The relevant components of the Kalb-Ramond field strength $H$ and of the torsionful spin connection $\Omega_{(-)}$ can be obtained straightforwardly,

$$H_{01g} = -\delta^5_g(\mathcal{Z}_0\mathcal{H})^{-1/2}\partial_r\log\mathcal{Z}_-\,, \tag{7.23}$$

$$\Omega_{(-)501} = \frac{1}{2}(\mathcal{Z}_0\mathcal{H})^{-1/2}\partial_r\log\left(\mathcal{Z}_-\mathcal{Z}_+\right), \tag{7.24}$$

$$H^{01g}\Omega_{(-)g}{}^{01} = H^{015}\Omega_{(-)5}{}^{01} = \frac{1}{2}\frac{\partial_r\log\left(\mathcal{Z}_-\mathcal{Z}_+\right)\partial_r\log\mathcal{Z}_-}{\mathcal{Z}_0\mathcal{H}}\,. \tag{7.25}$$

The last missing piece is the determinant of the 10-dimensional metric, which reads

$$\sqrt{|g_{(10)}|} = e^{3(\phi-\phi_\infty)}f^{-1}\frac{R}{2}\,r^2\mathcal{H}\sin\theta(k/k_\infty)^{-2/3}\,. \tag{7.26}$$

Putting everything together we get to

$$\mathbb{S}_{(4)} = \frac{R}{8G_N^{(10)}}\int d\theta d\phi d\Psi dz d^4y r^2\sin\theta\sqrt{\mathcal{Z}_0\mathcal{Z}_+\mathcal{Z}_-\mathcal{H}}\left(1+\frac{\alpha'}{8}\frac{\partial_r\log\left(\mathcal{Z}_-\mathcal{Z}_+\right)\partial_r\log\mathcal{Z}_-}{\mathcal{Z}_0\mathcal{H}}\right). \tag{7.27}$$

Once we substitute the explicit form of the functions $\mathcal{Z}_{0,\pm}$ given in eqs. (3.25) and $\mathcal{H}$ given in eq. (3.26) and integrate on the compact coordinates and on $\Sigma$ we arrive at the result

$$\mathbb{S}_{(4)} = \frac{\pi}{G_N^{(4)}}\sqrt{q_0\,q_+q_-q}\left(1+\frac{\alpha'}{4q_0q}\right), \tag{7.28}$$

with the 4-dimensional Newton constant given by

$$G_N^{(4)} = \frac{G_N^{(10)}}{(2\pi R)(2\pi R_u)(2\pi\ell_s)^4}\,. \tag{7.29}$$

Finally, with the identifications eqs. (5.13) and (4.11)

$$q_0 = \frac{\alpha'}{2R}N_{S5}\,,\qquad q_+ = \frac{\alpha'^2g_s^2}{2RR_u^2}N_W\,,\qquad q_- = \frac{\alpha'g_s^2}{2R}N_{F1}\,,\qquad q = \frac{nR}{2}\,, \tag{7.30}$$

[16] See [3, 51].



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and

$$G_N^{(10)} = 8\pi^6 \alpha'^4 g_s^2 \,, \tag{7.31}$$

the entropy can be finally rewritten as

$$\mathbb{S}_{(4)} = 2\pi \sqrt{N_{F1} N_W n N_{S5}} \left(1 + \frac{1}{n N_{S5}}\right) . \tag{7.32}$$

## 8 Discussion

The preceding result (7.32) coincides with the next to leading order approximation to

$$\mathbb{S}_{(4)} = 2\pi \sqrt{N_{F1} N_W \left(n N_{S5} + 2\right)} \,, \tag{8.1}$$

in a $n N_{S5} \gg 1$ expansion, which is the exact entropy formula computed previously in the literature using the entropy function formalism in ref. [51] — see also [52] — once the global effects of higher-curvature corrections on the charges are properly taken into account, as we are going to explain below. Since the near-horizon solution preserves the symmetries of $\mathrm{AdS}_3 \times \mathrm{S}^3$, at the horizon one has $R_{(-)}{}^a{}_b = 0$. Therefore all corrections to the area law in the Wald entropy arise from the variation of the Kalb-Ramond field strength with respect to the Riemann tensor, which suggests that it might be possible to compute them explicitly.

Direct comparison of (8.1) with the expression displayed in the previous literature seems to yield an apparent mismatch; previous studies based on near-horizon solutions obtain a correction factor of value +4, instead of +2, inside the square root. The obvious question is: why?

Before answering that question, let us notice that we can rewrite the expression eq. (8.1) substituting the number of solitonic 5-branes $N_{S5}$ by the total, asymptotic solitonic 5-brane charge $\mathcal{Q}_{S5} = N_{S5} + \frac{N_\lambda - 2}{n}$. In doing so, we get

$$\mathbb{S} = 2\pi \sqrt{N_{F1} N_W \left(n \mathcal{Q}_{S5} + 4 - N_\lambda\right)} \,, \tag{8.2}$$

where, we recall, $N_\lambda$ is the number of instantons in the solution. To get here we have used the formula for the shift in the charges that we obtained at first order in $\alpha'$. However, this expression is actually exact, as explained in [53]. Then, equation (8.2) is also exact. Since previous studies do not include non-trivial gauge fields, we see that setting $N_\lambda = 0$ we reproduce the aforementioned correction factor when the entropy is mostly expressed in terms of asymptotic charges instead of fundamental constituents, except for the momentum carried by the string. As we are going to argue, this is no coincidence.

In the entropy function formalism, the S5 charge is carried by an auxiliary Abelian 4-dimensional vector whose Bianchi identity is *uncorrected*, i.e. its field strength is a closed 2-form. This means that the charges carried by this auxiliary field is the same everywhere, asymptotically and at the horizon. In most of the preceding literature this charge was identified with $N_{S5}$, the number of S5-branes. On the other hand, it was observed in ref. [52] that the charge carried by this auxiliary field is actually the asymptotic S5 charge, $\mathcal{Q}_{S5}$, carried magnetically by the Kalb-Ramond field strength.



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This identification is in direct contradiction with the identification of total charges and sources which we have explained in full detail in the preceding sections. As we have argued repeatedly, the $N_{S5}$ S5-branes are responsible of the local S5 charge measured at the horizon, while the asymptotic charge results from adding to these the delocalized sources introduced by the higher curvature corrections. Observe that, if the asymptotic charge were $N_{S5}$, then the charge at the horizon would be given by $(N_{S5} + 2/n)$, with the shift caused by the higher curvature corrections. It seems hard to justify how this would be possible, specially taking into consideration that the curvature $R_{(-)}{}^{a}{}_{b}$ vanishes at the horizon.

Finally, we observe that, in order to reproduce the standard formula for the entropy we have preserved the near-horizon momentum charge $N_W$. This can be justified as follows. While in ref. [51] the Chern-Simons terms that carry the delocalized S5 charge are carefully taken into account, the term $T^{(0)}$ is not included in the action. This term is responsible for the appearance of delocalized momentum charge, as becomes manifest in the Einstein equation of motion (2.11). It is for this reason that we should not take the shift of this charge into account to perform this comparison.

For all these reasons we claim that the correct expression for the entropy expressed in terms of the number of fundamental constituents of the solution is eq. (8.1). This identification also matches the microscopic result eq. (1.1) ($k = n\mathcal{Q}_{S5} + 2 = nN_{S5}$).[17]

Notice that if we set $n = 1$ the value of the entropy coincides with that of the three-charge, five-dimensional black holes we described in ref. [3]. From our perspective this seems completely natural, since after setting $n = 1$ the event horizon of the black hole we get is identical to that of ref. [3], so both Wald entropies should match. Once again, this result would be in contradiction with the standard expression found in the literature, where the correction factor is $+3$ instead of $+2$. Yet again, in this case, we find the origin of the discrepancy might be on the identification of the charges. Notice that in the 5-dimensional black holes the hyperKähler space is simply $\mathbb{R}^4$, and the contribution to the asymptotic S5 charge coming from $\int R_{(-)}{}^{a}{}_{b} \wedge R_{(-)}{}^{b}{}_{a}$ is just $-1$. Hence, substituting $N_{S5} - 1 = \mathcal{Q}_{S5}$ in eq. (8.1) we reproduce the standard correction factor found in the literature.[18]

Therefore, in light of our results it seems that the appearance of a factor different from $+2$ is caused by a misidentification of the number of fundamental objects of the heterotic superstring and their relation with the asymptotic charges of the solution.

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[17] Observe that the identification of $k$ in ref. [13] was made comparing with a solution of the zeroth-order in $\alpha'$ effective action.

[18] At this stage, let us point out that the difference in the previously observed correction factors of $+3$ and $+4$ for these five- and four-dimensional black holes (sometimes referred as the $4D/5D$ *connection*) had already been identified as caused by the negative unit of S5 charge carried by the Kaluza-Klein monopole of unit charge (for generic charge $n$, the associated S5 charge is $-1/n$), see [54, 55]. The relevance of the S5 charge carried by the KK monopole had already been remarked in refs. [56, 57].



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## A Instanton number density as a Laplacian

Consider an arbitrary Gibbons-Hawking space, with metric

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta+\chi)^2 + \mathcal{H}dx^x dx^x\,, \qquad d\mathcal{H} = L\star_{(3)} d\chi\,, \tag{A.1}$$

and define on it an SU(2) field of the form

$$A^A = -\mathcal{H}^{-1}\Phi^A(d\eta+\chi) + \breve{A}^A\,, \tag{A.2}$$

where $\Phi^A$ and $\breve{A}^A$ are, respectively, a function and a 1-form defined on $\mathbb{E}^3$. Then the requirement of self-duality for the field strength

$$F^A = dA^A + \frac{1}{2}\epsilon^{ABC}A^B\wedge A^C \tag{A.3}$$

is equivalent to the Bogomol'nyi equation

$$F^A = +\star_{(4)} F^A\,, \qquad \Longleftrightarrow \qquad \breve{\mathcal{D}}\Phi^A = \star_3\breve{F}^A\,, \tag{A.4}$$

and one has

$$\begin{aligned} F^A &= -\breve{\mathcal{D}}\left(\mathcal{H}^{-1}\Phi^A\right)\wedge(d\eta+\chi) - \mathcal{H}^{-1}\Phi^A d\chi + \breve{F}^A \\ &= -\breve{\mathcal{D}}\left(\mathcal{H}^{-1}\Phi^A\right)\wedge(d\eta+\chi) + \star_3\breve{\mathcal{D}}\left(\mathcal{H}^{-1}\Phi^A\right)\,. \end{aligned} \tag{A.5}$$

The instanton number density is, then.

$$\begin{aligned} F^A\wedge F^A &= -2\breve{\mathcal{D}}(\mathcal{H}^{-1}\Phi^A)\wedge\star_3\breve{\mathcal{D}}(\mathcal{H}^{-1}\Phi^A)\wedge(d\eta+\chi) \\ &= -\mathcal{H}^{-1}\Big[2\breve{\mathcal{D}}(\mathcal{H}^{-1}\Phi^A)\wedge\star_3\breve{\mathcal{D}}\Phi^A - 2\mathcal{H}^{-1}\Phi^A\breve{\mathcal{D}}(\mathcal{H}^{-1}\Phi^A)\wedge\star_3\breve{\mathcal{D}}H \\ &\qquad + 2\mathcal{H}^{-1}\Phi^A\breve{\mathcal{D}}\star_3\breve{\mathcal{D}}\Phi^A - \mathcal{H}^{-2}\Phi^A\Phi^A d\star_3 d\mathcal{H}\Big]\wedge(d\eta+\chi) \\ &= -\mathcal{H}^{-1}\breve{\mathcal{D}}\star_3\breve{\mathcal{D}}\left(\mathcal{H}^{-1}\Phi^A\Phi^A\right)\wedge(d\eta+\chi) = \mathcal{H}^{-1}\partial_{\underline{x}}\partial_{\underline{x}}\left(\mathcal{H}^{-1}\Phi^A\Phi^A\right)|v|d^4x \\ &= \nabla^2_{(4)}\left(\Phi^A\Phi^A/\mathcal{H}\right)|v|d^4x\,, \end{aligned} \tag{A.6}$$

where we made use of the fact that $\mathcal{H}$ is harmonic, $d\star_3 d\mathcal{H} = 0$, and that $\breve{\mathcal{D}}\star_3\breve{\mathcal{D}}\Phi^A = \breve{\mathcal{D}}\breve{F}^A = 0$.

## B Regular instantons over Kaluza-Klein monopoles

Let us consider the monopoles introduced in section 4.2. We are going to see that, over the KK monopole of unit charge ($n=1$), all these monopoles give rise to regular instantons, giving an explicit construction of the instanton bundles. These instantons over a KK



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monopole with unit charge were first described in ref. [43]. Moreover, for higher KK monopole charge, $n > 1$, the corresponding instantons are also regular in the full spacetime metric provided the black hole horizon has non-vanishing area. From now on we consider the $n = 1$ case in detail for the sake of simplicity. The regularity condition for the $n > 1$ case is treated in section B.3.

WE are going to use the following convenient way of writing the gauge fields $A^A$ in eq. (4.19) in the case $n = 1$

$$A^A = -h(r)r^2 v_R^A + \frac{x^A}{r}(d\Psi + d\phi\cos\theta)\left(\frac{R}{2\mathcal{H}}rf(r) - h(r)r^2\right), \tag{B.1}$$

where $v_R^A$ are the right-invariant Maurer-Cartan forms

$$\begin{cases} v_R^1 = \sin\phi d\theta - \sin\theta\cos\Psi d\Psi\,, \\ v_R^2 = \cos\phi d\theta + \sin\theta\sin\phi d\Psi\,, \\ v_R^3 = d\phi + \cos\theta d\Psi\,, \end{cases} \qquad dv_R^A + \frac{1}{2}\epsilon_{ABC}v_R^B \wedge v_R^C = 0\,. \tag{B.2}$$

### B.1 Near-origin limit

When $r \to 0$, $\mathcal{H} \sim (R/2)/r$, and after the change of variables $r = \frac{\rho^2}{2R}$, we can rewrite the metric as

$$d\sigma^2 = d\rho^2 + \rho^2 d\Omega_{(3)}^2\,, \tag{B.3}$$

where

$$d\Omega_{(3)}^2 = \frac{1}{4}\left[(d\Psi + d\phi\cos\theta)^2 + d\Omega_{(2)}^2\right], \tag{B.4}$$

is the metric of the round $S^3$ of unit radius, so the space is locally $\mathbb{E}^4$.

On the other hand, at the origin $r = \rho = 0$, the gauge fields of the different instantons take the values

$$\lim_{r\to 0} A_{\mu,s} = \begin{cases} \frac{1}{g}v_R\,, & s \neq 0\,, \\ 0\,, & s = 0\,, \end{cases} \qquad \lim_{r\to 0} A_\lambda = \frac{1}{g}v_R \tag{B.5}$$

where $A = A^A T_A$ and $v_R = v_R^A T_A$ and $\{T_A\}$ is the basis of the $\mathfrak{su}(2)$ algebra

$$T_A = -\frac{i}{2}\sigma^A\,, \quad [T_A, T_B] = +\epsilon_{ABC}T_C\,. \tag{B.6}$$

Hence, except in the case $(\mu, s = 0)$, these vectors are singular at the origin, something which can be made explicit if we write $A(r = 0)$ in terms of the Vielbein basis. However, $A(r = 0)$ is pure gauge and the field strength $F$ is finite at the origin for $s \neq 0$. Furthermore, Kronheimer's conditions are met there. This indicates the existence, for $s \neq 0$, of another gauge in which the gauge field is regular at the origin.

An arbitrary gauge transformation of the vector reads

$$\hat{A} = VAV^{-1} - \frac{1}{g}dVV^{-1}\,, \tag{B.7}$$



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where $V \in \mathrm{SU}(2)$. If we choose $V = U$ where

$$U \equiv e^{-T^3\Psi} e^{-T^2\theta} e^{-T^1\phi}\,, \tag{B.8}$$

is the generic SU(2) group element parametrized in terms of the Euler angles $\Psi, \theta, \phi$, and we take into account the properties of the Maurer-Cartan forms[19]

$$v_R = U v_L U^{-1}\,, \tag{B.10}$$

we get the following transformed gauge field:[20]

$$g\hat{A} = -(1 + gh(r)r^2)v_L - g\left(\frac{R}{2\mathcal{H}} r f(r) - h(r)r^2\right)(d\Psi + \cos\theta d\phi)T_3\,. \tag{B.12}$$

If we apply this transformation to the $\lambda$ and $(\mu, s > 0)$-cases, we get, respectively:

$$g\hat{A}_\lambda = -\frac{\lambda^2 r}{1+\lambda^2 r}\left(v_L^1 T_1 + v_L^2 T_2\right) - \frac{r\left(1 + \lambda^2(r + R/2)\right)}{(R/2 + r)(1+\lambda^2 r)} v_L^3 T_3\,, \tag{B.13}$$

$$g\hat{A}_{\mu,s>0} = -\frac{\mu r}{\sinh(\mu r + s)}\left(v_L^1 T_1 + v_L^2 T_2\right) - \frac{r\left[1 + \mu R/2 \coth(\mu r + s)\right]}{R/2 + r} v_L^3 T_3\,. \tag{B.14}$$

Now, near the origin the vectors behave as $\hat{A} \sim \rho^2 v_L$ and we conclude that the gauge fields are regular at $r = 0$ for all values of $\lambda$ and $s > 0$. In the case $s = 0$ the vector is already regular and we do not need to perform any gauge transformation. In that case, it reads

$$g\hat{A}_{\mu,s=0} = \left(1 - \frac{\mu r}{\sinh \mu r}\right) v_R \tag{B.15}$$

$$+ \frac{x^A}{r} T_A (d\Psi + d\phi\cos\theta)\frac{r}{r + R/2}\left\{1 - \frac{\mu\left[r + R/2(1 - \cosh\mu r)\right]}{\sinh\mu r}\right\}\,. \tag{B.16}$$

### B.2 Asymptotic limit

In order to study the asymptotic limit, it is convenient to recall the global structure of the solution and to use cartesian coordinates $x^1, x^2, x^3$. In these coordinates, the two 1-forms $\chi^{(\pm)}$ read[21]

$$\chi^{(\pm)} = \frac{x^1 dx^2 - x^2 dx^1}{r(x^3 \pm r)}\,, \quad \text{where } r \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\,. \tag{B.17}$$

[19] These properties follow from their definition in terms of $U$:

$$v_L \equiv -U^{-1}dU\,, \quad v_R \equiv -dUU^{-1}\,. \tag{B.9}$$

[20] The left-invariant MC forms read

$$\begin{cases} v_L^1 = -\sin\Psi d\theta + \sin\theta\cos\Psi d\phi \\ v_L^2 = \cos\Psi d\theta + \sin\theta\sin\Psi d\phi \\ v_L^3 = d\Psi + \cos\theta d\phi \end{cases} \qquad dv_L^A - \frac{1}{2}\epsilon_{ABC} v_L^B \wedge v_L^C = 0\,. \tag{B.11}$$

[21] For convenience, in this section $\chi$ is defined locally as $\chi = d\phi\cos\theta$, without the factor of $q$ that appears in (4.4).



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We use $\chi^{(+)}$ in the upper space $x^3 \geq 0$ and $\chi^{(-)}$ in the lower one $x^3 \leq 0$ so that $\chi^{(\pm)}$ are regular in their respective regions. Moreover, we observe that

$$\lim_{r\to\infty} \chi^{(\pm)} = 0\,, \tag{B.18}$$

where the limit is again taken in the respective region. In this limit we also have $\mathcal{H} \sim 1$, and hence the space becomes the direct product $\mathbb{E}^3 \times \mathrm{S}^1$. The metric takes the form

$$d\sigma^2 = \frac{R^2}{4} d\Psi^2 + dx^i dx^i\,. \tag{B.19}$$

Let us now explore the asymptotic behavior of the gauge fields.

In the case of $\hat{A}_\lambda$ we obtain

$$\lim_{r\to\infty} \hat{A}_\lambda = \lim_{r\to\infty} \frac{-1}{g} v_L = -\frac{1}{g} T_3 d\Psi\,. \tag{B.20}$$

The first equality is obtained by using the explicit dependence in $r$ in eq. (B.13), while in the second equality we use the implicit dependence contained in the angular 1-forms inside the MC forms.[22] The vector at infinity is pure gauge, so that there is an element $V \in \mathrm{SU}(2)$ such that

$$\hat{A}_{\lambda\,\infty} = -\frac{1}{g} dV V^{-1}\,. \tag{B.21}$$

This element is

$$V = e^{\Psi T_3} = \begin{pmatrix} e^{-i\Psi/2} & 0 \\ 0 & e^{i\Psi/2} \end{pmatrix}, \tag{B.22}$$

which defines a map from the equator of the sphere to the subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Hence, $V$ is an homomorphism from the circle in itself which belongs to the first homotopy class.

This case is very simple because the asymptotic value of the vector does not depend on free parameters. The family $(\mu, s)$ is more interesting in this sense. In the case $s > 0$, we obtain from eq. (B.14)

$$\lim_{r\to\infty} \hat{A}_{\mu,s} = \lim_{r\to\infty} \frac{-(\mu R/2+1)}{g} (d\Psi + \chi) T_3 = -\frac{(\mu R/2+1)}{g} T_3 d\Psi\,, \tag{B.23}$$

where we used that $v_L^3 = (d\Psi + \chi)$ and eq. (B.18). Therefore, the vector at infinity is pure gauge, and the corresponding gauge transformation reads

$$V = e^{(\mu R/2+1)\Psi T_3} = \begin{pmatrix} e^{-i(\mu R/2+1)\Psi/2} & 0 \\ 0 & e^{i(\mu R/2+1)\Psi/2} \end{pmatrix}. \tag{B.24}$$

However, we must demand that $V$ is a single-valued map $V : \mathrm{S}^1 \subset \mathrm{S}^3 \to \mathrm{U}(1) \subset \mathrm{SU}(2)$. Since $\Psi$ has period $4\pi$ we see that the following quantization condition must hold so that $V(\Psi + 4\pi) = V(\Psi)$:

$$\frac{\mu R}{2} + 1 \in \mathbb{Z}\,. \tag{B.25}$$

[22] For example, $\lim_{r\to\infty} d\theta = 0$, which is clear if we take into account that the norm of $d\theta$ vanishes: $|d\theta|^2 = r^{-2}$ when $r \to \infty$. On the other hand, the norm of $d\Psi$ remains finite because it is a compact dimension: $|d\Psi|^2 \to 4/R^2$.



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The case $s = 0$ can be reduced to the previous one after we apply the gauge transformation $U^{-1}$ in (B.8) to the asymptotic value of eq. (B.15). Since we demand that $\mu \geq 0$, we can write

$$\mu = \frac{2m}{R}, \quad m = 0, 1, 2, \ldots \tag{B.26}$$

### B.3 Higher KK monopole charge with a horizon

When the charge of the KK monopole is bigger than one, the 4-dimensional manifold it describes contains an conical singularity at the origin due to a deficit in the angle covered by the coordinate $\Psi$. This property also affects the instanton field; see, e.g., the appearance of $n$ in eq. (4.19). Because of this factor, we have not been able to mimic the steps of the previous sections and find a gauge transformation rendering the fields zero at the origin, which seem to remain singular at this location of this GH space.

That the instantons are singular can be seen from the fact that the angular coordinates are ill-defined at $r = 0$, while in this basis some of the angular components of corresponding vectors 1-form are non-vanishing. However this problem is immediately solved in the black hole solutions that we consider if the event horizon has non-vanishing area. There, the angular coordinates are perfectly valid at $r = 0$ and the gauge fields are already regular in the original gauge (at this location the coordinate $r$ is not well defined, but the corresponding component of the 1-forms of all the instantons considered is zero).

Having said that, it is easy to check that the instantons of the $\lambda$-family are globally regular, since they vanish asymptotically. On the other hand, for the $\mu, s$-family we obtain the following asymptotic behavior

$$\lim_{r\to\infty} gA_{\mu,s} = \frac{\mu R}{2}(d\Psi + n\cos\theta d\phi)\frac{x^A}{r}T_A\,, \tag{B.27}$$

with $\chi = q\cos\theta d\phi$. Written in this manner, we see that the gauge field contains a string singularity at $\theta = 0, \pi$ that extends to infinity, while remaining regular elsewhere. The local expression around these locations reduces to

$$\lim_{\substack{r\to\infty \\ \theta\to 0,\pi}} gA_{\mu,s} = \mp\frac{\mu R}{2}(d\Psi \pm n d\phi)T_3\,, \tag{B.28}$$

for the upper and lower plane respectively.

We now perform a local gauge transformation (B.7) in a small open set around these two points with

$$V^{(\pm)} = e^{\mp\mu n R\phi T_3/2} = \begin{pmatrix} e^{\pm i\mu n R\phi/4} & 0 \\ 0 & e^{\mp i\mu n R\phi/4} \end{pmatrix}, \tag{B.29}$$

and get

$$\lim_{r\to\infty} g\hat{A}_{\mu,s} = \mp\left(\frac{\mu R}{2}d\Psi\right)T_3\,, \tag{B.30}$$

which have no string singularity whatsoever. As shown in ref. [42], the local gauge transformations eq. (B.29) can be extended through the sphere (except the poles) by replacing



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$T_3 \to x^A T_A/r$. Then, in the intersections $\hat{A}^{(+)}$ and $\hat{A}^{(-)}$ are related by the gauge transformation

$$V^{(\pm)} = e^{\mp \mu n R \phi (x^A/r) T_A}\,, \tag{B.31}$$

which is single valued in the coordinate $\phi$ if and only if $\mu n R/2$ is an integer number. Therefore, we get a generalization of the quantization condition eq. (B.26) for the $n > 1$ case,

$$\mu = \frac{2m}{Rn}\,, \quad m = 0, 1, 2, \ldots \tag{B.32}$$

### B.4 Contribution to the solution

These instantons contribute to the solution through the function $\mathcal{Z}_0$. Using the general result eq. (3.12), their contributions read

$$\Delta\mathcal{Z}_0\Big|_{\lambda} = -\frac{2\alpha'}{r(r+q)(\lambda^2 r+1)^2}\,, \tag{B.33}$$

$$\Delta\mathcal{Z}_0\Big|_{\mu,s>0} = -2\alpha'\frac{[1-\mu r\coth(\mu r+s)]^2}{r(r+q)}\,, \tag{B.34}$$

$$\Delta\mathcal{Z}_0\Big|_{\mu,s=0} = -2\alpha'\frac{[1-\mu r\coth \mu r]^2}{r(r+q)}\,, \tag{B.35}$$

where $q = nR/2$ and we set the gauge coupling constant in the Heterotic Superstring to $g = 1$.

Now the pole and asymptotic constant can be removed by adding an appropriate harmonic function $a + c/r$. After this transformation, the contributions read

$$\Delta\mathcal{Z}_0\Big|_{\lambda} = 2\alpha'\frac{(r+q)\lambda^2(\lambda^2 r+2)+1}{q(r+q)(\lambda^2 r+1)^2}\,, \tag{B.36}$$

$$\Delta\mathcal{Z}_0\Big|_{\mu,s>0} = 2\alpha'\left\{\mu^2 + \frac{q^{-1}}{r} - \frac{[1-\mu r\coth(\mu r+s)]^2}{r(r+q)}\right\}\,, \tag{B.37}$$

$$\Delta\mathcal{Z}_0\Big|_{\mu,s=0} = 2\alpha'\left\{\mu^2 - \frac{[1-\mu r\coth \mu r]^2}{r(r+q)}\right\}\,. \tag{B.38}$$



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