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# DIFFERENTIAL INCLUSION FOR THE EVOLUTION $p(x)$-LAPLACIANWITH MEMORY

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Abstract. We consider the evolution differential inclusion for a nonlocal operator that involves $p(x)$-Laplacian,

$$u_t - \Delta_{p(x)} u - \int_0^t g(t-s)\Delta_{p(x)} u(x,s)\,ds \in \mathbf{F}(u) \quad \text{in } Q_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with Lipschitz-continuous boundary. The exponent $p(x)$ is a given measurable function, $p^- \leq p(x) \leq p^+$ a.e. in $\Omega$ for some bounded constants $p^- > \max\{1, \frac{2n}{n+2}\}$ and $p^+ < \infty$. It is assumed that $g, g' \in L^2(0,T)$, and that the multivalued function $\mathbf{F}(\cdot)$ is globally Lipschitz, has convex closed values and $\mathbf{F}(0) \neq \emptyset$. We prove that the homogeneous Dirichlet problem has a local in time weak solution. Also we show that when $p^- > 2$ and $u\mathbf{F}(u) \subseteq \{v \in L^2(\Omega) : v \leq \epsilon u^2 \text{ a.e. in } \Omega\}$ with a sufficiently small $\epsilon > 0$ the weak solution possesses the property of finite speed of propagation of disturbances from the initial data and may exhibit the waiting time property. Estimates on the evolution of the null-set of the solution are presented.

## 1. Introduction

We study the differential inclusion for the $p(x)$-Laplace equation with the nonlocal memory term

$$\begin{gathered} u_t - \Delta_{p(x)} u - \int_0^t g(t-s)\Delta_{p(x)} u(x,s)\,ds \in \mathbf{F}(u) \quad \text{in } Q_T = \Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \times [0,T], \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz-continuous boundary, $g(s)$ is a given memory kernel. Here $\Delta_{p(x)}$ denotes the $p(x)$-Laplace operator

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

where $p(\cdot) : \Omega \to \mathbb{R}^n$ is a given measurable function such that

$$1 < p^- \leq p(x) \leq p^+ \quad \text{a.e. in } \Omega$$

with some bounded constant $p^{\pm}$. It is worth noting that the continuity of $p(x)$ is not required.

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Let $H = L^2(\Omega)$ and $\mathcal{B}(H)$ be the set of all bounded nonempty subsets of $H$. It will be assumed that the multivalued function $\mathbf{F}$ satisfies the following conditions.

(F1) The mapping $\mathbf{F} : H \to \mathcal{B}(H)$ is globally Lipschitz, there exists a constant $L > 0$ such that

$$\operatorname{dist}_{\mathcal{H}}(\mathbf{F}(u), \mathbf{F}(v)) \le L\|u - v\|_H, \quad u, v \in H, \tag{1.2}$$

where $\operatorname{dist}_{\mathcal{H}}(\cdot,\cdot)$ denotes the Hausdorff distance: given $A, B \subset H$,

$$\operatorname{dist}_{\mathcal{H}}(A, B) = \max\big\{\sup_{a\in A}\inf_{b\in B}\|a - b\|_H, \sup_{b\in B}\inf_{a\in A}\|a - b\|_H\big\}.$$

(F2) $\mathbf{F}$ has convex closed values.

An immediate byproduct of condition (F1) is that $\mathbf{F}(0) \ne \emptyset$. Two "ad hoc" examples of functions $\mathbf{F}$ satisfying these conditions are furnished by the functions $\mathbf{F}_1(u) = [0, |u|]$, where $[0, |u|] := \{v \in L^2(\Omega) : 0 \le v(x) \le |u(x)| \text{ for a.e. } x \in \Omega\}$ and

$$\begin{aligned}\mathbf{F}_2(u) &= \begin{cases} [\epsilon u, -u] & \text{if } u < 0,\\ [-u, \epsilon u] & \text{if } u \ge 0\end{cases}\\ &\equiv \Big\{v \in L^2(\Omega) : \epsilon u(x) \le v(x) \le -u(x) \text{if } u(x) < 0,\\ &\quad - u(x) \le v(x) \le \epsilon u(x) \text{ if } u(x) \ge 0, \text{ a.e. } x \in \Omega\Big\}\end{aligned}$$

with $\epsilon > 0$. Notice that for the function $\mathbf{F}_2(u)$ and every $u \in H$,

$$\begin{aligned}u\mathbf{F}_2(u) &= \begin{cases} [-u^2, \epsilon u^2] & \text{if } u < 0,\\ [-u^2, \epsilon u^2] & \text{if } u \ge 0\end{cases}\\ &= \big\{v \in L^2(\Omega) : -u^2(x) \le v(x) \le \epsilon u^2(x) \text{ a.e. in } \Omega\big\}.\end{aligned} \tag{1.3}$$

The mathematical models with differential inclusions appear in numerous applications such as the control of forest fires [8], the study of the processes of combustion in porous media [19], or conduction of electrical impulses in nerve axons [31, 32]. In climatology, the energy balance models may lead to evolution differential inclusions which involve the $p$-Laplacian [12, 13]. A degenerate parabolic-hyperbolic problem with a differential inclusion appears in a glaciology model [14].

The partial differential inclusions which involve the $p$-Laplace operator with constant $p$ were considered by many authors, see, e.g., [25, 26, 27, 30]. The inclusions for operators with variable nonlinearity, that is, the operators with $p(x)$-Laplacian, were considered in [21, 24, 28]. For an insight into the multivalued problems and differential inclusions we refer [5, 6, 33, 22] and references therein.

Parabolic equations with nonlocal terms appear in the mathematical description of the heat propagation in materials with memory where the heat flux may depend on the past history of the process - see, e.g., [20]. The specific features of problem (1.1) are the presence of the nonlocal memory term and the operator $\Delta_{p(x)}$ with variable exponent $p(x)$, which generalizes the classical $p$-Laplacian. The PDEs with variable nonlinearities are gaining ground in the mathematical modelling of the real life processes. The theory of such equations is developing very rapidly and already counts with a number of important results - see, e.g., [1, 17, 2] and the bibliographic review in [2]. However, in the presence of the nonlocal term the operator on the left-hand side of inclusion (1.1) is not monotone, which prevents one from a direct application of known results and methods in the study of problem (1.1). Our approach to problem (1.1) relies on a fixed point theorem in the form

[15] and the unique solvability of problem (1.1) for the nonlocal equation with the multivalued function $\mathbf{F}(u)$ substituted by an arbitrary given $f \in L^2(0,T;H)$.

In Section 2 we introduce the Sobolev spaces with variable exponents the solutions of problem (1.1) belong to and recall the main properties of these spaces used in the proofs.

In Section 3 we prove the global in time unique solvability of the homogeneous Dirichlet problem for the nonlocal equation

$$u_t - \Delta_{p(x)} u - \int_0^t g(t-s)\Delta_{p(x)} u(x,s)\, ds = f \tag{1.4}$$

with an arbitrary given $f \in L^2(0,T;H)$. This result is stated in Theorem 3.6. The global in time existence is proven under the assumptions

$$\begin{gathered} p : \Omega \to \mathbb{R} \text{ is measurable, } \max\big\{1, \frac{2n}{n+2}\big\} < p^- \le p(x) \le p^+, \\ g(t), g'(t) \in L^2(0,T), \quad |g(0)| < \infty, \quad |\nabla u_0|^{p(x)} \in L^1(\Omega), \quad f \in L^2(Q_T) \end{gathered}$$

without any restriction on the sign of the kernel $g(t)$. To prove solvability of the nonlocal equation (1.4) we use the method proposed in paper [3], which deals with the nonlocal equation (1.4) with constant $p$. The idea consists in splitting the nonlocal equation into a system of two equations composed of the evolution $p(x)$-Laplace equation with a specially chosen right-hand side and an integral equation of the Volterra type. Uniqueness of the global in time solution is proven in Theorem 4.2 of Section 4 under the weaker condition $p^- > 1$ and the same assumptions on the kernel $g$.

With Theorem 5.1, Section 5, we prove that problem (1.1) for the differential inclusion has at least one solution for a sufficiently small $T$, provided that the data satisfy the same conditions as in the existence theorem for equation (1.4) and the multivalued mapping $\mathbf{F}$ meets conditions (F1)-(F2). Following [15], we base the proof on the fixed point theorem for multivalued mappings from [34].

In the concluding Section 6 we show that the solutions of problem (1.1) may possess the property of finite speed of propagation of disturbances from the data and may display the waiting time effect. The properties of space localization hold for the solutions of inclusion (1.1) with the multivalued function $\mathbf{F}$ which satisfies the condition $u\mathbf{F}(u) \subseteq \{v \in L^2(\Omega) : v \le \epsilon u^2 \text{ a.e. in } \Omega\}$ with a sufficiently small $\epsilon > 0$, and under the assumptions that $p^- > 2$ and the oscillation of the exponent $p(x)$ is sufficiently small, namely, $p^+ < (1+\frac{2}{n})p^-$. We show that if $u_0 = 0$ in a ball $B_R$, then the solution of the inclusion (1.1) $u(x,t)$ equals zero in a co-centered ball $B_{\rho(t)}$ of a smaller radius $\rho(t)$ and estimate the rate of change of $\rho(t)$. In case that $u_0$ is vanishing rapidly near a part of the boundary of its support, then it is possible that $\rho(t) \equiv R$ on an interval $[0,t^*]$. The moment $t^*$ is termed the waiting time.

The effects of space localization are well-studied for the solutions of parabolic equations and systems of equations without nonlocal terms - see, e.g., [4, 2] for equations with constant and variable nonlinearity. To the best of our knowledge, the phenomenon of space localization in solutions of differential inclusions was studied only in the paper [11]. The results of the present work, as well as the results of [11], are obtained with the local energy method [4], which happens to be very convenient in the situations where the principle of comparison is unapplicable or no sub/super solutions are available for comparison.

## 2. Function spaces

Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega \in \text{Lip}$. Given a function $p : \Omega \to [p^-, p^+] \subset (1, \infty)$, $p^\pm = const$, we define the set

$$L^{p(\cdot)}(\Omega) = \big\{f : \Omega \to \mathbb{R} : f \text{ is measurable on } \Omega,\ \int_\Omega |f|^{p(x)}\,dx < \infty\big\}.$$

The set $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|f\|_{p(\cdot),\Omega} := \inf\big\{\alpha > 0 : \int_\Omega \big|\frac{f}{\alpha}\big|^{p(x)}\,dx \le 1\big\} \tag{2.1}$$

becomes a Banach space. Throughout the text we will repeatedly use the following properties of the spaces $L^{p(\cdot)}(\Omega)$.

- For every $f \in L^{r(\cdot)}(\Omega)$ and $g \in L^{r'(\cdot)}(\Omega)$, ($r' = \frac{r}{r-1}$ is the conjugate exponent of $r$) the generalized Hölder inequality holds

$$\int_\Omega |fg|\,dx \le \Big(\frac{1}{r^-} + \frac{1}{(r')^-}\Big)\|f\|_{r(\cdot),\Omega}\|g\|_{r'(\cdot),\Omega}\,, \tag{2.2}$$

$$\begin{aligned} &\min\Big\{\Big(\int_\Omega |f|^r\,dx\Big)^{1/r^+}, \Big(\int_\Omega |f|^r\,dx\Big)^{1/r^-}\Big\} \\ &\le \|f\|_{r(\cdot),\Omega} \\ &\le \max\Big\{\Big(\int_\Omega |f|^r\,dx\Big)^{1/r^+}, \Big(\int_\Omega |f|^r\,dx\Big)^{1/r^-}\Big\}. \end{aligned} \tag{2.3}$$

- If the domain $\Omega$ is bounded and $p_1(x) \ge p_2(x)$ a.e. in $\Omega$, there is a continuous inclusion $L^{p_1(\cdot)}(\Omega) \subset L^{p_2(\cdot)}(\Omega)$ and for all $u \in L^{p_1(\cdot)}(\Omega)$,

$$\|u\|_{p_2(\cdot),\Omega} \le C\|u\|_{p_1(\cdot),\Omega} \tag{2.4}$$

with a constant $C = C(|\Omega|, p_1^\pm, p_2^\pm)$.

Under the foregoing assumptions on the exponent $p(x)$, the variable Sobolev space $V = W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of the set of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{W^{1,p(\cdot)}(\Omega)} = \|v\|_{p(\cdot),\Omega} + \|\nabla v\|_{p(\cdot,\Omega)}.$$

The solution of the differential inclusion (1.1) will be sought as an element of the space

$$W = L^2(0,T;H) \cap L^\infty(0,T;V).$$

We refer the reader to the monograph [16] and also to [2, Ch.1] for further information about the variable Lebesgue and Sobolev spaces.

**Definition 2.1.** A function $u(x,t)$ is called weak solution of problem (1.1) if

1. $u \in C([0,T];H) \cap W$, $u_t \in L^2(0,T;H)$, $\Delta_{p(x)}u \in L^2(0,T;H)$, $u(x,0) = u_0(x)$;
2. there is a function $f : [0,T] \to H$ such that $f(t) \in \mathbf{F}(u(t))$, $t$-a.e. in $[0,T]$;
3. for every test-function $\phi \in W$,

$$\begin{aligned} &\int_{Q_T} \Big(u_t\phi + \nabla\phi \cdot \Big(|\nabla u|^{p(x)-2}\nabla u + \int_0^t g(t-s)|\nabla u(s)|^{p(x)-2}\nabla u(s)ds\Big)\Big)\,dx\,dt \\ &= \int_{Q_T} f\phi\,dx\,dt. \end{aligned}$$

## 3. Evolution $p(x)$-Laplace equation with memory

The proof of the existence of a solution to the evolution nonlocal differential inclusion (1.1) relies on the unique solvability of the Dirichlet problem for the nonlocal equation

$$\begin{gathered} u_t - \Delta_{p(x)} u = \int_0^t g(t-s)\Delta_{p(x)} u(x,s)\,ds + f \quad \text{in } Q_T = \Omega\times(0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega\times[0,T] \end{gathered} \tag{3.1}$$

with a given function $f$. We will assume that

$$g, g' \in L^2(0,T), \quad f\in L^2(Q_T), \quad u_0 \in L^2(\Omega)\cap W_0^{1,p(\cdot)}(\Omega). \tag{3.2}$$

**Definition 3.1.** A weak solution of problem (3.1) with a given function $f(z)$ is a function $u(z)$ satisfying items (1) and (3) of Definition 2.1.

3.1. **Auxiliary system.** To construct a solution of the nonlocal problem (3.1) we consider the auxiliary problem of finding the pair of functions $(u, Y)$ with the conditions

$$\begin{gathered} u_t - \Delta_{p(x)} u = Y + f(x,t) \quad \text{in } Q_T, \\ u = 0 \text{ on } \partial\Omega\times[0,T], \quad u(x,0) = u_0(x) \text{ in } \Omega, \end{gathered} \tag{3.3}$$

$$Y(t) = -\int_0^t g(t-s) Y(s)\,ds + F(x,t,u), \tag{3.4}$$

where $F$ is the nonlocal operator

$$\begin{aligned} F(x,t,u(x,t)) = u(x,t)g(0) - u_0(x)g(t) + \int_0^t g'(t-s)u(x,s)\,ds \\ - \int_0^t g(t-s) f(x,s)\,ds. \end{aligned} \tag{3.5}$$

Equation (3.4) is the classical Volterra equation.

**Lemma 3.2** ([3, Section 2])**.** *Let $g\in L^2(0,T)$. For every $T>0$ and every $F\in L^2(Q_T)$ equation (3.4) has a unique solution $Y\in L^2(Q_T)$ which satisfies the estimates*

$$\|Y\|^2_{2,Q_T} \le 4\mathrm{e}^{2T\|g\|^2_{2,(0,T)}} \|F\|^2_{2,Q_T}, \tag{3.6}$$

$$\|Y(t)\|^2_H \le 4(2\|g\|^2_{2,(0,T)} \mathrm{e}^{2T\|g\|^2_{2,(0,T)}} \|F\|^2_{2,Q_T} + \|F(t)\|^2_H) \quad \textit{a.e. in } (0,T). \tag{3.7}$$

**Galerkin's approximations.** A solution of problem (3.3)–(3.4) is obtained as the limit of the sequences $\{u_m\}$, $\{Y_m\}$,

$$u_m = \sum_{i=1}^m c_i(t)\psi_i(x), \quad Y_m = \sum_{i=1}^m d_i(t)\psi_i(x), \quad m\in\mathbb{N},$$

where $\{\psi_i\}$ is the system of eigenfunctions of the problem

$$(\psi,\phi)_{H_0^s(\Omega)} = \lambda(\phi,\psi)_H \quad \forall \phi\in H_0^s(\Omega) \tag{3.8}$$

with a natural $s \ge 1 + n\max\{0, \frac12 - \frac{1}{p^+}\}$, so that $H_0^s(\Omega)\hookrightarrow W_0^{1,p^+}(\Omega)$ with compact embedding. The set $\{\psi_i\}$ is orthogonal in $H_0^s(\Omega)$ and forms an orthonormal basis of $L^2(\Omega)$, $\{\lambda_i\}$ is a nondecreasing sequence of positive numbers. The functions

$v = \sum_{i=1}^m \phi_i(t)\psi_(x)$, $\phi_i(t) \in C[0,T]$, are dense in $W$. For every finite $m$ the coefficients $c_i(t)$ satisfy the system of the nonlinear ordinary differential equations

$$\begin{gathered} c_i'(t) = -\int_\Omega |\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla\psi_i\,dx + (Y_m,\psi_i)_H + (f,\psi_i)_H, \\ c_i(0) = (u_0^{(m)},\psi_i)_H, \quad i = 1,\dots,m, \end{gathered} \tag{3.9}$$

where $Y_m$ are solutions of the Volterra equations

$$Y_m(x,t) = -\int_0^t g(t-s)Y_m(x,s)\,ds + \sum_{i=1}^m (F(x,t,u_m),\psi_i)_H\psi_i. \tag{3.10}$$

**Proposition 3.3.** *For every $u_0 \in V = L^2(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$ there exists a sequence $\{u_0^{(m)}\}$, such that*

$$u_0^{(m)}(x) = \sum_{i=1}^m d_{i,m}\psi_i(x) \to u_0 \text{ in } V \text{ as } m\to\infty.$$

*Proof.* It suffices to show that for each $\epsilon > 0$ there is $v^{(m)} \in \mathcal{P}_m = \operatorname{span}\{\psi_1,\dots,\psi_m\}$ such that $\|u_0 - v^{(m)}\|_V < \epsilon$. Take an arbitrary $\epsilon > 0$. By the definition of $V$ there exists $w_\epsilon \in C_0^\infty(\Omega)$ such that

$$\|u_0 - w_\epsilon\|_V = \|\nabla(u_0 - w_\epsilon)\|_{p(\cdot),\Omega} < \frac{\epsilon}{2}. \tag{3.11}$$

Since $w_\epsilon \in C_0^\infty(\Omega) \subset H_0^s(\Omega)$, it follows that $\|w_\epsilon\|^2_{H_0^s(\Omega)} = \sum_{i=1}^\infty \lambda_i(w_\epsilon,\psi_i)_H^2 < \infty$, and it is necessary that

$$w_\epsilon^{(m)} = \sum_{i=1}^m \lambda_i(w_\epsilon,\psi)_H^2 \to w_\epsilon \quad \text{in } H_0^s(\Omega) \text{ as } m\to\infty.$$

There exists $k = k(\epsilon) \in \mathbb{N}$ such that $w_\epsilon^{(k)} \in \mathcal{P}_k$ and

$$\|w_\epsilon - w_\epsilon^{(k)}\|_V = \|\nabla(w_\epsilon - w_\epsilon^{(k)}\|_{p(\cdot),\Omega} \le C\|w_\epsilon - w_\epsilon^{(k)}\|_{H_0^s(\Omega)} < \frac{\epsilon}{2} \tag{3.12}$$

with the constant $C$ from the embedding inequality

$$\|v\|_V \le C(|\Omega|,p^\pm)\|\nabla v\|_{p^+,\Omega} \le C\|v\|_{H_0^s(\Omega)}.$$

Combining (3.11) with (3.12) we obtain

$$\begin{aligned} \|u_0 - w_\epsilon^{(k)}\|_V &\le \|u_0 - w_\epsilon\|_V + \|w_\epsilon - w_\epsilon^{(k)}\|_V \\ &\le \frac{\epsilon}{2} + C\|w_\epsilon - w_\epsilon^{(k)}\|_{H_0^s(\Omega)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

□

By Lemma 3.2 for every given $F \in L^2(Q_T)$ and $g \in L^2(0,T)$ equation (3.10) has a unique solution $Y_m \in L^2(Q_T)$, by the method of construction $Y_m \in \mathcal{P}_m$. Let us denote $\mathbf{c}(t) = (c_1(t),\dots,c_m(t))$, $\mathbf{d}(t) = (d_1(t),\dots,d_m(t))$, $\mathbf{f}(t) = (f_1(t),\dots,f_m(t))$ with $f_i(t) = (f,\psi_i)_H$, and set

$$-\int_\Omega |\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla\psi_i\,dx + (Y_m,\psi_i)_H = \mathcal{F}_i(t,\mathbf{c}(t),\mathbf{d}(t)).$$

System (3.9)–(3.10) can be written in the equivalent form

$$\begin{gathered}
c_i(t) = c_i(0) + \int_0^t \mathcal{F}_i(s, \mathbf{c}(s), \mathbf{d}(s))\,ds + \int_0^t f_i(s)\,ds, \quad i = 1, \dots, m, \\
d_i(t) = -\int_0^t g(t-s) d_i(s)\,ds + (F(x,t,u_m), \psi_i)_H.
\end{gathered} \tag{3.13}$$

The solutions of system (3.13) define the transformations

$$\mathbf{c}(t) = \mathcal{N}(\mathbf{d}(t)), \quad \mathbf{d}(t) = \mathcal{M}(\mathbf{c}(t))$$

so that $\mathbf{c}(t)$ is a fixed point of the transformation $\mathcal{L} = \mathcal{N} \circ \mathcal{M}$. Let us fix some $\alpha \in (0, 1/2)$ and consider the space of $m$-dimensional vectors with Hölder-continuous components

$$\mathcal{S}_{m,\alpha} = \{v(t) = (v_1(t), \dots, v_m(t)) : v_i \in C^\alpha[0,T]\}.$$

Set

$$\begin{gathered}
\|v(t)\|_{\mathcal{S}_{m,\alpha}} = \sum_{i=1}^m \|v_i\|_{C^\alpha[0,T]}, \\
\|v_i\|_{C^\alpha[0,T]} = \sup_{(0,T)} |v_i(t)| + \sup_{t,\tau \in (0,T),\, t \neq \tau} \frac{|v_i(t) - v_i(\tau)|}{|t-\tau|^\alpha}.
\end{gathered}$$

Using the Hölder inequality (2.2) and (2.3) we obtain

$$\begin{aligned}
&|(|\nabla u_m|^{p(x)-2} \nabla u_m, \nabla \psi_i)_H| \\
&\le \int_\Omega |\nabla u_m|^{p(x)-1} |\nabla \psi_i|\,dx \\
&\le C \| |\nabla u_m|^{p(x)-1} \|_{p'(\cdot),\Omega} \|\nabla \psi_i\|_{p(\cdot),\Omega} \\
&\le C \|\nabla \psi_i\|_{p(\cdot),\Omega} \max\Big\{ \Big( \int_\Omega |\nabla u_m|^{p(x)}\,dx \Big)^{1-\frac{1}{p^-}}, \Big( \int_\Omega |\nabla u_m|^{p(x)}\,dx \Big)^{1-\frac{1}{p^+}} \Big\}.
\end{aligned}$$

By the choice of the basis, for $i = 1, \dots, m$,

$$\begin{gathered}
\|\nabla \psi_i\|_{p(\cdot),\Omega} \le C(n, p^\pm, |\Omega|) \|\nabla \psi_i\|_{p^+,\Omega} \le C' \|\psi_i\|_{H_0^s(\Omega)} = C' \sqrt{\lambda_i} \le C' \sqrt{\lambda_m}, \\
\|u_m\|^2_{H_0^s(\Omega)} = \sum_{i=1}^m \lambda_i c_i^2(t) \le C \lambda_m \Big( \sum_{i=1}^m |c_i(t)| \Big)^2 \le C \sqrt{\lambda_m} \|\mathbf{c}\|_{\mathcal{S}_{m,\alpha}}, \\
\begin{aligned}
\int_\Omega |\nabla u_m|^{p(x)}\,dx &\le \max\Big\{ \|\nabla u_m\|^{p^+}_{p(\cdot),\Omega}, \|\nabla u_m\|^{p^-}_{p(\cdot),\Omega} \Big\} \\
&\le C \max\big\{ (\int_\Omega |\nabla u_m|^{p^+}\,dx)^{\frac{p^-}{p^+}}, \int_\Omega |\nabla u_m|^{p^+}\,dx \big\} \\
&= C \max\Big\{ \|\nabla u_m\|^{p^-}_{p^+,\Omega}, \|\nabla u_m\|^{p^+}_{p^+,\Omega} \Big\} \\
&\le C'' \max\Big\{ \|u_m\|^{p^-}_{H_0^s(\Omega)}, \|u_m\|^{p^+}_{H_0^s(\Omega)} \Big\}.
\end{aligned}
\end{gathered}$$

Gathering these estimates we find that

$$\begin{aligned}
\Big| \int_\Omega |\nabla u_m|^{p(x)-2} \nabla u_m \cdot \nabla \psi_i)\,dx \Big| &\le C \max\Big\{ \|u_m\|^{p^+-1}_{H_0^s(\Omega)}, \|u_m\|^{p^--1}_{H_0^s(\Omega)} \Big\} \|\psi_i\|_{H_0^s(\Omega)} \\
&\le \sqrt{\lambda_m} C \Big( \|\mathbf{c}(t)\|^{p^--1}_{\mathcal{S}_{m,\alpha}} + \|\mathbf{c}(t)\|^{p^+-1}_{\mathcal{S}_{m,\alpha}} \Big)
\end{aligned}$$

with a constant $C = C(m, n, p^{\pm}, |\Omega|)$. It is straightforward to check that

$$|(F(x, t, u_m), \psi_i)_H| \le C_1(\|\mathbf{c}(t)\|_{\mathcal{S}_{m,\alpha}} + 1)$$

with a constant $C_1$ depending on $m$, $n$, $T$, $\|g\|_{2,(0,T)}$ and $\|g'\|_{2,(0,T)}$. For every given $\mathbf{c} \in \mathcal{S}_{m,\alpha}$ the Volterra equations (3.13) for $d_i$ have solutions $d_i \in L^2(0, T)$ - see, e.g., [10, Ch.3]. For every $\mathbf{d} \in (L^2(0, T))^m$ the right-hand sides of the equations for $c_i$ in (3.13) define an operator that maps $\mathbf{c} \in \mathcal{S}_{m,\alpha}$ with $\alpha \in (0, 1/2)$ into $S_{m,1/2}$. Using the estimates on $\mathcal{F}_i(s, \mathbf{c}(s), \mathbf{d}(s))$, it is straightforward to check that $\mathcal{L}$ maps the closed ball $B = \{\mathbf{c}(t) : \|\mathbf{c}(t) - \mathbf{c}(0)\|_{\mathcal{S}_{m,\alpha}} \le 1\}$ into the set $\{\mathbf{c}(t) : \|\mathbf{c}(t) - \mathbf{c}(0)\|_{\mathcal{S}_{m,1/2}} \le R\}$, where $R \to 0$ as $T \to 0$. Since the embedding $\mathcal{S}_{m,1/2} \subset \mathcal{S}_{m,\alpha}$ is compact, and the function $F(x, t, r)$ is continuous with respect to $r$, it follows from the Schauder fixed point theorem that for a sufficiently small $T = T_m$ the operator $\mathcal{L}$ has at least one fixed point $\mathbf{c}^* \in B$. Once $\mathbf{c}^*(t)$ is found, the function $Y_m = \sum_{i=1}^m d_i(t)\psi_i(x)$ is recovered from the equations for $d_i$ in (3.13).

3.2. **A priori estimates.** For the sake of simplicity of notation, throughout this subsection we omit the subindex $m$ and denote

$$u := u_m, \quad Y := Y_m, \quad F(x, t, u) := \sum_{i=1}^m (F(x, t, u_m), \psi_i)_H \psi_i.$$

**Lemma 3.4.** *If conditions* (3.2) *are fulfilled, then for every $T < \infty$ and $t \in [0, T]$,*

$$\|u(t)\|_H^2 + \int_{Q_t} |\nabla u|^{p(x)}\, dz \le \|u_0\|_H^2 + (2+\beta(t)) \int_0^t \|f(s)\|_H^2\, ds + \alpha(t)\|u_0\|_H^2, \quad (3.14)$$

*where the functions $\alpha(t)$, $\beta(t)$ tend to zero as $t \to 0^+$. The functions $\alpha, \beta$ depend on $\|g\|_{2,(0,T)}$, $\|g'\|_{(0,T)}$, $|g(0)|$, but are independent of $u$ and $m$.*

*Proof.* Multiplying the equations for $c_i(t)$ in (3.13) by $c_i(t)$, $i = 1, 2, \dots, m$, summing up and integrating the result over the interval $(0, t)$ we arrive at the energy relations

$$\begin{aligned} \frac{1}{2}\|u(t)\|_H^2 + \int_{Q_t} |\nabla u|^{p(x)}\, dz &= \frac{1}{2}\|u_0\|_H^2 + \int_{Q_t} Y u\, dz + \int_{Q_t} f u\, dz \\ &\le \int_{Q_t} u^2\, dz + \frac{1}{2}(\|u_0\|_H^2 + \|Y\|_{2,Q_t}^2 + \|f\|_{2,Q_t}^2). \end{aligned} \quad (3.15)$$

Multiplication of equations for $d_i(t)$ by $d_i(t)$, $i = 1, 2, \dots, m$, gives

$$\|Y(t)\|_H^2 = -\Big(Y(t), \int_0^t g(t-s)Y(s)\, ds\Big)_H + (F(x, t, u), Y(t))_H. \quad (3.16)$$

Using (3.6) and the definition of $F$ we find that

$$\begin{aligned} \|Y\|_{2,Q_t}^2 &\le 4e^{2T\|g\|_{2,(0,T)}^2} \|F\|_{2,Q_t}^2 \\ &\le 8e^{2T\|g\|_{2,(0,T)}^2} \Big(|g(0)|^2\|u\|_{2,Q_t}^2 + \|g\|_{2,(0,T)}^2\|u_0\|_H^2 \\ &\quad + T\|g'\|_{2,(0,T)}^2\|u\|_{2,Q_t}^2 + T\|g\|_{2,(0,T)}^2\|f\|_{2,Q_t}^2\Big). \end{aligned}$$

Substituting this inequality into (3.15) and dropping the nonnegative term on the left-hand side we arrive at the following inequality for the function $y(t) = \|u(t)\|_H^2$:

$$\frac{1}{2}y(t) \le K \int_0^t y(s)\, ds + M + \phi(t) \quad (3.17)$$

with the constants

$$K = 1 + 8\mathrm{e}^{2T\|g\|^2_{2,(0,T)}}(|g(0)|^2 + T\|g'\|^2_{2,(0,T)}),$$
$$M = (\frac{1}{2} + 8\mathrm{e}^{2T\|g\|^2_{2,(0,T)}}\|g\|^2_{2,(0,T)})\|u_0\|^2_H$$

and the function

$$\phi(t) = (\frac{1}{2} + 8\mathrm{e}^{2T\|g\|^2_{2,(0,T)}}T\|g\|^2_{2,(0,T)})\|f\|^2_{2,Q_t}.$$

It follows that

$$\begin{aligned}\int_0^t y(s)\,ds &\le M\mathrm{e}^{Kt}\int_0^t \mathrm{e}^{-Ks}\,ds + \mathrm{e}^{Kt}\int_0^t \phi(s)\mathrm{e}^{-Ks}\,ds \\ &\le \frac{M}{K}(\mathrm{e}^{Kt}-1) \\ &\quad + \frac{1}{K}\big(\mathrm{e}^{Kt}-1\big)\Big(\frac{1}{2} + 8\mathrm{e}^{2T\|g\|^2_{2,(0,T)}}T\|g\|^2_{2,(0,T)}\Big)\int_0^t \|f(s)\|^2_H\,ds.\end{aligned} \tag{3.18}$$

Gathering (3.15), (3.17), (3.18) we obtain the estimate for $\|u(t)\|^2_H$:

$$\begin{aligned}\|u(t)\|^2_H &\le \|u_0\|^2_H + \int_0^t \|f(s)\|^2_H\,ds + 2(\mathrm{e}^{Kt}-1)\Big(M \\ &\quad + \Big(\frac{1}{2} + 8\mathrm{e}^{2T\|g\|^2_{2,(0,T)}}T\|g\|^2_{2,(0,T)}\Big)\int_0^t \|f(s)\|^2_H\,ds\Big).\end{aligned}$$

□

**Lemma 3.5.** *If conditions* (3.2) *are fulfilled, then for every $T < \infty$,*

$$\begin{aligned}&\|u_t\|^2_{2,Q_T} + \operatorname{ess\,sup}_{(0,T)}\int_\Omega |\nabla u(t)|^{p(x)}\,dx \\ &\le C\Big(\|f\|^2_{2,Q_T} + \int_\Omega |\nabla u_0|^{p(x)}\,dx + \|u_0\|^2_H + 1\Big).\end{aligned} \tag{3.19}$$

*The constant $C$ depends on $n$, $T$, $|\Omega|$, $p^\pm$, but is independent of $m$.*

*Proof.* The second energy equality follows after multiplication of the equations for $c_i(t)$ by $c_i'(t)$:

$$\|u_t(t)\|^2_H + \frac{d}{dt}(\frac{1}{p(x)}\int_\Omega |\nabla u(t)|^{p(x)}\,dx) = J_1 + J_2 \tag{3.20}$$

with $J_1 = (Y, u_t)_H$, $J_2 = (f, u_t)_H$. Taking into account (3.7) and (3.6), we estimate

$$|J_1 + J_2| \le \frac{1}{2}\|u_t(t)\|^2_H + C(\|Y\|^2_H + \|f\|^2_H), \tag{3.21}$$

where

$$\|Y(t)\|^2_H \le C(\|F(t)\|^2_H + \|F\|^2_{2,Q_t}) \le C(\|u\|^2_H + \|u\|^2_{2,Q_t} + \|u_0\|^2_H + \|f\|^2_{2,Q_t})$$

by Lemma 3.2. To obtain (3.19) we rewrite (3.20) in the form

$$\begin{aligned}&\frac{1}{2}\|u_t(t)\|^2_H + \frac{d}{dt}\Big(\frac{1}{p(x)}\int_\Omega |\nabla u(t)|^{p(x)}\,dx\Big) \\ &\le C\Big(\|u\|^2_H + \|u\|^2_{2,Q_t} + \|u_0\|^2_H + \|f\|^2_{2,Q_t} + \|f(t)\|^2_H\Big),\end{aligned} \tag{3.22}$$

integrate (3.22) in $t$ and apply (3.18) to estimate $\|u\|^2_{2,Q_t}$ on the right-hand side of the resulting inequality. □

The derived uniform estimates allow one to continue the sequences $\{u_m\}$, $\{Y_m\}$ to an arbitrary time interval $(0,T)$.

### 3.3. Existence of weak solutions.

**Theorem 3.6.** *If $f \in L^2(Q_T)$ and*

$$u_0 \in V, \quad g, g' \in L^2(0,T), \quad |g(0)| < \infty, \quad p^- > \max\{1, \frac{2n}{2+n}\}, \tag{3.23}$$

*then problem* (3.1) *has at least one global solution in the sense of Definition 3.1. This solution satisfies estimates* (3.14)*,* (3.19)*.*

*Proof.* By Lemma 3.4

(1) $u_m$ are uniformly bounded in $L^\infty(0,T;V) \cap L^\infty(0,T;H)$,
(2) $(u_m)_t$ are uniformly bounded in $L^2(0,T;H)$.

Since $V = W^{1,p(\cdot)}_0(\Omega) \subset W^{1,p^-}_0(\Omega)$ and $p^- > \max\{1, \frac{2n}{n+2}\}$, it follows from [23, Sec. 9, Cor. 6] that the sequence $\{u_m\}$ is relatively compact in $L^q(Q_T)$ with some $1 < q < \infty$. Thus, there exist $u \in L^2(Q_T)$, $Y \in L^2(Q_T)$ and $A \in (L^{p'(x)}(Q_T))^n$ such that

$$\begin{gathered} u_m \to u \text{ a.e. in } Q_T \text{ and } L^2(Q_T), \quad u_{mt} \rightharpoonup u_t \text{ in } L^2(Q_T), \\ |\nabla u_m|^{p(x)-2}\nabla u_m \rightharpoonup A \text{ in } (L^{p'(\cdot)}(Q_T))^n, \quad Y_m \rightharpoonup Y \text{ in } L^2(Q_T). \end{gathered} \tag{3.24}$$

By the construction of $u_m$, for every $\phi \in \mathcal{P}_N$ with $N \le m$,

$$\int_{Q_T} (u_{mt}\phi + \nabla\phi \cdot |\nabla u_m|^{p(x)-2}\nabla u_m - (Y_m + f)\phi)\,dx\,d\tau = 0 \tag{3.25}$$

and, in particular,

$$\int_{Q_T} [u_{mt}u_m + |\nabla u_m|^{p(x)} - (Y_m + f)u_m]dx\,d\tau = 0. \tag{3.26}$$

Letting in (3.25) $m \to \infty$, for every $\phi \in \mathcal{P}_N$

$$\int_{Q_T} [u_t\phi + \nabla\phi \cdot A - (Y + f)\phi]\,dx\,d\tau = 0. \tag{3.27}$$

Since the set $\{\phi_N\}_{N\ge1}$ is dense in $\mathbf{W}(Q_T)$, the previous equality is true for every $\phi \in \mathbf{W}(Q_T)$ and, in particular, for $\phi = u$:

$$\int_{Q_T} [u_t u + \nabla u \cdot A - (Y + f)u]\,dx\,d\tau = 0. \tag{3.28}$$

Now we need to prove that for every admissible test-function $\phi$,

$$\int_{Q_T} \nabla\phi \cdot |\nabla u_m|^{p(x)-2}\nabla u_m\,dt\,dx \to \int_{Q_T} \nabla\phi \cdot |\nabla u|^{p(x)-2}\nabla u\,dt\,dx.$$

By monotonicity, for every smooth function $\zeta$

$$\begin{aligned} |\nabla u_m|^p &= \Big(|\nabla u_m|^{p(x)-2}\nabla u_m - |\nabla\zeta|^{p(x)-2}\nabla\zeta\Big) \cdot \nabla(u_m - \zeta) \\ &\quad + |\nabla\zeta|^{p(x)-2}\nabla\zeta \cdot \nabla(u_m - \zeta) + |\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla\zeta \\ &\ge |\nabla\zeta|^{p(x)-2}\nabla\zeta \cdot \nabla(u_m - \zeta) + |\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla\zeta \end{aligned} \tag{3.29}$$

Subtracting (3.26) from (3.28) we obtain

$$\int_{Q_T} (-\nabla u \cdot A + |\nabla u_m|^{p(x)})\, dx\, d\tau = \sum_{i=1}^{3} I_{i,m} \tag{3.30}$$

with

$$I_{1,m} = -\int_{Q_T} (u_{mt}u_m - u_t u)\, dz, \quad I_{2,m} = -\int_{Q_T} (Yu - Y_m u_m)\, dz,$$
$$I_{3,m} = -\int_{Q_T} f(u - u_m)\, dz.$$

The integrals $I_{1,m}$, $I_{2,m}$, $I_{3,m}$ tend to zero as $m \to \infty$ because $u_m u_{mt}$, $Y_m u_m$, $fu_m$ are the products of weakly and strongly converging sequences. Due to the monotonicity condition (3.29), equality (3.30) yields

$$\begin{aligned}&\int_{Q_T} \Big(|\nabla \zeta|^{p(x)-2}\nabla\zeta \cdot \nabla(u_m - \zeta) + |\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla\zeta - \nabla u \cdot A\Big)\, dx\, d\tau \\ &\le \sum_{i=1}^{3} I_{i,m}.\end{aligned}$$

Letting $m \to \infty$ we obtain

$$\int_{Q_T} \Big(|\nabla\zeta|^{p(x)-2}\nabla\zeta - A\Big) \cdot \nabla(u - \zeta)\, dxd\tau \le 0$$

with an arbitrary test-function $\zeta \in \mathbf{W}(Q_T)$. Let $\zeta = u \pm \delta\eta$ with a positive parameter $\delta > 0$ and an arbitrary $\eta \in \mathbf{W}(Q_T)$. Simplifying the resulting inequality and letting $\delta \to 0$ we arrive at the inequalities

$$\pm \int_{Q_T} (|\nabla u|^{p(x)-2}\nabla u - A) \cdot \nabla\eta\ dx\, d\tau \ge 0 \quad \forall \eta \in \mathbf{W}(Q_T),$$

which is impossible unless $A = |\nabla u|^{p(x)-2}\nabla u$ a.e. in $Q_T$. Reverting to (3.27) we conclude that

$$\int_{Q_T} [u_t\phi + \nabla\phi \cdot |\nabla u|^{p(x)-2}\nabla u - (Y + f)\phi]\, dx\, d\tau = 0. \tag{3.31}$$

It follows that $\Delta_p u = -u_t + Y + f \in L^2(Q_T)$ and equation (3.3) is fulfilled a.e. in $Q_T$. Moreover, the inclusions $u$, $u_t \in L^2(Q_T)$ yield the inclusion $u \in C([0,T]; L^2(\Omega))$.

Applying the derived convergence properties of the sequence $\{u_m\}$ and (3.24) it is easy to see that for all $\chi \in W_0^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega F(x,t,u_m)\chi(x)\, dx \to \int_\Omega F(x,t,u)\chi\, dx \quad \text{as } m \to \infty$$

and that the limit function $Y$ satisfies

$$\int_\Omega Y(x,t)\chi(x)\, dx = -\int_{Q_t} \chi(x)g(t-s)Y(x,s)\, ds\, dx + \int_\Omega F(x,t,u)\chi(x)\, dx \tag{3.32}$$

for every $t \in (0,T)$. To identify $Y$ we test (3.31) in the cylinder $Q_t$ with $\phi(x,s) = \chi(x)g(t-s)$, $\chi(x) \in C_0^\infty(\Omega)$, and compare the result with (3.32): for a.e. $t \in (0,T)$,

$$\begin{aligned}
&-\int_\Omega \phi(x) \int_0^t g(t-s)\Delta_{p(x)} u(s)\, ds\, dx \\
&= \int_\Omega \phi(x) \int_0^t g(t-s)\, Y(s)\, ds\, dx + \int_\Omega \phi(x) \int_0^t g(t-s) f(x,s)\, ds\, dx \\
&\quad - \int_\Omega \phi(x) \int_0^t g(t-s) u_s(x,s)\, ds\, dx \qquad (3.33)\\
&= \int_\Omega \phi(x) \int_0^t g(t-s)\, Y(s)\, ds\, dx + \int_\Omega \phi(x) F(x,t,u)\, dx \\
&= -\int_\Omega \phi(x) Y(x,t)\, dx.
\end{aligned}$$

Since $t \in (0,T)$ is arbitrary, it follows that

$$Y(x,t) = \int_0^t g(t-s)\Delta_{p(x)} u(x,s)\, ds \quad \text{a.e. in } Q_T. \tag{3.34}$$

□

**Remark 3.7.** The energy estimates (3.14) and (3.19) remain true for the solution of the auxiliary problem (3.3).

## 4. Uniqueness of weak solutions

**Lemma 4.1.** *The nonlocal problem* (3.1) *is equivalent to system* (3.3)-(3.4).

*Proof.* Let us first check that every weak solution of problem (3.1) generates a solution of system (3.3)–(3.4) with $F$ defined by (3.5). Let $u$ be a weak solution of problem (3.1). Since the equation is fulfilled a.e. in $Q_T$, for every $\psi(x) \in C_0^\infty(\Omega)$ and a.e. $t \in (0,T)$ multiplication of equation (3.3) by $\psi(x)g(t-\tau)$ and integration over the cylinder $\Omega \times (0,t)$ gives

$$\begin{aligned}
&\int_\Omega \psi(x)\Big(\int_0^t g(t-\tau)\Delta_{p(x)} u(\tau)\, d\tau + \int_0^t g(t-\tau)\Big(\int_0^\tau g(\tau-s)\Delta_{p(x)} u(s)\, ds\Big)\, d\tau\Big)dx \\
&= \int_\Omega \psi(x)(u(t)g(0) - u_0 g(t))\, dx \\
&\quad + \int_\Omega \psi(x) \int_0^t (g'(t-\tau)u(\tau) - g(t-\tau)f(x,\tau))\, dx\, d\tau \\
&= \int_\Omega \psi(x) F(x,t,u(x,t))\, dx.
\end{aligned}$$

It follows that the function $Y = \int_0^t g(t-s)\Delta_{p(x)} u(s)\, ds$ is a solution of the Volterra equation (3.4). On the other hand, it is shown in the proof of Theorem 3.6 (see (3.34)) that if the pair $(u,Y)$ is a solution of system (3.3)-(3.4), then $u(x,t)$ is a solution of the nonlocal equation. □

**Theorem 4.2.** *Let us assume that $1 < p < \infty$, $g' \in L^2(0,T)$, $|g(0)| < \infty$. Then problem* (3.1) *has at most one weak solution in the sense of Definition 3.1.*

*Proof.* Let $u_1$, $u_2$ be two different solutions of problem (3.1) and $(u_1, Y_1)$, $(u_2, Y_2)$ be the corresponding solutions of system (3.3)–(3.4). Set $u = u_1 - u_2$ and $Y = Y_1 - Y_2$. Subtracting relations (3.31) for $u_i$ with the test-function $\phi = u$ we find that

$$\begin{aligned}&\frac{1}{2}\|u(t)\|_H^2 + \int_0^t \int_\Omega \nabla u \cdot (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2)\,dx\,d\tau \\ &= \int_0^t \int_\Omega Y u\,dx\,d\tau.\end{aligned} \tag{4.1}$$

On the other hand,

$$Y(x,t) = \int_0^t g(t-s)Y(x,s)ds + \widetilde{F}(x,t) \quad \text{a.e. in } Q_T \tag{4.2}$$

with

$$\widetilde{F}(x,t) = u(x,t)g(0) - \int_0^t g'(t-s)u(x,s)\,ds.$$

There is a constant $C = C(L, |g(0)|, \|g'\|_{2,(0,T)})$ such that

$$\|\widetilde{F}(t)\|_H^2 \le C\Big[\|u(t)\|_H^2 + \int_0^t \|u(s)\|_H^2\,ds\Big],$$

$$\|\widetilde{F}\|_{2,Q_T}^2 = \int_0^t \|\widetilde{F}(s)\|_H^2\,ds \le C(1+T)\int_0^t \|u(s)\|_H^2\,ds$$

whence, by Lemma 3.2,

$$\|Y(t)\|_H^2 \le C\|\widetilde{F}(t)\|_H^2 \le C\Big[\|u(t)\|_H^2 + \int_0^t \|u(s)\|_H^2 ds\Big].$$

Applying the last estimate and Young's inequality we obtain

$$\begin{aligned}\Big|\int_0^t \int_\Omega Y u\,dx\,d\tau\Big| &\le \int_0^t \|u(s)\|_H \|Y(s)\|_H\,ds \\ &\le C\int_0^t \|u(s)\|_H\Big[\|u(s)\|_H^2 + \int_0^s \|u(\tau)\|_H^2\,d\tau\Big]^{1/2} ds \\ &\le C\int_0^t \Big[\|u(s)\|_H^2 + \int_0^s \|u(\tau)\|_H^2\,d\tau\Big]\,ds,\end{aligned}$$

with a constant $C$ depending on $|g(0)|$, and $\|g'\|_{2,(0,T)}$. Plugging these inequalities into (4.1) and using the monotonicity of the second term on the left-hand side we arrive at the inequality

$$\|u(t)\|_H^2 \le C\int_0^t \Big[\|u(s)\|_H^2 + \int_0^s \|u(\tau)\|_H^2\,d\tau\Big]\,ds \le C(1+t)\int_0^t \|u(s)\|_H^2\,ds. \tag{4.3}$$

By Gronwall's inequality $\|u(t)\|_H^2 = 0$ for all $t \in (0,T)$. □

## 5. Existence of solution for inclusion (1.1)

**Theorem 5.1.** *Assume that conditions* (3.23) *are fulfilled and* $\mathbf{F}$ *satisfies conditions* (**F1**) – (**F2**). *Given a bounded set* $B_0 \subset V$, *there exists* $T_0 > 0$ *such that for each* $u_0 \in B_0$ *there exists at least one weak solution* $u$ *of problem* (1.1) *defined on* $[0, T_0]$.

For the proof of Theorem 5.1 we will rely on the following abstract results.

**Definition 5.2.** Let $M$ be a Lebesgue measurable subset of $\mathbb{R}^q$, $q \geq 1$. A selection of $E : M \to 2^H$ is a function $f : M \to H$ such that $f(y) \in E(y)$ for a.e. $y \in M$. We denote

$$\operatorname{Sel} E = \{f \mid f : M \to H \text{ is a measurable selection of } E\}.$$

**Definition 5.3.** Let $\mathcal{U}$ be a topological space and $E : \mathcal{U} \to 2^H$. $E$ is called weakly upper semicontinuous if for each $u \in \mathcal{U}$ $E(u)$ is nonempty, closed, and convex, and for each weakly closed subset $C$ in $\mathcal{U}$ the set

$$E^{-1}(C) = \{u \in \mathcal{U} : E(u) \cap C \neq \emptyset\}$$

is closed in $\mathcal{U}$.

**Theorem 5.4** ([15, Theorem 3.5])**.** *Let $K$ be a nonempty and weakly compact subset in a real Banach space $X$ and let $E : K \to 2^K \setminus \emptyset$ be such that for each $u \in K$, $E(u)$ is closed and convex. If the graph of $E$ is weakly $\times$ weakly sequentially closed, then $E$ has at least one fixed point, i.e., there exists at least one element $u \in K$ such that $u \in E(u)$.*

**Theorem 5.5** ([15, Theorem 3.3])**.** *Let $D$ be a nonempty, bounded and Lebesgue measurable subset of $\mathbb{R}^p$, $p \geq 1$, $\mathcal{U}$ a topological space, and $X$ a real Banach space. If $E : \mathcal{U} \to 2^X$ is weakly upper semicontinuous and $u_n : D \to \mathcal{U}$, $f_n \in \operatorname{Sel} E(u_n)$ for $n \in \mathbb{N}$ satisfy $f_n \rightharpoonup f$ weakly in $L^1(D; X)$ and $u_n \to u$ a.e. in $D$, then $f \in \operatorname{Sel} E(u)$.*

### 5.1. **Auxiliary results.**

**Lemma 5.6.** *Assume conditions* (3.23) *and let $K \subset L^2(0,T;H)$ be a nonempty weakly compact set. Then for every initial datum $u_0 \in V$ the set of solutions of problem* (3.1)*,*

$$M(K) := \{u_f : f \in K\}$$

*is relatively compact in $L^2(0,T;H)$.*

*Proof.* A weakly compact set $K \subset L^2(0,T;H)$ is bounded in the $L^2(0,T;H)$ norm. According to Theorem 3.6, the solution of problem (3.1) satisfies the estimates

$$\|u_t\|_{L^2(0,T;H)} + \operatorname{ess\,sup}_{(0,T)} \int_\Omega |\nabla u|^{p(x)}\,dx \leq C \tag{5.1}$$

with a constant $C$ depending on the data, but independent of $u$. By Young's inequality

$$\int_\Omega |\nabla u|^{p^-}\,dx \leq |\Omega| + \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Combining this inequality with (5.1) we conclude that for $f \in K$ the corresponding solutions of (3.1) satisfy the uniform estimates

$$\|\nabla u_f\|_{L^\infty(0,T;L^{p^-}(\Omega))} \leq C \quad \text{and} \quad \|\partial_t u_f\|_{L^2(0,T;H)} \leq C.$$

For $p^- > \max\{1, \frac{2n}{n+2}\}$ the embedding $W_0^{1,p^-}(\Omega) \subset H$ is compact and by [23, Th.5] the set $M(K)$ is relatively compact in $L^2(0,T;H)$. $\square$

**Lemma 5.7.** *If $f_n \rightharpoonup f$ in $L^2(0,T;H)$ and $u_n = u_{f_n} \to \bar{u}$ in $L^2(0,T;H)$ for some $\bar{u} \in L^2(0,T;H)$, then $\bar{u} = u_f$.*

*Proof.* Let $\{u_n\}$ be the sequence of solutions of problems (3.1) with the right-hand sides $\{f_n\}$. Since the sequence $\{f_n\}$ converges weakly in $L^2(0,T;H)$ to some $f \in L^2(0,T;H)$, it is uniformly bounded in $L^2(0,T;H)$, which means that the corresponding solutions of problem (3.1) $u_n$ satisfy the uniform estimates (3.14), (3.19) (see Remark 3.7). These estimates allow one to extract a subsequence (for which we shall use the same name) that possesses the convergence properties (3.24): there exist functions $u^* \in L^2(0,T;H)$ and $A \in (L^{p'(\cdot)}(Q_T))^n$ such that

$$\begin{gathered}
u_n \to u^* \text{ a.e. in } Q_T \text{ and } L^2(Q_T), \quad u_{nt} \rightharpoonup u^*_t \text{ in } L^2(Q_T), \\
|\nabla u_n|^{p(x)-2}\nabla u_n \rightharpoonup A \text{ in } (L^{p'}(Q_T))^n, \quad Y_n \rightharpoonup Y^* \text{ in } L^2(Q_T).
\end{gathered} \tag{5.2}$$

Let $\phi$ be an arbitrary smooth function. Using (5.2) we may pass to the limit in every term of the identities

$$\int_{Q_T} (\phi\partial_t u_n + |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\phi - (Y_n + f_n)\phi)\,dz = 0,$$

$$\int_{Q_T} (Y_n + \int_0^t g(t-s)Y_n(s)\,ds - F(x,t,u_n))\phi\,dz = 0$$

and arrive at the equality

$$\int_{Q_T} (\phi\partial_t u^* + A\cdot\nabla\phi - (Y^* + f)\phi)\,dz = 0.$$

To identify the limits $A$ and $Y$ we argue in exactly the same way as in the proof of Theorem 3.6. Thus, the sequence $\{u_n\}$ converges to the unique solution $u^*$ of problem (3.1). Since $u_n \to \overline{u}$ in $L^2(0,T;H)$ by assumption, it is necessary that $\overline{u} = u^*$ a.e. in $Q_T$ and that $\overline{u}$ is the solution of problem (3.1). $\square$

5.2. **Proof of Theorem 5.1.** Let us take $u_0 \in B_0$. Fix a number $m > 0$ such that

$$\|u_0\|_H + 1 \le m.$$

The proof of Theorem 5.1 consists in checking the fulfillment of the conditions of Theorem 5.4 and is split into several auxiliary steps.

**Proposition 5.8.** *There exists $r > 0$ such that if $\|w\|_H \le m$, then $\|z\|_H \le r$ for all $z \in \mathbf{F}(w)$.*

*Proof.* Recall that $\mathbf{F}(0) \neq \emptyset$ by assumptions (F1) and (F2). Take $z_0 \in \mathbf{F}(0)$ and let $w \in H$ be such that $\|w\|_H \le m$. Then for each $z \in \mathbf{F}(w)$

$$\|z\|_H = \|z - z_0 + z_0\|_H \le \|z - z_0\|_H + \|z_0\|_H \le L\|w\|_H + \|z_0\|_H \le Lm + \|z_0\|_H =: r.$$

$\square$

Without loss of generality we may assume that $r > 1$. By Remark 3.7, the solutions of problem (3.1) satisfy the estimate

$$\|u(t)\|_H^2 \le \|u_0\|_H^2 + 2\int_0^T \|f(s)\|_H^2\,ds + M(T),$$

being $M(T)$ a positive constant which tends to zero as $T \to 0^+$. The exact form of the constant $M$ is given in inequality (3.14). Let us take $T_0$ so small that $2T_0r^2 + M(T_0) < 1$ and consider the set

$$K := \{f \in L^2(0,T_0;H) : \|f\|_{L^\infty(0,T_0;H)} \le r\}. \tag{5.3}$$

The set $K \subset L^2(0,T_0;H)$ is weakly compact and nonempty. Let us define the mapping
$$P_{T_0} : K \to C([0,T_0];H), \quad P_{T_0}(f) = u,$$
where $u$ is the unique solution of problem (3.1) on the interval $[0,T_0]$.

**Proposition 5.9.** *$\|u(t)\|_H \le m$ for all $t \in [0,T_0]$.*

*Proof.* By (3.14), for all $t \in [0,T_0]$,
$$\begin{aligned}\|u(t)\|_H^2 &\le \|u_0\|_H^2 + 2\int_0^{T_0} \|f(s)\|_H^2\,ds + M(T_0)\\ &\le (m-1)^2 + 2T_0r^2 + M(T_0) < m^2 - 2m + 2 \le m^2\end{aligned}$$
because $m \ge 1$. It follows that $\|u(t)\|_H \le m$ for all $t \in [0,T_0]$. $\square$

We want to apply the fixed point theorem (Theorem 5.4) to the operator $\phi : K \to 2^K$ defined by $f \mapsto \phi(f) = \operatorname{Sel}\mathbf{F}(u)$, where $u = P_{T_0}(f)$. Let us check that $\phi$ is well-defined. According to [15, Theorem 3.2] $\operatorname{Sel}\mathbf{F}(u) \ne \emptyset$. Moreover, by Proposition 5.9 for $f \in K$ and $u = P_{T_0}(f)$ we have
$$\|u(t)\|_H \le m, \quad \forall t \in [0,T_0],$$
whence, by Proposition 5.8, for all $t \in [0,T_0]$,
$$\|z\|_H \le r, \quad \forall z \in \mathbf{F}(u(t)).$$
In particular, for all $\tilde f \in \operatorname{Sel}\mathbf{F}(u)$ we have $\|\tilde f(t)\|_H \le r$, $\forall t \in [0,T_0]$. It follows that
$$\|\tilde f\|_{L^\infty(0,T_0;H)} \le r,$$
which means that $\operatorname{Sel}\mathbf{F}(u) \subset K$, i.e., $\phi(f) \in 2^K$.

**Proposition 5.10.** *$\phi$ has closed values.*

*Proof.* Take $f \in K$ and consider a sequence $\{f_n\} \subset \phi(f) = \operatorname{Sel}\mathbf{F}(u)$ with $u = P_{T_0}(f)$. Let $f_n \to \bar f$ in $L^2(0,T_0;H)$. Then $\bar f$ is measurable and there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(t) \to \bar f(t)$ for a.e. $t \in [0,T_0]$. It follows that
$$\bar f(t) \in \overline{\mathbf{F}(u(t))} = \mathbf{F}(u(t)) \quad \text{for a.e. } t \in [0,T_0] \text{ and } \bar f \in \operatorname{Sel}\mathbf{F}(u).$$
$\square$

**Proposition 5.11.** *$\phi$ has convex values.*

*Proof.* Let $f \in K$. Take $f_1, f_2 \in \phi(f) = \operatorname{Sel}\mathbf{F}(u)$, $u = P_{T_0}(f)$ and $\alpha \in (0,1)$. Then $\alpha f_1 + (1-\alpha)f_2$ is measurable and $\alpha f_1(t) + (1-\alpha)f_2(t) \in \mathbf{F}(u(t))$ for a.e. $t \in [0,T_0]$, whence $\alpha f_1 + (1-\alpha)f_2 \in \phi(f) = \operatorname{Sel}\mathbf{F}(u)$ $\square$

**Proposition 5.12.** *The graph of $\phi$ is weakly$\times$weakly sequentially closed in $K$.*

*Proof.* Identifying the operator $\phi$ with its graph we write
$$\phi = \{(f,\bar f) : f \in K \text{ and } \bar f \in \phi(f)\}.$$
Let $\{(f_n,\bar f_n)\}$ be a sequence in $\phi$ such that $f_n \rightharpoonup f$ weakly in $L^2(0,T_0;H)$ and $\bar f_n \rightharpoonup \bar f$ weakly in $L^2(0,T_0;H)$. We have to prove that $(f,\bar f) \in \phi$. Since $K$ is weakly compact, then $f \in K$ and it follows that there exists a unique $u \in C([0,T_0];H)$ such that $u = P_{T_0}(f)$. Let us prove that $\bar f \in \phi(f)$, that is, $\bar f \in \operatorname{Sel}\mathbf{F}(u)$ where $u = P_{T_0}(f)$.

Since $(f_n, \bar{f}_n) \in \phi$, for each $n \in \mathbb{N}$ there exists $u_n \in C([0,T_0];H)$ such that $u_n = P_{T_0}(f_n)$ and $f_n \in \operatorname{Sel}\mathbf{F}(u_n)$. By Theorem 5.5, to guarantee that $\bar{f} \in \operatorname{Sel} F(u)$ it suffices to prove that $u_n \to u = P_{T_0}(f)$ a.e. in $[0,T_0]$ (up to a subsequence). By Lemma 5.6 the set $\{u_n\} = \{u_{f_n}\}$ is relatively compact in $L^2(0,T_0;H)$. Thus, there exist $\bar{u} \in L^2(0,T_0;H)$ and a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to \bar{u}$ in $L^2(0,T_0;H)$ as $k \to \infty$. By Lemma 5.7 we have $\bar{u} = u = P_{T_0}(f)$. It follows from Theorem 5.5 that $\bar{f} \in \operatorname{Sel}\mathbf{F}(u)$, that is, $\bar{f} \in \phi(f)$, which completes the proof. □

We are now in a position to complete the proof of Theorem 5.1: by Theorem 5.4 there exists $f \in K$ such that $f \in \phi(f)$, thence $u = P_{T_0}(f)$ is a solution of problem (1.1). □

**Remark 5.13.** In the special case $g \equiv 0$ the operator in equation (3.1) is maximal monotone, which allows one to use different techniques in the study of this problem - see, e.g., [24, 29] and references therein. We refer the reader to the monographs [34, 7, 9] for an insight into the theory of maximal monotone operators.

**Remark 5.14.** In this work, we do not discuss the challenging issue of uniqueness of the solution to problem (1.1). We refer the reader to papers [18, 19, 15, 13, 12] for further references and results on the uniqueness of solutions for the semilinear and quasilinear heat equations with discontinuous sources.

## 6. Finite speed of propagation and waiting time

6.1. **Energy functions and energy relation.** Let us fix a point $x_0 \in \Omega$, a number $0 < \rho_0 < \operatorname{dist}(x_0, \partial\Omega)$ and use the notation

$$
\begin{gathered}
B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}, \quad S_\rho = \partial B_\rho(x_0), \\
Q_{\rho,t} = B_\rho(x_0) \times (0,t), \quad S_{\rho,t} = S_\rho(x_0) \times (0,t).
\end{gathered}
$$

We are interested in the property of finite speed of propagation of disturbances from the initial data. This property is local and depends only on the nonlinear structure of the parabolic operator, for this reason we study the local weak solutions $u \in W$ of problem (1.1). Let us denote

$$
W(Q_{\rho_0,T}) = L^2(Q_{\rho_0,T}) \cap L^{p(\cdot)}(0,T;W^{1,p(\cdot)}(B_{\rho_0}(x_0)))
$$

and make the agreement to use, wherever it does not cause a confusion, the shorthand $B_\rho = B_\rho(x_0)$.

**Definition 6.1.** We say that a function $u \in C([0,T];L^2(B_{\rho_0})) \cap W(Q_{\rho_0,T})$ with $u_t \in L^2(Q_{\rho_0,T})$ is *a local solution* of problem (1.1) if

(1) there is a function $f : [0,T] \to L^2(B_{\rho_0})$ such that $f(t) \in \mathbf{F}(u(t))$, $t$-a.e. in $[0,T]$,
(2) for every test-function $\phi \in W(Q_{\rho_0,T})$ such that $\phi = 0$ on $S_{\rho_0} \times (0,T)$,

$$
\begin{aligned}
&\int_{Q_{\rho_0,T}} \Big(u_t\phi + \nabla\phi \cdot (|\nabla u|^{p-2}\nabla u + \int_0^t g(t-s)|\nabla u(s)|^{p-2}\nabla u(s)\,ds\Big)\,dz \\
&= \int_{Q_{\rho_0,T}} f\phi\,dz.
\end{aligned} \tag{6.1}
$$

It is clear that every weak solution of problem (1.1) constructed in Theorem 3.6 is a local weak solution. Given a local weak solution of problem (1.1), we introduce the energy functions

$$E(\rho,t)=\int_0^t\int_{B_\rho}|\nabla u|^{p(x)}dz,$$
$$b(\rho,t)=\|u(\cdot,t)\|^2_{2,B_\rho},\quad \bar{b}(\rho,t)=\operatorname{ess\,sup}_{\tau\le t}\|u(\cdot,\tau)\|^2_{2,B_\rho}.$$

By the Lebesgue differentiation theorem

$$E_\rho=\int_0^t\int_{S_\rho}|\nabla u|^{p(x)}\,dx\,dt\in L^1(0,\rho_0),$$
$$E_t=\int_{B_\rho}|\nabla u|^{p(x)}\,dx\in L^1(0,T),$$
$$b_\rho(\rho,t)=\|u(\cdot,t)\|^2_{2,S_\rho}\in L^1(0,\rho_0).$$

We will consider the local weak solutions with finite energy in $Q_{\rho_0,T}$:

$$\bar{b}(\rho_0,T)+E(\rho_0,T)=\operatorname{ess\,sup}_{(0,T)}\|u(\cdot,t)\|^2_{2,B_{\rho_0}}+\int_{Q_{\rho_0,T}}|\nabla u|^{p(x)}dx\,dt\le L \tag{6.2}$$

with a finite constant $L$.

**Lemma 6.2.** *Let $u\in W(Q_{\rho_0,T})\cap L^\infty(0,T;L^2(B_{\rho_0}))$. Assume that $u$ satisfies condition* (6.2) *in a cylinder $Q_{\rho_0,T}$. If*

$$\frac{\sup_{B_{\rho_0}}p(x)}{\inf_{B_{\rho_0}}p(x)}\equiv\frac{p^+}{p^-}<1+\frac{2}{n}, \tag{6.3}$$

*then for every cylinder $Q_{\rho,t}\subseteq Q_{\rho_0,T}$,*

$$\|u\|_{p(\cdot),Q_{\rho,t}}\le C,\quad C=C(L,p^\pm,n,T,\rho_0) \tag{6.4}$$

*Proof.* Let us assume first that $p^+\le 2$. In this case

$$\|u\|_{p(\cdot),Q_{\rho,t}}\le 2\|u\|_{2,Q_{\rho,t}}\|1\|_{\frac{2}{2-p(\cdot)},Q_{\rho,t}}\le t\|1\|_{\frac{2}{2-p(\cdot)},Q_{\rho,t}}\|u\|_{L^\infty(0,T),L^2(B_{\rho_0})}$$

by (2.2). If $p^+>2$ wee use the interpolation inequality [4]: for a.e. $t\in(0,T)$

$$\|u\|^{p^+}_{p^+,B_\rho}\le C(\|\nabla u\|_{p^-,B_\rho}+\|u\|_{2,B_\rho})^{p^+\theta}\|u\|^{p^+(1-\theta)}_{2,B_\rho},$$
$$0<\theta=\frac{\frac{1}{2}-\frac{1}{p^+}}{\frac{1}{2}-\frac{n-p^-}{np^-}}<\frac{p^-}{p^+}\le 1.$$

Since $\theta p^+<p^-$ by assumption (6.3), we may integrate this inequality in $t$ and apply Hölder's inequality, (2.4) and (2.3):

$$\begin{aligned}\|u\|^{p^+}_{p^+,Q_{\rho,t}}&\le C(T)(\|\nabla u\|^{\theta p^+}_{p^-,Q_{\rho,t}}+\|u\|^{\theta p^+}_{L^\infty(0,t;L^2(B_\rho)})\|u\|^{(1-\theta)p^+}_{L^\infty(0,t;L^2(B_\rho))}\\&\le C'(p^\pm,\rho_0,T)\Big(\|\nabla u\|^{\theta p^+}_{p(\cdot),Q_{\rho,t}}+L^{\frac{\theta p^+}{2}}\Big)L^{(1-\theta)\frac{p^+}{2}}\\&\le C''(p^\pm,\rho_0,T)\Big(\max\{L^\theta,L^{\frac{p^+}{p^-}\theta}\}+L^{\frac{\theta p^+}{2}}\Big)L^{(1-\theta)\frac{p^+}{2}}.\end{aligned}$$

By (2.4),

$$\|u\|^{p^+}_{p(\cdot),Q_{\rho,t}}\le C(p^\pm,\rho_0,T)\|u\|^{p^+}_{p^+,Q_{\rho,t}}$$

and (6.4) follows. □

**Remark 6.3.** The oscillation condition (6.3) is surely fulfilled if $p(x) \in C^0(\overline{B}_{\rho_0})$ and $\rho_0$ is sufficiently small - see [2, Lemma 1.32].

**Lemma 6.4.** *Let $B_{\rho_0}(x_0) \subset \Omega$ and $T \leq 1$. Assume that condition (6.3) is fulfilled and*

$$u_0(x) = 0 \ \textit{in} \ B_{\rho_0}, \quad g \in L^{p^+}(0,T). \tag{6.5}$$

*If a local weak solution $u(z)$ of problem (1.1) satisfies condition (6.2), then the following energy equality holds: for a.e. $\rho \in (0, \rho_0)$, $t \in (0,T)$,*

$$\frac{1}{2}b(\rho,t) + E(\rho,t) - \int_0^t \int_{B_\rho} fu\,dz = I_1 + I_2 + I_3, \tag{6.6}$$

*where*

$$\begin{gathered}
I_1 = \int_0^t \int_{S_\rho} u|\nabla u|^{p(x)-2}\nabla u \cdot \nu\,dS\,d\tau,\\
I_2 = -\int_0^t \int_{B_\rho} \nabla u(\tau) \int_0^\tau g(\tau - s)|\nabla u(s)|^{p(x)-2}\nabla u(s)\,ds\,dx\,d\tau,\\
I_3 = \int_0^t \int_{S_\rho} u(\tau) \int_0^\tau g(\tau - s)|\nabla u(s)|^{p(x)-2}\nabla u(s) \cdot \nu\,dsdSd\tau,
\end{gathered}$$

*and $\nu$ denotes the unit outer normal to $S_\rho$.*

*Proof.* Let us denote $r = |x - x_0|$ and introduce the functions

$$\zeta_k(r) = \begin{cases} 0 & \text{if } r \geq \rho,\\ k(\rho - r) & \text{if } r \in [\rho - 1/k, \rho],\\ 1 & \text{if } r < \rho - 1/k, \end{cases} \quad k \in \mathbb{N}.$$

Choosing $\phi = u\zeta_k(r)$ for the test-function in (6.1) we obtain the equality

$$\begin{aligned}
&\frac{1}{2}\int_{Q_{\rho_0,T}} (u^2)_t \zeta_k(r)\,dz\\
&+ \int_{Q_{\rho_0,T}} \zeta_k(r)\Big(|\nabla u(t)|^p + \nabla u(t) \int_0^t g(t-s)|\nabla u(s)|^{p(x)-2}\nabla u(s)\,ds\Big)\,dz\\
&= k\int_0^T \int_{\rho<|x-x_0|<\rho+\frac{1}{k}} u(t)\Big(|\nabla u(t)|^{p(x)-2}\nabla u(t)\\
&\quad + \int_0^t g(t-s)|\nabla u(s)|^{p(x)-2}\nabla u(s)\,ds\Big) \cdot \nu\,dz + \int_{Q_{\rho_0,T}} fu(t)\zeta_k(r)\,dz.
\end{aligned} \tag{6.7}$$

By the dominated convergence theorem every term on the left-hand side and the last term on the right-hand of (6.7) has a limit as $k \to \infty$. Let us denote

$$\begin{gathered}
J_1(x) = \int_0^T |u(t)||\nabla u(t)|^{p(\cdot)-1}\,dt,\\
J_2(x) = \int_0^T |\nabla u(t)|\Big(\int_0^t |g(t-s)||\nabla u(s)|^{p(x)-1}\,ds\Big)\,dt.
\end{gathered}$$

Taking into account (6.4), (6.2) and applying the Hölder inequality (2.2) and (2.3) we have

$$\int_{B_\rho} J_1(x)\,dx$$

$$
\begin{aligned}
&\le 2\|u\|_{p(\cdot),Q_{\rho,T}}\||\nabla u|^{p-1}\|_{p'(x),Q_{\rho,T}} \\
&\le 2\|u\|_{p(\cdot),Q_{\rho,T}} \max\Big\{\Big(\int_{Q_{\rho,T}} |\nabla u|^p\,dz\Big)^{1-\frac{1}{p^-}}, \Big(\int_{Q_{\rho,T}} |\nabla u|^p\,dz\Big)^{1-\frac{1}{p^+}}\Big\} < \infty.
\end{aligned}
$$

By Hölder's inequality, for every $x \in B_{\rho_0} \subset \Omega$ and $t \in (0,T) \subset (0,1)$,

$$
\begin{aligned}
\Big(\int_0^t |g(t-s)|^{p(x)}\,ds\Big)^{1/p(x)} &\le \|g\|_{p(x),(0,t)} \\
&\le t^{1-\frac{p(x)}{p^+}}\|g\|_{p^+,(0,T)} \\
&\le \|g\|_{p^+,(0,T)} := \sigma.
\end{aligned} \tag{6.8}
$$

Then

$$
\begin{aligned}
&\int_{B_\rho} J_2(x)\,dx \\
&\le \int_0^T \int_{B_\rho} |\nabla u(t)|\Big(\int_0^T |g(t-s)|^p\,ds\Big)^{1/p}\Big(\int_0^T |\nabla u(s)|^p\,ds\Big)^{\frac{p-1}{p}}\,dz \\
&\le (1+\sigma)^{\frac{1}{p^-}}\Big(\int_{B_\rho}(\int_0^T |\nabla u(t)|\,dt)\Big(\int_0^T |\nabla u(s)|^p ds\Big)^{\frac{p-1}{p}}\,dx\Big) \\
&\le (1+\sigma)^{\frac{1}{p^-}}\int_{B_\rho}((\int_0^T |\nabla u(t)|^p\,dt)^{1/p}T^{1-\frac{1}{p}}\Big(\int_0^T |\nabla u(s)|^p ds\Big)^{\frac{p-1}{p}})\,dx \\
&\le (1+\sigma)^{\frac{1}{p^-}}(1+T)^{1-\frac{1}{p^+}}\int_{Q_{\rho,T}} |\nabla u|^p\,dz < \infty.
\end{aligned}
$$

It follows from the Lebesgue differentiation theorem that for a.e. $\rho \in (0,\rho_0)$ there exists

$$
\begin{aligned}
&\lim_{k\to\infty} k\int_0^T \int_{\rho<|x-x_0|<\rho+\frac{1}{k}} u(t)\Big(|\nabla u(t)|^{p-2}\nabla u(t) \\
&+\int_0^t g(t-s)|\nabla u(s)|^{p-2}\nabla u(s)\,ds\Big)\cdot\nu\,dx\,dt = I_1 + I_3.
\end{aligned}
$$

Letting in (6.7) $k\to\infty$ we obtain (6.6) with $t=T$. The arguments remain valid if we substitute $T$ by any $t\in(0,T)$. $\square$

### 6.2. **Finite speed of propagation.**

**Theorem 6.5.** *Let $u(z)$ be a local weak solution of problem* (1.1) *in the sense of Definition 6.1 in a cylinder $Q_{\rho_0,T}$, satisfying condition* (6.2). *Assume that*

$$
\begin{aligned}
&\text{(i) } \rho_0 \le 1,\ T\le 1,\ L\le 1, \\
&\text{(ii) } g\in L^{p^+}(0,T),\ 2<p^-\le p(x), \\
&\text{(iii) } sf(s)\le \epsilon s^2 \text{ for all } f\in \operatorname{Sel}(\mathbf{F}),\text{ every } s\in\mathbb{R} \text{ and some } 0<\epsilon<\tfrac{1}{4T}.
\end{aligned} \tag{6.9}
$$

*Let $u_0=0$ in $B_{\rho_0}$. If $p(x)$ satisfies in $B_{\rho_0}$ the oscillation condition*

$$
\frac{p^+}{p^-} < 1+\frac{2}{n}\big(1-\frac{1}{p^-}\big), \quad p^+ = \sup_{B_{\rho_0}} p(x), \quad p^- = \inf_{B_{\rho_0}} p(x), \tag{6.10}
$$

*then there exists $t^* \in (0,T)$ such that $u(z)$ possesses the property of finite speed of propagation on the interval $t \in (0,t^*)$: $u(z) = 0$ in $B_{\rho(t)}(x_0)$ with $0 \le t < t^*$, where the variable radius $\rho(t)$ of the null set is given by*

$$\rho^{1+\nu}(t) = \max\{0,\ \rho_0^{1+\nu} - C\, t^{\alpha} L^{1-\beta}\} \tag{6.11}$$

*with a positive constant $C$ and the exponents $\alpha$, $\beta$, $\nu$ defined in formulas* (6.22).

**Remark 6.6.** (i) For small $t$ the function $\rho(t)$ defined by (6.11) is strictly positive and the set $B_{\rho(t)}(x_0)$ is nonempty.

(ii) The oscillation condition (6.10) is stronger than condition (6.3).

(iii) If we know that the energy is finite in some cylinder $Q_{R,\theta}$, the restriction $L \le 1$ is fulfilled for the sufficiently small parameters $\rho_0 < R$ and $T \le \theta$.

(iv) The proofs of the existence of a weak solution of problem (1.1) and the property of finite speed of propagation are practically independent. To prove the latter we only deal with selections, for this reason the assumptions on the multivalued function $\mathbf{F}$ are formulated as the uniform estimate (6.9) (iii) on $uf(u)$ for all possible selections $f \in \operatorname{Sel}(\mathbf{F})$. An example of a multivalued function $\mathbf{F}$ that satisfies this condition is furnished by the function $\mathbf{F}_2$ defined in the introduction.

*Proof of Theorem 6.5.* Let us transform the energy equality (6.6) into a nonlinear differential inequality for the energy function $E + b$. Applying Hölder's inequality (2.2) and (2.3) we have

$$\begin{aligned} |I_1(\rho,t)| &\le 2\int_{S_{\rho,t}(x_0)} |\nabla u|^{p(x)-1}|u|\, dSdt \le 2\||\nabla u|^{p-1}\|_{p'(\cdot),S_{\rho,t}}\|u\|_{p(\cdot),S_{\rho,t}} \\ &\le C' \max\big\{E_\rho^{1-\frac{1}{p^+}},\, E_\rho^{1-\frac{1}{p^-}}\big\}\|u\|_{p^+,\ S_{\rho,t}}, \end{aligned} \tag{6.12}$$

with the constant

$$\begin{aligned} 2\|1\|_{\frac{p^+}{p^+-p(\cdot)},S_{\rho,t}} &= 2\max\{1, |S_{\rho,t}|^{1-\frac{p^-}{p^+}}\} = 2(|\omega_{n-1}|\rho^{n-1}t)^{1-\frac{p^-}{p^+}} \\ &\le 2|\omega_{n-1}|^{1-\frac{p^-}{p^+}} = C', \end{aligned}$$

being $|\omega_{n-1}|$ the surface of the unit ball in $\mathbb{R}^n$. By Hölder's inequality

$$\begin{aligned} |I_2(\rho,t)| &\le \int_0^t \Big(\int_{B_\rho} |\nabla u(\tau)| \int_0^\tau |g(\tau-s)||\nabla u(s)|^{p(x)-1}\, ds\, dx\Big) d\tau \\ &\le \sigma \int_{B_\rho} \Big(\int_0^t |\nabla u(\tau)|\, d\tau (\int_0^t |\nabla u(s)|^{p(x)}\, ds)^{\frac{p(x)-1}{p(x)}} d\tau\Big)\, dx \\ &\le \sigma \int_{B_\rho} t^{1-\frac{1}{p(x)}} \Big(\int_0^t |\nabla u|^{p(x)}\, d\tau\Big)^{1/p(x)} (\int_0^t |\nabla u(s)|^{p(x)}\, ds)^{1-\frac{1}{p(x)}}\, dx \\ &= \sigma \int_{B_\rho} t^{1-\frac{1}{p(x)}} \int_0^t |\nabla u|^{p(x)}\, d\tau\, dx. \end{aligned}$$

with the constant $\sigma$ from (6.8). Since $t \in (0,1]$ by assumption, the estimate on $I_2$ takes the form

$$|I_2(\rho,t)| \le \sigma t^{1-\frac{1}{p^-}} E(\rho,t).$$

The estimate on $I_3$ is obtained in a similar way:

$$|I_3| \le \int_0^t \int_{S_\rho} |u(\tau)| \Big(\int_0^t |g(t-s)|^{p(x)}\, ds\Big)^{1/p(x)} \Big(\int_0^t |\nabla u(s)|^{p(x)}\, ds\Big)^{\frac{1-p(x)}{p(x)}}\, dSd\tau$$

$$
\begin{aligned}
&\le \sigma \int_{S_{\rho,t}} |u(\tau)| \Big( \int_0^t |\nabla u(s)|^{p(x)}\, ds \Big)^{1-\frac{1}{p(x)}} \, dS d\tau \\
&\le 2\sigma \|u\|_{p(\cdot),S_{\rho,t}} \|\Theta\|_{p'(\cdot),S_{\rho,t}}, \quad \Theta = \Big( \int_0^t |\nabla u(s)|^{p(x)}\, ds \Big)^{1-\frac{1}{p(x)}}.
\end{aligned}
$$

Since

$$
\int_{S_{\rho,t}} \int_0^t |\nabla u(\tau)|^{p(x)}\, dS d\tau \le t E_\rho(\rho,t)
$$

and $t \in (0,1]$. We continue the estimate on $I_3$ as follows:

$$
\begin{aligned}
|I_3| &\le 2\sigma \|u\|_{p(\cdot),S_{\rho,t}} \max\Big\{ \Big( t \int_{S_{\rho,t}} |\nabla u|^{p(x)}\, dS\, ds \Big)^{1-\frac{1}{p^-}}, \Big( t\int_{S_{\rho,t}} \dots \Big)^{1-\frac{1}{p^+}} \Big\} \\
&\le 2\sigma \|u\|_{p(\cdot),S_{\rho,t}} t^{1-\frac{1}{p^-}} \max\big\{ E_\rho^{1-\frac{1}{p^-}}, E_\rho^{1-\frac{1}{p^+}} \big\} \\
&\le 4\sigma C' \|u\|_{p^+,S_{\rho,t}} t^{1-\frac{1}{p^-}} \max\big\{ E_\rho^{1-\frac{1}{p^-}}, E_\rho^{1-\frac{1}{p^+}} \big\}
\end{aligned}
$$

with the constant $C'$ from (6.12). Substitution of the estimates on $I_j$, $j = 1,2,3$, into (6.6) leads to the inequality

$$
\begin{aligned}
\frac{1}{2} b(\rho,t) + E(\rho,t) &\le (1 + 4\sigma t^{1-\frac{1}{p^-}}) C' \max\big\{ E_\rho^{1-\frac{1}{p^+}},\, E_\rho^{1-\frac{1}{p^-}} \big\} \|u\|_{p^+,S_{\rho,t}} \\
&\quad + \sigma t^{1-\frac{1}{p^-}} E(\rho,t) + \int_{Q_{\rho,t}} f u\, dz.
\end{aligned}
$$

Let us choose $t_* \in (0,1]$ so small that $2\sigma t_*^{1-\frac{1}{p^-}} \le 1$ and recall that by assumption (6.9) (iii) we have $f(x,t)u(x,t) \in u(x,t)\mathbf{F}(u(x,t))$ for a.e. $t \in (0,T)$ and $x \in \Omega$, and

$$
u(x,t) f(u(x,t)) \le \epsilon u^2(x,t).
$$

Then for a.e. $t \in (0,t_*]$,

$$
b(\rho,t) + E(\rho,t) \le C \max\big\{ E_\rho^{1-\frac{1}{p^+}},\, E_\rho^{1-\frac{1}{p^-}} \big\} \|u\|_{p^+,S_{\rho,t}} + 2\epsilon t \bar{b}(\rho,t).
$$

The right-hand side of this inequality is a nondecreasing function of $t$, for this reason the inequality remains true if the left-hand side is substituted by $\frac{1}{2}(\bar{b} + E)$. If $4\epsilon T < 1$, the resulting inequality reduces to

$$
\frac{1}{4}(\bar{b} + E) \le C \max\big\{ E_\rho^{1-\frac{1}{p^+}},\, E_\rho^{1-\frac{1}{p^-}} \big\} \|u\|_{p^+,S_{\rho,t}} \tag{6.13}
$$

with an absolute constant $C$.

Let us use the trace-interpolation inequality, (see, e.g., [4, p. 298]):

$$
\|u\|_{p^+,S_\rho} \le C(\|\nabla u\|_{p^-,B_\rho} + \rho^\delta \|u\|_{2,B_\rho})^\theta \|u\|_{2,B_\rho}^{1-\theta} \tag{6.14}
$$

with the exponents

$$
\theta = \frac{p^-}{p^+} \frac{n(p^+ - 2) + 2}{n(p^- - 2) + 2p^-} \in (0,1), \quad \delta = \frac{n(p^- - 2) + 2p^-}{2p^-} > 1
$$

and an independent of $u$ constant $C$. Applying (6.14) we find that

$$\begin{aligned}
\|u\|^{p^+}_{p^+,S_{\rho,t}} &= \int_0^t \|u\|^{p^+}_{p^+,S_\rho} dt \\
&\le C\int_0^t (\|\nabla u\|_{p^-,B_\rho} + \rho^{-\delta}\|u\|_{2,B_\rho})^{p^+\theta}\|u\|^{p^+(1-\theta)}_{2,B_\rho}\, dt \\
&\le C\max\{1,\rho^{-p^+\delta\theta}\}\int_0^t \Big(\|\nabla u\|^{p^-}_{p^-,B_\rho} + b^{\frac{p^-}{2}}\Big)^{\theta\frac{p^+}{p^-}}\bar{b}^{p^+\frac{1-\theta}{2}}\, dt \qquad (6.15)\\
&\le C\rho^{-p^+\delta\theta}\bar{b}^{p^+\frac{1-\theta}{2}}\int_0^t \Big(\|\nabla u\|^{p^-}_{p^-,B_\rho} + \bar{b}^{\frac{p^-}{2}-1}b\Big)^{\theta\frac{p^+}{p^-}}\, dt \\
&\le C(1+L)^{\frac{p^-}{2}-1}\rho^{-p^+\delta\theta}\bar{b}^{p^+\frac{1-\theta}{2}}\int_0^t \big(\|\nabla u\|^{p^-}_{p^-,B_\rho} + \bar{b}\big)^{\theta\frac{p^+}{p^-}}\, dt,
\end{aligned}$$

where $L$ is the constant from (6.2). The oscillation condition (6.10) yields the inequality $\theta p^+ < p^-$. Applying Hölder's inequality from (6.15) we obtain

$$\begin{aligned}
&\|u\|_{p^+,S_{\rho,t}} \\
&\le C\,(1+L)^{\frac{p^-}{2}-1}t^{\frac{1}{p^+}-\frac{\theta}{p^-}}\rho^{-\delta\theta}\Big(\int_0^t (\|\nabla u\|^{p^-}_{p^-,B_\rho} + b)dt\Big)^{\frac{\theta}{p^-}}\bar{b}^{\frac{1-\theta}{2}} \\
&\le C(1+L)^{\frac{p^-}{2}-1}\,t^{\frac{1}{p^+}-\frac{\theta}{p^-}}\rho^{-\delta\theta}\Big(\|\nabla u\|^{p^-}_{p^-,Q_{\rho,t}} + t\,\bar{b}\Big)^{\frac{\theta}{p^-}}\bar{b}^{\frac{1-\theta}{2}} \qquad (6.16)\\
&\le C(1+L)^{\frac{p^-}{2}-1}(1+t)^{\frac{\theta}{p^-}}t^{\frac{1}{p^+}-\frac{\theta}{p^-}}\rho^{-\delta\theta}\big(\|\nabla u\|^{p^-}_{p^-,Q_{\rho,t}} + \bar{b}\big)^{\frac{\theta}{p^-}+\frac{1-\theta}{2}}.
\end{aligned}$$

Noting that

$$\|1\|_{\frac{p(\cdot)}{p(\cdot)-p^-},Q_{\rho,t}} \le \max\big\{1,|Q_{\rho,t}|^{1-\frac{p^-}{p^+}}\big\} \le |\omega_{n-1}|^{1-\frac{p^-}{p^+}},$$

and using Hölder's inequality (2.2) and (2.3), we have

$$\begin{aligned}
&\int_{Q_{\rho,t}} |\nabla u|^{p^-}\,dx\,dt \\
&\le 2\|1\|_{\frac{p(\cdot)}{p(\cdot)-p^-},Q_{\rho,t}}\||\nabla u|^{p^-}\|_{\frac{p(\cdot)}{p^-},Q_{\rho,t}} \\
&\le 2\|1\|_{\frac{p(\cdot)}{p(\cdot)-p^-},Q_{\rho,t}}\max\Big\{\int_{Q_{\rho,t}}|\nabla u|^{p(x)}\,dx\,dt,\,\Big(\int_{Q_{\rho,t}}|\nabla u|^{p(x)}\,dx\,dt\Big)^{\frac{p^+}{p^-}}\Big\} \\
&\le C(n,p^\pm)\max\{E,\,E^{\frac{p^-}{p^+}}\} \\
&\le C(n,p^\pm)E^{\frac{p^-}{p^+}}.
\end{aligned}$$

It follows from (6.16) that

$$\begin{aligned}
\|u\|_{p^+,S_{\rho,t}} &\le Ct^{\frac{1}{p^+}-\frac{\theta}{p^-}}\rho^{-\delta\theta}(E^{\frac{p^-}{p^+}} + \bar{b}^{1-\frac{p^-}{p^+}}\bar{b}^{\frac{p^-}{p^+}})^{\frac{\theta}{p^-}+\frac{1-\theta}{2}} \\
&\le C't^{\frac{1}{p^+}-\frac{\theta}{p^-}}\rho^{-\delta\theta}(E+\bar{b})^{\frac{p^-}{p^+}(\frac{\theta}{p^-}+\frac{1-\theta}{2})}.
\end{aligned}$$

Substituting the result into (6.13), we arrive at the inequality

$$\frac{1}{4}(\bar{b}+E) \le C\phi^{1/\lambda'}(\rho,t)\max\big\{E_\rho^{1-\frac{1}{p^+}},\,E_\rho^{1-\frac{1}{p^-}}\big\}(E+\bar{b})^{1/\lambda} \qquad (6.17)$$

with

$$\phi^{1/\lambda'}(\rho,t)=t^{\frac{1}{p^+}-\frac{\theta}{p^-}}\rho^{-\delta\theta},\quad \frac{1}{\lambda}=\frac{p^-}{p^+}\Big(\frac{1-\theta}{2}+\frac{\theta}{p^-}\Big)\in(0,1) \tag{6.18}$$

and an absolute constant $C$. Applying to the right-hand side Young's inequality and simplifying, we reduce (6.17) to the inequality

$$E\le E+b\le E+\overline{b}\le C\phi(\rho,t)\max\left\{E_\rho^{1/\mu^+},\,E_\rho^{1/\mu^-}\right\} \tag{6.19}$$

with

$$\mu^\pm=\frac{p^\pm}{p^\pm-1}\big(1-\frac{1}{\lambda}\big), \tag{6.20}$$

which can be written in the form

$$E_\rho\ge\begin{cases}(C'\phi)^{-\mu^+}E^{\mu^+} & \text{if } E_\rho\le 1,\\ (C'\phi)^{-\mu^-}E^{\mu^-} & \text{if } E_\rho>1.\end{cases}$$

Since $\rho_0$ and $T$ are already chosen so small that $E+b\le 1$ in, the last inequality yields

$$\begin{aligned}E_\rho&\ge\min\{(C'\phi)^{-\mu^+},(C'\phi)^{-\mu^-}\}E^{\mu^-}\\&\ge C''t^{-\mu^-(\frac{1}{p^+}-\frac{\theta}{p^-})}\rho^{\delta\theta\mu^-}E^{\mu^-}\\&=C''t^{-\alpha}\rho^{\nu}E^{\beta}\end{aligned} \tag{6.21}$$

with the exponents

$$\begin{gathered}\beta=\mu^-=(1-\frac{p^-}{p^+}(\frac{1-\theta}{2}+\frac{\theta}{p^-}))\frac{p^-}{p^--1}\in(0,1),\\ \alpha=\beta(\frac{p^--\theta p^+}{p^+p^-})>0,\\ \nu=\frac{1}{2}\delta\theta(1+\frac{p^--2}{p^-})\frac{p^-}{p^--1}>0.\end{gathered} \tag{6.22}$$

Integrating (6.21) with respect to $\rho$ over the interval $(\rho,\rho_0)$ where $E$ remains positive we obtain

$$\begin{aligned}E^{1-\beta}(\rho,t)&\le E^{1-\beta}(\rho_0,t)-C''t^{-\alpha}\frac{1-\beta}{1+\nu}(\rho_0^{1+\nu}-\rho^{1+\nu})\\&\le L^{1-\beta}-C''t^{-\alpha}\frac{1-\beta}{1+\nu}(\rho_0^{1+\nu}-\rho^{1+\nu}).\end{aligned}$$

The right-hand side of this inequality becomes negative if $\rho<\rho(t)$, where $\rho(t)$ is defined by the equality

$$\rho^{1+\nu}(t)=\rho_0^{1+\nu}-L^{1-\beta}t^{\alpha}\frac{1+\nu}{C''(1-\beta)}=\rho_0^{1+\nu}-t^{\alpha}CL^{1-\beta}.$$

Since $E(\rho,t)$ is nonnegative by definition, this means that necessarily $E(\rho,t)=0$ for $\rho\le\rho(t)$. The assumption $u_0=0$ in $B_{\rho_0}$ yields the equality $u(z)=0$ a.e. in the cylinder $B_{\rho(t)}\times(0,t)$. □

**Remark 6.7.** We do not claim optimality of condition (6.9) (iii) on the growth of the selection $f(u(t))$. Our aim is to prove that the finite speed of propagation takes place for the solutions whose existence is guaranteed by Theorem 5.1. It is worth noting that the assertion of Theorem 6.5 remains true for any $f$, provided that the energy $E$ satisfies the nonlinear differential inequality of the type (6.21) - see, e.g., [4, Ch.3] for the examples of local parabolic equations with constant nonlinearities or [11] for the solutions of differential inclusions.

6.3. **Waiting time property.** Let us consider the following situation: there exist $R > 0$ and $\rho_0 \in (0, R)$ such that for some point $x_0 \in \Omega$

$$B_R \subseteq \Omega, \quad u_0 \equiv 0 \text{ in } B_{\rho_0} \subset B_R. \tag{6.23}$$

The support of $u_0$ is contained in $\Omega \setminus B_{\rho_0}$. We will assume that $u_0$ is sufficiently "flat" near the boundary of its support: there exist $\epsilon > 0$ such that

$$\|u_0\|^2_{2,B_\rho} \le \epsilon(\rho - \rho_0)_+^{\frac{1}{1-\nu}} \quad \forall \rho \in (\rho_0, R) \tag{6.24}$$

with the exponent

$$\nu = 1 - \frac{\mu^+(1-\mu^-)}{\mu^-} \in (0,1) \tag{6.25}$$

and $\mu^\pm$ from condition (6.20).

**Theorem 6.8.** *Let the conditions of Theorem 6.5 be fulfilled, assume that $u_0$ satisfy condition (6.24) with a constant $\epsilon$. If a local weak solution $u(z)$ of problem (1.1) satisfy condition (6.2) with a constant $L \le 1$ and $\epsilon$ is sufficiently small, there exists $t_* \in (0, T)$, which depends on $L$, $\epsilon$ and $\|g\|_{p^+,(0,T)}$, such that $u(z) = 0$ a.e. in $B_{\rho_0} \times (0, t_*)$.*

*Proof.* Following the proof of Lemma 6.4 we derive the equality: for a.e. $\rho \in (\rho_0, R)$, $t \in (0, T)$

$$\frac{1}{2} b(\rho, t) + E(\rho, t) - \int_0^t \int_{B_\rho} uf \, dx \, dt = I_1 + I_2 + I_3 + I_4, \tag{6.26}$$

where $I_1$, $I_2$, $I_3$ are defined in Lemma 6.4, and

$$I_4 = \frac{1}{2}\|u_0\|^2_{2,B_\rho} \le \frac{\epsilon}{2}(\rho - \rho_0)_+^{\frac{1}{1-\nu}}.$$

Following the proof of Theorem 6.5 we derive the nonhomogeneous counterpart of the differential inequality (6.19),

$$E + b \le E + \bar{b} \le C\phi(\rho, t) \max\left\{E_\rho^{1/\mu^+}, E_\rho^{1/\mu^-}\right\} + C\epsilon(\rho - \rho_0)_+^{\frac{1}{1-\nu}} \tag{6.27}$$

with the exponents $\mu^\pm$ from (6.20). Inequality (6.27) can be written in the equivalent form

$$E + b \le E + \bar{b} \le C \begin{cases} \phi(\rho, t) E_\rho^{1/\mu^+} + \epsilon(\rho - \rho_0)_+^{\frac{1}{1-\nu}} & \text{if } E_\rho > 1, \\ \phi(\rho, t) E_\rho^{1/\mu^-} + \epsilon(\rho - \rho_0)_+^{\frac{1}{1-\nu}} & \text{if } E_\rho \le 1. \end{cases}$$

Raising both parts to the power $\mu^\pm$ we can rewrite the last inequality into the form

$$\begin{aligned} E^{\mu^-} &\le (E + b)^{\mu^-} \le (\bar{b} + E)^{\mu^-} \\ &\le C \max\{\phi^{\mu^-}, \phi^{\mu^+}\} E_\rho + C \max\left\{\left(\epsilon(\rho - \rho_0)_+^{\frac{1}{1-\nu}}\right)^{\mu^+}, \left(\epsilon(\rho - \rho_0)_+^{\frac{1}{1-\nu}}\right)^{\mu^-}\right\}. \end{aligned} \tag{6.28}$$

Since the data satisfy conditions (6.9), inequality (6.28) yields the ordinary differential inequality

$$E^{\mu^-} \le (E + b)^{\mu^-} \le C_*\left(t^\alpha E_\rho + \epsilon^{\nu\mu^+}(\rho - \rho_0)_+^{\mu^+/(1-\nu)}\right),$$

$$0 \le b + E \le L \text{ for } \rho \in (\rho_0, R), \quad \alpha = \frac{p^- - \theta p^+}{p^- p^+} > 0.$$

Let us fix $t_* > 0$, $\epsilon_* > 0$ and consider the problem

$$w^{\mu^-} = C_* \Big( t_*^\alpha w_\rho + \epsilon_*^{\nu\mu^+} (\rho - \rho_0)_+^{\mu^+/(1-\nu)} \Big), \quad \rho \in (\rho_0, R),$$
$$w(\rho_0) = 0, \quad w(R) = L$$

with the constant $L$ from condition (6.2). It is straightforward to check that this problem admits a solution

$$w(\rho) = A(\rho - \rho_0)_+^{\frac{1}{1-\mu^-}}, \quad A = L(R - \rho_0)^{-\frac{1}{1-\mu^-}}, \quad A = \text{positive constant},$$

provided that $t_*$ is the solution of the equation

$$A^{\mu^-} = C_* \Big( t_*^\alpha \frac{A}{1 - \mu^-} + \epsilon_*^{\nu\mu^-} \Big). \tag{6.29}$$

This equation has a solution for the sufficiently small $A$ and $\epsilon_* > 0$. It is proven in [4, p. 129] that $\epsilon \in (0, \epsilon^*]$ the function $w(\rho)$ majorates $E(\rho, t_*)$ on the interval $[\rho_0, R]$.

Since $E(\rho, t)$ is monotone increasing in $\rho$ and $t$, and $E(\rho, t) \le w(\rho)$ on $(\rho_0, R)$, it is necessary that $E(\rho, t) = 0$ for $\rho \in (0, \rho_0]$ and $t \in (0, t_*]$. It follows that $u = const$ in $Q_{\rho_0,T}$, whence $u = 0$ therein because $u_0 = 0$ in $B_{\rho_0}$ by assumption. $\square$

**Remark 6.9.** Condition (6.29) connects the three characteristic parameters of the problem: the total energy $L$, the waiting time $t_*$ and the threshold value of the source intensity $\epsilon_*$. For this reason, given an arbitrary intensity $0 < \epsilon_* < \infty$, the effect of waiting time of the solution can be provided by an appropriate choice of $t_*$ and $L$.

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