Choquet Theorem for Random Sets in Polish Spaces and Beyond*

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Abstract. A fundamental long-standing problem in the theory of random sets is concerned with the possible characterization of the distributions of random closed sets in Polish spaces via capacities. Such a characterization is known in the locally compact case (the Choquet theorem) in two equivalent forms: using the compact sets and the open sets as test sets. The general case has remained elusive. We solve the problem in the affirmative using open test sets.

1 The problem

The Choquet theorem is a central result in the theory of random sets, allowing one to 'pack' all the information from a probability distribution (acting on sets of sets) in a simpler function, a capacity (acting on sets of points). That is similar to the way the cumulative distribution function contains the distributional information of a random variable. And, like in that case, the harder part is to find the essential properties characterizing those functions which can actually be 'unpacked' to recover a whole distribution.

The *hitting functional* of a random closed set X is given by

$$T_X(A) = P(X \cap A \neq \emptyset).$$

The random set is reconstructed from the information whether it hits (intersects) the sets A in a family of *test sets*. The standard presentation of the Choquet theorem assumes that the carrier space is a locally compact, second countable, Hausdorff space. Those spaces contain $\mathbb{R}^n$ and enjoy a number of its nice topological properties, e.g. they admit a separable complete metric (i.e. they are *Polish spaces*) and are σ-compact (in fact, hemicompact).

The Choquet theorem in locally compact, second countable, Hausdorff spaces was established in 1972 by Matheron [6,7] who provided a proof based on traditional measure extension tools after Choquet's pioneering

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work [3, Theorem 51.1] which was not explicitly concerned with the problem of characterizing the distributions of random sets. In 1989, Norberg [12], by entirely different order-theoretical methods, extended it to locally compact, second countable, sober spaces. In 2014, a fourth method allowed us to give a Choquet theorem in locally compact, σ-compact, Hausdorff spaces [17].

But local compactness is a problematic requirement in probability theory. A rather more natural setting, as already established in the 1960s in books like Parthasarathy's [13] and Billingsley's [2], is that of general Polish spaces, to the point that a measurable space whose σ-algebra is isomorphic to the Borel σ-algebra of a Polish space is nowadays known as a *standard measurable space.*

Reflections on the need for a Choquet theorem in Polish spaces date back at least to Goodman *et al.* [5, Chapter 3, p. 93], who wrote

> *In this Chapter, we consider ... locally compact, Hausdorff and separable spaces. The reason is this. The foundations of random closed sets are based on Choquet theorem on such spaces. It should be noted that the natural domain of probability theory is Polish spaces ... These spaces might not be locally compact, for example, infinite dimensional Banach Spaces. Also ... in other applications, such as optimal control of distributed systems ... it is necessary to consider infinite dimensional topological spaces.*

Since, it has explicitly been stated as an open problem in [8, Open Problem 2.29, p. 41], [10, Remark, p. 128], and [9, Open Problem 1.3.24, p. 69].

The theory of stochastic processes involves state or path spaces which are not locally compact. The advent of ever more complex forms of data, such as fuzzy and functional data, also draws attention to carrier spaces which fail to be locally compact or even Polish spaces. Confidence regions, depth functions, and statistics defined as solutions of optimization problems (like M-estimators) all lead naturally to random sets in those spaces.

In this communication, we will solve the problem in the affirmative by extending the Choquet theorem to metrizable Lusin spaces. That generality is sufficient to solve the open problem as stated as well as to additionally cover some examples of state, path and fuzzy set spaces which are not actually Polish but metrizable and Lusin.

2 Preliminaries

As mentioned in Section 1, a *Polish space* is a separable space whose topology is compatible with a complete metric. A topological space is called a *Lusin space* if it is the image of a Polish space by a continuous bijective mapping. In other words, its topology is weaker than some Polish topology in the same space (for example, the norm topology of a separable Banach space is Polish while its weak topology is Lusin).

Let $\mathbb{E}$ be a topological space. We will denote by $\mathcal{P}(\mathbb{E})$ the class of parts of $\mathbb{E}$, by $\mathcal{B}(\mathbb{E})$ its Borel σ-algebra (the σ-algebra generated by the open sets), by $\mathcal{F}(\mathbb{E})$ its non-empty closed sets, by $\mathcal{F}'(\mathbb{E})$ its closed sets, by $\mathcal{K}(\mathbb{E})$

its non-empty compact sets, by $\mathcal{K}'(\mathbb{E})$ its compact sets and by $\mathcal{G}(\mathbb{E})$ its open sets.
A subset of $\mathbb{E}$ is called *universally measurable* if it is in the completion of the Borel σ-algebra for every probability measure on $\mathcal{B}(\mathbb{E})$. This defines a larger σ-algebra $\mathcal{B}_u(\mathbb{E})$ called the *universal completion* of $\mathcal{B}(\mathbb{E})$.
A *capacity* in $\mathbb{E}$ is a set function $c : \mathcal{L} \to [0,1]$ on a lattice of sets $\mathcal{L} \subset \mathcal{P}(\mathbb{E})$ such that $\emptyset, \mathbb{E} \in \mathcal{L}$, $c(\emptyset) = 0$ and $c(\mathbb{E}) = 1$ hold, and moreover $c(A) \leq c(B)$ whenever $A \subset B$. It will be called:
-inner continuous, if $c(A_n) \to c(A)$ whenever $A_n \nearrow A$;
-outer continuous, if $c(A_n) \to c(A)$ whenever $A_n \searrow A$;
-completely alternating, if

$$c(\bigcap_{i=1}^{n} A_i) \leq \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} c(\bigcup_{i \in I} A_i)$$

for any $n \in \mathbb{N}$;
-completely monotone, if

$$c(\bigcup_{i=1}^{n} A_i) \geq \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} c(\bigcap_{i \in I} A_i)$$

for any $n \in \mathbb{N}$.
Often the definitions of complete alternation and monotony are expressed equivalently in terms of successive differences, see [8,10].
A *random closed set* on a probability space $(\Omega, \mathcal{A}, P)$ is a mapping $X : \Omega \to \mathcal{F}(\mathbb{E})$ such that $\{X \cap G \neq \emptyset\} \in \mathcal{A}$ for all $G \in \mathcal{G}(\mathbb{E})$. Equivalently, X is measurable when $\mathcal{F}(\mathbb{E})$ is endowed with the *Effros σ-algebra* $\mathcal{E}(\mathcal{F}(\mathbb{E}))$ generated by all subsets of the form $\{A \in \mathcal{F}(\mathbb{E}) \mid A \cap G \neq \emptyset\}$ where G ranges over $\mathcal{G}(\mathbb{E})$. The distribution P_X of X is the induced probability measure $P_X : \mathcal{E}(\mathcal{F}(\mathbb{E})) \to [0,1]$ given by $P_X(A) = P(X \in A)$.
If X satisfies the generally stronger requirement that $\{X \cap B \neq \emptyset\} \in \mathcal{A}$ for all $B \in \mathcal{B}(\mathbb{E})$, then it is called *strongly measurable*. In that case, also the sets $\{X \subset B\}$ are measurable, since they can easily be obtained using complementation as $\{X \subset B\} = \{X \cap B^c \neq \emptyset\}^c$.

3 Support results

We collect here the theorems which will be used in the proof of the main result. The following is well known, see e.g. [16, Lemma 9.1.4].

Lemma 1. *Let $\mathbb{E}$ be a separable metrizable space. Then there exists a totally bounded metric which is compatible with the topology of $\mathbb{E}$.*

We will use an old observation of Shafer [15, p. 829].

Lemma 2. *Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices of subsets of $\mathbb{E}_1, \mathbb{E}_2$ respectively, such that $\emptyset, \mathbb{E}_1 \in \mathcal{L}_1$ and $\emptyset, \mathbb{E}_2 \in \mathcal{L}_2$. If $c : \mathcal{L}_1 \to [0,1]$ is a completely monotone capacity and $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ is an $\cap$-homomorphism, i.e. $\varphi(\emptyset) = \emptyset$, $\varphi(\mathbb{E}_1) = \mathbb{E}_2$ and $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$ for all $A, B \in \mathcal{L}_1$, then $c \circ \varphi : \mathcal{L}_2 \to [0,1]$ is a completely monotone capacity.*

We also need an adaptation to containment functionals of the Choquet theorem.

Lemma 3. *Let* $\mathbb{E}$ *be a locally compact, second countable, Hausdorff space. Then, the formula*

$$P(X \subset F) = C(F) \quad \forall F \in \mathcal{F}'(\mathbb{E})$$

defines a bijection between the distributions P_X *of random closed sets in* $\mathbb{E}$ *and the outer continuous, completely monotone capacities* T *on* $\mathcal{F}'(\mathbb{E})$.

Proof. The Choquet theorem in the form given in, e.g., [10, p.122–123] identifies distributions of random closed sets X and inner continuous, completely alternating capacities T on $\mathcal{G}(\mathbb{E})$ by

$$P(X \cap G \neq \emptyset) = T(G).$$

It suffices to consider the dual capacity given by $C(A) = 1 - T(A^c)$ since

$$P(X \subset F) = 1 - P(X \cap F^c \neq \emptyset) = 1 - T(F^c).$$

This transformation maps inner continuous, completely alternating capacities to their dual capacities which are outer continuous and completely monotone instead. □

One can also retrieve Lemma 3 as a particular case of [17, Theorem 3.2]. Recall the *myopic topology* of $\mathcal{K}(\mathbb{E})$ is generated by the sets $\{A \in \mathcal{K}(\mathbb{E}) \mid A \cap G \neq \emptyset\}$ and $\{A \in \mathcal{K}(\mathbb{E}) \mid A \subset G\}$ for all $G \in \mathcal{G}(\mathbb{E})$.

Lemma 4. *Let* $\mathbb{E}$ *be a compact metric space. Then* $\mathcal{E}(\mathcal{F}(\mathbb{E}))$ *is the Borel* σ*-algebra generated by the myopic topology.*

Proof. Combine [8, Theorem 2.7.(iii), p. 29] with [8, Theorem C.5.(iii), p. 403].

Our interest in the myopic topology is due to the following theorem.

Lemma 5. *Let* $\mathbb{E}$ *be a Polish space. Let* P *be a probability measure on* $(\mathcal{K}(\mathbb{E}), \mathcal{B}(\mathcal{K}(\mathbb{E})))$ *(with the Borel* σ*-algebra of the myopic topology). Then, for each* $B \in \mathcal{B}(\mathbb{E})$*, the set* $\{K \in \mathcal{K}(\mathbb{E}) \mid K \subset B\}$ *is universally measurable. Also, letting* P_u *be the natural extension of* P *to* $\mathcal{B}_u(\mathcal{K}(\mathbb{E}))$*, the identity*

$$c(B) = P_u(\{K \in \mathcal{K}(\mathbb{E}) \mid K \subset B\})$$

defines an outer continuous, completely monotone capacity $c : \mathcal{B}(\mathbb{E}) \to [0,1]$ *such that*

$$c(B) = \sup_{\substack{K \in \mathcal{K}(\mathbb{E}) \\ K \subset B}} c(K) = \inf_{\substack{G \in \mathcal{G}(\mathbb{E}) \\ B \subset G}} c(G). \tag{1}$$

Moreover, the restriction of c *to* $\mathcal{G}(\mathbb{E})$ *is inner continuous.*

Proof. The result is a combination of material from [14], which in turn relies heavily on [3]. Universal measurability is [14, Lemma 1], complete monotony and outer continuity follow from [14, Theorem 2], and (1) from [14, Theorem 1]. Inner continuity on open sets follows then as observed in [14, Proposition 1]. □

Finally, we will also use the following result of Frolík [4, Proposition 7.11].

Lemma 6. *Let* $\mathbb{E}$ *be a Lusin space. If* $\mathbb{F}$ *is a metric space and* $e : \mathbb{E} \to \mathbb{F}$ *embeds* $\mathbb{E}$ *homeomorphically into* $\mathbb{F}$*, then* $e(\mathbb{E}) \in \mathcal{B}(\mathbb{F})$.

4 Main result

In this section, we state and prove the Choquet theorem.

Theorem 7. *Let* $\mathbb{E}$ *be a metrizable Lusin space. Then, the formula*

$$P(X \cap G \neq \emptyset) = T(G) \quad \forall G \in \mathcal{G}(\mathbb{E})$$

establishes a bijection between the distributions P_X *of random closed sets in* $\mathbb{E}$ *and the inner continuous, completely alternating capacities* T *on* $\mathcal{G}(\mathbb{E})$.

Proof. Showing that the hitting functional of a random closed set is an inner continuous, completely alternating capacity involves basic properties of probabilities and is standard. The fact that it is a capacity is clear. Inner continuity is a consequence of the identity

$$\{X \cap \bigcup_n G_n \neq \emptyset\} = \bigcup_n \{X \cap G_n \neq \emptyset\}$$

and the continuity of probability measures for monotone sequences. Complete alternation is obtained by rewriting it as the statement that the probability of certain events is non-negative (see e.g. [10, p. 116] for details).

For the converse, let $T : \mathcal{G}(\mathbb{E}) \to [0,1]$ be an inner continuous, completely alternating capacity, and define the dual capacity $C : \mathcal{F}'(\mathbb{E}) \to [0,1]$ by $C(F) = 1 - T(F^c)$. It is outer continuous and completely monotone, instead of inner continuous and completely alternating.

Being the continuous image of a separable space, $\mathbb{E}$ is separable. By Lemma 1, it admits a totally bounded metric. The completion $\overline{\mathbb{E}}$ of $\mathbb{E}$ with that metric, being both totally bounded and complete, is a compact metric space.

For the sake of greater clarity, subsets of $\overline{\mathbb{E}}$ will be written in boldface and the complement of $\mathbf{A} \subset \overline{\mathbb{E}}$ will be denoted by $\overline{\mathbb{E}} \backslash \mathbf{A}$.

The natural embedding $e : \mathbb{E} \to \overline{\mathbb{E}}$ identifies homeomorphically $\mathbb{E}$ with $e(\mathbb{E})$. Let $e^{\leftarrow} : \mathcal{P}(\overline{\mathbb{E}}) \to \mathcal{P}(\mathbb{E})$ be the pre-image mapping given by

$$e^{\leftarrow}(\mathbf{A}) = \{x \in \mathbb{E} \mid e(x) \in \mathbf{A}\}.$$

By the continuity of e, we have $e^{\leftarrow}(\mathbf{F}) \in \mathcal{F}'(\mathbb{E})$ for each $\mathbf{F} \in \mathcal{F}'(\overline{\mathbb{E}})$.

Let $\overline{C} : \mathcal{F}'(\overline{\mathbb{E}}) \to [0,1]$ be given by $\overline{C}(\mathbf{F}) = C(e^{\leftarrow}(\mathbf{F}))$ for any $\mathbf{F} \in \mathcal{F}'(\overline{\mathbb{E}})$.

Clearly $\overline{C} = C \circ e^{\leftarrow}$ is outer continuous, and it is a completely monotone capacity by Lemma 2 since $e^{\leftarrow}$ is an $\cap$-homomorphism. Indeed, $e^{\leftarrow}(\emptyset) = \emptyset$, $e^{\leftarrow}(\mathcal{F}'(\overline{\mathbb{E}})) = \mathcal{F}'(\mathbb{E})$ (because each $F \in \mathcal{F}'(\mathbb{E})$ is $e^{\leftarrow}(\mathrm{cl}_{\overline{\mathbb{E}}} F)$) and

$$e^{\leftarrow}(\mathbf{F}_1 \cap \mathbf{F}_2) = \{x \in \mathbb{E} \mid e(x) \in \mathbf{F}_1 \cap \mathbf{F}_2\} = e^{\leftarrow}(\mathbf{F}_1) \cap e^{\leftarrow}(\mathbf{F}_2)$$

for all $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{F}'(\overline{\mathbb{E}})$.
The proof will proceed now by subsequently defining mappings X''', X'', X' and X, of which X will be the random closed set we need.
By Lemma 3, there is a random closed set X''' in $\overline{\mathbb{E}}$, defined on a measurable space endowed with a probability measure Q, such that

$$Q(X''' \subset \mathbf{F}) = \overline{C}(\mathbf{F}) \quad \forall \mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}).$$

It induces the distribution $Q_{X'''}$ on the measurable space $(\mathcal{F}(\overline{\mathbb{E}}), \mathcal{E}(\mathcal{F}(\overline{\mathbb{E}})))$. Endow $\mathcal{F}(\overline{\mathbb{E}})$ with the myopic topology (recall compact sets and closed sets coincide in $\overline{\mathbb{E}}$). By Lemma 4, $\mathcal{E}(\mathcal{F}(\overline{\mathbb{E}})) = \mathcal{B}(\mathcal{F}(\overline{\mathbb{E}}))$ so the Effros σ-algebra admits a universal completion $\mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}}))$. Let $(Q_{X'''})_u$ be the natural extension of $Q_{X'''}$ to $\mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}}))$ (note we are extending $Q_{X'''}$ by adding to $\mathcal{B}(\mathcal{F}(\overline{\mathbb{E}}))$ the universally null, thus $Q_{X'''}$-null, sets). Consider the identity mapping

$$X'' = \mathrm{id} : (\mathcal{F}(\overline{\mathbb{E}}), \mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}})), (Q_{X'''})_u) \to \mathcal{F}(\overline{\mathbb{E}}),$$

which is obviously a random closed set since $\mathcal{E}(\mathcal{F}(\overline{\mathbb{E}})) \subset \mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}}))$.
By Lemma 5, $\{X'' \subset \mathbf{B}\}$ is a measurable event for each $\mathbf{B} \in \mathcal{B}(\overline{\mathbb{E}})$, and there exists an outer continuous, inner continuous on $\mathcal{G}(\mathbb{E})$, completely monotone capacity $\hat{C} : \mathcal{B}(\overline{\mathbb{E}}) \to [0,1]$, such that

$$(Q_{X'''})_u(X'' \subset \mathbf{B}) = \hat{C}(\mathbf{B}) = \sup_{\substack{\mathbf{K} \in \mathcal{K}(\overline{\mathbb{E}}) \\ \mathbf{K} \subset \mathbf{B}}} \hat{C}(\mathbf{K})$$

$$= \sup_{\substack{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \\ \mathbf{F} \subset \mathbf{B}}} \hat{C}(\mathbf{F}) = \inf_{\substack{\mathbf{G} \in \mathcal{G}(\overline{\mathbb{E}}) \\ \mathbf{B} \subset \mathbf{G}}} \hat{C}(\mathbf{G})$$

for each $\mathbf{B} \in \mathcal{B}(\overline{\mathbb{E}})$.
We still have, for each $\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}})$,

$$\hat{C}(\mathbf{F}) = (Q_{X'''})_u(X'' \subset \mathbf{F}) = (Q_{X'''})_u(\{\mathrm{id} \cap \mathbf{F}^c \neq \emptyset\}^c)$$
$$= Q_{X'''}(\{\mathrm{id} \cap \mathbf{F}^c \neq \emptyset\}^c) = Q_{X'''}(\mathrm{id} \subset \mathbf{F}) = Q(X''' \subset \mathbf{F})$$
$$= \overline{C}(\mathbf{F}) = C(e^{\leftarrow}(\mathbf{F})).$$

Now define

$$X' = e^{\leftarrow}|_{\mathcal{F}(\overline{\mathbb{E}})} : (\mathcal{F}(\overline{\mathbb{E}}), \mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}})), (Q_{X'''})_u) \to \mathcal{F}'(\mathbb{E}).$$

Let us use Y to prove that X' is strongly measurable and $(Q_{X'''})_u(X' \subset F) = C(F)$ for each $F \in \mathcal{F}(\mathbb{E})$. Let $B \in \mathcal{B}(\mathbb{E})$. Then

$$\{X' \subset B\} = \{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \mid e^{\leftarrow}(\mathbf{F}) \subset B\} = \{X'' \subset \overline{\mathbb{E}} \backslash e(B^c)\}$$

(since e is injective, $e^{\leftarrow}(\mathbf{F}) \subset B$ if and only if $\mathbf{F}$ is disjoint from $e(B^c)$ i.e. a subset of $\overline{\mathbb{E}} \backslash e(B^c)$).
Now $e(B^c) \in \mathcal{B}(e(\mathbb{E}))$ because e is a homeomorphism and $B^c \in \mathcal{B}(\mathbb{E})$. By Lemma 6, $e(\mathbb{E}) \in \mathcal{B}(\overline{\mathbb{E}})$. We deduce $e(B^c) \in \mathcal{B}(\overline{\mathbb{E}})$, also $\overline{\mathbb{E}} \backslash e(B^c) \in \mathcal{B}(\overline{\mathbb{E}})$ and thus $\{X'' \subset \overline{\mathbb{E}} \backslash e(B^c)\} \in \mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}}))$. By the arbitrariness of B, the mapping X' is strongly measurable.

Fix an arbitrary $F \in \mathcal{F}(\mathbb{E})$. Then

$$(Q_{X'''})_u(X' \subset F) = (Q_{X'''})_u(X'' \subset \overline{\mathbb{E}} \backslash e(F^c)) = \sup_{\substack{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \\ \mathbf{F} \subset \overline{\mathbb{E}} \backslash e(F^c)}} C(e^{\leftarrow}(\mathbf{F})).$$

Since the quantity in the supremum depends on $\mathbf{F}$ only through $e^{\leftarrow}(\mathbf{F})$, we have

$$\sup_{\substack{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \\ \mathbf{F} \subset \overline{\mathbb{E}} \backslash e(F^c)}} C(e^{\leftarrow}(\mathbf{F})) = \sup_{\substack{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \\ e(e^{\leftarrow}(\mathbf{F})) \subset \overline{\mathbb{E}} \backslash e(F^c)}} C(e^{\leftarrow}(\mathbf{F})) = \sup_{\substack{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \\ e^{\leftarrow}(\mathbf{F}) \subset F}} C(e^{\leftarrow}(\mathbf{F})).$$

Since $\{\mathbf{F} \cap e(\mathbb{E})\}_{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}})} = \mathcal{F}(e(\mathbb{E}))$ and e is an homeomorphism onto its image, $\{e^{\leftarrow}(\mathbf{F})\}_{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}})} = \mathcal{F}(\mathbb{E})$ whence there is some $\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}})$ for which $e^{\leftarrow}(\mathbf{F})$ is exactly F (precisely, we can take $\mathbf{F}$ to be the closure in $\overline{\mathbb{E}}$ of $e(F)$). Since C is monotone, the supremum must be attained at that $\mathbf{F}$. Therefore

$$\sup_{\substack{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \\ e^{\leftarrow}(\mathbf{F}) \subset F}} C(e^{\leftarrow}(\mathbf{F})) = C(F).$$

Through that chain of identities we have proved $(Q_{X'''})_u(X' \subset F) = C(F)$ for an arbitrary non-empty closed $F \subset \mathbb{E}$, or equivalently

$$\begin{aligned}(Q_{X'''})_u(X' \cap G \neq \emptyset) &= 1 - (Q_{X'''})_u(X' \subset G^c) \\ &= 1 - C(G^c) = 1 - (1 - T((G^c)^c)) = T(G)\end{aligned}$$

for all $G \in \mathcal{G}(\mathbb{E})$.

Unfortunately X' may take on empty values, so we are still not done. Since

$$\begin{aligned}\{X' \neq \emptyset\} &= \{\mathbf{F} \in \mathcal{F}(\overline{\mathbb{E}}) \mid \mathbf{F} \cap e(\mathbb{E}) \neq \emptyset\} \\ &= \{X'' \cap e(\mathbb{E}) \neq \emptyset\} = \{X'' \subset \overline{\mathbb{E}} \backslash e(\mathbb{E})\}^c,\end{aligned}$$

the facts that X'' is strongly measurable and $\overline{\mathbb{E}} \backslash e(\mathbb{E}) \in \mathcal{B}(\overline{\mathbb{E}})$ (remember Lemma 6) imply $\{X' \neq \emptyset\} \in \mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}}))$. We can therefore take the trace measure space $(\Omega, \mathcal{A}, P)$ with the sample space $\Omega = \{X' \neq \emptyset\}$, the σ-algebra $\mathcal{A} = \{A \cap \Omega \mid A \in \mathcal{B}_u(\mathcal{F}(\overline{\mathbb{E}}))\}$ and the measure $P = (Q_{X'''})_u|_{\mathcal{A}}$. Indeed P is a probability measure since

$$P(\Omega) = (Q_{X'''})_u(X' \neq \emptyset) = (Q_{X'''})_u(X' \cap \mathbb{E} \neq \emptyset) = T(\mathbb{E}) = 1.$$

And still

$$P(X \cap G \neq \emptyset) = (Q_{X'''})_u(X' \cap G \neq \emptyset) = T(G)$$

for all $G \in \mathcal{G}(\mathbb{E})$, whence the proof is complete. □

5 Discussion

Since every Polish space is metrizable and Lusin, Theorem 7 solves the open problem discussed in Section 1.

In [11], Nguyen and Nguyen have presented a ‘negative version’ of the Choquet theorem in Polish spaces. They present a completely alternating

capacity on open sets of the Polish (separable Banach) space ℓ_2 which satisfies a limited variant of inner continuity but does not correspond to any random closed set in ℓ_2.
That is in fact compatible with Theorem 7, since their capacity only satisfies $G_n \nearrow G \Rightarrow T(G_n) \to T(G)$ under the additional assumption that $G_n \to G$ in the Hausdorff pseudometric (Nguyen and Nguyen emphasize that their result is *not* a counterexample to the Choquet theorem in Polish spaces with open test sets).
An example of a metric space which is Lusin but not Polish, relevant in the theory of fuzzy sets, is the levelwise L^p-type metric d_p in the space of fuzzy numbers [1]. Therefore, an M-estimator in that space [**?**] is an example of a random closed set covered by our version of the Choquet theorem.

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