Knotted solutions for linear and nonlinear theories: Electromagnetism and fluid dynamics

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ABSTRACT

We examine knotted solutions, the most simple of which is the "Hopfion", from the point of view of relations between electromagnetism and ideal fluid dynamics. A map between fluid dynamics and electromagnetism works for initial conditions or for linear perturbations, allowing us to find new knotted fluid solutions. Knotted solutions are also found to be solutions of nonlinear generalizations of electromagnetism, and of quantum-corrected actions for electromagnetism coupled to other modes. For null configurations, electromagnetism can be described as a null pressureless fluid, for which we can find solutions from the knotted solutions of electromagnetism. We also map them to solutions of Euler's equations, obtained from a type of nonrelativistic reduction of the relativistic fluid equations.

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1. Introduction

Solutions with knotted topological structures play an important role in various areas of physics, but in this paper we will concern ourselves with two, electromagnetism and fluid dynamics. Despite the simplicity of the theory, the basic knotted solution of free Maxwell electromagnetism was only found in [1,2], following an earlier work in [3]. There are null solutions, $\vec{E}^2 - \vec{B}^2 = 0$, as well as generically non-null solutions in Rañada's construction, both of which are explicitly time dependent. In [4], it was shown that we can construct more general null solutions obtaining (m, n) knotted structures, and in [5], new knotted solutions were found using conformal $SO(4, 2)$ transformations with complex parameters on known ones. For a review of this subject, and more complete references, see [6].

On the other hand, the theory of knots was actually developed in the 19th century based on knotted fluid lines, whose topological robustness was already discovered by Lord Kelvin, following the work of Helmholtz in 1858. The abstract study of knots and their evolution [7] is a fertile subject, for reviews see [8–10] and the book [11]. Remarkably, however, explicit theoretical solutions of fluid equations were very scarce, whereas experimental creation

of knots waited until a few years ago [12] (see also [13] for some numerical construction). Moffat [14] finally defined a "helicity" $\mathcal{H}_v$ for the fluid flow similar, as we will see, to a magnetic helicity for electromagnetism, and wrote some explicit solutions with $\mathcal{H}_v \neq 0$. The properties of these were studied in [15,16]. More solutions were found in [17–19]. A map of magnetohydrodynamics to just fluid dynamics was used extensively (for instance, [20–22]), and one also was able to show that in magnetohydrodynamics a defined "velocity of lines of force" $\vec{v}_p = (\vec{E} \times \vec{H})/H^2$ [23] can be measured, and in some cases ("frozen field condition") coincides with the velocity of the fluid transporting it (see for instance [22,24], the last also considering transporting the electromagnetic Hopfion).

In this letter, we will use connections between electromagnetism and ideal fluid dynamics to find both new knotted solutions in electromagnetism, as well as new (time dependent) knotted solutions in fluid dynamics, that we believe have not been written explicitly before.

2. Knots in electromagnetism

In this section we review electromagnetic knotted solutions and some of their properties. Using the Riemann–Sielberstein (RS) vector $\vec{F} = \vec{E} + i\vec{B}$ ($c = 1$), the source-free Maxwell's equations are

$$\vec{\nabla} \times \vec{F} = i\frac{\partial}{\partial t}\vec{F}; \quad \vec{\nabla} \cdot \vec{F} = 0. \tag{1}$$

Electromagnetic duality is manifest as a rotation $\vec{F} \rightarrow e^{i\phi}\vec{F}$. We will define electric and magnetic potentials as $\vec{F} = \vec{B}_e + i\vec{B}_m = \vec{\nabla} \times (\vec{A}_e + i\vec{A}_m)$. To characterize the non-trivial topology of electromagnetic fields, a common set of observables are the helicities, that give a measure of the mean value of the linking number of the electromagnetic field lines. The helicities $\mathcal{H}_{ab}$, $a, b = e, m$ are defined as

$$\mathcal{H}_{ab} = \int d^3x \vec{A}_a \cdot \vec{B}_b . \tag{2}$$

Calculating their time derivatives, we find that $\mathcal{H}_{ee}$ and $\mathcal{H}_{mm}$ are conserved if $\vec{E} \cdot \vec{B} = 0$ and $\mathcal{H}_{em}$ and $\mathcal{H}_{me}$ are conserved if $\vec{E}^2 - \vec{B}^2 = 0$. One can find solutions that have conserved helicities and contain "knotted" structures for the electric and magnetic fields, characterized by Hopf or winding number invariants of the field structures.

2.1. Knotted solutions in Bateman's construction

One way to obtain knotted solutions is in Bateman's construction, imposing an ansatz

$$\vec{F} = \vec{\nabla}\alpha \times \vec{\nabla}\beta , \tag{3}$$

with $\alpha, \beta \in \mathbb{C}$. One Maxwell equation, $\vec{\nabla} \cdot \vec{F} = 0$, is automatically satisfied, and the other imposes the constraint $\vec{F}^2 = 0 \Rightarrow \vec{E}^2 - \vec{B}^2 = 0$ and $\vec{E} \cdot \vec{B} = 0$, so these are null solutions. Using this construction one can show that there is a "Hopfion" solution

$$\alpha = \frac{r^2 - t^2 - 1 + 2iz}{r^2 - t^2 + 1 + 2it}; \quad \beta = \frac{2(x - iy)}{r^2 - t^2 + 1 + 2it}. \tag{4}$$

Where (t, x, y, z) are the spacetime coordinates and $r^2 = x^2 + y^2 + z^2$. The Hopfion has $\mathcal{H}_{em} = \mathcal{H}_{me} = 0$ and nonzero helicities $\mathcal{H}_{mm} = \mathcal{H}_{ee}$, but their value depends on the amplitude of the electromagnetic field $|\vec{E}|$. An invariant that only depends on the topology can be constructed by introducing a map to unit quaternions $q = (\alpha + \beta j)/\sqrt{|\alpha|^2 + |\beta|^2} \in \mathbb{H}$ [5]. Using $\mathbb{R}^3 \cup \{\infty\} \cong S^3$ and $\mathbb{H}_{|q|^2=1} \cong SU(2) \cong S^3$, at any fixed time the "Hopfion" solution maps $S^3 \rightarrow S^3$ with unit winding number $w = 1$. By replacing α with α^m and β with β^n in (3), we find a (m, n) knot solution with winding number $w = mn$.

2.2. Knotted solutions in Rañada's construction

There are more general, non-null ($\vec{E}^2 - \vec{B}^2 \neq 0$), knotted solutions. Consider the ansatz for $F_{\mu\nu}$ and $*F_{\mu\nu}$ of the type

$$F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{1}{(1 + \bar{\phi}\phi)^2} (\partial_\mu \bar{\phi} \partial_\nu \phi - \partial_\nu \bar{\phi} \partial_\mu \phi) ,$$

$$*F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{1}{(1 + \bar{\theta}\theta)^2} (\partial_\mu \bar{\theta} \partial_\nu \theta - \partial_\nu \bar{\theta} \partial_\mu \theta). \tag{5}$$

The condition $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, determines the equations for ϕ, θ. The solutions then solve Maxwell's equations, and by construction $F_{\mu\nu} * F^{\mu\nu} \propto \vec{E} \cdot \vec{B} = 0$. Moreover, the 2-form F decomposes as $F = dq \wedge dp$ (Clebsch representation), where p, q are real functions, and in particular

$$\vec{B} = \vec{\nabla}p \times \vec{\nabla}q . \tag{6}$$

The relation between p, q and ϕ is

$$p = \frac{1}{1 + |\phi|^2}; \quad q = \frac{arg(\phi)}{2\pi} , \tag{7}$$

and there is a similar one for $*F = du \wedge dv$, with u, v in terms of θ. If p and q are single-valued and well-defined in the whole of space, then the magnetic helicity $\mathcal{H}_{mm}$ is zero. If not, we have

$$\vec{A} = p\vec{\nabla}q + \vec{\nabla}\chi , \tag{8}$$

where χ is such that $\vec{A}$ is well defined. The magnetic helicity is

$$\mathcal{H}_{mm} = \int d^3x \vec{\nabla}\chi \cdot (\vec{\nabla}p \times \vec{\nabla}q). \tag{9}$$

Defining the function

$$\phi_H(x, y, z) = \frac{2(x + iz)}{2z + i(r^2 - 1)}, \tag{10}$$

the "Hopfion" solution in Rañada's construction is such that, at $t = 0$,

$$\phi(x, y, z) = \phi_H(z, -y, x), \quad \theta(x, y, z) = \phi_H(x, z, -y). \tag{11}$$

The solution has $\mathcal{H}_{ee} = \mathcal{H}_{mm} \neq 0$ and $\mathcal{H}_{em} = \mathcal{H}_{me} = 0$. A more general set of solutions with non-zero helicities [25] are, at $t = 0$,

$$\phi = \frac{(x + iy)^{(n)}}{(z + i(r^2 - 1)/2)^{(m)}} \quad \theta = \frac{(y + iz)^{(l)}}{(z + i(r^2 - 1)/2)^{(s)}} , \tag{12}$$

where the (m) index means we leave the modulus intact, but we raise the phase to the m-th power.

2.3. Hopf index

Using $\mathbb{R}^3 \cup \{\infty\} \cong S^3$ and $\mathbb{C} \cup \{\infty\} \cong S^2$, at any fixed time ϕ can be seen as a map $S^3 \rightarrow S^2$, and the magnetic helicity is its Hopf index $\mathcal{H}(\phi)$. Indeed, we can define an area 2-form on S^2,

$$\omega = \frac{1}{2\pi i} \frac{d\phi^* \wedge d\phi}{(1 + |\phi|^2)^2} , \tag{13}$$

whose pullback onto S^3 is a 2-form $\mathcal{F}$ whose components are the spatial components in (5) (taking $a = 1/4$) $\mathcal{F}_{ij} = F_{ij}$. The Hopf index then equals the magnetic helicity

$$\mathcal{H}(\phi) = \int_{S^3} \mathcal{A} \wedge \mathcal{F} = \int d^3x \vec{A} \cdot \vec{B} = \mathcal{H}_{mm} . \tag{14}$$

A Hopf index $\mathcal{H}(\phi_H) = 1$ is found for instance for (10).

3. Solutions to nonlinear theories

We now show that the knotted solutions in Bateman's construction are also solutions of any nonlinear electromagnetism theory that reduces to the Maxwell case for small fields. This is true in general for any null configurations (for recent discussions see e.g. [26,27]). Since $\vec{F}^2 = 0$, both the two possible Lorentz invariants constructed out of $\vec{E}$ and $\vec{B}$ vanish on the solutions. Let us define

$$L \equiv \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2} = \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2);$$

$$P \equiv \frac{1}{8b^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{1}{b^2}\vec{E} \cdot \vec{B} , \tag{15}$$

where b is a dimension 2 constant. We assume that fields vary on distances much larger than $b^{-1/2}$, so that we can ignore possible derivatives on $F_{\mu\nu}$. Then, nonlinear generalizations of electromagnetism are described by actions of the form

$$\mathcal{L} = b^2 \left[-\frac{L}{2} + \sum_{n \geq 2} \sum_{m \geq 0} c_{n,m} L^n P^m \right] , \tag{16}$$

i.e., which reduces to Maxwell at small fields, and has nonlinear corrections written solely in terms of L and P. This includes actions like the Born–Infeld Lagrangian, $\mathcal{L} = -b^2[\sqrt{1+L-P^2}-1]$ [28], introduced to get rid of diverging electric fields in electromagnetism, as well as actions obtained by integrating out other fields, like in the case of the one-loop Euler–Heisenberg Lagrangian for QED, where the fermions have been integrated out, $\mathcal{L} = b^2\left[-\frac{L}{2} + \beta b^2[L^2 + 7P^2]\right]$, where $b = m^2$ and $\beta = 2\alpha^2/45$ (m is the fermion mass and α the fine structure constant). This form also applies for *any higher loop* integration of the coupling to *any field* (see [29] for a review). Therefore knotted solutions are valid in the full quantum theory after integrating out the other fields. Moreover, this is not only true for usual quantum fields, but also string modes in string theory can be integrated out to give the same result. Indeed, for electromagnetism confined to a D-brane in string theory, the integrating out of higher modes (in $\alpha' = l_s^2$, the string scale) results in the BI action with $b = 1/\alpha'$. Note that in this case, the action is α' exact at leading order in g_s, and there are no $\partial_\mu F_{\nu\rho}$ terms.

In terms of quantities analogous to the ones of electromagnetism in a medium (see [30]),

$$\vec{H} \equiv -\frac{\partial \mathcal{L}}{\partial \vec{B}}, \quad \vec{D} \equiv +\frac{\partial \mathcal{L}}{\partial \vec{E}}, \tag{17}$$

the equations of motion look formally the same as Maxwell's in a medium,

$$\begin{aligned} &\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0, \ \vec{\nabla} \cdot \vec{B} = 0 \\ &\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = 0, \ \vec{\nabla} \cdot \vec{D} = 0. \end{aligned} \tag{18}$$

For solutions such that $L = P = 0$, $\vec{H}$ reduces to $\vec{B}$ and $\vec{D}$ to $\vec{E}$, so we have Maxwell's equations in vacuum and thus knotted solutions found using Bateman's construction in Maxwell theory are solutions of the nonlinear theory as well.

4. Mapping electromagnetic to fluid knots

An ideal fluid, for an adiabatic flow with potential per unit mass π is governed by the Euler's and continuity equations,

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\pi, \tag{19}$$

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0.$$

The right hand side of the Euler's equations (19) equals $-\vec{\nabla}h$ ($\delta h = \frac{\delta p}{\rho} + \delta\pi$), where h is the enthalpy per unit mass. The continuity equation can also be rewritten as

$$\partial_t h + \vec{v} \cdot \vec{\nabla}h + c_s^2 \vec{\nabla} \cdot \vec{v} = 0, \tag{20}$$

where $c_s = \sqrt{\partial p/\partial \rho}$ is the sound speed.

For incompressible fluids $\vec{\nabla} \cdot \vec{v} = 0$ and compressible barotropic fluids $p = p(\rho)$, there is also a conserved helicity. From the Euler's equations (19), one finds

$$\partial_t(\vec{v} \cdot \vec{\omega}) - \vec{\nabla} \cdot \left[\vec{\omega}\left(\frac{\vec{v}^2}{2} - \int \frac{dp}{\rho} - \pi\right) - \vec{v}(\vec{v} \cdot \vec{\omega})\right] = 0, \tag{21}$$

where $\vec{\omega} = \vec{\nabla} \times \vec{v}$ is the vorticity, which means that we have the conserved *fluid helicity*, the integral of the velocity Chern–Simons term,

$$\mathcal{H}_v = \int d^3x \, \vec{v} \cdot \vec{\omega} = \int d^3x \, \vec{v} \cdot (\vec{\nabla} \times \vec{v}). \tag{22}$$

Knotted fluid solutions are solutions for which there is a linking of $\vec{v}(t, \vec{x})$ at fixed time t, i.e. nonzero fluid helicity.

We see the analogy with electromagnetism [31,32]: $\vec{v}$ is the analog of $\vec{A}$, so $\vec{B}$ is the analog of the vorticity $\vec{\omega}$, and $\mathcal{H}_v$ is the analog of $\mathcal{H}_{mm}$. One can define Clebsch variables λ and μ for the fluid, as in [33], giving the velocity field

$$\vec{v} = \lambda \vec{\nabla}\mu + \vec{\nabla}\Phi, \tag{23}$$

where Φ is the fluid potential, just like the decomposition for $\vec{A}$ in (8) in the Rañada construction. Note that this is not the much-used map from magneto-hydrodynamics (fluid coupled to electromagnetism) to hydrodynamics, where $\vec{v}$ is mapped to $\vec{v}$, but $\vec{\omega}$ is mapped to $\vec{B}$, and one restricts the configurations to the ones with $\vec{B} = \vec{\omega} = \vec{\nabla} \times \vec{v}$. Instead, we can define a full map, from fluid to electromagnetism, by

$$\vec{B} = \vec{\omega} = \vec{\nabla} \times \vec{v}, \quad \vec{E} = -\partial_t \vec{v} - \vec{\nabla}h, \tag{24}$$

which means that really $\vec{A} = \vec{v}$; $A_t = h$, though we have no gauge invariance now, since $\vec{v}$ is physical (observable). Two of Maxwell's equations $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0$ are automatically satisfied, but from the Euler's equations (19) we find the condition

$$\vec{E} = (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\omega} \times \vec{v} + \vec{\nabla}\left(\frac{\vec{v}^2}{2}\right) = \vec{B} \times \vec{A} + \vec{\nabla}\left(\frac{\vec{A}^2}{2}\right), \tag{25}$$

which does not hold for knotted solutions of Maxwell's equations. At the linearized level however, it implies $\vec{E} = 0$, i.e., pure magnetism, and the continuity equation, in electromagnetic variables

$$\partial_t A_t + (\vec{A} \cdot \vec{\nabla})A_t + c_s^2 \vec{\nabla} \cdot \vec{A} = 0, \tag{26}$$

becomes the Lorenz gauge condition, identifying c_s with the speed of light.

Rather than mapping the full solution, we will use the map between electromagnetism and fluid variables at a fixed time

$$\vec{v}(t=0, \vec{x}) = \vec{A}(t=0, \vec{x}), \ h(t=0, \vec{x}) = h_0. \tag{27}$$

Where h_0 is an arbitrary constant. In this case the knotted solutions supply initial conditions for Euler's and continuity equations. The time-dependent solutions have non-zero helicity and correspond to a fluid configurations of non-trivial topology.

In [33], knotted solutions for an incompressible fluid were (implicitly) found by giving an initial condition of the form (23), where $\lambda = \cos\vartheta(\vec{x}), \mu = \varphi(\vec{x})$, with (ϑ, φ) the polar and azimuthal angles of a S^2. Φ would be determined by the incompressibility condition, but finding an explicit solution is the main obstacle to obtaining an analytic expression for the velocity. We will avoid this issue by considering more general cases of compressible barotropic fluids, so the only constraint on Φ is that the velocity should be a smooth function of the spatial coordinates. Knotted solutions in Rañada's construction can be mapped for instance using the Clebsch decompositions (8) and (23), making $\lambda = p$, $\mu = q$ and $\Phi = \chi$, where p, q are given by (7) and ϕ can be taken to be (11) or (12). In terms of ϕ, the velocity is

$$\vec{v} = \frac{1}{4\pi i(1+|\phi|^2)}\left(\frac{\vec{\nabla}\phi}{\phi} - \frac{\vec{\nabla}\phi^*}{\phi^*}\right) + \vec{\nabla}\Phi. \tag{28}$$

In addition, we present here an additional set of initial conditions for knotted solutions. Consider the stereographic projection of S^3 on $\mathbb{R}^3$:

$$X_i = \frac{2x_i}{1+r^2}, \quad X_4 = \frac{1-r^2}{1+r^2}. \tag{29}$$

We define the complex coordinates $Z_1 = X_1 + iX_2$, $Z_2 = X_3 + iX_4$, that then we use them to define a ϕ

$$\phi = \frac{Z_1^n}{Z_2^m}, \; n, m \geq 1, \;\; n, m \in \mathbb{Z} \,. \qquad (30)$$

A non-singular velocity is obtained for

$$\Phi = -\frac{n}{2\pi} \arctan\left(\frac{y}{x}\right). \qquad (31)$$

The helicity of these configurations is $\mathcal{H}_v = -nm$.

5. Electromagnetism as a fluid and its knotted solutions

Any gapless quantum system is expected to have an effective fluid description at low energies. For a relativistic system this means that the energy–momentum tensor can be put in the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(\eta_{\mu\nu} + u_\mu u_\nu) + \pi_{\mu\nu} \,, \qquad (32)$$

where u_μ is the 4-velocity of the fluid, ρ is the energy density and p the pressure. $\pi_{\mu\nu}$ depends on derivatives of u_μ, ρ and p. From $\partial_\mu T^{\mu\nu} = 0$ we obtain the relativistic fluid equations. This program can be applied to any quantum system as in [34–37]. For a perfect fluid $\pi_{\mu\nu} = 0$ we obtain in the non-relativistic limit $u^\mu \simeq (1, \vec{v})$, $|\vec{v}|^2 \ll 1$, $p \ll \rho$ the continuity and Euler's equations. However, in electromagnetism we can also find another, more unusual, fluid, a "null pressureless fluid", or "null dust", with $p = 0$ and $u^\mu u_\mu = 0$.

As has been observed in [38], this is the case for null configurations with $\vec{E}^2 - \vec{B}^2 = 0$ and $\vec{E} \cdot \vec{B} = 0$. In this case the electromagnetic energy–momentum tensor takes the form $T_{\mu\nu} = \rho u_\mu u_\nu$, with $u_\mu = (1, \vec{v})$ ($\vec{v}^2 = 1$) and

$$\rho = \frac{1}{2}(\vec{E}^2 + \vec{B}^2); \quad \vec{v} = \frac{1}{\rho}\vec{E} \times \vec{B} \,. \qquad (33)$$

The Hopfion solution determined by (3) and (4) is null, so we can map it to fluid dynamics. We find an energy density

$$\rho = \frac{16\left((t-z)^2 + x^2 + y^2 + 1\right)^2}{\left(\left(r^2 - t^2 + 1\right)^2 + 4t^2\right)^3}. \qquad (34)$$

The velocity field is

$$v_x = \frac{2(y + x(t-z))}{1 + x^2 + y^2 + (t-z)^2}, \; v_y = \frac{-2(x - y(t-z))}{1 + x^2 + y^2 + (t-z)^2}, \quad (35)$$

and $v_z^2 = 1 - v_x^2 - v_y^2$. The topological structure can be seen in Fig. 1. Although the fluid helicity $\mathcal{H}_v$ diverges due to non-vanishing contributions at spatial infinity, these can be subtracted to produce a finite nonzero result. The velocity of the null fluid can be rewritten in terms of the RS vector $\vec{F}$ and its complex conjugate $\vec{F}^*$, as

$$\vec{v} = 2i\vec{F} \times \vec{F}^*/\vec{F} \cdot \vec{F}^*. \qquad (36)$$

This implies that (m, n) knots, and more general null solutions found by the replacement in (3) $\alpha \to f(\alpha)$, $\beta \to g(\beta)$, have the same velocity as the Hopfion. On the other hand, the energy density ρ is different for each configuration.

5.1. Maps to non-relativistic fluids

Switching to lightcone coordinates, $x^\pm = t \pm z$, the velocity of the Hopfion solution (35), is independent of x^+, in such a way that it can be mapped to a solution of a non-relativistic $2 + 1$-dimensional system, with $\tau = x^-$ playing the role of time coordinate. Defining β^a by $\beta^a = \frac{v^a}{1 - v^z}$, $a = x, y$, for any configuration satisfying $\partial_+ v^a = 0$, the relativistic fluid equations become the
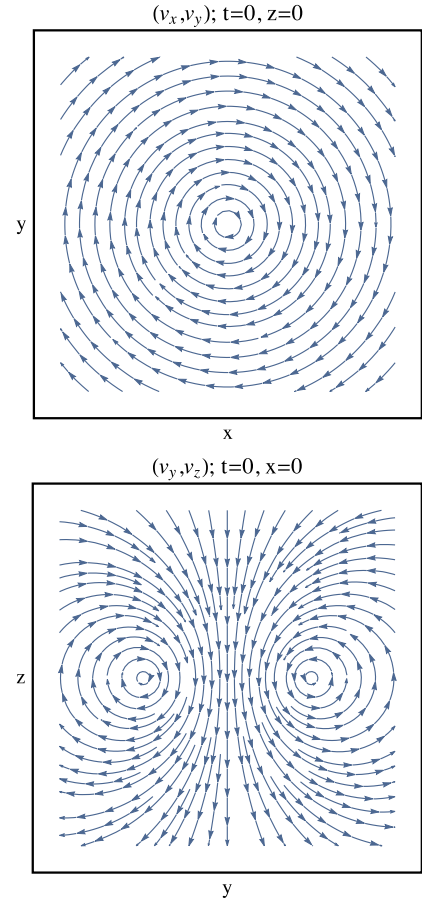

Fig. 1. Orthogonal sections of the velocity field for the Hopfion solution, on the (x, y) plane (top) and (y, z) plane (bottom). Using rotational symmetry in the (x, y) directions the linked torus structure is apparent.

$2 + 1$-dimensional Euler's equations of a compressible fluid with velocity β^a and constant pressure[1]:

$$\partial_\tau \beta^a + \beta^b \partial_b \beta^a = 0. \qquad (37)$$

In this description, the Hopfion solution is a spherical bouncing shock

$$\beta^a = \epsilon^{ab} \partial_b \tilde{\psi} + \tau \partial^a \tilde{\psi}, \quad \tilde{\psi} = \log(x^2 + y^2 - 1 - \tau^2). \qquad (38)$$

If $\partial_+ v^a = 0$, it is also possible to establish a map to an *incompressible* fluid. The relativistic fluid equations are

$$(1 - v^z)\partial_\tau v^a + v^b \partial_b v^a = 0. \qquad (39)$$

At any fixed τ we can make an identification of the first term with the pressure in Euler's equations

$$\partial^a p \equiv (1 - v^z)\partial_\tau v^a \,. \qquad (40)$$

Then v^a, p map to *steady state* (time-independent) solutions of Euler's equations, with τ seen as a parameter. The Hopfion solution can be written as

$$v^a = \epsilon^{ab} \partial_b \psi + \tau \partial^a \psi, \quad \psi = \log(1 + x^2 + y^2 + \tau^2) \,. \qquad (41)$$

At $\tau = 0$ the velocity satisfies the incompressibility condition $\partial_a v^a = 0$. In this case, an explicit solution to (40) can be found

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[1] This map should not be confused with other non-relativistic limits of electromagnetism where the speed of light is taken to infinity [39].

$$p = p_\infty - \frac{2}{1 + x^2 + y^2}. \tag{42}$$

This solution is a smooth vortex, it can be obtained by the stereographic projection of a constant vorticity configuration on the sphere (see e.g. [18]). This map can also be used to give initial conditions, at $\tau = 0$, for the null fluid, given a steady state (v_S^a, p_S) solution in $2 + 1$ dimensions, by

$$v^a(\tau = 0) = v_S^a, \quad \partial_\tau v^a(\tau = 0) = \frac{\partial^a p_S}{1 - v_S^2}. \tag{43}$$

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References

[1] A.F. Ranada, A topological theory of the electromagnetic field, Lett. Math. Phys. 18 (1989) 97.
[2] A.F. Ranada, Knotted solutions of the maxwell equations in vacuum, J. Phys. A, Math. Gen. 23 (16) (1990) L815.
[3] A. Trautman, Solutions of the Maxwell and Yang–Mills equations associated with Hopf fibrings, Int. J. Theor. Phys. 16 (1977) 561.
[4] H. Kedia, I. Bialynicki-Birula, D. Peralta-Salas, W.T.M. Irvine, Tying knots in light fields, Phys. Rev. Lett. 111 (2013) 150404, arXiv:1302.0342 [math-ph].
[5] C. Hoyos, N. Sircar, J. Sonnenschein, New knotted solutions of Maxwell's equations, J. Phys. A 48 (25) (2015) 255204, arXiv:1502.01382 [hep-th].
[6] M. Arrayás, D. Bouwmeester, J. Trueba, Knots in electromagnetism, Phys. Rep. 667 (2017) 1–61.
[7] T. Kambe, T. Takao, Motion of distorted vortex rings, J. Phys. Soc. Jpn. 31 (2) (1971) 591–599.
[8] R.L. Ricca, M.A. Berger, Topological ideas and fluid mechanics, Phys. Today 49 (12) (1996) 28–34.
[9] R.L. Ricca, Applications of knot theory in fluid mechanics, Banach Cent. Publ. 42 (1998) 321–346.
[10] R.L. Ricca, New developments in topological fluid mechanics, Nuovo Cimento Soc. Ital. Fis. C 32 (1) (2009) 185.
[11] V.I. Arnold, B.A. Khesin, Topological Methods in Hydrodynamics, vol. 125, Springer Science & Business Media, 1999.
[12] D. Kleckner, W.T. Irvine, Creation and dynamics of knotted vortices, Nat. Phys. 9 (4) (2013) 253–258.
[13] D. Proment, M. Onorato, C.F. Barenghi, Vortex knots in a Bose–Einstein condensate, Phys. Rev. E 85 (3) (2012) 036306.
[14] H.K. Moffatt, The degree of knottedness of tangled vortex lines, J. Fluid Mech. 35 (01) (1969) 117–129.
[15] H. Moffatt, A. Tsinober, Helicity in laminar and turbulent flow, Annu. Rev. Fluid Mech. 24 (1) (1992) 281–312.
[16] H. Moffatt, R.L. Ricca, Helicity and the Calugareanu invariant, Proc. R. Soc. Lond. A Math. Phys. Eng. Sci. 439 (1992) 411–429.
[17] A. Morgulis, V.I. Yudovich, G.M. Zaslavsky, Compressible helical flows, Commun. Pure Appl. Math. 48 (5) (1995) 571–582.
[18] D.G. Crowdy, Stuart vortices on a sphere, J. Fluid Mech. 498 (2004) 381402.
[19] E. Ifidon, E. Oghre, Vortical structures on spherical surfaces, J. Niger. Math. Soc. 34 (2) (2015) 216–226.
[20] A.L. Kholodenko, Optical knots and contact geometry I. From Arnold inequality to Ranada's dyons, Anal. Math. Phys. 6 (2) (2016) 163–198, arXiv:1402.1793 [math-ph].
[21] A.L. Kholodenko, Optical knots and contact geometry II. From Ranada dyons to transverse and cosmetic knots, Ann. Phys. 371 (2016) 77–124, arXiv:1406.6108 [math-ph].
[22] A.H. Boozer, Mathematics and Maxwell's equations, Plasma Phys. Control. Fusion 52 (12) (2010) 124002.
[23] W.A. Newcomb, Motion of magnetic lines of force, Ann. Phys. 3 (Apr. 1958) 347–385.
[24] W.T.M. Irvine, Linked and knotted beams of light, conservation of helicity and the flow of null electromagnetic fields, J. Phys. A, Math. Theor. 43 (38) (2010) 385203.
[25] M. Arrayas, J.L. Trueba, A class of non-null toroidal electromagnetic fields and its relation to the model of electromagnetic knots, J. Phys. A, Math. Theor. 48 (2015) 025203, arXiv:1106.1122 [hep-th].
[26] M. Ortaggio, V. Pravda, Class. Quantum Gravity 33 (11) (2016) 115010, arXiv:1506.04538 [gr-qc].
[27] E. Goulart, Nonlinear electrodynamics is skilled with knots, Europhys. Lett. 115 (1) (2016) 10004, arXiv:1602.05071 [gr-qc].
[28] M. Born, L. Infeld, Foundations of the new field theory, Proc. Roy. Soc. Lond. A 144 (1934) 425–451.
[29] G.V. Dunne, Heisenberg–Euler effective agrangians: basics and extensions, in: M. Shifman, A. Vainshtein, J. Wheater (Eds.), From Fields to Strings: Circum-navigating Theoretical Physics. Ian Kogan Memorial Collection (3 volume set), 2004, pp. 445–522, arXiv:hep-th/0406216.
[30] H. Nastase, J. Sonnenschein, More on Heisenberg's model for high energy nucleon–nucleon scattering, Phys. Rev. D 92 (2015) 105028, arXiv:1504.01328 [hep-th].
[31] H. Marmanis, Analogy between the Navier–Stokes equations and Maxwell's equations: application to turbulence, Phys. Fluids 10 (1998) 1428–1437.
[32] S. Sridhar, Turbulent transport of a tracer: an electromagnetic formulation, Phys. Rev. E 58 (1998) 522–525.
[33] E. Kuznetsov, A. Mikhailov, On the topological meaning of canonical Clebsch variables, Phys. Lett. A 77 (1) (1980) 37–38.
[34] H. Nastase, DBI scalar field theory for QGP hydrodynamics, Phys. Rev. D 94 (2) (2016) 025014, arXiv:1512.05257 [hep-th].
[35] S. Endlich, A. Nicolis, R. Rattazzi, J. Wang, The quantum mechanics of perfect fluids, J. High Energy Phys. 04 (2011) 102, arXiv:1011.6396 [hep-th].
[36] S. Dubovsky, L. Hui, A. Nicolis, D.T. Son, Effective field theory for hydrodynamics: thermodynamics, and the derivative expansion, Phys. Rev. D 85 (2012) 085029, arXiv:1107.0731 [hep-th].
[37] L. Berezhiani, J. Khoury, J. Wang, A Universe without dark energy: cosmic acceleration from dark matter–baryon interactions, arXiv:1612.00453 [hep-th].
[38] I. Bialynicki-Birula, Field theory of photon dust, Acta Phys. Pol. B 23 (1992) 553–559.
[39] G. Rousseaux, Forty years of Galilean Electromagnetism (1973–2013), Eur. Phys. J. Plus 128 (2013) 81.