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Teorías Gauge en 5 y menos dimensiones, holografía y resultados exactos

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RESUMEN (en español)

Durante los últimos años grandes progresos sobre la comprensión de teorías cuánticas de campos en un numero diferente de dimensiones han sido hechos. Esto ha sido posible haciendo uso de nuevas herramientas matemáticas que han sido descubiertas en Teoría de Cuerdas. En particular la correspondencia AdS/CFT nos ha permitido estudiar una teoría de campo en régimen de acoplo-fuerte usando su dual gravitatorio. Por otra parte, ha sido descubierto que es posible relacionar una teoría cuántica de campos con una específica configuración de Dp branas, dándonos un modo completamente distinto de estudiar dichas teorías.

En esta tesis vamos a aplicar las herramientas matemáticas anteriores a teorías cuánticas de campos en 5d con N= y teorías de campos supersimétrica en un número menor de dimensiones. Las teorías de campos en 5d son muy interesantes. De hecho, aunque dichas teorías no son renormalizable, para una elección del grupo gauge y materia pueden estar en punto fijo, el cual puede exhibir exóticos fenómenos como simetrías excepcionales.

Empezamos el estudio de los flujos del grupo de renormalización entre diferentes teorías en 5d con N=1 haciendo uso del correspondiente dual gravitacional. En particular podemos identificar dos tipologías distintas de flujos dependientes del operador particular que toma un valor esperado en la teoría de campo dual. De hecho, podemos ver que cuándo movemos las branas de la singularidad presente en la métrica el flujo está provocado por un operador mesonico, mientras, cuando eliminamos la singularidad haciendo un *blow-up*, el flujo esta provocado por un operador bariónico.

Por otra parte, hemos estudiado el límite de Nekrasov-Shatashvili (NS) del índice superconforme en 5d. En general dicho límite es singular. Por este motivo introducimos una prescripción que nos asegura que todos los coeficientes del índice, una vez que ha sido tomado el límite, son números enteros. Por otra parte, aplicamos la prescripción anterior a diferentes teorías con grupo de gauge U (1) y diferentes grupos de simetría global. Hemos visto que la nuestra prescripción para el limite NS reproduce el índice de Schur en 4d.

Estudiamos el límite rígido de la supergravedad euclídea conforme en 5d con N=2 sobre una variedad de Riemann. Mostramos que la condición necesaria y suficiente que asegura la existencia de una solución es la existencia de un vector de Killing conforme.

En la última parte examinamos el moduli space de instantones self-duales sobre CP^2. Por hacerlo usamos la correspondiente construcción ADHM, que puede ser puesta en una teoría de campo en 3d con N=2. En particular nos concentramos sobre los siguientes aspectos:

1) En general el *moduli space* de los instantones sobre CP^2 tiene unas cuantas direcciones compactas que no pueden ser caracterizadas haciendo uso del método usual basado en la computación del Serie de Hilbert para los operadores gauge invariantes de la teoría. Empezamos el estudio de dichas direcciones usando una *ungauging techinique*.

2) Haciendo uso de la correspondencia AdS/CFT obtenemos parte del *moduli space* de los instantones en el dual gravitacional.

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3) Aportamos la construcción ADHM para instantones sobre CP^2/Zn y hacemos la computación de la Serie de Hilbert correspondiente para grupo de simetría local unitario, ortogonal y simpléctico. Por otra parte, después de una oportuna identificación de los nodos del diagrama a quiver, obtenemos la Serie de Hilbert para el *moduli space* de instantones sobre C^2/Zn.

RESUMEN (en Inglés)

Great improvements in our understanding of quantum field theories in different numbers of dimensions have been performed during the last years. This has been possible using new mathematical tools that have been discovered in a String Theory context. First of all the AdS/CFT correspondence provided us the possibility to study a strongly-coupled QFT using its gravity dual. Moreover it was discovered that we can associate a particular Dp branes configuration to a QFT, providing us a completely new way to study these theories
In this thesis we apply the above mathematical tools in the context of 5d N=1 and lower dimensional supersymmetric QFTs. Among the others 5d theories are particularly interesting. As a matter of fact, even if these theories are not renormalizable, they exhibit a rich structure of symmetries and for a proper choice of the gauge group and of the flavour group can be at fixed point.
We begin the study of RG flows between different 5d N=1 theories using their gravity dual. In particular we are able to identify two different kinds of flow depending on the particular operator that acquires a VEV in the dual QFT. As a matter of fact we see that when we move the branes away from the singularity present in the metric the flow is triggered by a mesonic operator. On the other hand when we perform a blow-up of the singularity the flow is triggered by a baryonic operator.
Moreover we study the Nekrasov-Shatashvili (NS) limit of the 5d superconformal index. In general this limit is singular. Therefore we introduce a consistent prescription, such that all the coefficients of the expansion of the superconformal index, once the limit has taken, are integers. Moreover we apply the above prescription to various U(1) gauge theories with different global symmetry group. We find that our limit matches the 4d Schur index.
We study the rigid limit of 5d N=2 euclidean conformal supergravity on Riemannian manifolds. We show that the necessary and sufficient condition for the existence of a solution is the existence of a conformal Killing vector.
Finally we examine the moduli space of self-dual instantons on CP^2. In order to do this we use the corresponding ADHM-like construction, which can be embedded in a 3d N=2 QFT. In particular we focus on the following aspects:

1) In general the moduli space of self-dual instantons on CP^2 presents some compact directions that can not be characterized using the usual approach based on the computation of the Hilbert Series for the gauge invariant operators of the theory. We begin the study of the above directions using *an ungauging technique.*

2) Using the AdS/CFT we realize part of the instanton moduli space in the corresponding gravity dual.

3) We provide the ADHM-like construction for instantons on CP^2/Zn and we perform the computation of the corresponding Hilbert Series for unitary, orthogonal and symplectic gauge groups. Moreover, after a proper identifications of the nodes of the quiver, we recover the Hilbert Series for the moduli space of instantons on C^2/Zn.

SR. DIRECTOR DE DEPARTAMENTO DE FÍSICA
SR. PRESIDENTE DE LA COMISIÓN ACADÉMICA DEL PROGRAMA DE DOCTORADO EN FÍSICA FUNDAMENTAL Y APLICADA

Dedicated to my father.

Teorías Gauge en 5 y menos dimensiones, holografía y resultados exactos

Summary:

Great improvements in our understanding of quantum field theories in different numbers of dimensions have been performed during the last years. This has been possible using new mathematical tools that have been discovered in a String Theory context. First of all the AdS/CFT correspondence provided us the possibility to study a strongly-coupled QFT using its gravity dual. Moreover it was discovered that we can associate a particular Dp branes configuration to a QFT, providing us a completely new way to study these theories.

In this thesis we apply the above mathematical tools in the context of $5d$ $\mathcal{N} = 1$ and lower dimensional supersymmetric QFTs. Among the others $5d$ theories are particularly interesting. As a matter of fact, even if these theories are not renormalizable, they exhibit a rich structure of symmetries and for a proper choice of the gauge group and of the flavour group can be at fixed point.

We begin the study of RG flows between different $5d$ $\mathcal{N} = 1$ theories using their gravity dual. In particular we are able to identify two different kinds of flow depending on the particular operator that acquires a VEV in the dual QFT. As a matter of fact we see that when we move the branes away from the singularity present in the metric the flow is triggered by a mesonic operator. On the other hand when we perform a blow-up of the singularity the flow is triggered by a baryonic operator.

Moreover we study the Nekrasov-Shatashvili (NS) limit of the $5d$ superconformal index. In general this limit is singular. Therefore we introduce a consistent prescription, such that all the coefficients of the expansion of the superconformal index, once the limit has been taken, are integers. Moreover we apply the above prescription to various $U(1)$ gauge theories with different global symmetry group. We find that our limit matches the $4d$ Schur index.

We study the rigid limit of $5d$ $\mathcal{N} = 2$ euclidean conformal supergravity on Riemannian manifolds. We show that the necessary and sufficient condition for the existence of a solution is the existence of a conformal Killing vector.

Finally we examine the moduli space of self-dual instantons on $\mathbb{C}P^2$. In order to do this we use the corresponding ADHM-like construction, which can be embedded in a $3d$ $\mathcal{N} = 2$ QFT. In particular we focus on the following

aspects:

1. In general the moduli space of self-dual instantons on $\mathbb{C}P^2$ presents some compact directions that can not be characterized using the usual approach based on the computation of the Hilbert Series for the gauge invariant operators of the theory. We begin the study of the above directions using an *ungauging technique*.

2. Using the AdS_4/CFT_3 we realize part of the instanton moduli space in the corresponding gravity dual.

3. We provide the ADHM-like construction for instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ and we perform the computation of the corresponding Hilbert Series for unitary, orthogonal and symplectic gauge groups. Moreover, after a proper identifications of the nodes of the quiver, we recover the Hilbert Series for the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_n$.

Resumen:

Durante los últimos años han sido hechos grandes progresos sobre la comprensión de teorías cuánticas de campos en un numero diferente de dimensiones. Esto ha sido posible haciendo uso de nuevas herramientas matemáticas que han sido descubiertas en Teoría de Cuerdas. En particular la correspondencia AdS/CFT nos ha permitido estudiar una teoría de campo en régimen de acoplo-fuerte usando su dual gravitatorio. Por otra parte, ha sido descubierto que es posible relacionar una teoría cuántica de campos con una específica configuración de Dp branas, dándonos un modo completamente distinto de estudiar dichas teorías.

En esta tesis vamos a aplicar las herramientas matemáticas anteriores a teorías cuánticas de campos en $5d$ con $\mathcal{N} = 1$ y teorías de campos supersimétrica en un número menor de dimensiones. Las teorías de campos en $5d$ son muy interesantes. De hecho, aunque dichas teorías no son renormalizable, para una elección del grupo gauge y materia pueden estar en punto fijo, el cual puede exhibir exóticos fenómenos como simetrías excepcionales.

Empezamos el estudio de los flujos del grupo de renormalización entre diferentes teorías en $5d$ con $\mathcal{N} = 1$ haciendo uso del correspondiente dual gravitacional. En particular podemos identificar dos tipologías distintas de flujos dependientes del operador particular que toma un valor esperado en la teoría de campos dual. De hecho, podemos ver que cuándo movemos las branas de la singularidad presente en la métrica el flujo está provocado por un operador mesonico, mientras, cuando eliminamos la singularidad haciendo un *blow-up*, el flujo esta provocado por un operador bariónico.

Por otra parte, hemos estudiado el límite de Nekrasov-Shatashvili (NS) del índice superconforme en $5d$. En general dicho límite es singular. Por este motivo introducimos una prescripción que nos asegura que todos los coeficientes del índice, una vez que ha sido tomado el límite, son números enteros. Por otra parte, aplicamos la prescripción anterior a diferentes teorías con grupo de gauge $U(1)$ y diferentes grupos de simetría global. Hemos visto que la nuestra prescripción para el limite NS reproduce el índice de Schur en $4d$.

Estudiamos el límite rígido de la supergravedad euclídea conforme en $5d$ con $\mathcal{N} = 2$ sobre una variedad de Riemann. Mostramos que la condición necesaria y suficiente que asegura la existencia de una solución es la existencia de un vector de Killing conforme.

En la última parte examinamos el *moduli space* de los instantones self-duales sobre $\mathbb{C}P^2$. Por hacerlo usamos la correspondiente construcción ADHM, que puede ser puesta en una teoría de campos en $3d$ con $\mathcal{N} = 2$. En particular nos concentramos sobre los siguientes aspectos.

1. En general el *moduli space* de los instantones sobre $\mathbb{C}P^2$ tiene unas cuantas direcciones compactas que no pueden ser caracterizadas haciendo uso del método usual basado en la computación del Serie de Hilbert para los operadores gauge invariantes de la teoría. Empezamos el estudio de dichas direcciones usando una *ungauging techinique*.

2. Haciendo uso de la correspondencia AdS_4/CFT_3 obtenemos parte del *moduli space* de los instantones en el dual gravitacional.

3. Aportamos la construcción ADHM para instantones sobre $\mathbb{C}P^2/\mathbb{Z}_n$ y hacemos la computación de la Serie de Hilbert correspondiente para grupo de simetría local unitario, ortogonal y simpléctico. Por otra parte, después de una oportuna identificación de los nodos del diagrama a quiver, obtenemos la Serie de Hilbert para el *moduli space* de los instantones sobre $\mathbb{C}^2/\mathbb{Z}_n$

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Preface

This thesis has been submitted to the Faculty of Science, University of Oviedo, as a partial fulfilment of the requirements to obtain the PhD degree. The work presented here has been developed during the years 2013-2017 under the supervision of Dr. Diego Rodríguez-Gómez at the Department of Physics University of Oviedo. Moreover I also spent two months at Queen Mary University of London and one month at Theoretical Physics Department of the Imperial College of London during the Spring of 2015. This collaboration was very productive and resulted in the article number 4 (reported in the list of papers of the present thesis).

Thesis Objectives

The objectives of the present thesis concern the study of different aspects of quantum field theories in five and lower dimensions ($d = 3, 4$). In particular we focused on:

- The study of RG flows in $5d$ $\mathcal{N} = 1$ theories using their holographic duals.

- The computation of the Nekrasov-Shatashvili limit of the superconformal index of a $5d$ $\mathcal{N} = 1$ QFT. We discovered that, for a proper choice of the gauge and flavour groups, the previous limit matches the Schur limit of the superconformal index in $4d$.

- The characterization of the moduli space of self-dual instantons on $\mathbb{C}P^2$ whose dual ADHM-like construction is provided by a $3d$ $\mathcal{N} = 2$ QFT.

- The study of the rigid limit of $5d$ conformal supergravity on a Riemann manifold.

List of papers

1. A. Pini and D. Rodriguez-Gomez, *"Gauge/gravity duality and RG flows in 5d gauge theories,"* Nucl. Phys. B **884** (2014) 612[arXiv:1402.6155 [hep-th]].
2. A. Pini and D. Rodriguez-Gomez, "Aspects of the moduli space of instantons on $\mathbb{C}P^2$ and its orbifolds," Phys. Rev. D **93** (2016) [arXiv:1502.07876 [hep-th]].
3. A. Pini, D. Rodriguez-Gomez and J. Schmude, *"Rigid Supersymmetry from Conformal Supergravity in Five Dimensions",* JHEP **1509**, 118 (2015) [arXiv:1504.04340 [hep-th]].
4. C. Papageorgakis, A. Pini and D. Rodriguez-Gomez, *"Nekrasov-Shatashvili limit of the 5D superconformal index,"* Phys. Rev. D **94**, no. 4, 045007 (2016)[arXiv:1602.02647 [hep-th]]

The impact factors of the scientific journals where the above articles have been published are included in Table 1.

List of papers (Impact factors)

Journal	Year	Impact Factor	Area
Nuclear Physics B	2014	3.9292	Physics, Particles and Fields
Physical Review D	2015	4.506	Astronomy, Astrophysics, Physics, Particles and Fields
JHEP	2015	6.023	Physics, Particles and Fields

Table 1: Impact factors of the scientific journals where the articles of this thesis have been published. Source: Journal Citation Reports © 2016.

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1. Introduction

"Aveva sopra il capo [...] l'unico riscatto alla dannazione del panta rei, e pensava che fossero affari Suoi, e non suoi"

Umberto Eco, "Il pendolo di Foucault".

During the 20 th century two new physical theories completely changed our understanding of nature. These are General Relativity (GR) and Quantum Mechanics (QM). The first provided us a very good description of the Universe at a very large scale, and it drastically changed our understanding of the gravitational interaction, that has been reinterpreted as a curvature of the space-time. Furthermore it led to important physical predictions such as gravitational waves[1] and black holes.

On the other hand QM changed our understanding of nature at a very small distance. We had to change our mind for what concern the description of the subatomic world. Moreover it led to important experimental predictions and to the description of atomic spectra.

However, despite their empirical success, these two theories seem to be incompatible. A theory that is able to incorporate QM and GR in a unified way does not seem easy to achieve. The main problem that arises in the quantization of gravity is that the resulting theory is not renormalizable. In order to see this let's consider the Einstein-Hilbert action in four dimensions

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R. \tag{1.1}$$

If we require that the action (1.1) is dimensionless we see that the coupling constant that regulates the gravitational interaction G_N must have the dimension of $(\text{length})^2$. This has an importance consequence. Let's consider the physical processes reported in figure 1.1, that respectively describe the exchange of one graviton (figure 1.1 (a)) and two gravitons (figure 1.1 (b))

[1]Remarkably very recently there have been an experimental confirmation regarding the existence of gravitational waves [1].

between fermions. The amplitude for the first process, since it must be dimensionless, is proportional to G_N multiplied by the square of the energy scale E at which the process takes place. [2] On the other hand the second process (with the exchange of two gravitons) is proportional to G_N^2 multiplied by E^4. Therefore the amplitude for the first process is proportional to $(E/M_p)^2$, while the amplitude for the second process is proportional to $(E/M_p)^4$. This has two important consequences. First of all it means that at ordinary energy scale (i.e. $E \ll M_p$) gravity is irrelevant and can be neglected compared to other fundamental interactions (that have dimensionless coupling constants). That's why the Standard Model of Particle Physics, even if it does not include gravity has a great empirically success. However when $E/M_p \sim 1$ perturbation theory breaks down and we are not able to take care of the gravitational interaction! Historically this situation is analogous to the Fermi theory of weak interactions [2]. As a matter of fact also this theory was regulated by a dimensional coupling constant called G_F. Consequently this theory could be used to make predictions only at low energies, i.e. when $G_F/E \ll 1$. However in order to makes sense of this theory at high energies it was necessary to introduce an UV cut-off. This problem was solved with the discovery of new more fundamental particles: the W^+, W^- and Z^0 bosons and led to the current formulation of the Weak interactions. This analogy could suggest that also Einstein theory of general relativity could be considered as a low-energy effective theory and should be replaced by a new more fundamental theory at higher energy scales. Among the others String Theory is one of the most successful attempts that have been formulated in order to solve this problem. [3]

Historically string theory[4] arose in the 1960s as an attempt to describe strong interactions. The basic idea of the theory is that all the particles appear as different oscillation modes of a unique one-dimensional fundamental object, a *string*. In 1970 due to the discovery of Quantum Chromodynamics (QCD) string theory was ruled out as possible explanation of strong interactions. However, as it was observed, among other particles in the spectrum there was also a massless spin-2 particle, that can be identified with the graviton (the particle mediating the gravitational interaction). However, despite the possibility to incorporate gravity with the other interactions in a natural way, a consistent string theory was not easy to develop. First of all it was

[2]Note that for simplicity we work in natural unit with $c = \hbar = 1$. Moreover $G_N^{-1/2} = M_p = 1.22 \times 10^{19} GeV$.

[3]In this thesis we only consider the String Theory approach. We refer the interested reader to [3] for a overview of the different attempts that have been formulated in order to find a consistent theory of the gravitational interaction.

[4]For an exhaustive introduction to string theory we refer the reader to [4, 5, 6].

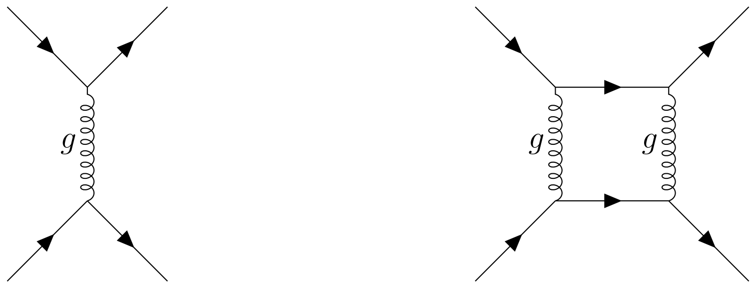

Figure 1.1: Feynamn diagrams showing the exchange of one gravition (a) and two gravitons (b).

understood that in order to reproduce the known spectrum of particles (including fermions) string theory requires supersymmetry. Moreover in order to be consistent string theory had to be define in $1 + 9$ dimensions.

In 1984 the so called *first string revolution* took place. It was discovered that a $\mathcal{N}$=1 consistent string theory was possible in ten space-time dimensions provided that the gauge group was chosen to be $SO(32)$ or $E_8 \times E_8$. These are called *Heterotic* string theories. At a later time it was understood that it was possible to construct consistent string theories in ten dimensions with the double of supercharges (i.e. $\mathcal{N} = 2$ theories). These are the so called *Type II A string theory* and *Type II B string theory*. Finally the last possible consistent theory is called *Type I string theory* and can be obtained starting from Type II B performing an *orientifold projection*.

At a first time the existence of five different consistent theories in $10d$ was a bit puzzling. However it was discovered that these theories were not independent, but were related by the so called *duality relations*. For example the type II A string theory and Type II B string theory as well as the two heterotic string theories are related by $T - duality$. T-duality implies that two different string theories defined on two different geometries can be physically equivalent. In the most simple example of T-duality a circle of radius R is equivalent to a circle of radius l_s^2/R (where l_s is the string length). Moreover during the so called *second superstring revolution* another kind of duality called *S-duality* was discovered. An S-duality transformation relates a string theory with coupling constant g_s [5] with a string theory whose coupling constant is $1/g_s$ and the two theories are physically equivalent. In particular S -duality relates Type I string theory to $SO(32)$ heterotic string theory and Type II B string theory to itself. Therefore, due to these duality transforma-

[5]The string coupling constant g_s is given by the vacuum expectation value of e^{ϕ}, where ϕ is a scalar field called the *dilaton*. Moreover string theory is regulated by the parameter α' that is proportional to the inverse of the string tension.

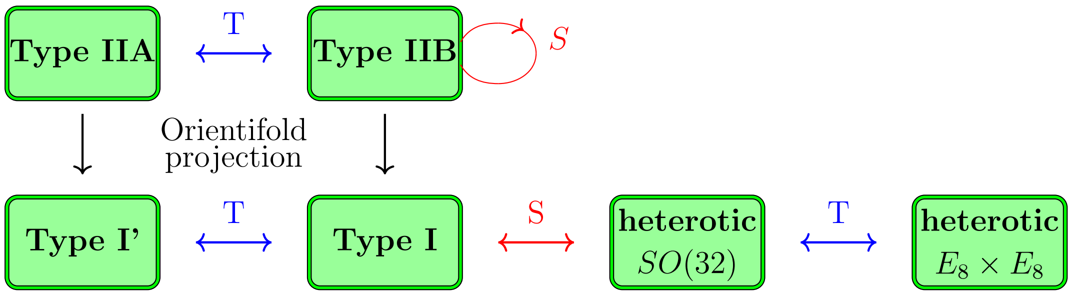

Figure 1.2: Connections between the different superstring theories using the dualities transformations.

tions, the five string theories that have been discovered should not be really regarded as distinct theories. We summarize the relations between the different string theories in figure 1.2. Moreover all these five string theories can be seen as different limits of a more fundamental, but still mysterious theory, called *M-theory*. The low-energy limit of M-theory is the eleven dimensional supergravity.

Moreover, in addition to strings, superstring theories contains also non perturbative objects called *Dp-branes*. All the branes decouple at low-energy, i.e. when $g_s \mapsto 0$. Therefore they do not appear in perturbation theory. A Dp brane can be defined as an object on which strings, that satisfy Dirichlet boundary conditions, can end. This implies that QFT can be realized on the world-volume of the branes. This led to the possibility to study a QFT in completely different way. As a matter fact, at least in some specific cases, we can characterize a given QFT studying the corresponding dual brane configuration (see e.g. [7] for a review of this topic).

Another important discovery arising from string theory is the *AdS/CFT correspondence* [8, 9]. In its strongest formulation it is a holographic strong-weak correspondence between string theory on an Anti de Sitter (AdS) d-dimensional background and a conformal field theory (CFT) in $d-1$ dimensions. A dictionary between the gravity theory and the QFT has been established. Fields on the gravity side are related to operators on the QFT side and local symmetry on the gravity side are related to global symmetry in the dual QFT. Moreover the correspondence has been extended also in the context of non conformal field theories.

In this thesis we apply some of the mathematical tools and physical results outlined above in order to study quantum fields theories in different numbers of dimensions. First of all we focus on $5d$ quantum field theories with $\mathcal{N}=1$ supersymmetry. Remarkably the mere existence of these theories is surprising. As a matter of fact these theories are non renormalizable and therefore it was believed that could be properly defined only through the introduction of an UV cut-off. However it was discovered in [10] that , for a proper choice of the gauge group and the matter content of the theory, some $5d$ supersymmetric field theories can be at fixed point. Moreover it was provided a brane construction that allows to embed and study the above theories in Hetoritic string theory [10, 11]. Alternatively these theories can be realized starting from M-theory and performing a dimensional reduction over a Calabi-Yau three-fold [12]. A surprising feature of these theories is that they exhibit an enhancement of the global symmetry group at the fixed point. In particular it was conjectured [10] that for a theory with $SU(2)$ gauge group and $N_f \leq 8$ flavours the global symmetry group is enhanced to E_{N_f+1} at the UV fixed point. Moreover the previous theories admit a gravity dual that allows to study some of their properties in a holographic way. In section 3 we review the main aspects related to these theories. In particular we examine how we can construct their gravity dual starting from a proper brane configuration and how the theories proposed in [10] can be generalized taking an orbifold. An interesting feature concerning these theories is the study of their RG flow. This can be triggered perturbing the initial theory by a relevant operator. In particular in [13] we begin the study of the holographic RG flow between different $5d$ $\mathcal{N} = 1$ QFTs classifying the operators that are taking a VEV according to their conformal dimension and other quantum numbers. We refer the reader to the article reported in section 6.1 for the full analysis of such RG flows.

The $5d$ theories introduced in [10] are defined on a flat background. Therefore it's natural to try to understand if the above theories can also be placed on a generic curved background preserving at least some part of the initial supersymmetry. A systemic procedure that allows to define a SUSY QFT on a curved manifold has been introduced by Seiberg and Festuccia [14]. We review the main aspects of this technique in section 4.3 and in [15] we applied it in the context of $5d$ $\mathcal{N} = 1$ QFTs. We found that the necessary and sufficient condition that must be satisfied in order to have a well defined supersymmetric theory on a Riemann manifold is the existence of a conformal Killing vector. The full analysis has been carried out in the article reported in section 6.3.

Among the others a particular interesting background on which we can define a $5d$ supersymmetric QFT is $\mathbb{S}^1 \times \mathbb{S}^4$. This manifold arises naturally in the computation of the superconformal index (SCI) of the theory, that can be expressed as a path integral over $\mathbb{S}^1 \times \mathbb{S}^4$. As we review in section 4.1 the SCI allows to count BPS operators of the theory according to their global symmetry quantum numbers, providing in this way a lot of information. In general it's also useful to consider particular limits of the SCI that allow to extract information arising form a particular subset of operators of the theory [16]. For example we review the so-called *Schur limit* in section 4.2.2. Among the others a particular interesting limit is the *Nekrasov-Shatashvili limit* of the $5d$ SCI. This limit is implemented turning off one of the two epsilon parameters defining the Ω *background* [17]. In [18] we examine such limit and we see that, at least in the case of Abelian theories, we recover the Schur limit of the $4d\,\mathcal{N}=2$ index. This computation has been performed in the article reported in section 6.4.

Moreover, as we review in section 4.1, the computation of the SCI using localization includes instanton corrections that can be studied characterizing the corresponding moduli space. We review the basic information regarding the instantons moduli space and how we can embed the mathematical ADHM construction [19] in string theory in section 5. Furthermore in section 5.1.3 we analyse how we can characterize the corresponding moduli space using the so called "Plethystic programe" [20]. In the particular case of the $5d$ $\mathcal{N}=1$ QFT considered in [10] the instanton contribution to the index can be studied through the characterization of the moduli space of instantons on $\mathbb{R}^4$. However as we discovered before we can define a consistent supersymmetric $5d$ theory also on a different kind of background. Therefore the study of the moduli space of instantons on different kind of four-dimensional manifold becomes of great importance. A particular interesting manifold, that has been considered in [21], is $\mathbb{C}P^2 \times \mathbb{S}^1$. In particular in [22] it was considered the ADHM construction for the moduli space of self-dual instantons on $\mathbb{C}P^2$. In [23] we extend such analysis making use of the notion of *resolved moduli space* [24, 25, 26] and we begin its its characterization from a physical point of view. Moreover we also consider a $\mathbb{Z}_n$ orbifold of the initial space and we provide the ADHM-like construction for instantons on $\mathbb{C}P^2/\mathbb{Z}_n$. The full description is reported in the article in section 6.2.

We report the various articles, on which this thesis is based in section 6. Finally we end up with some conclusions and possible future work in section 7.

2. Introducción

"Aveva sopra il capo [...] l'unico riscatto alla dannazione del panta rei, e pensava che fossero affari Suoi, e non suoi"

Umberto Eco, "Il pendolo di Foucault".

Durante el siglo veinte dos nuevas teorías han modificado completamente nuestra comprensión de la naturaleza. Dichas teorías son la Relatividad General (RG) y la Mecánica Cuántica (MC). La primera nos ha dado una comprensión muy precisa del Universo a una escala muy grande, y ha cambiado completamente nuestra interpretación de la interacción gravitacional, que es reinterpretada como curvatura del espacio-tiempo. Además, ha proporcionado nuevas importantes predicciones físicas como las ondas gravitacionales [1] y agujeros negros.

Por otra parte, la MC ha cambiado nuestra comprensión de la naturaleza a escala muy pequeña, modificando nuestra visión del mundo subatómico. Además, ha aportado importantes predicciones experimentales, como la descripción de los espectros de los átomos.

Sin embargo, a pesar del éxito experimental, dichas teorías parecen ser incompatibles. Una teoría que pueda incorporar MC y RG en un modo completamente unificado no parece que se pueda obtener de modo sencillo. El problema principal que se manifiesta cuando se intenta cuantizar la gravedad es que la correspondiente teoría no es renormalizable. Para entender esto consideramos la acción de Hilbert-Einstein en cuatro dimensiones

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R. \tag{2.1}$$

Si imponemos que la acción (2.1) sea adimensional podemos ver que la constante de acoplamiento que regula la interacción gravitacional G_N tiene que

[1] Notablemente recientemente tuvo lugar un importante confirma experimental sobre la existencia de ondas gravitacionales [1].

tener dimensión de $(longitud)^2$. Esto tiene una consecuencia muy importante. Consideremos el proceso físico reportado en la figura 2.1, que describe respectivamente el intercambio de un gravitón (figura 2.1 (a)) y dos gravitónes (figura 2.1 (b)) entre dos fermiones. La amplitud del primer proceso, dado que tiene que ser adimensional, es proporcional a G_N multiplicado por el cuadrado de la escala de energía E del proceso. [2]. En cambio el segundo proceso (con el intercambio de dos gravitones) es proporcional a G_N^2 multiplicado por E^4. Consecuentemente la amplitud del primer proceso es proporcional a $(E/M_p)^2$, mientras la amplitud para el segundo proceso a $(E/M_p)^4$. Esto tiene dos consecuencias muy importantes. La primera es que a una escala de energía ordinaria (es decir $E \ll M_p$) la gravedad es irrelevante y puede ser despreciada en comparación con las otras interacciones fundamentales (que tienen una constante de acoplamiento adimensional). Por este motivo el Modelo Estándar de la Física de Partículas, aunque no incluya la gravedad, tiene un gran éxito a nivel experimental. Sin embargo, cuando $E/M_p \sim 1$ no se puede hacer un desarrollo perturbativo de modo que no se puede estudiar la gravedad en el mismo modo que las otras interacciones! Históricamente esta situación es análoga a la teoría de Fermi de la interacción débil [2]. De hecho, esta teoría estaba controlada por una constante de acoplamiento dimensional llamada G_F. Consecuentemente esta teoría podía ser usada solo para hacer predicciones a baja energía, es decir cuando $G_F/E \ll 1$. Sin embargo, para poder usar esta teoría a energías más altas es necesario introducir un *cut-off* UV. Al final este problema fue solucionado con el descubrimiento de dos nuevas partículas fundamentales: los bosones W^+, W^- y Z^0 y ha llevado a la formulación actual de las interacciones débiles. Esta analogía puede sugerir que también la teoría de la relatividad general de Einstein pueda ser considerada como una teoría efectiva y pueda ser remplazada por una nueva teoría mas fundamental a una escala de energía más alta. Entre otras, la Teoría de Cuerdas es una de las propuestas que tuvieron más éxito para solucionar este problema. [3]

De un punto de vista histórico la teoría de cuerdas fue formulada en los años sesenta como tentativa para describir las interacciones fuertes. La idea de base de la teoría es que todas las partículas aparecen como diferentes modos de oscilación de un único objeto con una sola dimensión: una cuerda. En el 1970 debido al descubrimiento de la Cromo Dinámica Cuántica (QCD) la teoría de cuerdas fue excluida como posible explicación de las interacciones

[2] Por simplicidad usamos las unidades naturales con $c = \hbar = 1$ Por otra parte $G_N^{-1/2} = M_p = 1.22 \times 10^{19} GeV$.

[3] En esta tesis tomamos en consideración solo la Teoría de Cuerdas. Pero enviamos el lector interesado a [3] para una visión global de las diferentes teorías que han sido formuladas para explicar la interacción gravitacional.

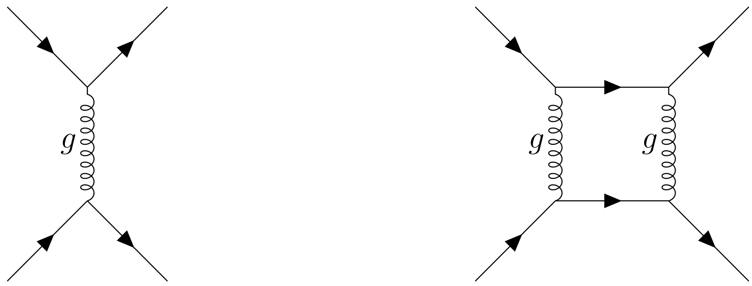

Figure 2.1: Diagramas de Feynman que describen el intercambio de un gravitón (a) y dos gravitones (b).

fuertes. Sin embargo entre otras partículas en el espectro había también una partícula con masa igual a cero y spin 2, y que puede ser identificada con el gravitón (la partícula que media la interacción gravitatoria). No obstante, encontrar una teoría de cuerdas completamente consistente a nivel cuántico no fue sencillo. En primer lugar, resulta que para reproducir el espectro de las partículas conocidas (incluyendo también los fermiones) la teoría de cuerdas necesitaba supersimetría. Por otra parte, para ser consistente respetando todas las simetrías la teoría tenía que ser definida en $1+9$ dimensiones.

En el 1984 tuvo lugar la primera revolución de la teoría de cuerdas. Fue descubierto que una teoría de cuerdas con $\mathcal{N}=1$ podía ser definida en diez dimensiones a condición que el grupo de simetría local de la teoría fuese $SO(32)$ o $E_8 \times E_8$. Dichas teorías se llaman *teorías de cuerdas Heteróticas de Tipo I*. Más adelante se entendió que era posible construir una teoría de cuerda consistente en diez dimensiones con un numero doble de supercargas (es decir $\mathcal{N}=2$). Dichas teorías se llaman teorías de cuerdas de *Tipo II A* y teorías de cuerdas de *Tipo II B*. Por ultimo, la teoría de cuerdas de *Tipo I* se puede obtener a partir de la teoría de cuerda de Tipo IIB haciendo una *orientifold projection*.

Al principio la existencia de cinco teorías de cuerdas consistentes en 10 dimensiones parecía problemática. Sin embargo, se vio que estas teorías no son independientes y están relacionadas por relaciones de dualidad. Por ejemplo, la teoría de Tipo IIA y la teoría de Tipo IIB así como las dos teorías de cuerdas heteróticas están relacionadas por *T-dualidad*. La T dualidad implica que dos teorías diferentes de cuerdas definidas sobre dos geometrías diferentes pueden ser equivalentes a nivel físico. En el caso más sencillo de T-dualidad, un circulo de radio R es equivalente a un circulo de radio l_s^2/R (donde l_s es la longitud de la cuerda). Por otra parte durante la segunda revolución de las supercuerdas ha sido descubierta otro tipo de dualidad llamada *S-dualidad*. Una transformación de S-dualidad pone en relación una

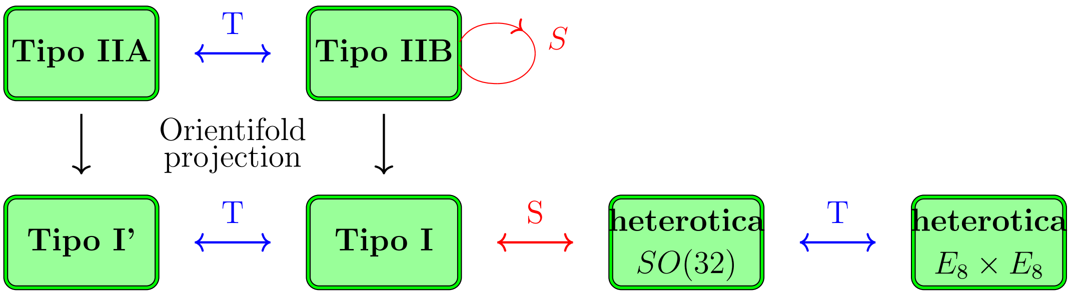

Figure 2.2: Conexiones entre las diferentes teorías de cuerdas usando las transformaciones de dualidad.

teoría de cuerda con constante de acoplamiento g_s[4] con una teoría de cuerda con constante de acoplamiento $1/g_s$ y las dos teorías son equivalentes de un punto de vista físico. En particular la S-dualidad pone en relación la teoría de Tipo I con la teoría heterótica con grupo de simetría local $SO(32)$ y la teoría de Tipo IIB con se misma. Por lo tanto, debido a dichas relaciones de dualidad, las cinco teorías de cuerdas que han sido descubiertas no tienen que ser consideradas como teorías distintas. Resumimos las relaciones entre las diferentes teorías de cuerdas en la figura 2.2. Por otra parte, las cinco teorías de cuerdas pueden ser vistas como diferentes límites de una teoría más fundamental, y aún misteriosa, conocida como *teoría M*. El límite de baja energía de la teoría M es la supergravedad en once dimensiones.

Por otra parte, además de las cuerdas, las teorías de supercuerdas tienen también objetos no perturbativos conocidos como Dp-branas. Todas las branas se desacoplan a bajas energías, es decir cuando $g_s \to 0$. Por lo tanto, no aparecen en teoría de perturbaciones. Una Dp brana se puede definir como un objeto donde las cuerdas, que satisfacen condiciones de contorno de tipo Dirichlet, pueden acabar. Esto implica que una teoría cuántica de campos puede vivir sobre el *world-volume* de la brana. Esto ha llevado a la posibilidad de estudiar teorías de campos de un modo completamente diferente. De hecho, por los menos en algunos casos específicos, podemos caracterizar una dada teoría de campo estudiando la configuración de branas dual a dicha teoría (ver por ejemplo [7] para un a review sobre este argumento).

Otro importante descubrimiento debido a la teoría de cuerdas ha sido la *correspondencia AdS/CFT* [8, 9]. En su formulación más fuerte es una correspondencia entre una teoría de cuerdas sobre un espacio Anti de Sitter (AdS) en d-dimensiones y una teoría de campo conforme (CFT) en $d-1$

[4]La constante de acoplamiento en teoría de cuerdas g_s es igual al valor esperado de e^{ϕ}, donde ϕ es un campo escalar llamado dilatón. Además la teoría de cuerdas está regulada por el parámetro α' que es proporcional al inverso de la tension de la cuerda.

dimensiones. Ha sido encontrado un diccionario entre la teoría de gravedad y la teoría de campos. En particular campos en lado gravitacional están relacionados con operadores de la teoría de campos y simetrías local en lado de gravedad están puestas en relación con simetrías globales en la teoría de campo dual. Además, recientemente la correspondencia ha sido extendida en el contexto de teorías de campos non conformes.

En esta tesis aplicamos herramientas matemáticas y resultados físicos de teorías de cuerdas para estudiar teorías cuánticas de campos en varias dimensiones. En primer lugar nos enfocamos sobre el estudio de teorías cuántica de campos en $5d$ con $\mathcal{N} = 1$ de supersimetría. Extraordinariamente la mera existencia de dichas teorías es sorprendente. De hecho estas teorías no son renormalizable y por ello se pensaba que podían ser definidas sólo con la introducción de un *cut-off* ultravioleta. Sin embargo en [10] fue descubierto que, con una oportuna elección del grupo de simetría local y del contenido de materia de la teoría, ciertas teorías de campos en $5d$ podían admitir un punto fijo. Además se encontró una construcción con branas que permite integrar y estudiar dichas teorías en teoría de cuerdas heterótica [10, 11]. Alternativamente estas teorías se pueden realizar en teoría M haciendo una reducción dimensional sobre una variedad de Calabi-Yau[12]. Una característica sorprendente de estas teorías es que muestran un aumento del grupo de simetría global al punto fijo. En particular fue conjeturado por Seiberg [10] que para una teoría con grupo de simetría local $SU(2)$ y numero de flavor $N_f \leq 8$ el grupo de simetría global es aumentado a E_{N_f+1} en el punto fijo ultravioleta. Además las teorías anteriores admiten un dual gravitatorio que permite de estudiar unas de sus propiedades en modo holográfico. En sección 3 revisamos los aspectos principales de dichas teorías. En particular examinamos como podemos construir el dual gravitatorio empezando por la correspondiente configuración en termino de branas y como las teorías introducida en [10] se puedan generalizar tomando un *orbifold* de la teoría inicial. Una característica interesante relativa a estas teorías es le estudio del flujo del Grupo de Renormalización (GR). Esto se puede generar perturbando la teoría por un operador relevante. En particular en [13] empezamos el estudio del flujo del GR holografíco entre diferentes teoría cuántica de campos en $5d$ con $\mathcal{N} = 1$ clasificando los operadores que toman un valor esperado según sus dimension conforme y otros números cuánticos. Enviamos el lector al artículo reportado en la sección 6.1 para una análisis completa de dichos flujos del GR.

Las teorías de campos introducidas en [10] están definidas sobre un *background* plano. Por lo tanto es natural estudiar si dichas teorías se puedan poner también sobre una variedad curva genérica preservando por lo menos una parte de la supersimetría inicial. Un procedimiento que permite definir una teoría cuántica de campos supersimetríca sobre una variedad curva fue introducido por Seiberg y Festuccia en [14]. Revisamos los principales aspectos de esta técnica en la sección 4.3. En [15] aplicamos dicho procedimiento para el caso de teorías cuántica de campos en $5d$ con $\mathcal{N} = 1$. Hemos descubierto que la condición necesaria y suficiente que tiene que estar satisfecha para poder tener una teoría de campos supersimetríca bien definida sobre una variedad de Riemann es la existencia de un vector de KIlling conforme. La análisis completa ha sido desarrollada en el artículo reportado en la sección 6.3.

Entre los otros un *background* particularmente interesante, sobre el cual podemos definir una teoría cuántica de campos en $5d$, es $\mathbb{S}^1 \times \mathbb{S}^4$. Dicha variedad se presenta en modo natural para el calculo del índice superconforme (SCI) de la teoría, que se puede exprimir como un *path-integral* sobre $\mathbb{S}^1 \times \mathbb{S}^4$. En la sección 4.1 revisamos cómo el índice superconfrome permita de contar los operadores BPS de la teoría de acuerdo con los sus números cuánticos de simetría global, dando en este modo mucha información sobre la teoría considerada. Es también útil considerar limites particulares del índice que permiten de extraer información proveniente por un particular sector de operadores de la teoría. Por ejemplo en la sección 4.2.2. revisamos el limite de Schur del indice superconforme en $4d$. Entre los otros un limite particularmente interesante es *limite de Nekrasov-Shatashvili* del índice superconforme. Este limite está implementado poniendo a cero uno de los dos parámetros epsilon que definen el *background* Ω [17]. En [18] examinamos dicho limite y vemos como, por lo menos en el caso de teorías Abelianas, se pueda recuperar el limite de Schur del indice superconforme de una teoría en $4d$ con $\mathcal{N} = 2$. Esta computación ha sido desarrollada en el artículo reportado en la sección 6.4.

Además, como revisamos en la sección 4.1, la computación del índice superconforme usando la técnica de calculo llamada "*localization*" incluye correcciónes no perturbativas debidas a instantones y dichas correcciónes se pueden estudiar caracterizando el *moduli space* correspondiente. Por lo tanto el estudio de dicho espacio es muy relevante. Revisamos las informaciones básicas sobre el *moduli space* de los istantones y como la construcción ADHM [19] se pueda realizar en teoría de cuerdas en la sección 5. Además en la sección 5.1.3 examinamos como podemos caracterizar el *moduli space* correspondiente usando el "*Plethystic programe*" [20]. En el caso particular de una teoría en $5d$ con $\mathcal{N} = 1$ del tipo considerado en [10] la contribución de lo instantones se puede estudiar a través de la caracterización del *moduli space* de los instantones sobre $\mathbb{R}^4$. Sin embargo hemos descubierto que se puede definir una teoría cuántica de campos consistente en $5d$ también sobre otros tipos de variedad. Por lo tanto el estudio del *moduli space* de los instantones sobre diferentes tipos de variedades en cuatro dimensiones llega a ser de gran importancia. Un tipo de variedad particularmente interesante que fue considerado en [21] es $\mathbb{C}P^2 \times \mathbb{S}^1$. En particular en [22] fue considerada la construcción ADHM para instantones sobre $\mathbb{C}P^2$. En [23] ampliamos dicha análisis haciendo uso de la noción de *resolved moduli space* [24, 25, 26] y empezamos el estudio de dicho espacio desde un prospectiva física. Además también consideramos un orbifold $\mathbb{Z}_n$ del espacio inicial y aportamos la construcción ADHM para instantones sobre $\mathbb{C}P^2/\mathbb{Z}_n$. Reportamos los diferentes artículos, sobre los cuales esta tesis está basada en la sección 6. Al final acabamos con unas conclusiones y posibles planes de trabajo futuro en la sección 8.

3. $5d$ quantum field theories

In general $5d$ quantum field theories are non-renormalizible. In order to see this it is enough to consider the Yang-Mills term of the action

$$\frac{1}{g_{YM}^2}\int d^5x \mathrm{Tr} F_{\mu\nu}F^{\mu\nu}, \tag{3.1}$$

performing dimensional analysis it turns out that the coupling constant g_{YM} must have the dimension of length, i.e. $[g_{YM}^2] = L$. Therefore naively the only way to make sense of these kind of theories is to introduce an UV cut-off. Surprisingly it has been discovered [10, 12, 11] that some of these theories, for a specific choice of the matter content and of the gauge group, are well defined via UV fixed points. Moreover these theories can be embedded in string theory using an appropriate brane construction and admit a gravity dual. In this chapter we review the basic aspects related to these theories, their holographic duals and their possible generalization to quiver gauge theories. The material of this chapter is necessary for the articles in sections 6.16.2 and 6.4.

3.1 General features of $5d$ QFT

3.1.1 $5d$ QFT with $\mathcal{N} = 1$

The $\mathcal{N} = 1$ supersymmetric theory in $5d$ enjoys $F(4)$ superconformal symmetry whose bosonic subgroup is $SO(2,5) \times SU(2)$; where $SO(2,5)$ is the conformal group in five dimensions, while $SU(2)$ is an R-symmetry group under which the central charges transform as a doublet. The corresponding superconformal algebra is related by dimensional reduction to $\mathcal{N} = 2$ algebra in four dimensions and to $\mathcal{N} = 4$ algebra in three dimensions. The massless representation of the $5d$ $\mathcal{N} = 1$ algebra are

- The **hypermultiplet** $\mathcal{H}$ consisting of a complex scalar q^A and a complex fermion ψ.

- The **vectormultiplet** $\mathcal{A}$ consisting of a vector field A_μ, a real scalar field ϕ and a fermion λ_A [1].

The $U(1)_I$ topological symmetry

A common characteristic of $5d$ QFT is that the current

$$J = \, ^\star(F \wedge F), \tag{3.2}$$

is always conserved. Therefore these theories have a further $U(1)_I$ global symmetry. The charge of this symmetry is the instanton number I.

3.1.2 Higgs branch and Coulomb branch

Inside a $5d$ $\mathcal{N}$=1 QFT we can identify two different kinds of branches

- $\mathcal{M}_C$ **Coulomb branch** is associated with the vacuum expectation values of the scalar fields ϕ^i in the vector multiplets. In general on the Coulomb branch the gauge group G of the theory is broken to its maximal torus

  $$G \quad \mapsto \quad U(1)^r, \qquad \text{where} \;\; r = \text{rank}(G)$$

  Away from the origin of the Coulomb branch the low-energy theory is characterized by a prepotential $\mathcal{F}(\mathcal{A}^i)$, that is locally a function of the vectormultiplets.

- $\mathcal{M}_H$ **Higgs branch** is associated with the vacuum expectation values of the scalar fields in the hypermultiples. This space is an hyper-Kähler manifold and is protected against quantum corrections.[2]

In the particular case in which the gauge group of the theory is $Sp(N)$ without massless matter fields in the fundamental representations we have to specify another discrete parameter in order to completely characterize the theory. This parameter can be interpreted as a $\mathbb{Z}_2$ valued theta angle related with $\pi_4(Sp(N)) = \mathbb{Z}_2$.

3.1.3 Well defined $5d$ QFT and the prepotential $\mathcal{F}$

The Coulomb branch of the theory is described by a prepotential $\mathcal{F}(\mathcal{A}^i)$ that, in order to preserve gauge invariance, can be at most cubic in the vector

[1] Where A denotes an $SU(2)_R$ symmetry index

[2] This is due to the fact that, like in the four dimensional case [27], the gauge coupling can be thought as the vacuum expectation value of a vector multiplet.

multiplet. Moreover the prepotential is completely determined by a one-loop computation. For an arbitrary gauge group G and matter multiplets with mass m_f the most general expression that can be assumed by the prepotential on the Coulomb branch is [12]

$$\mathcal{F} = \frac{1}{2g_0^2} h_{ij}\phi^i\phi^j + \frac{c_0}{6} d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}\left(\sum_R \mid R\cdot\phi\mid^3 - \sum_f \sum_{w\in W_f} \mid w\cdot\phi + m_f \mid^3\right), \tag{3.3}$$

being

$$h_{ij} = \mathrm{Tr}(T_iT_j), \qquad d_{ijk} = \frac{1}{2}\mathrm{Tr}(T_i(T_jT_k + T_kT_j)),$$

and where the T_i are the Cartan generators of G. Moreover $\mathbf{R}$ are the roots of G, while $\mathbf{W}_f$ are the weights of G in the representation $\mathbf{r}_f$. Finally c_0 is quantized parameter related to a five dimensional Chern-Simons term. The first two terms appearing in (3.3) are the classical prepotential while the last two terms are the quantum corrections due to vector multipltes and matter multiplets. Once the prepotential is known we can compute the metric on the moduli space and the effective gauge coupling g_{eff}. That reads

$$t(\phi)_{ij} = \left(\frac{1}{g_{eff}^2}\right)_{ij} = \partial_i\partial_j\mathcal{F}, \tag{3.4}$$

while the metric reads

$$ds^2 = t_{ij}(\phi)d\phi^i d\phi^j. \tag{3.5}$$

Nontrivial RG fixed points

As discussed in [12] if the metric (3.5) is not negative throughout the moduli space we can take the *strong coupling limit* sending $m_0 = g_0^{-2} \to 0$, so that we have a fixed point. On the other hand, if this is not case, the fact that the metric becomes negative is a reflection of the non renormalizability of the theory.

The condition $t(\phi)_{ij} \geq 0$ can also be reformulated as a condition on the prepotential. As matter a fact is equivalent to require that prepotential (3.3) is a convex function along the Coulomb branch, ie. it must satisfy

$$\mathcal{F}(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) \leq \lambda\mathcal{F}(\mathbf{x})+(1-\lambda)\mathcal{F}(\mathbf{y}) \text{ for } 0 \leq \lambda \leq 1 \quad \forall\, \mathbf{x},\mathbf{y} \in \mathcal{M}_C, \tag{3.6}$$

since $\mathcal{F}$ and $\partial_i\mathcal{F}$ vanish at the origin a consequence of the condition (3.6) is that $\mathcal{F} \geq 0$ along all the Coulomb branch. An inspection to the expression (3.3) allows to see that the vector multiplet contribution is purely convex while the hypermultiplet term, due to the minus sign in front of it, leads to a purely concave contribution. Finally the c_0 has not a completely defined behaviour. Therefore naively the prepotential is a convex function provided there is not too much matter inside the theory. This means that, choosing in a proper way the matter content of the theory, it's possible to construct 5d QFT at fixed point.

Example: $G = SU(2)$ with matter

Let's make an example and let's consider the particular case in which $G = SU(2)$ with N_f matter hypermultiplets in the fundamental representation. This case has been considered in [10]. The Coulomb branch is $\mathcal{M}_C = \mathbb{R}^+$ and it is parametrized by the scalar field ϕ in the vector multiplet. The effective gauge coupling g_{eff}, using the expression (3.3), reads

$$\frac{1}{g_{eff}^2} = \frac{1}{g_0^2} + 16 \mid \phi \mid - \sum_{i=1}^{N_f} \mid \phi - m_i \mid - \sum_{i=1}^{N_f} \mid \phi + m_i \mid \tag{3.7}$$

Therefore if $N_f < 8$ the metric is positive and we can take the strong coupling limit sending $g_0 \to 0$. This theory has also an $SO(2N_f) \times U(1)_I$ global symmetry. The $SO(2N_f)$ factor is associated to the hypermultiplets while the $U(1)_I$ is the topological symmetry associated to the instanton number current (3.2).

3.1.4 Brane construction

A possible generalization of the theory considered in the previous example is a quantum field theory with gauge group $G = Sp(N)$ and a matter hypermultiplet A in the antisymmetric representation. Following [10] we can embed this theory in Type I'[3] string theory using a system of N_f D8 branes, N D4 branes and an orientifold plane $O8^-$. The corresponding brane configuration is summarized in table 3.1.

The quantum field theory under consideration lives on the stack of the N D4 branes, the open string between the D4 and D8 give the N_f fundamental hypermultipltes of the theory, while the locations of the D8 branes along the interval corresponds to their masses. The Coulomb branch of this theory is parametrized by the scalar fields ϕ^i and it corresponds to the position of the D4 branes. On the other hand the N_f fundamental hypermultiplet and the antisymmetric hypermultiplet (which is in the trivial representation for

[3]The Type I' theory is obtained starting from Type II A string theory and performing an orientifold projection.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$O8^-$	x	x	x	x	x	x	x	x	x	
N_f $D8$	x	x	x	x	x	x	x	x	x	
N $D4$	x	x	x	x	x					

Table 3.1: Branes configuration for a theory with gauge group $G = Sp(N)$ with N_f matter hypermultipltes.

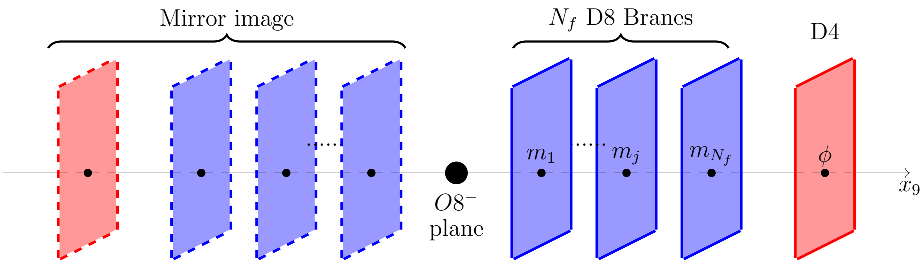

Figure 3.1: Graphical representation of the brane configuration reported in table 3.1, when $N = 1$ on the Coulomb branch of the theory.

$N = 1$) parametrize the Higgs branch, that is the moduli space of $SO(2N_f)$ one-instanton[4]. The full global symmetry group of this theory is

$$SU(2)_R \times SU(2) \times SO(2N_f) \times U(1)_I \tag{3.8}$$

where $SU(2)_R$ is due to the R-symmetry, the second $SU(2)$ factor is associated with the massless antisymmetric hypermultiplet, $SO(2N_f)$ factor with the N_f fundamental hypermultiplets and the $U(1)_I$ factor with the topological current (3.2).

Only one D4 brane

Let's consider initially the simplest case setting $N = 1$, i.e. let's consider only one D4 brane. Therefore the gauge group is $G = Sp(1) \simeq SU(2)$.

Let's now derive the coupling constant of the theory living on the world-volume of the D4 brane. The D8 background metric reads

$$ds^2 = H_8(x_9)^{-1/2}(-dt^2 + dx_1^2 + ... + dx_8^2) + H_8(x_9)^{1/2}dx_9^2, \quad e^{-\phi} = H_8(x_9)^{5/4}, \tag{3.9}$$

[4]see section 5 for a brief review of this topic

moreover let's assume that the orientifold plane $O8^-$ is located at $x_9 = 0$, while the D8 branes are located at various points along the x_9 direction, i.e. at $x_9 = 0 \leq x_9^1 \leq x_9^2 \leq ... \leq x_{N_f}^9$, this configuration is summarized in figure 3.1. The function $H_8(x_9)$ is given by

$$H_8(x_9) = c + 16\frac{x_9}{l_s} - \sum_{i=1}^{N_f} \frac{| x_9 - x_9^i |}{l_s} - \sum_{i=1}^{N_f} \frac{| x_9 + x_9^i |}{l_s} \tag{3.10}$$

Let's assume that the D4 brane is a probe for the D8 branes. The gauge coupling g_{eff} of the worldvolume QFT theory of the D4 brane and the function $H_8(x_9)$ are related as [5]

$$g_{eff}^2 = \frac{l_s}{H_8(x_9)} \tag{3.11}$$

If we take the field theory limit, i.e

$$g_{eff}^2 = \text{fixed}, \quad \phi = \frac{x_9}{l_s^2} = \text{fixed}, \quad l_s \mapsto 0 \tag{3.12}$$

we get

$$\frac{1}{g_{eff}^2} = \frac{1}{g_{cl}^2} + 16\phi - \sum_{i=1}^{N_f} | \phi - m_i | - \sum_{i=1}^{N_f} | \phi + m_i |, \tag{3.13}$$

where $g_{cl} = c/l_s$. This way we obtain again the relation (3.7).

It has been conjectured that at fixed point the global symmetry is enhanced to $SU(2)_R \times SU(2) \times E_{N_f+1}$ [10]. The Higgs branch of the corresponding theory is conjectured to be the moduli space of 1 instanton of E_{N_f+1} [6] From a brane picture perspective we reach the fixed point when all the D8 branes and the D4 brane have been located at the orientifold plane, moreover we have to require that $c \to 0$ in the equation (3.10) and that the dilaton blows up at the orientifold plane. This corresponds to take the strong coupling limit sending $g_{cl} \to \infty$ and $\phi \to 0$. The further degree of freedom, that are responsible of the symmetry enhancement, are due to instantons that correspond to D0 branes. These instantons become massless at the UV fixed point since their masses m_I are proportional to $1/g_{eff}^2$. See figure 3.2 for a graphical representation of the strong coupling limit.

[5]This relation can be derived expanding the DBI action of the D4 brane in the background (3.9).

[6]We refer the reader to section 4.1 for a test of the global symmetry enhancement using the superconformal index.

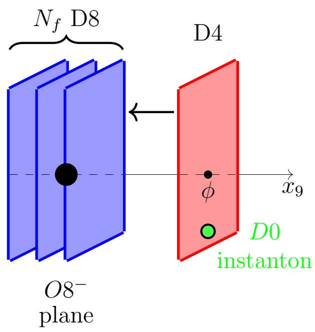

Figure 3.2: Brane configuration corresponding to the symmetry enhancement at the UV fixed point

The general case

The previous construction can be easily generalized for the case in which N D4 branes are considered. This time the gauge group is $Sp(N)$ while the global symmetry group is still given by (3.8). The Higgs branch, away from the UV fixed point, is expected to be the moduli space of $SO(2N_f)$ N-instantons. When we reach the fixed point the global symmetry is enhanced as before, while the corresponding Higgs branch is conjectured to be the moduli space of E_{N_f+1} N-instantons.

3.1.5 The supergravity description

As initially suggested in [28] and then shown in [29] it's possible to construct an holographic gravity dual to the quantum field theory analysed in the previous section. Here we review such construction. As outlined above we consider N_f D8 branes located at the orientifold plane $O8^-$. The bosonic part of type II A supergravity that we need to take into account is

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(e^{-2\Phi}R + 4\partial_\mu\Phi\partial^\mu\Phi) - \frac{1}{2\cdot 6!} \mid F_6 \mid^2 - \frac{1}{2}m^2\right). \quad (3.14)$$

As shown in [29], it' possible to find a solution of the equations of motion such that the near horizon limit of the metric reads

$$ds^2 = \hat{\Omega}(\alpha)^2\left(Q_4^{-1/2}u^2dx_{1,4}^2 + \frac{9}{4}Q_4^{1/2}\frac{du^2}{u^2}Q_4^{1/2}d\Omega_4^2\right), \quad (3.15)$$

where

$$Q_4 = \left(\frac{2^{11}\pi^4}{3^4(8-N_f)}\right)^{1/3} N, \quad \hat{\Omega}(\alpha) = \left(\frac{3}{4\pi}(8-N_f)\text{sin}\alpha\right)^{-1/6}, \quad (3.16)$$

and

$$d\Omega_4^2 = d\alpha^2 + \frac{1}{4}\cos^2\alpha[(d\psi + \cos\theta d\phi)^2 + d\theta^2\sin^2\theta d\phi^2], \tag{3.17}$$

with

$$0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi, \quad 0 \le \psi \le 4\pi \quad \text{and} \quad 0 \le \alpha \le \pi/2. \tag{3.18}$$

The space-time described by the metric (3.15) is a warped product between an AdS_6 space and S^4. This space has a boundary located at $\alpha = 0$, that corresponds to the location of the $O8^-$ ($z = 0$) orientifold plane. Therefore the topology of the boundary is $AdS_6 \times S^3$.

At the boundary the dilaton blows up as

$$e^\Phi = Q_4^{-1/4}\hat{\Omega}^5(\alpha) \tag{3.19}$$

Moreover also the Ricci scalar diverges at the boundary. As stated in [29] it's possible to establish a duality relation between $TypeI'$ string theory on the background (3.15) with a 4-form flux of N units on S^4 a $\mathcal{N} = 1$ supersymmetric 5d QFT at a fixed point.

Finally let's analyse the symmetries on both sides of the correspondence

- The $SO(2,5)$ isomtrey group of the AdS_6 space corresponds to the conformal symmetry of the 5d theory.
- The $SO(4) \sim SU(2) \times SU(2)$ subgroup of the isometry group of the sphere S^3 preserved by the warping is related to the $SU(2)_R$ R-symmetry and to the $SU(2)$ symmetry associated with the massless hypermultiplet in the antisymmetric representation.
- The flavor symmetry $SO(2N_f)$ and the enhancement to E_{N_f+1} can not be seen in the gravity dual background. This is due to the fact that it takes place at the boundary where the dual gravity description is not valid. However we can identify the instantonic $U(1)_I$ symmetry. This is dual to the RR 1-form potential C_1 in the bulk. Moreover there are not wrapped D-branes in this background corresponding to the fact that there are not baryonic operators [30].

3.1.6 Orbifolds of the initial theory

Following [31] we can generalize the previous class of theories taking orbifold of the internal space. In order to achieve this aim we replace the R^4 space transverse to the D4 branes by an ALE space asymptotic to $\mathbb{C}^2/\mathbb{Z}_n$. The corresponding brane configuration is summarized in table 3.2

	x_0	x_1	x_2	x_3	x_4	(x_5)	(x_6)	(x_7)	(x_8)	x_9
$O8^-$	x	x	x	x	x	x	x	x	x	
N_f $D8$	x	x	x	x	x	x	x	x	x	
N $D4$	x	x	x	x	x					

Table 3.2: Brane configuration for a theory with gauge group $G = Sp(N)$ with matter and orbifold action. The directions in the circle are subjected to the orbifold action.

Such that the orbifold action is defined as

$$Z_n : (z_1, z_2) \sim (e^{2\pi i/n} z_1, e^{-2\pi i/n} z_2), \quad z_1 = x_5 + ix_6, \quad z_2 = x_7 + ix_8,$$

Following [31] we review the matter content of the corresponding theory. We begin in Type IIA string theory on $\mathbb{C}^2/\mathbb{Z}_n$ that preserves $SO(1,5) \times SU(2)_R \times U(1)$ global symmetry and $\mathcal{N} = (1,1)$ supersymmetry in six dimensions. Then we perform the orientifold projection $I_9\Omega$. Where I_9 is the reflection along the x^9 spatial direction, while Ω is the worldsheet parity. This gives raise to $SO(1,4) \times SU(2)_R \times U(1)$ space-time symmetry and to $5d$ $\mathcal{N} = 1$ supersymmetry. After this operations we get

- $\mathcal{N} = 1$ gravity multiplet, one vector multiplet and one hypermultiplet in the untwisted sector of the theory.
- While in the twisted sector the result depends on the number n that characterize the orbifold action:
  - If $n = 2k+1$ we obtain k vector multiplets and k hypermultipltes.
  - If $n = 2k$ we obtain $k-1$ vector multiplets, $k-1$ hypermultiples. While for the kth twisted sector there are two possibilities [32]. The so called *No Vector Structure* (NVS), in which we keep an hypermultiplet and the *Vector Structure* (VS) in which we keep a vector multiplet.

If we do not take into account the N_f contributions we get three different families of quiver gauge theories, that have been summarized in the following sections.

Odd orbifolds

The gauge group of this theory is $G = Sp(2N) \times SU(2N)^k$. While we have a $U(1)_I$ instantonic symmetry factor for each gauge group and $U(1)$ factor for each bifundamental and for the antisymmetric field. The corresponding quiver gauge theory is reported in figure 3.3. The global non R-symmetry group is

$$U(1)^{k+1} \times U(1)_I^{k+1} \tag{3.20}$$

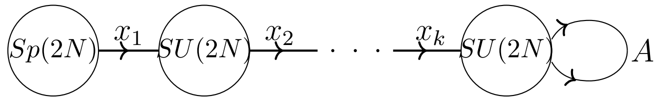

Figure 3.3: quiver diagram for the $\mathbb{C}^2/\mathbb{Z}_{2k+1}$ case.

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Even orbifold, $k = 2n$ with vector structure

The corresponding quiver gauge theory is described by the quiver reported in figure 3.4. The gauge group is $G = Sp(2N) \times SU(2N)^{k-1} \times Sp(2N)$. Moreover we have $k+1$ instantonic symmetries, and $U(1)$ symmetry for each bi-fundamental x_i[7] The global non R-symmetry group is

$$U(1)^k \times U(1)_I^{k+1} \tag{3.21}$$

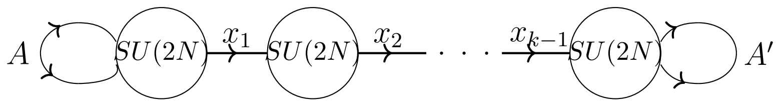

Figure 3.4: quiver diagram for the $\mathbb{C}^2/\mathbb{Z}_{2k}$ with no vector structure case.

Even orbifold, $k = 2n$ without vector structure

The gauge group is $G = SU(2N)^k$. Moreover also in this case we have a $U(1)$ instantonic symmetry for each gauge group and $U(1)$ global symmetry for each bifundamental [8] The corresponding quiver gauge theory is described by the quiver reported in figure 3.4. The global non R-symmetry group is

$$U(1)^{k+1} \times U(1)_I^{k} \tag{3.22}$$

[7]Note that if $k = 1$ this symmetry is enhanced to $SU(2)$ since the bi-fundamental of $Sp(2N) \times SU(2N)$ is pseudoreal.

[8]The case $k = 1$ is different since the matter global symmetry is enhanced to $SU(2) \times U(1)$

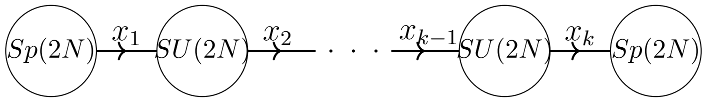

Figure 3.5: quiver diagram for the $\mathbb{C}^2/\mathbb{Z}_{2k}$ case with vector structure.

The supergravity dual

The geometry of the gravity dual, after the action of the orbifold, is $AdS_6 \times S^4/Z_n$. The metric is still given by the expression (3.15), however the periodicity of the angle ψ is changed as follows

$$\psi : [0, 4\pi] \mapsto [0, 4\pi/n].$$

Also in this case we can identify the symmetries between the two sides of the correspondence. The internal space $S^3/\mathbb{Z}_n$ has an $SU(2) \times U(1)$ global symmetry. Where the $SU(2)$ factor comes from the S^2 base of the space while the $U(1)$ factor comes from the S^1 ψ fiber. The above $SU(2)$ factor is identified with the $SU(2)_R$ symmetry of the dual quantum field theory. While the $U(1)$ factor is identified with the $U(1)_M$ mesonic symmetry. In the particular case in which $n = 2$ the previous symmetry is enhanced to an $SU(2) \times SU(2)$ symmetry in accordance with the enhancement of the mesonic global symmetry discussed in 3.1.6.

3.1.7 RG flows in 5d gauge theories

Let's focus for simplicity on a $\mathbb{Z}_2$ orbifold of the initial theory. As discussed in section 3.1.6 this leads to two different gauge theories

- The **first theory** has gauge group $G = Sp(2N) \times Sp(2N)$ with bifundamentals matters. Its global symmetry group is $SU(2)_M \times U(1)_{I_1} \times U(1)_{I_2} \times SU(2)_R$, where the mesonic symmetry $SU(2)_M$ acts on the hypermultiplets. Moreover $U(1)_{I_1}$ and $U(1)_{I_2}$ are two topological symmetries associated to the two gauge groups.

- The **second theory** has gauge group $G = SU(2N)$. Its global symmetry group is $SU(2)_M \times U(1)_I \times U(1)_B \times SU(2)_R$, where the mesonic symmetry $SU(2)_M$ acts on the antisymmetric hypermultipltes, $U(1)_I$ is a topological symmetry associated to the gauge group . Finally $U(1)_B$ is a baryonic symmetry under which one of the two antisymmetric hypermultiplet, let's say A_1, has charge $+1$ and the other antisymmetric hypermultiplet A_2 has charge -1.

Following the analysis performed in [33] for the $4d$ with $\mathcal{N} = 1$ case we can examine two different kinds of RG flow. First of all we we can imagine an RG flow triggered by the VEV of a mesonic operator. This kind of RG flow, in terms of the brane description, corresponds to move the D4 branes away from the singularity present in the metric. On the other hand, only when the second theory is is concerned, we can consider an RG flow triggered by the VEV of a baryonic operator. This corresponds to a blow-up of the singularity present in the metric.

This picture can be tested solving the equations of motion for the warp factor near the boundary and classifying the operators that are a taking a VEV in the dual QFT according to their conformal dimension and other global symmetry quantum numbers. Moreover the fact that, for the second kind of RG flow, the $U(1)_B$ baryonic symmetry is broken can be tested performing the computation of the DBI action for a D2 brane wrapping a 3-cycle in the dual geometry. This must be equivalent to the modulus of the baryonic condensate [31]. On the other hand, following the procedure outlined in [34] for the $4d$ case, we can also determine the phase of the condensate and study the Goldstone boson arising from the breaking of the baryonic global symmetry.

In the article reported in section 6.1 we study these kinds of holographic RG flows. We examine, as a warm-up, the case of $4d$ $\mathcal{N} = 2$ theories and then we move to consider the $5d$ $\mathcal{N} = 1$ case. Even if the two cases are quite similar there is an remarkable difference. As a matter of fact in the $5d$ case the boundary is reached when both the bulk radial coordinate r and the angle α (3.18) that parametrizes the internal $\mathbb{S}^3$ sphere goes to zero. As discussed in section 3.1.5, both the dilaton and the Ricci scalar diverge at $\alpha = 0$. Therefore we had to fix in a proper way the boundary condition there. We refer the reader to the corresponding article for further details.

4. The Superconformal Index

In this chapter we briefly review *supersymmetric localization* and we examine how its application in the context of supersymmetric quantum field theories drastically simplify the computation of the partition function of the theory. Then we introduce the *Superconformal Index* (SCI) for $5d$ $\mathcal{N} = 1$ and $4d$ $\mathcal{N} = 2$ quantum field theories and we see how we can compute it using localization. Moreover we examine how the SCI can be used to text the global symmetry enhancement discussed in chapter 3 in the context of $5d$ $\mathcal{N} = 1$ theories.

4.1 Supersymmetric localization

After its introduction in the seminal paper [35] in the context of QFTs there have been a lot of application of the computational technique called *supersymmetric localization*. In the following section we review how localization can be applied for the computation of VEV of observables and of the partition function of a supersymmetric quantum field theory. The main result is that the path integral can be reduced to a lower dimensional integral over a a *localization locus* weighted by the classical action and a 1-loop superdeterminat of the fluctuations transverse to the localization locus.[1].

4.1.1 Supersymmetry and the partition function

Let's consider a QFT with a fermionic symmetry generated by the charge Q (in particular Q can be a supercharge), such that it squares to a bosonic charge

$$Q^2 = B \tag{4.1}$$

[1]For a full review regarding localisation and its application in the context of supersymmetric gauge theories we refer the reader to [36, 37, 38]

Moreover let's focus on *BPS gauge invariant operators* $\mathcal{O}_{BPS}$. These operators are annihilated by the supercharge Q

$$Q\mathcal{O}_{BPS} = 0. \tag{4.2}$$

We want to compute exactly the vacuum expectation value of such observables

$$\langle \mathcal{O}_{BPS} \rangle = \int_{\mathcal{F}} [\mathcal{D}X] \mathcal{O}_{BPS} e^{-S[X]} \tag{4.3}$$

where the symbol X denotes the various fields present in the theory and $\mathcal{F}$ denotes the *field space*, i.e. the set over which the various fields take a value. In order to compute the VEV (4.3) we observe that the expectation value of a Q-exact observable vanishes

$$\langle Q\mathcal{O} \rangle = \int_{\mathcal{F}} [\mathcal{D}X] (Q\mathcal{O}) e^{-S[X]} = \int_{\mathcal{F}} [\mathcal{D}X] Q(\mathcal{O} e^{-S[X]}) = 0 \tag{4.4}$$

where we assumed that the measure $[\mathcal{D}X]$ and the action $S[X]$ are invariant under the Q-action. We found the integral of a total derivative, that vanishes if there are not boundary terms. Therefore in general we have

$$\langle \mathcal{O}_{BPS} \rangle = \langle \mathcal{O}_{BPS} + Q\mathcal{O} \rangle \tag{4.5}$$

This means that we can deform the path integral (4.3) adding the Q variation of B-invariant fermionic functional $\mathcal{P}_F[X]$ without changing the VEV of the BPS operator. If we do so the path-integral (4.3) now reads

$$\langle \mathcal{O}_{BPS} \rangle = \int_{\mathcal{F}} [\mathcal{D}X] \mathcal{O}_{BPS} e^{-S[X] - tQ\mathcal{P}_F[X]} \qquad \forall \;\; t \tag{4.6}$$

As a check we can take the t-derivative of the expression (4.6) and we get

$$\begin{aligned} &\frac{d}{dt} \int_{\mathcal{F}} [\mathcal{D}X] \mathcal{O}_{BPS} e^{-S[X] - tQ\mathcal{P}_F[X]} = - \int_{\mathcal{F}} [\mathcal{D}X] (Q\mathcal{P}_F) \mathcal{O}_{BPS} e^{-S[X] - tQ\mathcal{P}_F[X]} = \\ &- \int_{\mathcal{F}} [\mathcal{D}X] Q(P_F \mathcal{O}_{BPS} e^{-S[X] - tQ\mathcal{P}_F[X]}) = 0 \end{aligned} \tag{4.7}$$

Therefore, if there are not boundary terms, the previous integral does not depend on t. This means that we can evaluate the previous integral taking the limit

$$\langle \mathcal{O}_{BPS} \rangle = \lim_{t \mapsto \infty} \int_{\mathcal{F}} [\mathcal{D}X] \mathcal{O}_{BPS} e^{-S[X] - tQ\mathcal{P}_F[X]} \tag{4.8}$$

In the above limit the path integral will be dominated by the saddle points of the localising action

$$S_{loc}[X] = Q\mathcal{P}_F[X] \tag{4.9}$$

In order to evaluate (4.8) let's expand the fields X around the saddle point configurations of S_{loc} and then we take the limit $t \mapsto \infty$. [2]

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X \tag{4.10}$$

where X_0 denotes the classical configuration. The semi-classical expansion of the action reads

$$S[X_0] + \frac{1}{2}\int\int \frac{\delta^2 S_{loc}[X]}{\delta X^2}\,|_{X=X_0}\,(\delta X)^2 \tag{4.11}$$

It's important to note that this result is 1-loop exact since higher terms of the expansion are eliminated once the limit $t \mapsto \infty$ is taken. Inserting the above expansion in the path-integral (4.8) and integrating out the fluctuations we obtain

$$\langle \mathcal{O}_{BPS}\rangle = \int_{\mathcal{F}_Q} [\mathcal{D}X_0]\mathcal{O}_{BPS}\,|_{X=X_0}\, e^{-S[X_0]}\text{SDet}\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right]^{-1} \tag{4.12}$$

We observe that is possible to proof that the original path integral over the full field space $\mathcal{F}$ has been localized to a new lower dimensional integral over the *BPS locus* $\mathcal{F}_Q$, that is the space

$$\mathcal{F}_Q = \{[X] \in \mathcal{F} \mid \text{fermions} = 0, \quad Q(\text{fermions}) = 0\} \tag{4.13}$$

The VEV (4.12) can be computed performing the integration of the classical action contribution $S[X_0]$ that must be corrected, due to the integration of the fluctuations δX, with the 1-loop Super-Determinant. The Super-Determinant is given by the ratio of the determinants of the operators that give a contributions at the quadratic order in the bosonic and fermionic fluctuations.

4.2 The Superconformal Index in different dimensions

Historically the precursor of the SCI is the so called *Witten index* [39]

$$\text{Tr}[(-1)^F e^{-\beta H}], \tag{4.14}$$

[2]Note that the inverse of the parameter t plays the role of an auxiliary Planck constant, i.e. $\hbar_{aux} = 1/t$.

where F is the fermion number operator, that is equal to one for a fermionic state and zero for a bosonic state. In a supersymmetric QFT each state with an energy eigenvalue different from zero contains the same number of fermionic and bosonic degree of freedom. Therefore the Witten index (4.14) does not depend on the temperature. Physically it gives the number of zero energy bosonic states minus the number of zero energy fermionic vacuum states.

The superconformal index has been in introduced [40] in the context of $4d$ $\mathcal{N}=1$ QFT and subsequently generalized for supersymmetric theories in different numbers of dimensions and with a different amount of supersymmetry [41]. This quantity encodes the information regarding the protected spectrum of a SCFTs. Basically it counts gauge invariants operators inside the QFT taken into consideration. These gauge invariant operators are classified according to representations of the superconformal algebra of the theory and correspond to physical states in the radial quantization of the theory. Schematically it takes the form [40]

$$I = \mathrm{Tr}[(-1)^F e^{-\mu_i T_i} e^{-\beta\delta}], \qquad \delta = \{Q, Q^\dagger\}, \tag{4.15}$$

where, as before, F is the fermion number operator. $\{T_i\}$ denotes a complete set of generators (with corresponding fugacities μ_i) that commutes with the supercharge Q with respect the index is computed. The trace is taken over the states of the theory on $\mathbb{S}^{d-1}$, after radial quantization. The index counts only the states with $\delta = 0$ (since with $\delta \neq 0$ cancel pairwise). Moreover it is possible to prove that different choice of Q leads to indices that are physically equivalent [16]. In the following we concentrate only to the $5d$ $\mathcal{N}=1$ and the $4d$ $\mathcal{N}=2$ case.

4.2.1 Superconformal Index in $5d$

In the $5d$ case we can radially quantize theory on $\mathbb{R}\times\mathbb{S}^4$. Then the physical states are then labelled by the charges of the Cartan generators of the bosonic sublagebra $SO(2,5)\times SU(2)_R$. The energy ϵ_0 corresponds to dilatation, the angular momenta j_1 and j_2 are the charges of $SU(2)_1 \times SU(2)_2 \subset SO(5)$ and finally j_R is the $SU(2)_R$ charge. Moreover in the radial quantized theory the supercharges Q^A_m and the superconformal supercharges S^m_A are conjugate. Their commutator reads [42, 41]

$$\{Q^A_m, S^m_A\} = \delta^n_m \delta^A_B D + 2\delta^A_B M^n_m - 3\delta^n_m R^A_B, \tag{4.16}$$

where D is the dilatation, M^m_n is the $SO(5)$ rotation and R^A_B are the $SU(2)_R$ symmetry. The states satisfying the bound (4.16) are BPS and live in short multiplets. The above bound allows to count the BPS spectrum of a CFT.

Cartan generators		Chemical potential
Δ	$\leftrightarrow$	$e^{-\beta}$
j_1+R	$\leftrightarrow$	$x = e^{-\gamma_1}$
j_2	$\leftrightarrow$	$y = e^{-\gamma_2}$
k	$\leftrightarrow$	q
$H_i \; i=1,...,N_f$	$\leftrightarrow$	e^{-m_i}

Table 4.1: Cartan generators and corresponding chemical potentials .

Features of the $5d$ SCI

Following [43] we select the supercharge $Q = Q^{A=1}_{m=2}$. This way we can count states that are annihilated by Q and $S = Q^\dagger$. Therefore the states that satisfy the above property are 1/8 BPS states. We use the Cartan generators reported in table 4.1 to label the physical states of the theory. All these Cartans commute with Q and S. The $5d$ superconformal index reads [43]

$$I(x,y,m_i,q) = \text{Tr}[(-1)^F e^{-\beta\{Q,S\}} x^{2(J_1+R)} y^{2J_2} e^{-i\sum_j m_j H_j} q^k], \tag{4.17}$$

where the trace is taken over the Hilbert space on $\mathbb{S}^4$. After a Wick rotation and a compactification along the euclidean time direction we can express the superconfromal index (4.17) as a path integral over $\mathbb{S}^1 \times \mathbb{S}^4$, with periodic boundary conditions for both bosonic and fermionic fields along the $\mathbb{S}^1$ time direction. So that the expression (4.17) becomes

$$I(x,y,m_i,q) = \int_{S^1\times S^4} D\Psi e^{-S_E[\Psi]}. \tag{4.18}$$

The localization technique outlined in section 4.1 has been applied in [43] for the evaluation of the path integral (4.18). The final result is that the SCI (4.17) can be expressed as

$$I(x,y,m_i,q) = \int \mathcal{D}\alpha I_{pert} I_{inst} \tag{4.19}$$

where $\mathcal{D}\alpha$ is the Haar measure of the gauge group of the theory.

- The *perturbative contribution* I_{pert} is obtained taking the Plethystic Exponential (PE)[3] of the single letter contribution, due to the vector

[3]The Plethystic Exponential of a function $f(x_1,...,x_n)$ such that $f(0,...,0) = 0$ is defined as

$$\text{PE}[f(x_1,...,x_N)] = \text{Exp}\left[\sum_{n=1}^{\infty} \frac{f(x_1^n,...x_N^n)}{n}\right] \tag{4.20}$$

multiplet and half-hypermultiplet present in the theory

$$i_v = -\frac{x(y+y^{-1})}{(1-xy)(1-\frac{x}{y})}\chi_{Adj}, \quad i_M = \frac{x}{(1-xy)(1-\frac{x}{y})}\chi_M \tag{4.21}$$

where χ_{Adj} are the characters of the adjoint representation of the gauge group while χ_M are the characters of the representation M under which the half-hypermultiplet transforms.

- The *instanton contribution* I_{inst} arises due to the fact the path integral (4.18) localizes on instantonic solutions around the north pole (anti-instantons) and south pole (instantons) of the $\mathbb{S}^4$. Therefore the instanton partition function is given by the product of a factor $I^{(N)}$ from the north pole and a factor $I^{(S)}$ from the south pole. The result reads [43]

$$I_{inst} = I^{(N)}(q)I^{(S)}(q) = I^{(S)}(q^{-1})I^{(S)}(q), \quad \text{where} \quad I^{(S)}(q) = \sum_{k=0}^{\infty} I_k q^k \tag{4.22}$$

It has been shown in [44] that the instanton partition function I_k, in the particular case of pure gauge theories, is equal to the Hilbert for the moduli space of instantons on R^4.

The instanton partition function (4.22) for a $SU(2)$ gauge theory with N_f hypermultiplet has been computed in [43] and independently in [45] using the topological vertex.

SCI and enhancement of the global symmetry

The enhancement of the global symmetry conjectured by Seiberg [10] can take place if at the UV fixed point there are additional conserved currents, such that, if are taken together with the currents generating $SO(2N_f) \times U(1)_I$ symmetry, constitute the current algebra of E_{N_f+1}. Moreover these currents are non perturbative and therefore the enhancement of the global symmetry can only seen examining observables that contain some amount of nonperturbative information about the theory. For this reason the SCI is a suitable quantity that can be used to text the enhancement of global symmetry at the UV fixed point. As a matter of fact in [43] the computation of the SCI for a theory with $SU(2)$ gauge group and number of flavours $N_f \leq 5$ has been performed. It has been found that, at the lowest order in the x chemical potential, the SCI for a theory with N_f flavours always takes the form

$$I_{N_f} = 1 + \chi_{Adj}^{EN_f+1} x^2 + \mathcal{O}(x^3) \tag{4.23}$$

where $\chi_{Adj}^{E_{N_f+1}}$ are the characters of the adjoint representation of E_{N_f+1}. The appearance of the adjoint characters $\chi_{Adj}^{E_{N_f+1}}$ leads to a global symmetry enhancement at every order of the expansion of the SCI [46]. In order to show this we have to examine the coefficient of the x^2 in the above expansion (4.23). As shown in [46] only scalar fields in the adjoint representation of $SU(2)_R$ and with conformal dimension $\Delta = 3$ contribute to this term. This implies that in all these theories taken into account and labelled by different value of N_f there is a superconformal multiplet. The primaries operators of this multiplet are scalar fields that are a triplet under $SU(2)_R$ and that furthermore are in the representation of $SO(2N_f) \times U(1)_I$ obtained from the adjoint representation of E_{N_f+1}. Moreover it also possible to show that the above multiplet is equal to the superconformal multiplet of flavour currents that leads to the enhancement of the global symmetry [4].

4.2.2 Superconformal index in four dimensions

Following the same conventions employed in in [16] we can express the $4d$ $\mathcal{N} = 2$ index (4.15) as

$$I(p,q,t) = \mathrm{Tr}[(-1)^F p^{\frac{1}{2}\delta_{1+}} q^{\frac{1}{2}\delta_{1-}} t^{R+r} e^{-\beta\tilde{\delta}_{1\dot{-}}}] \tag{4.24}$$

where

$$\delta_{1+} = E + 2j_1 - 2R - r, \quad \delta_{1-} = E - 2j_1 - 2R - r, \quad \tilde{\delta}_{1\dot{-}} = E - 2j_1 - 2R + r$$

and where E is the conformal dimension, j_1 and j_2 are the Cartan generators of the $SU(2)_1 \times SU(2)_2$ isometry group, while R and r are the Cartan generators of the $SU(2)_R \times U(1)_r$ R-symmetry group. Moreover we observe that the index (4.24) depends on three superconformal fugacities, since the subalgebra commuting with a single supercharge has rank three

The Schur limit

Among the others a particular limit of the index (4.24) that can be considered is the *Schur limit*. This is obtained setting $q = t$ and leaving p arbitrary. This index receives contribution from states with both $\delta_{1+} = \delta_{1\dot{-}} = 0$. Therefore the *Schur Index* I_S reads

$$I_S = \mathrm{Tr}[-(1)^F q^{E-R}]. \tag{4.25}$$

[4]We refer the interested reader to [46] for a detailed proof of the above statement.

4.2.3 Nekrasov-Shatashvili limit of $5d$ superconformal index

As in the four dimensional case it would be interesting to take a particular limit of the $5d$ SCI that can isolate the contributions arising only from some particular operators. In particular we can consider the Nekrasov-Shatashvili (NS)[5] of the index. This limit is implemented turning off one of the two epsilon parameters (ϵ_1, ϵ_2) that characterize the Ω background [17]. These parameters are related to the (x, y) fugacities introduced in section 4.2.1 as

$$xy = e^{-\epsilon_1}, \quad \frac{x}{y} = e^{-\epsilon_2}. \tag{4.26}$$

Therefore the naive implementation of the NS limit reads

$$xy \to 1, \quad \frac{x}{y} \to \text{fixed}. \tag{4.27}$$

However a look to the expressions (4.21) for the single letter partition functions shows that this naive implementation of the NS limit leads to a singularity. In the article reported in section (6.4) we overcome this problem introducing a prescription that allows to take the NS limit giving a finite result and ensuring that all the coefficients of the expansion of the index, once the limit has been taken, are given by integer numbers. [6]

Moreover in [48] it was outlined a connection between the Schur limit of the $4d$ SCI (introduced in section 4.2.2) and an algebraic quantity associated with the BPS spectrum on the Coulomb branch. More explicitly in the case of a rank-r theory it was conjectured that the index I_{KS} is given by the trace of the Kontsevich-Soibelman (KS) operator $\mathcal{O}$. [7]

$$I_{KS} = (q)_\infty^{2r}\text{Tr}[\mathcal{O}] \tag{4.29}$$

The particular expression of the operator $\mathcal{O}$ can be read off starting from the BPS quiver of theory taken in consideration. In the article reported in section 6.4 we consider the NS limit of different $5d$ $\mathcal{N} = 1$ abelian gauge theories with

[5] This limit was introduced for the first time in a four-dimensional context in [47].

[6] This last condition is motivated by the fact that the coefficients of the expansion of the index, as discussed above, are the numbers of operators with a given set of quantum numbers.

[7] where the *Pochhammer symbol* is defined as

$$(q)_0 = 1, \qquad (q)_n = \prod_{k=1}^{n}(1 - q^k) \tag{4.28}$$

different flavour groups. We construct the corresponding $5d$ BPS quiver and we extract the trace of the corresponding KS operator. Finally we compute the quantity expressed by the formula (4.29). An important difference with the four dimensional case is the presence of a further global fugacity related with the topological charge. In terms of BPS quiver data we see that this further global symmetry translates in the presence of a further node of the diagram. We refer the reader to the article 6.4 for further details.

4.3 Rigid supersymmetry in curved space time

In this section we outline the problem to define a supersymmetric QFT on a curved space time M endowed with a non flat metric $g_{\mu\nu}$ in a way that allows to preserve at least a partial amount of the initial supersymmetry.

Following [36] let's consider a supersymmetric QFT defined on a flat space-time. Let's assume that this QFT is described by an action $S^{(0)}$

$$S^{(0)} = \int d^d x \mathcal{L}^{(0)}. \tag{4.30}$$

Moreover let's assume that it is endowed with a SUSY algebra generated by infinitesimal transformations $\delta^{(0)}$. This algebra schematically acts as follows on the bosonic and fermionic degree of freedom of the theory

$$\delta^{(0)}(boson) = fermion, \quad \delta^{(0)}(fermion) = boson, \tag{4.31}$$

in order to preserve supersymmetry we must require that

$$\delta^{(0)}\mathcal{L}^{(0)} = \partial_\mu(...)^\mu \tag{4.32}$$

this means that the variation of the langragian under a SUSY transformation must be a total derivative.

In order to obtain a QFT defined on a curved background we have to replace to flat metric $\eta_{\mu\nu}$ with the curved metric $g_{\mu\nu}$ that defines the background. Moreover we have to replace all the ordinary derivatives with covariant derivatives

$$\eta_{\mu\nu} \mapsto g_{\mu\nu}, \quad \partial_\mu \mapsto \nabla_\mu. \tag{4.33}$$

However in general the new theory (even if it will be well defined on the curved background $g_{\mu\nu}$) will not be any more supersymmetric since

$$\delta^{(0)} \Big|_{\substack{\eta \to g \\ \partial \to \nabla}} \mathcal{L}^{(0)} \Big|_{\substack{\eta \to g \\ \partial \to \nabla}} \neq \nabla_\mu(...)^\mu. \tag{4.34}$$

However, as we review in the next section, a systematic procedure that allows to define a supersymmetric QFT on a curved background has been developed in [14].

Supersymmetric field theories on curved background

The method introduced by Seiberg and Festuccia in [14] is a two step procedure

1. Firs of all we couple the supersymmetric field theory taken in consideration with supergravity. The supergravity multiplet in general contains the metric $g_{\mu\nu}$, the gravitino $\psi_{\mu\alpha}$ and some auxiliary fields (that of course depend on the particular supergravity that have been selected).
2. Then we take the *rigid limit*, i.e. we send $G_N \mapsto 0$. Once this limit has been taken gravity decouples and the metric is sent to a fixed background metric. Moreover also the auxiliary fields of the supergravity multiplet now assume a fixed background value.

In order to preserve supersymmetry we have to impose that the supersymmetic variations of the gravitini as well as of al the fermionic fields in the gravity multiplet vanish. This requirements lead to a set of *Killing spinor equations*. This is a set of first order differential equations which are the conditions that must be satisfied in order to preserve supersymmetry on the manifold M that we are considering. Finally, once a a solution a of the above Killing spinor equations is known, we can find the lagrangian of the supersymmetric field theory and the supersymmetric transformations of the fields on the curved manifold M.

4.3.1 $5d$ $\mathcal{N} = 1$ theories on curved backgrounds

As we saw in the previous chapters a $5d$ supersymmetric QFT can be very interesting object to study due to its properties and its applications related to the computation of some observables of the theory. As a matter of fact the partition function of a $5d$ $\mathcal{N} = 1$ theory can be expressed as a path-integral over $\mathbb{S}^5$ [49, 50], while the superconformal index as a path integral over $\mathbb{S}^1 \times \mathbb{S}^4$. Therefore supported by these motivations a natural question is to understand which conditions must be satisfied in order to define a $5d$ theory on a curved manifold preserving at least some part of the initial supersymmetry.

In the article reported in section 6.3 we address this question and we apply the technique outlined in section 4.3 in the context of $5d$ $\mathcal{N} = 1$ quantum field theory. In order to this we consider $5d$ $\mathcal{N} = 2$ conformal euclidean supergravity coupled to $5d$ conformal matter consisting of both vector and hypermultiplets. Following the same procedure outlined in [51] in the context of $4d$ $\mathcal{N} = 2$ supergravity we look for a solution of the gravitino and dilatino equations that must be satisfied in order to preserve supersymmetry. We find that necessary and sufficient condition for existence of a solution is the existence of a conformal Killing vector v.

Then we study under which conditions we can turn on a VEV for scalar field in the backgrounds vector multiplet. Such a VEV breaks the initial conformal invariance of the theory and leads to a flow to a *standard gauge theory*, characterized by the term $g_{YM}^{-2}F^2$. Where the Yang-Mills coupling constant g_{YM} as the VEV for the scalar field. We find that this can takes place if the vector v is Killing and not only conformal Killing. Furthermore we find that most of the backgrounds exhibit a more involved geometric structure, that is called *transversally holomorphic foliation* (THF). We examine which conditions must be satisfied in order to ensure that a generic background of conformal supergravity admits a THF. We find that the necessary and sufficient condition is the existence of a global section of an $su(2)/\mathbb{R}$ bundle that is covariantly constant with respect to a connection $\mathcal{D}^Q$ that arises from the intrinsic torsion parametrizing the spinor. We refer the reader to the corresponding article for further details.

5. Moduli space of instantons and Hilbert Series

In this chapter we introduce the notion of *moduli space of instantons*. We analyse its basic property for the case of instantons on $\mathbb{R}^4$. Finally we examine how we can embed it in string theory using an ADHM-like construction [52, 53, 54]. Moreover we examine how the problem regarding the characterization of the moduli space of instantons can be replaced with the study of the moduli space of vacua of supersymmetric quantum field theory.

5.1 Instantons on $\mathbb{R}^4$

Instantons can be defined as classical solutions of the equation of motion with a finite action [1]. Let's consider the action for a QFT with gauge group G.

$$S = \frac{1}{2g^2} \int d^4x \text{Tr}[F^{\mu\nu} F_{\mu\nu}], \tag{5.1}$$

where g is the gauge coupling of theory. The equation of motions are

$$D^{\mu} F_{\mu\nu} = 0, \tag{5.2}$$

where D^{μ} is the covariant derivative. The requirement that the action is finite implies a constraint on the gauge potential A_{μ} at spatial infinity of $\mathbb{R}^4$, where we must require that the gauge potential is a pure gauge, i.e.

$$A_{\mu} \to i g^{-1} \partial_{\mu} g \qquad r \to \infty, \tag{5.3}$$

where $g(x) = e^{T(x)}$ and the $T(x)$ are the generators of the gauge group G. It's possible to prove that these solutions are classified according to the *instanton*

[1]There are different reviews regarding instantons and the corresponding moduli space. For a more detailed analysis of this topic we refer the reader to [55].

number k

$$k = \frac{1}{24\pi^2} \int_{S^3_\infty} d^3 S_\mu Tr(\partial_\nu g) g^{-1} (\partial_\rho g) g^{-1} (\partial_\sigma g) g^{-1}. \qquad (5.4)$$

Therefore the space of all possible solutions is divided in different sectors each of one is labelled by a different valued of the integer k. Moreover it can be shown that the value of the action of an instanton in given sector is bounded, this is called the *Bogomonln'yi bound*

$$S \geq \frac{8\pi^2}{g^2} \mid k \mid, \qquad (5.5)$$

and the equality holds if and only if

$$\begin{aligned} F_{\mu\nu} &= {}^*F_{\mu\nu} \quad \text{with} \;\; k > 0 \\ F_{\mu\nu} &= -{}^*F_{\mu\nu} \quad \text{with} \;\; k < 0 \end{aligned} \qquad (5.6)$$

where

$$^*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \qquad (5.7)$$

a solution of the equations (5.6) must solve the full equations of motion, since it minimizes the action in a given sector.

5.1.1 The moduli space of instantons

The moduli space $\mathcal{M}_{k,N}$ is defined to be the set of all solutions to equations (5.6) with a given instanton number k and with gauge group $G = SU(N)$ [2]. The dimension of this space can be calculated using the Atiyah-Singer index theorem, that counts the number of *zero modes*. A zero mode $\delta_\alpha A_\mu$ can be defined as follows. Given a solution of the self-dual equation $A_\mu(x_\mu, X_\alpha)$ (5.6) (where X_α is the set of all collective coordinates) we define the corresponding zero modes as

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha} + D_\mu \Omega_\alpha, \qquad (5.8)$$

where we note that each zero mode is defined up to an infinitesimal gauge transformation $D_\mu \Omega_\alpha$. In order to fix the gauge we choose to impose the gauge fixing condition

$$D_\mu (\delta_\alpha A_\mu) = 0. \qquad (5.9)$$

[2]For the moment let's restrict our analysis to the particular case in which the gauge group is $SU(N)$.

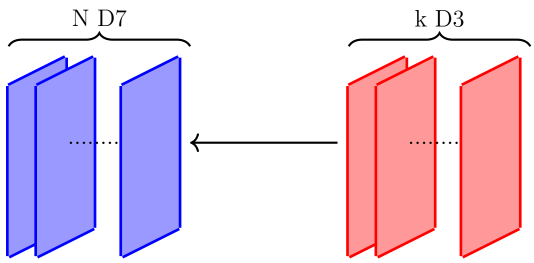

Figure 5.1: Pictorial representation of the ADHM-like construction for the moduli space of instantons on $\mathbb{R}^4$ (when $p = 7$). The Higgs branch of this configuration is reached when we put the stack of D3 breanes inside the stack of D7 branes.

It turns out that the dimension of the moduli space $\mathcal{M}_{k,N}$ is

$$\dim(\mathcal{M}_{k,N}) = 4kN. \tag{5.10}$$

Moreover the moduli space can be endowed with a metric $g_{\alpha\beta}$

$$g_{\alpha\beta} = \frac{1}{2g^2}\int d^4x \text{Tr}[(\delta_\alpha A_\mu)(\delta_\beta A_\mu)]. \tag{5.11}$$

It turns out that the moduli space of instantons is a smooth Hyper-Kähler manifold, away from the points in which the size of the instantons goes to zero (corresponding to *small instantons*).

5.1.2 The ADHM construction of the moduli space

The ADHM construction is a mathematical construction introduced by Atihay, Drinfeld, Hitchin and Manin in [19]. This algorithm allows to construct all the solution to the self-dual equations (5.6). Subsequently it has been understood that such construction can be embedded in string theory, providing this way an *ADHM-like construction*. [52, 53, 54]. In the following we briefly review how the ADHM-like construction works.

Let's consider a configuration with a stack of N Dp branes, and another stack of k $D(p-4)$ branes in Type II string theory. This configuration is graphically summarized in figure 5.1. We place all the Dp branes at same point in spacetime. The QFT living on the worldvolume of the coincident Dp branes is SYM with gauge group $G = U(N)$. This worldvolume theory includes also the couplings with the different RR-fields in the bulk. In particular it includes the term

$$\text{Tr}\int_{Dp} d^{p+1}x\; C_{p-3}\wedge F\wedge F, \tag{5.12}$$

where F is the gauge field of the theory living on the stack of coincident Dp branes, while C_{p-3} is the RR-form that couples to the $D(p-4)$ branes. The important observation is that an instanton on the Dp branes with a non zero $F \wedge F$ gives rise to the source term

$$\frac{8\pi^2}{g^2}\int d^{p-3}x C_{p-3}, \tag{5.13}$$

however this is the same source induced by a $D(p-4)$ brane [56]. Since the factor $8\pi^2/g^2$ can be reinterpreted as the mass and the charge of the brane. Therefore we can identify the instantons on the worldvolume theory of the Dp branes with the $D(p-4)$ branes. In order to derive the ADHM -like construction let's consider the QFT living on the worldvolume of the $D(p-4)$ branes. For instance, without loss of generality, let's fix $p = 7$. Then let's denote the scalar fields that parametrize the D3 branes position as

$$(X^\mu, \hat{X}^m) \qquad \mu = 1, 2, 3, 4 \qquad m = 5, ..., 10, \tag{5.14}$$

such that $\hat{X}^\mu$ parametrize the directions transverse to the branes while X^μ the position of the branes. All these fields can be represented by a $k \times k$ matrix $(X^\mu)^\alpha{}_\beta$ (with $\alpha, \beta = 1, ..., k$). We can define

$$Z = X_1 + iX_2, \quad W = X_3 - iX_4 \tag{5.15}$$

Moreover we have further fields arising from the open strings stretched between the two stacks of branes. We denote these fields with ψ and $\tilde{\psi}$

$$\psi^\alpha{}_a, \quad \tilde{\psi}^a{}_\alpha, \qquad \text{whit} \qquad \alpha = 1, ..., k \ \text{ and } \ a = 1, ..., N. \tag{5.16}$$

The scalar potential of this theory reads

$$V = \frac{1}{g^2}\sum_{m,n}[\hat{X}_m, \hat{X}_n]^2 + \sum_{m,\mu}[\hat{X}_m, X_\mu]^2 + \sum_a((\psi^a)^\dagger \hat{X}_m^2 \psi_a + \tilde{\psi}^a \hat{X}_m^2 (\tilde{\psi}_a)^\dagger) +$$
$$g^2\mathrm{Tr}(\sum_a \psi_a(\psi^a)^\dagger - (\tilde{\psi}_a)^\dagger\tilde{\psi}^a + [Z, Z^\dagger] + [W, W^\dagger])^2 + g^2\mathrm{Tr} \mid \sum_a \psi_a\tilde{\psi}^a + [Z, W] \mid \tag{5.17}$$

We consider a configuration that minimizes the previous scalar potential (5.17), i.e. $V = 0$ and where the D3 branes are inside the D7 branes (i.e. $\hat{X}_m = 0$). This configuration corresponds to the Higgs branch $\mathcal{M}_H$ of the previous theory. Formally is defined as

$$\mathcal{M}_H = \{V = 0, \hat{X}_m = 0\}/U(k). \tag{5.18}$$

where we take the quotient by the gauge group of the theory. The remarkable result is that the Higgs branch coincides with the moduli space of k-$SU(N)$ instantons. This means that the metric on the Higgs branch is equal to the metric on the moduli space of instantons. Moreover the equivalence implies that the Higgs branch is an HyperKähler manifold.

We can check that the dimension of the Higgs branch (5.18) matches the dimension of the moduli space of instantons (5.10). As a matter of fact we have $4kN$ degree of freedom due to ψ^a and $\tilde{\psi}^a$, $4k^2$ in Z and W while the F-terms gives $2k^2$ real constraints, the D-term k^2 constraints and since we are are taking the quotient by $U(k)$ gauge group we have also further k^2 constraints. Therefore summing up the different contributions the dimension of the Higgs branch reads

$$\dim(\mathcal{M}_H) = 4kN \tag{5.19}$$

The previous construction ADHM-like construction can also be generalized for the case of symplectic and orthogonal instantons. This is generalization is obtained adding an orientifold plane of the appropriate charge.

Therefore due to this ADHM-like construction the problem concerning the characterization of instanton moduli space has been reformulated into a problem regarding the characterization of the moduli space of vacua of a supersymmetric QFT. We examine in the next section how this aim can be carried out systematically using the so called *"Plethystic program"* and the a mathematical tool called *Hilbert Series*.

5.1.3 Moduli space of vacua of $4d$ $\mathcal{N} = 1$

In this section we review how we can characterize the moduli space of supersymmetric QFT (and consequently how we can study the moduli space of instantons) counting gauge invariants chiral operators (GIO). As we will se the Hilbert Series will be a very useful tool. [3]

In general a supersymmetric QFT in $4d$ with $\mathcal{N} = 1$ supersymmetry endowed with a superpotential W has a scalar potential V that takes the form

$$V = \sum_i \mid F_i \mid^2 + \frac{g^2}{2} \sum_a (D^a)^2, \tag{5.20}$$

[3] For a full review regarding the Plethystic program and its use in string theory we refer the reader to the seminal papers [20, 57]. While for a review regarding the Hilbert Series and its application in the context of $4d$ and $3d$ with $\mathcal{N} \geq 2$ we refer the reader to the review [58]. In the present section we summarize only the most relevant aspects of this mathematical constructions that have been employed in the paper reported in section 6.2.

Given by the sum of the squares of two distinct contributions called F-terms and D-terms

- *F-terms* are given by
$$F_i(X) = \frac{\partial W}{\partial X^i} \tag{5.21}$$
- *the D-terms* read
$$D^a(X, X^\dagger) = \sum_i X_i^\dagger (T^a)^i_j X^j \tag{5.22}$$
where the T^a are the generators of the gauge group of the QFT. While X^i is the scalar field inside the $\mathcal{N} = 1$ chiral multiplet.

The moduli space of vacua $\mathcal{M}$ is obtained when we set to zero the scalar potential (5.20), this means that

$$\begin{aligned} \mathcal{M} &= \{(X, X^\dagger) \mid F_i(X) = 0 \; \forall \; i, \; D^a(X, X^\dagger) = 0 \; \forall \; a\}/G \\ &\cong \{(X) \mid \partial W(X) = 0\}/G^{\mathbb{C}} \end{aligned} \tag{5.23}$$

where $G^{\mathbb{C}}$ is the complexified gauge group.

The usual strategy that is employed in order to characterize the space (5.23) is to consider *chiral gauge invariant operators* (GIO). These in $4d$ $\mathcal{N} = 1$, $\mathcal{O}_i(x)$. These are 1/2 BPS operators annihilated by all the supercharges with positive R-symmetry eigenvalue, i.e.

$$\bar{Q}_{\dot{\alpha}} \mathcal{O}_i(x) = 0 \;\; \forall \;\; \dot{\alpha} = 1, 2 \tag{5.24}$$

The chiral operators form a commutative ring, that is called *chiral ring* $\mathcal{R}$. Moreover these operators satisfy

$$\mathcal{O}_i \mathcal{O}_j = c^k_{ij} \mathcal{O}_k, \tag{5.25}$$

up to $\bar{Q}$-exact term. The chiral ring is determined when a basis $\{\mathcal{O}\}_{\{i\}}$ has been specified and when the structure constants c^k_{ij} are given. There is a one to one correspondence between vacuum expectation values of GIO $\langle \mathcal{O}_i \rangle$ and holomorphic functions on $\mathcal{M}$, once also possible relations between the generators of the chiral ring have been taken into account. This means that a complete characterization of the chiral ring $\mathcal{R}$ allows to completely determine the manifold $\mathcal{M}$. The chiral Ring in general is a quotient ring

$$\mathcal{R} = \mathbb{C}[\mathcal{O}_1, ..., \mathcal{O}_n]/\mathcal{I}, \tag{5.26}$$

where $\mathbb{C}[\mathcal{O}_1, ..., \mathcal{O}_n]$ denotes the polynomial ring with complex coefficients, while the ideal $\mathcal{I}$ encodes the possible relations between the generators of the chiral ring. For $4d$ $\mathcal{N} = 1$ theories the chiral operators are gauge invariant polynomials constructed using the chiral fields X.

The Hilbert Series

A very useful mathematical tool that can be employed for the characterization of the chiral ring (5.26) is the Hilbert Series. This is a generating function that counts scalar gauge invariants chiral operators according to their conformal dimension and other quantum numbers. Formally the Hilbert Series reads

$$\mathrm{HS}(t, k_i) = \mathrm{Tr}_{\mathcal{H}} \left[t^{\Delta} \prod_{i=1}^{N} k_i^{q_i} \right] \tag{5.27}$$

where the trace is taken over,

$$\mathcal{H} = \{\mathcal{O}_i \mid \bar{Q}_{\dot{\alpha}} \mathcal{O}_i = 0, \quad M_{\mu\nu} O_i = 0\}, \tag{5.28}$$

that is the space of scalar gauge invariant chiral operators. Moreover t denotes the chemical potential related with the $U(1)_R$ R-symmetry, while $(k_1, ...k_N)$ are the chemical potentials related with other global quantum numbers $(q_1, ...q_N)$. Using the Hilbert Series we can extract useful information regarding the moduli space of vacua $\mathcal{M}$

- The complex dimension of the moduli space $\mathcal{M}$ is equal to the dimension of the pole at $t = 1$ of the unrefined Hilbert Series $\mathrm{HS}(t, 1)$.
- Moreover if $\mathcal{M}$ has the topology of a cone (which is the case for a superconformal quantum field theory) the coefficient of $(1 - t)^{-d}$ is proportional to the volume of the base of the cone.
- Finally taking the *Plethystic Logarithm* (PLog)[4] of the Hilbert Series we can extract the charges of the generators of the chiral ring and the relations among them.

Using the properties of the Hilbert Series outlined above we can characterize the moduli space $\mathcal{M}$ as an algebraic variety.

5.2 The moduli space of instantons on $\mathbb{C}P^2$

In the article reported in section 6.2 we move a further step in the characterization of the moduli space of self-dual instantons on $\mathbb{C}P^2$. The ADHM

[4]Given a function $f(t)$, such that $f(0) = 0$. The corresponding Plethystic Logarithm is defined as

$$\mathrm{PLog}[f(t)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} log(f(t^k)),$$

where $\mu(k)$ is the Möbius function.

construction for this moduli space has been introduced by a King 1989 [59]. Then, at a later time, it was understood that such ADHM construction can be embedded in a $3d$ $\mathcal{N}=2$ quiver gauge theory summarized in figure 5.2. Using this ADHM-like construction it was possible to begin the study of this

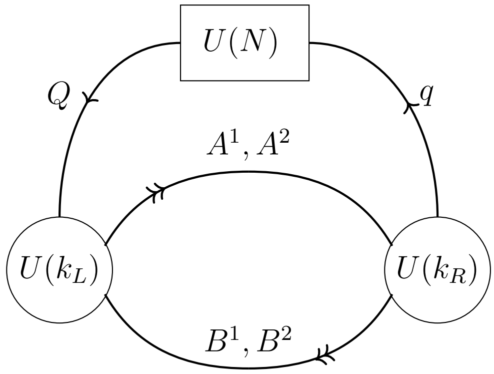

Figure 5.2: Quiver for the $3d$ $\mathcal{N}=2$ theory encoding the ADHM-like construction for self-dual instantons on $\mathbb{C}P^2$.

moduli space [22] from a physical perspective using the technique outlined in section 5.1.3. This moduli space can be characterized by two quantum numbers, namely the *instanton number* $\hat{k}$ and, since $\mathbb{C}P^2$ is a non trivial topological space, also by the *first Chern number* $\hat{c}$. These can be expressed in terms of quiver data in the following way

$$\hat{k} = \frac{1}{2}(k_L + k_R) - \frac{1}{2N}(k_L - k_R)^2, \qquad \hat{c} = k_L - k_R, \tag{5.29}$$

where k_L, k_R and N are the ranks of the gauge and flavour groups of the $3d$ theory. Moreover it turn out that the Hilbert Series for the moduli space of instantons on $\mathbb{C}P^2$ is equal to the Hilbert Series for the moduli space of instantons $\mathbb{C}^2$ [22]. [5] This result naively suggests that the complex dimension of the moduli space of instantons on $\mathbb{C}P^2$ $M^{SU(N)}_{\mathbb{C}P^2}$ should be equal to

$$\dim_{\mathbb{C}} M^{SU(N)}_{\mathbb{C}P^2} = 2N\min(k_L, k_R), \tag{5.31}$$

however the above quantity depends only on two of three integers N, k_L and k_R that must be specified in order to completely determine a given instanton

[5] As shown in [22] the equality between the two Hilbert Series is obtained after an identification of the rank of the flavor group of the two theories and imposing that K the rank of the gauge node of the $\mathbb{C}^2$ is equal to the minimum of the ranks of the gauge nodes of the $\mathbb{C}P^2$ theory, i.e.

$$K = \min(k_L, k_R). \tag{5.30}$$

configuration. This observation suggests that the moduli space of instantons on $\mathbb{C}P^2$ has extra directions (associated to all three quantum numbers) and that in general the Hilbert Series is blind to these directions. In order to resolve these directions it's useful to introduce the notion of *resolved moduli space* $\hat{M}^{SU(N)}_{\mathbb{C}P^2}$, whose complex dimension reads [24, 25, 26]

$$\dim_{\mathbb{C}}\hat{M}^{SU(N)}_{\mathbb{C}P^2} = 2\hat{k}N = \dim_{\mathbb{C}}M^{SU(N)}_{\mathbb{C}P^2} + \hat{c}(N-\hat{c}) \tag{5.32}$$

We can observe that when $\hat{c} = 0$ the dimension of $M^{SU(N)}_{\mathbb{C}P^2}$ is equal to the dimension of $\hat{M}^{SU(N)}_{\mathbb{C}P^2}$.

In the article reported in section 6.2 we begin the study and the characterization of the above directions from a physical point of view. We explicitly analyse the simplest configurations. These have an instanton number $\hat{k}$ equal to zero but a non-vanishing first Chern class $\hat{c}$. In particular using an *ungauging technique* we perform the computation of the Hilbert Series for such configurations and we find an agreement with the dimension of the moduli space predicted by the formula (5.32). In more details we rewrite the full $U(k)$ gauge group of the theory as $U(1) \times SU(k)$ and we promote the $U(1)$ part to a global baryonic symmetry. This way we are able to construct gauge invariant operators and we get a non trivial result for the corresponding Hilbert Series.

Moreover, in the particular case in which $k_L = k_R$ and $N = 1$ (and in the large k_L limit), the $3d$ $\mathcal{N} = 2$ QFT that describes the moduli space of instantons on $\mathbb{C}P^2$ admits a gravity dual [60]. Therefore using the AdS_4/CFT_3 correspondence and a dual giant graviton provided by an M2 branes wrapping the temporal direction and the $\mathbb{S}^2$-sphere inside the AdS_4 space we realize part of the instanton moduli space in the dual geometry. As in other cases already analysed in the literature (see e.g.[61]) we see that the gravity dual is able to capture only the mesonic subranch of the full moduli space.

Finally we also provide the ADHM-like construction as well as the computation of the Hilbert Series of the corresponding moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$. We worked out the case of unitary, orthogonal and symplectic instantons. We conjecture a possible relation between the quiver data of the ADHM-like construction with the instanton quantum numbers. Moreover we outline the correspondence between the Hilbert Series for the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ and the Hilbert Series for the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_n$, that was previously studied in [62]. We found that, after a proper identification of the gauge and flavour nodes of the quiver diagrams describing the two theories, the Hilbert Series of the two moduli space are the same. We refer the reader to the corresponding article 6.2 for further details.

6. Articles

In this section we report the various articles on which the present thesis work is based.

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6.1 Gauge/gravity duality and RG flows in $5d$ gauge theories

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Gauge/gravity duality and RG flows in 5d gauge theories

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Abstract

We discuss RG flows in 5d gauge theories triggered by VEVs of either mesonic or baryonic operators. As a warm-up, we explicitly discuss the counterpart of these flows in 4d gauge theories with $\mathcal{N}=2$ supersymmetry by focusing on the A_1 theory. As opposed to the $\mathcal{N}=1$ case, in cases with 8 supercharges we need to solve a more involved PDE. In the 5d case the boundary conditions for such equation play a crucial role in order to reproduce the expected spectrum.

1. Introduction

The basic version of the AdS_{d+1}/CFT_d correspondence equates gravity in an AdS_{d+1} background with a certain CFT_d living on its boundary. On general grounds, the AdS_{d+1} space arises as the near brane geometry of a stack of branes at the tip of a certain – typically singular – cone whose radial coordinate becomes the AdS_{d+1} radius while its base encodes the details of the CFT_d. While the CFT_d might have a rich structure of vacua, the AdS_{d+1} dual describes just the trivial one where no operator is taking a VEV. In the following we will assume the CFT_d to have a moduli space of vacua. Hence, it is natural to probe its structure by moving among vacua upon considering operator VEVs in the CFT_d. On general grounds, operator VEVs will trigger

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RG flows from the original theory to a new IR fixed point. Conversely, the dual gravitational description will not anymore be exactly AdS_{d+1} but only an asymptotically AdS_{d+1}. In fact, since we are considering moving on the moduli space of the CFT_d rather than adding a deformation, the asymptotics should be just the same. Then, the internal structure of the space as one goes from the boundary to the bulk encodes the precise details of the RG flow, in particular which operators are taking a VEV as well as their dimensions and other quantum numbers. Of course, deep in the interior, one expects a different AdS_{d+1} throat to locally develop standing for the IR fixed point theory.

In this paper we study aspects of these RG flows in theories with 8 supercharges in 4 and 5 dimensions. Our flows come in two broad types, namely one in which a meson-like operator acquires a VEV, and another in which a baryon-like operator acquires a VEV. The former corresponds to locating the branes sourcing the geometry away from the tip of the singular cone – hence we dub them singular flows – while the later correspond to blow-ups of the cone – we refer to them as baryonic flows. As the name indicates, in the baryonic flows a baryonic symmetry undergoes spontaneous breaking by the VEV. In the 4d case, similar flows have been considered, mostly for the $\mathcal{N}=1$ case (see e.g. [1–6]). Here we consider in detail the $\mathcal{N}=2$ case, which presents some particularities such as the need to solve a rather involved PDE. Furthermore, this case serves as warm-up for the basically unexplored 5d case. In the later we find an interesting interplay among boundary conditions which allows to find the correct dimensions for the operators in the gravity side.

We will particularize our discussion to branes probing the A_1 singularity. While the 4d case corresponds to D3 branes probing $\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}$, the 5d case corresponds to D4 branes probing the A_1 singularity wrapped by an $O8^-$ plane with N_f D8. Then, the organization of the paper is as follows. In Section 2 we study the gravitational aspects of $d=4$ flows, both in the singular and resolved cases. We then turn to the gauge theory description, discussing first the wavefunctions on the A_1 singularity. Through these we can explicitly identify the dual operators, both in the singular and resolved case. The latter nicely fits as a broken baryon symmetry phase. Indeed, using standard techniques, we identify both the VEV of the baryon condensate and the Goldstone boson. In Section 3 we turn to the 5d case. This case is a bit more subtle, as two possible theories, due to the orientifold action, are possible. After describing them, we turn to study flows in the singular and resolved spaces. A singularity in the background, already present in the trivial AdS_6 vacuum and with a clear string theory interpretation, plays a crucial role in selecting the dimensions of the operators taking VEV. In the broken baryon symmetry case we also identify the VEV of the condensate as well as the Goldstone boson. We finish in Section 4 with some conclusions.

2. 4d $\mathcal{N}=2$ flows

In this section we study RG flows in a 4d $\mathcal{N}=2$ gauge theory through its holographic dual. The simplest example is the so-called A_1 gauge theory, which can be engineered by placing N D3 branes probing a $\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}$ singularity [7]. The CFT is a $SU(N)\times SU(N)$ gauge theory with global non-R symmetry $SU(2)_M\times U(1)_B$, being $SU(2)_M$ a mesonic global symmetry and $U(1)_B$ a baryonic symmetry.

The types of flows which we will consider are triggered by motion on the moduli space, that is, by the VEV of certain operators. On general grounds we can imagine two types of such flows: one "mesonic" type where all operators acquiring a VEV are neutral under the $U(1)_B$ and another "baryonic" type where an operator charged under the $U(1)_B$ acquires a VEV. The former possibility corresponds to the case where the stack of D3 branes sourcing the geometry is located away from the tip of the cone, while the later possibility corresponds to the blow-up of the cone. We stress that in both cases the backgrounds corresponding to the flows asymptote to the same geometry, namely that of the singular cone. Hence both geometries indeed correspond to motion on the moduli space of the dual gauge theory – rather than turning on a deformation.

In the gravity side, we consider a stack of D3 branes probing the – possibly resolved – $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ geometry (we collect some useful details on the geometry of the $\mathbb{C}^2/\mathbb{Z}_2$ singularity in Appendix A). On general grounds we can write the standard Freund–Rubin ansatz for the background

$$ds^2 = h^{-\frac{1}{2}} dx_{1,3}^2 + h^{\frac{1}{2}} ds_6^2, \qquad F_5 = (1+\star) d\left(h^{-1}\right) \wedge dx^0 \wedge \cdots \wedge dx^3; \tag{1}$$

where h is only a function of the internal coordinates and the internal space metric ds_6^2 is

$$ds_6^2 = \frac{dr_1^2}{f(r_1)} + \frac{r_1^2}{4} f(r_1)(d\psi + \cos\theta d\phi)^2 + \frac{r_1^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + dr_2^2 + r_2^2 d\chi^2 \tag{2}$$

being

$$f(r_1) = 1 - \frac{c^4}{r_1^4}. \tag{3}$$

The equation of motion of the 5-form field strength yields to

$$d \star_6 dh = \mathcal{C}\delta, \tag{4}$$

being $\star_6$ the Hodge dual with respect the metric in the internal space, $\mathcal{C}$ a normalization constant and δ the source term – in the end we have D3 branes somewhere in the cone. In the following we will particularize this general equation to the cases of interest.

2.1. Flows on the singular cone

We consider a stack of D3 branes at a certain point away from the tip of the singular $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ in the internal space. The equation of motion (4) is just the Laplace equation in the internal space, which can be written as

$$\frac{1}{r_1^3}\partial_{r_1}\left(r_1^3 \partial_{r_1} h\right) + \frac{4}{r_1^2}\Delta h + \frac{4}{r_1^2}\partial_\psi^2 h + \frac{1}{r_2}\partial_{r_2}(r_2 \partial_{r_2} h) + \frac{1}{r_2^2}\partial_\chi^2 h = \frac{\mathcal{C}}{\sqrt{\det g_6}}\delta(X - X_0), \tag{5}$$

where X is a generic label for the internal coordinates and X_0 is the position where the stack is; and where $\mathcal{C}$ is a constant related to the AdS_5 radius L as

$$L^4 = \frac{\mathcal{C}}{4\,\mathrm{vol}(S^5/\mathbb{Z}_2)}. \tag{6}$$

Besides

$$\Delta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \left(\frac{\partial_\phi}{\sin\theta} - \cot\theta\partial_\psi\right)^2. \tag{7}$$

Let us collectively denote the $\{\psi, \theta, \phi, \chi\}$ coordinates by ξ. Then the Laplacian reads schematically $(\Delta_{r_1} + \Delta_{r_2} + \Delta_\xi)h = \mathcal{C}\delta(r_1 - r_1^0)\delta(r_2 - r_2^0)\delta(\xi - \xi_0)$, being Δ_k the appropriate Laplacians along the X^k directions.

Introduce now the functions $\tilde{Y}_{R,l,m} = e^{im\phi} e^{iR\psi} J_{R,l,m}(\theta)$ satisfying[1] [8]

$$\Delta \tilde{Y}_{R,l,m} = -\big(l(l+1) - R^2\big)\tilde{Y}_{R,l,m}. \tag{8}$$

We can then construct

$$Y_I = e^{in\chi}\tilde{Y}_{R,l,m}, \tag{9}$$

where I collectively stands for all indices. As reviewed in Appendix B, in order to have a well-defined solution, $R, l, m \in \mathbb{Z}$ (with $l \geq |m|$, $l \geq |R|$). Besides, since $\chi \in [0, 2\pi]$, it is clear that $n \in \mathbb{Z}$. Upon expanding

$$h = \sum h_I(r_1, r_2) Y_I^\star(\xi_0) Y_I(\xi), \tag{10}$$

then the equation of motion becomes of the form $\sum(\Delta_{r_1} + \Delta_{r_2} + f(r_1, r_2))h_I Y_I^\star(\xi_0) Y_I(\xi) = \mathcal{C}\delta(r_1 - r_1^0)\delta(r_2 - r_2^0)\delta(\xi - \xi_0)$. Since $\tilde{Y}_{R,l,m}$ form a complete set of eigenfunctions, so do the Y_I. Hence $\sum_I Y_I^\star(\xi_0)Y_I(\xi) = \delta(\xi - \xi_0)$. Thus, in order to find a solution of the complete equation we need to demand $(\Delta_{r_1} + \Delta_{r_2} + f(r_1, r_2))h_I = \mathcal{C}\delta(r_1 - r_1^0)\delta(r_2 - r_2^0)$, which explicitly reads

$$\begin{aligned}
&\frac{1}{r_1^3}\partial_{r_1}\big(r_1^3\partial_{r_1} h_I\big) - \frac{4l(l+1)}{r_1^2} h_I + \frac{1}{r_2}\partial_{r_2}(r_2\partial_{r_2} h_I) - \frac{n^2}{r_2^2} h_I \\
&\quad = \frac{\mathcal{C}}{r_1^3 r_2}\delta\big(r_1 - r_1^0\big)\delta\big(r_2 - r_2^0\big).
\end{aligned} \tag{11}$$

In order to further proceed, it is useful to introduce polar coordinates as $r_1 = \rho\cos\alpha$ and $r_2 = \rho\sin\alpha$. In these coordinates the branes will be at $\{\rho_0, \alpha_0\}$. It is easy to check that the regular solutions are

$$\begin{aligned}
h_I^< &= \frac{\mathcal{C}}{4l + 2n + 4}\frac{1}{\rho_0^4}\left(\frac{\rho}{\rho_0}\right)^{2l+n} \cos^{2l}\alpha \sin^n\alpha \leftrightarrow \rho < \rho_0, \\
h_I^> &= \frac{\mathcal{C}}{4l + 2n + 4}\frac{1}{\rho^4}\left(\frac{\rho_0}{\rho}\right)^{2l+n} \cos^{2l}\alpha \sin^n\alpha \leftrightarrow \rho > \rho_0.
\end{aligned} \tag{12}$$

Collecting all the pieces, we can write the warp factor in the $\rho > \rho_0$ region as

$$h = \frac{\mathcal{C}}{4\rho^4} + \sum_{l,n>0} \frac{\mathcal{C}}{4l + 2n + 4}\frac{1}{\rho^4}\left(\frac{\rho_0}{\rho}\right)^{2l+n} \mathcal{Y}_I(\alpha_0, \xi_0)^\star \mathcal{Y}_I(\alpha, \xi); \tag{13}$$

while in the $\rho < \rho_0$ region it reads

$$h = \frac{\mathcal{C}}{4\rho_0^4} + \sum_{l,n>0} \frac{\mathcal{C}}{4l + 2n + 4}\frac{1}{\rho_0^4}\left(\frac{\rho}{\rho_0}\right)^{2l+n} \mathcal{Y}_I(\alpha_0, \xi_0)^\star \mathcal{Y}_I(\alpha, \xi); \tag{14}$$

being

$$\mathcal{Y}_I(\alpha, \xi) = \cos^{2l}\alpha \sin^n\alpha Y_I(\psi, \theta, \phi, \chi). \tag{15}$$

Note in particular that, as advertised above, the geometry asymptotes just like the singular cone. Indeed h, in the $\rho > \rho_0$ region, starts as ρ^{-4}.

[1] See the Appendix B for an introduction to these eigenfunctions.

2.1.1. An alternative computation

The coordinates used in the previous section are adapted to the orbifold, as the A_1 singularity is explicitly separated from the $\mathbb{C}$ factor. However, upon a change of coordinates, the metric in the internal space can be written as

$$ds_6^2 = d\rho^2 + \rho^2 ds^2_{S^5/\mathbb{Z}_2}. \tag{16}$$

The equation for h is just

$$\frac{1}{\rho^5}\partial_\rho\left(\rho^5\partial_\rho h\right) + \frac{1}{\rho^2}\Delta_{S^5/\mathbb{Z}_2} h = \mathcal{C}\delta, \tag{17}$$

being $\Delta_{S^5/\mathbb{Z}_2}$ the Laplacian on the $S^5/\mathbb{Z}_2$. Since that space is locally identical to S^5, the equation of motion looks exactly like the S^5 one. Denoting the S^5 angular coordinates by $\bar{\xi}$ and setting

$$h = \sum h_I(\rho)\mathcal{Y}_I^\star(\bar{\xi}_0)\mathcal{Y}_I(\bar{\xi}), \tag{18}$$

with $\mathcal{Y}_I$ the S^5 spherical harmonics, the $h_I(\rho)$ eom is

$$\frac{1}{\rho^5}\partial_\rho\left(\rho^5\partial_\rho h_I\right) - \frac{\ell(\ell+4)}{\rho^2}h_I = \frac{\mathcal{C}}{\rho^5}\delta(\rho-\rho_0); \tag{19}$$

whose solution is

$$h_I^< = \frac{\mathcal{C}}{4+2\ell}\frac{1}{\rho_0^4}\left(\frac{\rho}{\rho_0}\right)^\ell \quad \leftrightarrow \quad \rho<\rho_0,$$
$$h_I^> = \frac{\mathcal{C}}{4+2\ell}\frac{1}{\rho^4}\left(\frac{\rho_0}{\rho}\right)^\ell \quad \leftrightarrow \quad \rho>\rho_0. \tag{20}$$

We have not yet taken into account the orbifold. Prior to orbifolding, we see that modes are classified into representations of spin ℓ of $SO(6)$. Decomposing such representations into $U(1)_\chi \times SU(2)_M \times SU(2)_R$, the orbifold selects a subset of representations with even spin, that is $\ell = 2l + n$, where $2l$ stands for the orbifold, hence recovering the results in the previous analysis.

2.2. Flows on the resolved cone

We now consider the resolution of the cone (see Appendix A). Then Eq. (4) becomes

$$r_1^{-3}\partial_{r_1}\left(r_1^3 f\partial_{r_1}h\right) + \frac{4}{r_1^2}\Delta h + \frac{4}{r_1^2 f}\partial_\psi^2 h + \frac{1}{r_2}\partial_{r_2}(r_2\partial_{r_2}h) + \frac{1}{r_2^2}\partial_\chi^2 h = \mathcal{C}\delta, \tag{21}$$

where we collectively denote the sources by δ and where

$$\Delta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \left(\frac{\partial_\phi}{\sin\theta} - \cot\theta\partial_\psi\right)^2. \tag{22}$$

Just as in the singular case we can write

$$h = \sum_I h_I(r_1,r_2)Y_I^\star(\xi_0)Y_I(\xi). \tag{23}$$

Hence, the equation for h_I (away from the sources) is

$$r_1^{-3}\partial_{r_1}\big(r_1^3 f\partial_{r_1}h_I\big) - \frac{4(l(l+1)-R^2)}{r_1^2}h_I - \frac{4R^2}{r_1^2 f}h_I + \frac{1}{r_2}\partial_{r_2}(r_2\partial_{r_2}h_I) - \frac{n^2}{r_2^2}h_I = 0. \tag{24}$$

Since the D3 branes will be located at $r_1 = c$ and $r_2 = 0$, where the χ circle and the ψ circle shrink, it's reasonable to truncate to $R = n = 0$. Then, the equation to solve is

$$\frac{1}{r_1^3}\partial_{r_1}\big(r_1^3 f\partial_{r_1}h_I\big) - \frac{4l(l+1)}{r_1^2}h_I + \frac{1}{r_2}\partial_{r_2}(r_2\partial_{r_2}h_I) = 0. \tag{25}$$

Let us now switch to the $\{\rho, \alpha\}$ coordinates, so that the above equation becomes

$$\begin{aligned}&\big(-4l(1+l)\rho^4 h_I \sec(\alpha)^2\tan(\alpha) + \big(\rho^4 - \big(3\rho^4 + c^4\big(2+\cos(2\alpha)\big)\sec(\alpha)^4\big)\tan(\alpha)^2\big)\partial_\alpha h_I\\ &\quad + \tan(\alpha)\big(\big(\rho^4 - c^4\sec(\alpha)^2\tan(\alpha)^2\big)\partial_\alpha^2 h_I + \rho\big(\big(5\rho^4 + c^4\sec(\alpha)^2\big)\partial_\rho h(\rho,\alpha)\\ &\quad + 2c^4\sec(\alpha)^2\tan(\alpha)\partial_\alpha\partial_\rho h_I + \rho\big(\rho^4 - c^4\sec(\alpha)^2\big)\partial_\rho^2 h_I\big)\big)\big)\tan(\alpha) = 0.\end{aligned} \tag{26}$$

We have not been able to solve exactly the previous equation. Nevertheless, we can study its large-ρ asymptotic properties. To that matter, we set to first order

$$h_I = \left(\frac{c}{\rho}\right)^a f(\alpha), \tag{27}$$

and expand in powers of $\frac{c}{\rho}$. The leading term determines $f(\alpha)$ as

$$\big(-4a + a^2 - 2m^2 + (-4+a)a\cos(2\alpha)\big)f\sin\alpha + 2\big(\cos(3\alpha)f' + \cos^2\alpha\sin\alpha f''\big) = 0; \tag{28}$$

where for simplicity we have set $m^2 = 4l(l+1)$. The solution to this is

$$\begin{aligned}f = \cos^{-1+\sqrt{1+m^2}}\alpha\,{}_2F^1\bigg[&\frac{3}{2} - \frac{a}{2} + \frac{\sqrt{1+m^2}}{2},\\ &-\frac{1}{2} + \frac{a}{2} + \frac{\sqrt{1+m^2}}{2}, 1+\sqrt{1+m^2}, \cos^2\alpha\bigg].\end{aligned} \tag{29}$$

Regularity at $\alpha = 0$ demands

$$a = 4 + \sqrt{1+m^2} - 1 + 2q = 4 + 2l + 2q, \tag{30}$$

for $q \in \mathbb{Z}$. This integer arises since regularity at $\alpha = 0$ demands a certain $\Gamma(x)^{-1}$ function to vanish, which happens for $x = -q$. This integer should not be confused with the n in Section 2.1, as the later is related to the $U(1)$ charge conjugate to χ, while q is not related to any charge.

We can directly read off the dimension of the modes from here, which is just $a-4$. Hence $\Delta = 2l + 2q$. Note that, since $a = 4 + \Delta$, we again have that, as promised, the geometry asymptotically becomes just the same as the singular cone. This explicitly reflects the fact that also the blow-up geometry corresponds to a flow triggered not by a deformation of the gauge theory but by a VEV.

2.3. Gauge theory

The *AdS*/*CFT* duality implies that the above geometries describe two different RG flows along which certain operators $\mathcal{O}_I$ acquire a VEV. Such operators correspond to the wavefunctions

$$\langle\mathcal{O}_I\rangle = \rho_0^{2l+mn}\mathcal{Y}_I(\xi_0)^\star, \tag{31}$$

being $m = 1$ for the singular case and $m = 2$ for the resolved case.

In order to identify the gauge theory operators corresponding to the $\mathcal{Y}_I(\xi_0)^\star$, let us first turn to the algebraic geometry of the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$. Introducing complex coordinates z_i related to the real ones as

$$z_1 = \rho \cos\alpha e^{i\frac{\psi+\phi}{2}} \cos\frac{\theta}{2}, \qquad z_2 = \rho \cos\alpha e^{i\frac{\psi-\phi}{2}} \sin\frac{\theta}{2}, \qquad z_3 = \rho \sin\alpha e^{i\chi}, \tag{32}$$

we have that the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ space is defined by the z_i coordinates together with the identification $(z_1, z_2) \sim (-z_1, -z_2)$.

Let us strip off the $\mathbb{C}$ factor and concentrate on the $\mathbb{C}^2/\mathbb{Z}_2$ space. In complex coordinates, the metric is simply $ds^2_{A_1} = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$. This is invariant under $SU(2)_M \times SU(2)_R$. The $SU(2)_M$ symmetry acts on the (z_1, z_2) doublet, while the $SU(2)_R$ symmetry acts on the $(z_1, \bar{z}_2)$ doublet. Both these actions are compatible with the orbifold projection for this $\mathbb{Z}_2$ case – since it acts diagonally by a -1 on either doublet.

Let us first forget about the orbifold projection. Out of the $SU(2)_R$, only its $U(1)_R$ subgroup is manifest. Hence, it is useful to consider the manifest symmetry subgroup $SU(2)_M \times U(1)_R$. The charges of the $\{z_i, \bar{z}_i\}$ under its $U(1)_M \times U(1)_R$ Cartan are

	z_1	z_2	$\bar{z}_1$	$\bar{z}_2$
$U(1)_M$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$U(1)_R$	1	1	-1	-1

(33)

Following [1], we consider forming the states

$$\left|\frac{n}{2}, m\right\rangle = \sqrt{\frac{1}{(\frac{n}{2}+m)!(\frac{n}{2}-m)!}}\, z_1^{\frac{n}{2}+m} z_2^{\frac{n}{2}-m},$$
$$\overline{\left|\frac{n}{2}, m\right\rangle} = \sqrt{\frac{1}{(\frac{n}{2}+m)!(\frac{n}{2}-m)!}}\, \bar{z}_2^{\frac{n}{2}+m} \bar{z}_1^{\frac{n}{2}-m}. \tag{34}$$

Here $m \in [-\frac{n}{2}, \frac{n}{2}]$. Such states have the following charges

	$\lvert\frac{n}{2}, m\rangle$	$\overline{\lvert\frac{n}{2}, m\rangle}$
$U(1)_M$	m	m
$U(1)_R$	n	$-n$

(35)

Thus these states are the states in a spin $l = \frac{n}{2}$ representation of $SU(2)$ – hence the labels of the states. On the other hand, for the $|\frac{n}{2}, m\rangle$, since $l = \frac{n}{2}$ and $R = n$, it is clear that $\frac{n}{2} = \frac{l}{2} + \frac{R}{4}$, while for the $\overline{|\frac{n}{2}, m\rangle}$, since $l = \frac{n}{2}$ and $R = -n$, $\frac{n}{2} = \frac{l}{2} - \frac{R}{4}$. Hence, we can label the states as

$$\left|\frac{n}{2}, m\right\rangle = \left|\frac{l}{2} + \frac{R}{4}, m\right\rangle, \qquad \overline{\left|\frac{n}{2}, m\right\rangle} = \left|\frac{l}{2} - \frac{R}{4}, m\right\rangle. \tag{36}$$

With these we can now construct arbitrary $SU(2)_M$ and $U(1)_R$ states. These are labeled by the $SU(2)_M$ quantum numbers $\{l, m\}$ and by the R charge as $|l, m; R\rangle$, and are constructed as

$$|l, m; R\rangle = \sum C^{\{l,m\}}_{\{\{\frac{l}{2}+\frac{R}{4}, m_1\}, \{\frac{l}{2}-\frac{R}{4}, m_2\}\}} \left|\frac{l}{2} + \frac{R}{4}, m_1\right\rangle \left|\frac{l}{2} - \frac{R}{4}, m_2\right\rangle, \tag{37}$$

where $C^{\{j,m\}}_{\{\{j_1,m_1\},\{j_2,m_2\}\}}$ are the top spin Clebsch–Gordan coefficients. One can then easily check that the states $|l,m;R\rangle$, when written in terms of the real coordinates, become just the $\tilde{Y}_{R,l,m}$ functions.

We are still not done, as we need to recover the $\mathbb{C}$ factor. It is parametrized by the complex coordinate z_3, and the metric is $ds^2_{\mathbb{C}} = dz_3 d\bar{z}_3$, which is obviously invariant under a $U(1)_\chi$ symmetry. The relevant wavefunctions are simply, up to a normalization, $|n\rangle = z_3^n$, which have $U(1)_\chi$ charge n. It then follows that

$$\mathcal{Y}_I = |n\rangle|l,m;R\rangle. \tag{38}$$

Note that z_3^n adds a factor of $\sin^n\alpha$, so that the $\mathcal{Y}_I$ recover the expression of the wavefunctions obtained from the gravity side.

We still need to consider the effect of the orbifold projection. Upon acting with the orbifold, the states in Eq. (34) pick a factor $(-1)^{\pm n}$. Hence, the surviving states are those for which n is even. This translates into the fact that the only allowed states are those with even R.

We now turn to the gauge theory. It is a $\mathcal{N}=2$ $SU(N)\times SU(N)$ gauge theory with 2 hypers which can be broken in $\mathcal{N}=1$ language into two bifundamentals in the $(\square,\bar{\square})$ and two bifundamentals in the $(\bar{\square},\square)$. The W is

$$W = \phi_1 A_i B_j \epsilon^{ij} + \phi_2 B_i A_j \epsilon^{ij}. \tag{39}$$

The theory shows an $SU(2)_M$ symmetry rotating A_i and B_i. Besides, it has a $U(1)_r \times SU(2)_R$ R-symmetry. The Cartan of the non-abelian factor of the R-symmetry will be denoted $U(1)_R$, while the Cartan of the $SU(2)_M$ will be denoted by $U(1)_M$. The charges of each field under such Cartans are

	ϕ_1	ϕ_2	A_1	A_2	B_1	B_2
$U(1)_M$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$U(1)_R$	0	0	1	1	1	1
$U(1)_r$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

(40)

Note that we can combine $U(1)_R$ and $U(1)_r$ into a new $U(1)_\chi$ such that only the ϕ_i are charged.

The F-terms are

$$A_1B_2 = A_2B_1, \qquad B_1A_2 = B_2A_1, \qquad B_i\phi_1 = \phi_2 B_i, \qquad \phi_1 A_i = A_i\phi_2. \tag{41}$$

Let us start considering operators purely in the Higgs branch, *i.e.* those made out of A_i and B_j with no adjoints. Imposing the F-terms, we can construct the following three operators $u = A_1B_1$, $v = A_2B_2$ and $w = A_1B_2$ subject to the relation $uv = w^2$ defining the $\mathbb{C}^2/\mathbb{Z}_2$ singularity. We can solve this relation introducing $u = z_1^2$, $v = z_2^2$ and $w = z_1z_2$. Clearly $(z_1,z_2) \sim -(z_1,z_2)$. Hence this way we have an explicit mapping between the states constructed above and the operators in field theory, which we will collectively denote by $\mathcal{O}_I$.

Denoting by $J_3^{M,R}$ the third component of the $SU(2)_{M,R}$, one can check that the states with the maximal $J_3^{M,R}$ are of the form z_1^{2l}, which becomes $(A_1B_1)^{2l}$. Lower $J_3^{M,R}$ operators generically involve both z_i and $\bar{z}_i$, and hence correspond to non-holomorphic operators. However, they can be thought as descendants obtained by repeatedly acting with $SU(2)_{M,R}$ raising/lowering operators, acting on the doublets (A_i,B_i) and $(A_i,\epsilon_{ij}B_j^\dagger)$ respectively. Hence, every operator can be regarded as belonging to the same multiplet as a purely holomorphic operator, hence having the same dimension equal to the classical one.

Let's now look at operators in the Coulomb branch. These will involve insertions of ϕ_i^n, and hence will have n charge under $U(1)_\chi$. Note also that the theory has a $\mathbb{Z}_2$ symmetry exchanging the two gauge groups while at the same time doing

$$\phi_1 \leftrightarrow \phi_2, \qquad A_i \leftrightarrow B_i. \tag{42}$$

This is just the quantum symmetry of the orbifold. We can construct operators which are not invariant under the symmetry. Consider for example

$$\mathcal{O}_+ = \mathrm{Tr}\,\phi_1^2 + \mathrm{Tr}\,\phi_2^2, \qquad \mathcal{O}_- = \mathrm{Tr}\,\phi_1^2 - \mathrm{Tr}\,\phi_2^2. \tag{43}$$

Clearly, only $\mathcal{O}_+$ is invariant, while $\mathcal{O}_-$ picks a minus sign. As only twisted sector states are charged under the quantum symmetry of the orbifold, we must conclude that the $\mathcal{O}_-$ field comes from the twisted sector. On the other hand, it is natural to expect that the operators capturing the motion of the branes are neutral under this symmetry. This naturally leads to consider the branch where $\phi_1 = \phi_2$, so that ϕ^n naturally corresponds to the z_3^n above. In fact, the $U(1)_\chi$ charge is precisely the expected n.

2.3.1. The singular case

Since in the singular flows we placed the branes away from the tip of the singular cone, it is natural to expect that no twisted sector field will get a VEV. Hence, let us consider the branch of the moduli space which freezes twisted sector fields by setting $\phi = \phi_1 = \phi_2$. Then, the F-term relations imply that $\phi_{1,2}$ commute with A_i and B_i. Therefore a generic operator can be written as $\mathrm{Tr}\,\phi^n \mathcal{O}_I$. This exactly mimics the structure of the wavefunctions which we have found, as indeed the $|n\rangle$ state corresponds to the ϕ^n. Such operators come in spin l representations of $SU(2)_M$ and have dimension $2l + n$, exactly as the modes found in supergravity. Thus, since only mesonic fields take a VEV, the dual gauge theory is in a mesonic branch.

Note that we could place the stack of branes at the origin of the $\mathbb{C}$ plane, which would force us to impose $n = 0$. Then the operators taking a VEV are purely mesonic operators on the Higgs branch, which is then isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. Alternatively we could place the branes at the origin of the $\mathbb{C}^2/\mathbb{Z}_2$, which would demand to set $l = 0$. Hence the operators taking a VEV would be those of the form ϕ^n.

2.3.2. The resolved case and baryonic symmetry breaking

Recall that in this case we truncated $R = 0$ and $n = 0$. Hence the wavefunctions for the modes can be translated into operators following the general discussion above upon setting $R = 0$ and $n = 0$. The dimension of these operators is $\Delta = 2l + 2q$. Setting for the moment $q = 0$, we see that we get a tower of operators with spin l under $SU(2)_M$ and neutral under $SU(2)_R$ – because $R = 0$. The lowest operator is $l = 1$, which has dimension 2 and includes, as its $m = 0$ component, the familiar $\mathcal{U} \sim A_i^\dagger A_i - B_i B_i^\dagger + [\phi_1, \bar{\phi_1}] - [\phi_2, \bar{\phi_2}]$ operator, which is part of the baryonic current multiplet, just as in [1], thus supporting the claim that the resolved background corresponds to a spontaneously broken baryonic symmetry phase.

For higher l modes one can simply use the dictionary in our general discussion above and find the corresponding operator $\mathcal{O}_I$. In addition, analogously to the conifold case [1], one can check that going one order further in $\frac{c}{\rho}$, the dimension of the operator is $\Delta + 2$, which suggests that these higher order terms corresponds to insertions of powers of $\mathcal{U}$, that is, to operators of the form $\mathcal{U}^n \mathcal{O}_I$ for $n = 1, 2, \cdots$.

Let us now turn to the integer q. Again the wavefunctions can be read off from our general discussion just by setting $R = 0$ and $n = 0$. It thus seems that there is no room for the extra integer q. Nevertheless one can imagine constructing an operator of the form $\mathcal{T} \sim \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger$ which is only charged under $U(1)_r$ and has dimension 2, at least classically. Even though this is a non-chiral field, based on the findings on the holographic dual, we conjecture that the classical dimension continues to hold, so that the modes with higher q correspond to insertions of this operator, that is, to $\mathcal{T}^q\mathcal{O}_I$. Note that this is a twisted sector field, which fits naturally within the blow-up scenario.

As we have argued, among the lowest dimension operators taking a VEV we find the scalars in the conserved current multiplet. Hence it is natural to identify this background with a phase of the gauge theory where the baryonic symmetry has been spontaneously broken by giving a VEV to a single chiral field, which, with no loss of generality we can take as $A_1 = c\mathbb{1}$. This gives a VEV to a dimension $\Delta = N$ baryonic operator $\mathcal{B}_A = A_1^N \sim c^N$. Such VEV is captured holographically by the so-called baryonic condensate, namely an Euclidean D3 brane wrapping $X = \{r_1, r_2, \psi, \chi\}$. Its DBI action is

$$S_{DBI} = \frac{T_3}{2}\int_X r_1 r_2 h. \tag{44}$$

Plugging in the expression for the warp factor we find[2]

$$S_{DBI} = \frac{T_3}{2}(2\pi)^2 \sum_l \tilde{Y}^\star_{0,l,0}(\xi_0)\tilde{Y}_{0,l,0}(\xi)\int r_1 r_2 h_l(r_1, r_2). \tag{45}$$

The integrand scales asymptotically as ρ^{-1-2l}. Hence all the $l \neq 0$ terms will give some finite ρ integral, while the $l = 0$ will give a logarithmically divergent integral which has to be cut-off at some ρ_c. So we can separate the $l = 0$ term and write

$$\begin{aligned} S_{DBI} = {} & \frac{T_3}{2}(2\pi)^2 \int r_1 r_2 h_0(r_1, r_2) \\ & + \frac{T_3}{2}(2\pi)^2 \sum_{l>0} \tilde{Y}^\star_{0,l,0}(\xi_0)\tilde{Y}_{0,l,0}(\xi)\int r_1 r_2 h_l(r_1, r_2). \end{aligned} \tag{46}$$

It is easy to show that the divergent term diverges like $S_{DBI} = N \log \rho_c$, being ρ_c a UV cut-off. This corresponds to the VEV of dimension $\Delta = N$ operator, as expected for a baryon VEV.

Alternatively, we can consider a $D3$ brane wrapping the blown-up S^2. Its action is simply

$$S_{DBI} = -T_3 c^2 \pi \int dx^0 dx^1. \tag{47}$$

This finite-tension object, which looks like a cosmic string from the field theory perspective and is "electric–magnetic" dual to the baryon condensate, would source a δC_4 such that

$$\delta F_5 = (1 + \star) da_2 \wedge W, \tag{48}$$

being W a 2-form in the internal space which, for $r \sim c$, becomes the volume form of the blown-up cycle while a_2 is a 2-form in the Minkowski directions whose existence, along the lines in [4], we assume here. The δF_5 then contains a piece with $\delta F_5 \supset \star_4 da_2 \wedge (h \star_6 W)$. We can locally write

[2] Note that one can re-write this warped volume in terms of the Kähler form of the cone, finding $S_{DBI} = \frac{T_3}{4}\int_X h\frac{\omega^2}{2!}$, and hence adapt the proof in [8] to the CY_3 case to conclude that VEV of the baryon condensate will have the expected form.

$\star_4 da_2 = dp$, so a local integration is $\delta C_4 = p(h \star_6 W)$. Note that $\star_6 W$ is a 4-form transverse, in the internal space, to the blown-up 2-cycle, that is, precisely along the directions wrapped by the baryonic condensate brane. Thus, its WZ action is of the form $e^{iT_3 p \int h \star_6 W}$, that is, becomes the phase of the VEV of the baryon. Then the Minkowski scalar p is naturally identified with the Goldstone boson of the broken global symmetry. Furthermore, the equations of motion of δF_5 demand that p satisfies $d \star_4 dp = \delta(x - x_0)$ begin x_0 the position of the cosmic string – D3 brane – in the Minkowski, thus showing how the Goldstone boson is an axionic field winding around the defect arising from spontaneous symmetry breaking [4].

3. 5d RG flows

We now turn to the 5d case. Naively, gauge theories in 5d are non-renormalizable. Nevertheless it was shown in [9] that, under certain circumstances, upon appropriately choosing gauge group and matter content they can be at fixed points. As these theories admit a large N limit [10], it is natural to look for a gravity dual.

For the class of theories discussed in [9] the gravity dual was found in [11] by considering N_f D8 branes on top of an $O8^-$ probed by N D4 branes. If $N_f < 8$, and upon tuning the dilaton to diverge on top of the 8-brane stack, the near brane geometry becomes $AdS_6 \times \hat{S}^4$, being $\hat{S}^4$ half of a 4-sphere due to the left-over of the orientifold projection in the near brane region. On the other hand, the dual field theory is a $USp(2N)$ gauge theory with one antisymmetric hypermultiplet.

As opposed to lower dimensionalities, AdS_6 geometries are very scarce [12].[3] However, the basic AdS_6/CFT_5 duality can be naturally extended by orbifolding the internal space [15]. The geometry becomes $AdS_6 \times \hat{S}_4/\mathbb{Z}_k$, while the dual fixed point theory is a quiver gauge theory whose details depend on whether k is even or odd and, in the former case, how exactly the orientifold projection acts.

In order to ease the discussion, let us concentrate in the following on the case of a $\mathbb{Z}_2$ orbifold and set, for simplicity, $N_f = 0$. As discussed in [15], there are two possible orientifold projections, depending on their action on the twisted sector of the orbifold projection. In the so-called vector structure (VS) case the orientifold projection keeps a 5d hypermultiplet from the orbifold twisted sector so that the resulting theory is a $USp(2N) \times USp(2N)$ gauge theory with bifundamental matter. In turn, the so-called no-vector structure (NVS) projection keeps a 5d vector multiplet and leads to a $SU(2N)$ gauge theory with 2 antisymmetric hypermultiplets.

Both theories correspond to N D4 branes probing an $O8^-$ wrapping $\mathbb{C}^2/\mathbb{Z}_2$ and are dual to exactly the same $AdS_6 \times \hat{S}^4/\mathbb{Z}_2$ background. Reassuringly, in both cases the classical flavor-blind sector of the Higgs branch – probed by dual giant gravitons spinning in $\hat{S}^4/\mathbb{Z}_2$ – is precisely the A_1 singularity [16] (see also [17] for further holographic checks regarding operator counting in orbifold theories).

We are interested in exploring flows in these theories triggered by VEVs of operators. Just as in the 4d case discussed in the previous section, in the gravity dual we can imagine either moving the stack of branes sourcing the geometry away from the orbifold singularity or alternatively blowing up the singularity. Note that, on general grounds, if the singularity is blown-up a brane behaving as a cosmic string will be allowed [15]. Hence exactly as in the 4d case it is natural to expect the flows in the singular cone to be triggered by mesonic VEVs while those on the resolved cone to be triggered by baryonic VEVs. However, while the NVS has a $U(1)_B$ baryonic

[3] See however [13,14] for new AdS_6 geometries.

symmetry the VS case does not. This is inherited from the orientifold projection, which in the VS case removes the blow-up mode for the orbifold singularity while keeps it in the NVS case. Hence, even though the same $AdS_6 \times \hat{S}_4$ background is dual to both the VS and the NVS theories, the spectrum of allowed fluctuations will be different. In particular the resolution mode is only present in the NVS case. Thus, when dealing with flows in the resolved conifold, we must refer to the NVS case, where the blow-up mode is allowed and the dual CFT has a $U(1)_B$ symmetry.

On general grounds, the geometry sourced by N D4 branes probing a stack of N_f D8 branes on top of an O8$^-$ plane must be of the form

$$ds^2 = \Omega^2(z)\{h^{-\frac{1}{2}}dx_{1,4}^2 + h^{\frac{1}{2}}[ds_4^2 + dz^2]\}, \tag{49}$$

where

$$\Omega(z) = \left(\frac{3}{2}mz\right)^{-\frac{1}{6}}. \tag{50}$$

In addition to the Romans mass $F_0 = m$, there is a 6-form field strength given by the standard Freund–Rubin ansatz

$$F_6 = d(h^{-1})dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \tag{51}$$

The equations of motion of the 6-form fix h to satisfy

$$d(\Omega^{-2} \star_5 dh) = \mathcal{C}\delta; \tag{52}$$

where $\star_5$ is the Hodge-star operator with respect to the $ds_4^2 + dz^2$ metric, $\mathcal{C}$ a normalization constant proportional to N and δ a source term supported at the location of the D4 branes.

The ds_4^2 stands for the metric on the transverse space to the D4s inside the 8-branes. In the singular case it is just the singular A_1 space. Locating the D4s at the tip we obtain the warped $AdS_6 \times \hat{S}^4/\mathbb{Z}_2$ geometries of [15]. The warp factor of the AdS_6 is a function of the azimuthal angle α of the $\hat{S}^4$. Moreover, one can see that both curvature and dilaton diverge at the north pole of the $\hat{S}^4$ when $\alpha = 0$, which is then identified with the location of the orientifold. This should expected since, in order to find the fixed point theory, we need to remove the bare YM coupling, which in the gravity side amounts to tune the dilaton to diverge on top of the orientifold plane.[4] Note that, even though the geometry blows-up at $\alpha = 0$ and in principle the full string theory should be used to describe the physics there, at least certain quantities are well-defined in the supergravity solution. In fact, the part of the Higgs branch not involving fundamental fields is well-captured by giant gravitons spinning at $\alpha = 0$ [16].

In the following we will be interested on moving the branes away from the singularity as well as on replacing the singular A_1 space by its resolution. On general grounds we expect the first type of geometries to correspond to RG flows triggered by mesonic VEVs – and hence possible in both the VS and NVS cases – while the second ones to correspond to flows triggered by baryonic VEVs. Of course, as described above, the latter can only happen in the NVS case.

3.1. *Flows on the singular space*

Let us concentrate on flows on the singular space. The equation of motion for h is

[4] In fact, due to string duality, this suggests the emergence of enhanced global symmetries in these fixed point theories [9].

$$\frac{1}{r^3}\partial_r\big(r^3\partial_r h\big)+\frac{1}{z^{\frac{1}{3}}}\partial_z\big(z^{\frac{1}{3}}\partial_z h\big)+\frac{4}{r^2}\Delta h+\frac{4}{r^2}\partial_\psi^2 h=\mathcal{C}\delta. \tag{53}$$

We localize the branes at a given point ξ_0 in the angular coordinates $\xi=\{\psi,\theta,\phi\}$. Following the same procedure as above we expand

$$h=\sum h_I(r,z)\tilde{Y}_I^\star(\xi_0)\tilde{Y}_I(\xi), \tag{54}$$

where the $\tilde{Y}_I$ functions are the eigenfunctions of the Δ Laplacian introduced above. Note that now we do not have the circle χ – the transverse space to the A_1 singularity in the internal directions is just a line. Hence our multi-index I only involves $\{R,l,m\}$. We find the following equation for the function h_I

$$\frac{1}{r^3}\partial_r\big(r^3\partial_r h_I\big)+\frac{1}{z^{\frac{1}{3}}}\partial_z\big(z^{\frac{1}{3}}\partial_z h_I\big)-\frac{4l(l+1)}{r^2}h_I=\mathcal{C}\delta. \tag{55}$$

It is useful to switch to polar coordinates in the $\{r,z\}$ plane defining $r=\rho\cos\alpha$ and $z=\rho\sin\alpha$. We then find

$$\begin{aligned}
h_I^< &= \frac{1}{\rho^{\frac{10}{3}}}\left(\frac{\rho}{\rho_0}\right)^a\cos^{2l}\alpha\,{}_2F^1\left[-\frac{a}{2}+l,\frac{5}{3}+\frac{a}{2}+l,2+2l,\cos^2\alpha\right]\leftrightarrow\rho<\rho_0,\\
h_I^> &= \frac{1}{\rho^{\frac{10}{3}}}\left(\frac{\rho_0}{\rho}\right)^a\cos^{2l}\alpha\,{}_2F^1\left[-\frac{a}{2}+l,\frac{5}{3}+\frac{a}{2}+l,2+2l,\cos^2\alpha\right]\leftrightarrow\rho>\rho_0.
\end{aligned} \tag{56}$$

As described above, the north pole of the S^4 has to be treated with care in this case, as both dilaton and curvature diverge there. Hence, we need to carefully discuss the boundary conditions to be imposed on α. To that matter, let us consider a generalized version of the eom in the singular case

$$\frac{1}{r_1^3}\partial_{r_1}\big(r_1^3\partial_{r_1}\varphi\big)+\frac{1}{r_2^a}\partial_{r_2}\big(r_2^a\partial_{r_2}\varphi\big)-\frac{4l(l+1)}{r_1^2}\varphi=0, \tag{57}$$

so that $a=1$ is the eom for the AdS_5 case and $a=\frac{1}{3}$ the eom for the AdS_6 case.

The equation depends on the eigenvalue l. We can consider two copies of the equation for modes for different eigenvalues $l_{1,2}$. Manipulating them, we find

$$\begin{aligned}
&\partial_{r_1}\big(r_1^3r_2^a(\chi\partial_{r_1}\varphi-\varphi\partial_{r_1}\chi)\big)+\partial_{r_2}\big(r_1^3r_2^a(\chi\partial_{r_2}\varphi-\varphi\partial_{r_2}\chi)\big)\\
&\quad=\big[4l_1(l_1+1)-4l_2(l_2+1)\big]r_1r_2^a\varphi\chi.
\end{aligned} \tag{58}$$

So defining the "current"

$$j_i=r_1^3r_2^a(\chi\partial_{r_i}\varphi-\varphi\partial_{r_i}\chi), \tag{59}$$

we can write

$$\partial_{r_i}j_{r_i}=\big[4l_1(l_1+1)-4l_2(l_2+1)\big]r_1r_2^a\varphi\chi. \tag{60}$$

Hence we find a good candidate for internal product for our wavefunctions, namely

$$\langle\chi|\varphi\rangle\sim\int r_1r_2^a\chi\varphi, \tag{61}$$

provided we impose boundary conditions such that

$$\int \partial_{r_i} j_{r_i} = \int_{\partial} j_{r_i} dr_i = 0. \tag{62}$$

We will be interested on normalizable wavefunctions for which the above inner product is well-defined. This can be done by demanding the current to vanish on the $\alpha = 0$ singularity. One can easily check that this is satisfied provided that $a = 2l$, so that finally, and upon fixing the correct normalization, we have

$$h_I^{<} = \frac{\mathcal{C}}{\frac{10}{3} + 4l} \frac{1}{\rho_0^{\frac{10}{3}}} \left(\frac{\rho}{\rho_0}\right)^{2l} \cos^{2l}\alpha \leftrightarrow \rho < \rho_0,$$
$$h_I^{>} = \frac{\mathcal{C}}{\frac{10}{3} + 4l} \frac{1}{\rho^{\frac{10}{3}}} \left(\frac{\rho_0}{\rho}\right)^{2l} \cos^{2l}\alpha \leftrightarrow \rho > \rho_0. \tag{63}$$

The leading term in the asymptotic region is $\rho^{-\frac{10}{3}}$. Recalling that the AdS_6 radial coordinate is $\varrho = \rho^{\frac{2}{3}}$ it is easy to see that, together with the contribution from the overall Ω^2 warping, we recover, in the asymptotic region, the warped $AdS_6 \times \hat{S}^4/\mathbb{Z}_2$ geometry. This strongly suggests that this geometry again corresponds to a flow in the original gauge theory triggered by a VEV.

We could again check our results against a computation performed upon changing from the beginning into polar coordinates. Just as in the AdS_5 case one easily obtains the same spectrum upon imposing the appropriate quantization conditions on the quantum numbers.

3.2. Flows on the resolved space

We now change the ds_4^2 metric for that of the Eguchi–Hanson space with resolution parameter c. As stressed above, $c \neq 0$ is only possible in the NVS case, as the VS projection kills this mode. The equation for h is now

$$\frac{1}{r^3}\partial_r\big(r^3 f \partial_r h\big) + \frac{1}{z^{\frac{1}{3}}}\partial_z\big(z^{\frac{1}{3}}\partial_z h\big) + \frac{4}{r^2}\Delta h + \frac{4}{r^2 f}\partial_\psi^2 h = \mathcal{C}\delta. \tag{64}$$

Since we will place our stack of D4 branes on the blow-up S^2 where the ψ circle shrinks, we will set $R = 0$. Then, expanding $h = \sum_I h_I \tilde{Y}_I^\star(\xi_0)\tilde{Y}_I(\xi)$, we find an equation for h_I in the $\{r, z\}$ plane. As in the AdS_5 case, this equation is fairly complicated, so we will content ourselves with the analysis of the asymptotic properties. Switching to the $\{\rho, \alpha\}$ polar coordinates we write

$$h_I = \left(\frac{c}{\rho}\right)^a f(\alpha). \tag{65}$$

We then find the equation

$$\big(-10a + 3a^2 - 6m^2 + a(-10 + 3a)\cos(2\alpha)\big) f \sin\alpha$$
$$+ \cos\alpha\big(2\big(-4 + 5\cos(2\alpha)\big) f' + 3\sin(2\alpha) f''\big) = 0, \tag{66}$$

where $m^2 = 4l(l + 1)$. The solution to this equation is

$$f = \cos^{2l}\alpha\, {}_2F^1\left[\frac{5}{3} - \frac{a}{2} + l, \frac{a}{2} + l, 2 + 2l, \cos^2\alpha\right]. \tag{67}$$

Demanding the current to vanish at $\alpha = 0$ sets

$$a = \frac{10}{3} + 2l. \tag{68}$$

Hence

$$h_I = \frac{1}{\rho^{\frac{10}{3}}} \left(\frac{c}{\rho}\right)^{2l} \cos^{2l}\alpha. \tag{69}$$

Therefore, when written in the ϱ radial coordinate and taking into account the overall warp factor Ω, this geometry is again asymptotically the same $AdS_6 \times \hat{S}^4/\mathbb{Z}_2$ cone, thus showing that it must correspond to a VEV deformation of the original – recall, NVS – CFT.

3.3. Gauge theory

Let us now turn to the gauge theory operators. Borrowing the discussion in Section 2.3, it is clear that the $\tilde{Y}_I$ wavefunctions must correspond to the states in Eq. (37).

On the other hand, the two gauge theories relevant to our discussion are, respectively, a $USp(2N) \times USp(2N)$ gauge theory with bifundamental matter in the VS case and a $SU(2N)$ theory with an antisymmetric hypermultiplet in the NVS case. In both cases we can add up to 8 fundamental hypermultiplets. Nevertheless our wavefunctions do not involve any flavor quantum number, signaling that the geometry is blind to flavor degrees of freedom. Recall that in [16] the geometry was not able to capture operators on the Higgs branch involving fundamental matter, since the open string sector corresponding to D8–D8 strings which would be responsible for those operators is just not present in the near brane limit geometry (this is a generic feature when including flavor as non-compact larger branes into the gauge/gravity duality, as the decoupling limit freezes such modes). Hence it comes as no surprise that in this case the same happens. Because of this we will set $N_f = 0$.

For future reference, let us spell the symmetries in each case and the representations of each field. In the VS case the global non-R symmetry is $SU(2)_M \times U(1)_{I_1} \times U(1)_{I_2}$, being $SU(2)_M$ a global mesonic symmetry acting on the hypermultiplet. Besides, $U(1)_{I_{1,2}}$ are the topological symmetries associated to each gauge group. On the other hand, in the NVS case, the global non-R symmetry is $SU(2)_M \times U(1)_B \times U(1)_I$, where again $SU(2)_M$ is a global mesonic symmetry acting on the antisymmetric hypermultiplet. Besides $U(1)_B$ is a baryonic symmetry under which one complex doublet in the antisymmetric hyper – call it A_1 – has charge 1 and the other – call it A_2 – has charge -1. Finally $U(1)_I$ is the topological symmetry associated to the gauge group. In addition, in both cases there is a global $SU(2)_R$ symmetry.

It is important to recall that the correct AdS coordinate is ϱ. Hence the modes both in the singular and resolved cases scale like ϱ^{3l}, and thus correspond to $\Delta = 3l$ operators. Note that indicates no large anomalous dimension, even though, just as in the 4d case, some of the operators taking VEV will be non-chiral (in the 4d sense). This is again due to the combination of $SU(2)_M$ and $SU(2)_R$, which allows to place any operator in the same multiplet as a chiral operator.

3.3.1. The singular case

As described above, the geometries corresponding to placing the stack away from the singularity correspond to flows triggered by VEVs of mesonic operators. These flows exist in both VS and NVS theories, which, consequently, have an identical spectrum of mesonic operators [15–17]. In fact, in both cases the mesonic moduli space is classified under the $SU(2)_M \times SU(2)_R$ global symmetry, common to both VS and NVS. Using this, since the $\tilde{Y}_I$ correspond to the states in

Eq. (37) whose quantum numbers under all relevant symmetries are known, it is easy to translate among fluctuations and their corresponding operators – basically those listed in [16,17] – finding a perfect matching. Exactly as in the 4d case, the $SU(2)_R \times SU(2)_M$ allows to relate non-holomorphic operators to holomorphic ones, hence ensuring that the dimensions should equal the classical ones. Note in particular that $SU(2)_M$ spin l operators involve $2l$ scalars – which in 5d are dimension $\frac{3}{2}$, and hence they have the expected $\Delta = 3l$.

Note that the operators so constructed belong to the Higgs branch. As opposed to the case of 4d gauge theories, in 5d the R-symmetry does not contain a $U(1)_r$. In particular this stands for the fact that the scalar ϕ in the 5d vector multiplet is a real field. Hence the n quantum number analogous to that in Section 2.1 is absent. Nevertheless one might wonder about operators including q powers of ϕ, which in this case seems to be absent. Indeed, the boundary conditions setting the current at $\alpha = 0$ to zero don't allow for any other quantum number in any similar way to the q in Section 2.2. In support of this, analogously to Section 2.1.1, one can repeat the computation in polar coordinates from the beginning and make use of angular eigenfunctions on the S^4 to impose the correct quantization conditions. However, on the S^4 the eigenfunctions are classified into $SO(5)$ reps. The two Cartans of $SO(5)$ must correspond to l_3 and R, so there is no room for another quantum number, in agreement with the discussion above.

3.3.2. *The resolved case*

We now turn to the resolved case. As emphasized above, the resolution mode is only allowed in the NVS case. Hence, the fluctuations we have obtained correspond to the states in Eq. (37) upon setting $R = 0$.

The lowest-dimensional operators taking a VEV is a triplet of scalars involving $\mathcal{U} \sim A_1 A_1^\dagger - A_2 A_2^\dagger$ at $m = 0$. These correspond to the scalars in the $U(1)_B$ conserved current multiplet as expected for a flow triggered by a baryonic VEV.

With no loss of generality we can assume a VEV for the A_1 field proportional to c. Then, the baryon-like operator $\mathcal{B} = A_1^N$ of dimension $\Delta = \frac{3}{2}N$ would acquire a VEV. On the other hand, we can consider an Euclidean D2 brane wrapping $X = \{r, z, \psi\}$, which stands for the baryonic condensate [15]. The DBI action for the brane is

$$S_{DBI} = i\frac{T_2}{2}(2\pi)\left(\frac{2}{3m}\right)^{-\frac{1}{3}} \int\limits_X r z^{\frac{1}{3}} h. \tag{70}$$

Asymptotically the integrals for the h_l modes are

$$\int d\rho\, \rho^{-1-2l} \tag{71}$$

so again the $l > 0$ yield finite integrals, while the $l = 0$ term leads to a logarithmically divergent term which has to be regulated by a cut-off. It is easy to see that, upon using the correct AdS radial coordinate ϱ, the leading divergence goes like $\frac{3}{2}N \log \varrho_c$, being ϱ_c the UV cut-off. Hence this corresponds to the dimension of the expected baryon operator VEV, namely $\Delta = \frac{3}{2}N$. Furthermore, we can consider a $D4$ brane wrapping the blown-up S^2 and describing a cosmic string in the field theory directions – which is a $(1+2)$-dimensional defect in 5d. Its DBI action is

$$S_{DBI} = -T_4 \int e^{-\Phi} \Omega^5 h^{-\frac{1}{4}} \frac{c^2}{4} \sin\theta = -T_4 c^2 \pi \int dx^0 dx^1 dx^2. \tag{72}$$

So we find a finite-tension object "electric–magnetic" dual to the baryon condensate. The δF_6 sourced by the brane is of the form

$$\delta F_6 = da_3 \wedge W; \tag{73}$$

being W a closed 2-form in the internal space which asymptotes to the volume form of the blown-up S^2. The dual δF_4 is of the form

$$\delta F_4 = \star_5 da_3 \wedge \left(\Omega^{-2} h \star_{internal} W\right). \tag{74}$$

Writing $\star_5 da_3 = dp$, a local integration is

$$\delta C_3 = p \wedge \left(\Omega^{-2} h \star_{internal} W\right). \tag{75}$$

It is clear that $(\Omega^{-2} h \star_{internal} W)$ threads the cycle wrapped by the baryonic condensate, so that its WZ action is proportional to ip. Therefore we see that indeed the baryon condensate captures the baryonic VEV including the Goldstone boson of the broken symmetry. The D4 brane is nothing but the cosmic string around which the Goldstone boson of the – spontaneously broken – baryon symmetry winds.

4. Conclusions

In this paper we have studied geometries dual to flows triggered either by mesonic and baryonic VEVs in gauge theories with 8 supercharges in 4 and 5 dimensions. As opposed to the $\mathcal{N} = 1$ flows well studied in the literature such as e.g. [1], in this case we need to solve an involved PDE, whose general solution in the resolved cases we have not been able to find.

At the bottom of the geometries, close to the source branes, we expect an AdS throat to emerge, corresponding to the IR fixed point. Unfortunately, due to the lack of explicit solution to the PDE in the resolved cases, we have not been able to explicitly show it.

Quite remarkably, even though some of the operators taking VEVs in our flows are non-chiral, the dimensions are those of the free field theory. As explained above, this is due to the $SU(2)_M \times SU(2)_R$ symmetry, whose combination allows to regard any non-chiral operator as in the same multiplet of a chiral operator. It is interesting to note that this seems to be a property of the $\mathbb{Z}_2$ orbifold theories, as for a generic $\mathbb{Z}_p$ orbifold the mesonic symmetry is just $U(1)_M$. We leave a detailed study for the future.

In the 5d case the singularity at $\alpha = 0$ plays an important role in providing the correct quantization conditions. Indeed, without such conditions one seems to obtain regular modes elsewhere for an arbitrary a – basically the scaling dimension of the dual operator, since we consider $h_I \sim \rho^{-a} f(\alpha)$ –. However, demanding the vanishing of the current at $\alpha = 0$ yields to the correct AdS_6 asymptotics – which demand an overall $\rho^{-\frac{10}{3}}$ – and gives the correct – and discrete – dimension to the operators. As raised in the text, one slightly puzzling feature is that the 5d operators taking a VEV are purely Higgs branch operators with no vector multiplet scalar insertions. Note that, despite its singularity at $\alpha = 0$, the SUGRA background captures well the CFT properties, as it also happens in the case of [16].

The resolved geometries correspond to flows triggered by VEVs of baryonic operators. Indeed we find a fully consistent picture, with the baryonic VEV being identified with the appropriate baryonic condensate brane. Its DBI action gives the modulus of the VEV with the expected dimensional scaling. While we have not been able to compute the finite, wavefunction, part of the action since we don't have the exact form for the warp factor, we expect a similar result as in

[1]. In fact, in the AdS_5 case, where the internal space is the base of a CY_3 cone, we can simply borrow the general result in [8] appropriately adapted to the CY_3 case. On the other hand, the DBI action for the baryon condensate brane is nicely proportional to the Goldstone boson, sourced by a finite-tension cosmic string brane. While we have not been able to check the existence and properties of the required W-form since, for a start, the exact h_I are not known, we believe that it will exist along the lines of [4].

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Appendix A. Some explicit details about the CY_3 structure of $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$

Being a direct product, the metric on the singular $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ can be easily written as

$$\begin{aligned} ds_6^2 = dX_I g_6^{IJ} dX_J = dr_1^2 + \frac{r_1^2}{4}(d\psi + \cos\theta d\phi)^2 \\ + \frac{r_1^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + dr_2^2 + r_2^2 d\chi^2, \end{aligned} \tag{76}$$

where $\psi \in [0, 2\pi]$. Note that, upon introducing $r_1 = \rho\cos\alpha$, $r_2 = \rho\sin\alpha$ the metric becomes just $d\rho^2 + \rho^2 ds^2_{S^5/\mathbb{Z}_2}$.

We can write the metric on the resolved $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ simply by plugging the Eguchi–Hanson metric in the $\mathbb{C}^2/\mathbb{Z}_2$ piece as

$$ds_6^2 = \frac{dr_1^2}{f(r_1)} + \frac{r_1^2}{4} f(r_1)(d\psi + \cos\theta d\phi)^2 + \frac{r_1^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + dr_2^2 + r_2^2 d\chi^2, \tag{77}$$

being

$$f(r_1) = 1 - \frac{c^4}{r_1^4}. \tag{78}$$

Introducing complex coordinates $\{z_1, z_2, z_3\}$ defined as

$$z_1 = \left(r_1^4 - c^4\right)^{\frac{1}{4}} e^{i\frac{\psi+\phi}{2}} \cos\frac{\theta}{2}, \qquad z_2 = \left(r_1^4 - c^4\right)^{\frac{1}{4}} e^{i\frac{\psi-\phi}{2}} \sin\frac{\theta}{2}, \qquad z_3 = r_2 e^{i\chi}, \tag{79}$$

it is easy to check that the Kähler potential reads

$$\begin{aligned} F = c^2\sqrt{1 + \frac{z_1\bar{z}_1 + z_2\bar{z}_2}{c^4}} - c^2\log\left(1 + \sqrt{1 + \frac{z_1\bar{z}_1 + z_2\bar{z}_2}{c^4}}\right) \\ + c^2\log\left(1 + \frac{z_1\bar{z}_1}{z_2\bar{z}_2}\right) + z_3\bar{z}_3. \end{aligned} \tag{80}$$

Furthermore, introducing the natural complex 1-forms

$$e_1 = \frac{dr_1}{\sqrt{1-\frac{c^4}{r_1^4}}} + i\frac{r_1}{2}\sqrt{1-\frac{c^4}{r_1^4}}g_5,$$
$$e_2 = \frac{r_1}{2}(d\theta - i\sin\theta d\phi), \qquad e_3 = dr_2 + ir_2 d\chi, \tag{81}$$

where $g_5 = d\psi + \cos\theta d\phi$, one can easily verify that the Kähler form arising from the Kähler potential is just $\omega = \sum_{i=1}^{3} e_i \wedge \bar{e}_i$. The holomorphic 3-form is

$$\Omega = e^{i(\psi+\chi)} e_1 \wedge e_2 \wedge e_3. \tag{82}$$

One can easily verify that

$$d\Omega = 0, \qquad d\omega = 0, \qquad \Omega \wedge \omega = 0. \tag{83}$$

This explicitly shows the CY_3 structure of the resolved $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.

Appendix B. The eigenfunctions $\tilde{Y}_{R,l,m}$

In Section 2.1 we used the eigenfunctions $Y_I(\chi, \theta, \psi, \phi)$ of the Laplacian on the singular cone. In this appendix we provide a short review of their construction following [8].

The Laplacian on $\mathbb{C}^2/\mathbb{Z}_2$ is given by

$$\frac{1}{r_1^3}\partial_{r_1}\left(r_1^3\partial_{r_1}\right) + \frac{1}{r_1^2}\left(4\Delta + 4\partial_\psi^2\right), \tag{84}$$

being

$$\Delta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \left(\frac{\partial_\phi}{\sin\theta} - \cot\theta\partial_\psi\right)^2. \tag{85}$$

Hence, the angular Laplace equation on $\mathbb{C}^2/\mathbb{Z}_2$ is

$$\left(4\Delta + 4\partial_\psi^2\right)\tilde{Y}_{R,l,m} = -\hat{E}_{R,l,m}\tilde{Y}_{R,l,m}. \tag{86}$$

Let us start by considering the reduced problem

$$4\Delta\tilde{Y}_{R,l,m}(\theta, \psi, \phi) = -E_{R,l,m}\tilde{Y}_{R,l,m}(\theta, \psi, \phi). \tag{87}$$

Inspection of the Δ Laplacian suggests that the $\{\psi, \phi\}$ part can be diagonalized by writing

$$\tilde{Y}_{R,l,m}(\theta, \psi, \phi) = e^{iR\psi} e^{im\phi} J_{l,m}(\theta). \tag{88}$$

Since both ψ, ϕ have period 2π it follows that both R, m must be integers in order to ensure single-valuedness. Furthermore, $J_{l,m}(\theta)$ must satisfy

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta J_{l,m}(\theta)\right) - \left(\frac{m}{\sin\theta} - R\cot\theta\right)^2 J_{l,m}(\theta) = -\frac{E_{R,l,m}}{4} J_{l,m}(\theta). \tag{89}$$

The solutions to this equation are

$$J_{l,m}^A(\theta) = \sin^m\theta\cot^R\frac{\theta}{2}\,{}_2F^1\left[-l+m, 1+l+m, 1+m-R, \sin^2\frac{\theta}{2}\right], \tag{90}$$

and

$$J^B_{l,m}(\theta) = \sin^R\theta \cot^m \frac{\theta}{2} \, {}_2F^1\left[-l+R, 1+l+R, 1-m+R, \sin^2\frac{\theta}{2}\right], \tag{91}$$

in both cases the eigenvalues are

$$E_{R,l,m} = 4\big(l(l+1) - R^2\big). \tag{92}$$

Hence it immediately follows that $\Delta\tilde{Y}_{R,l,m} = -(l(l+1) - R^2)\tilde{Y}_{R,l,m}$.

It is easy to see that if $R > m$ the solution $J^A_{l,m}(\theta)$ is singular around $\theta = 0$; while, if $R < m$ it is the solution $J^B_{l,m}(\theta)$ that becomes singular. Therefore, depending on the value of R, we should select only one of the two solutions. Furthermore, regularity around $\theta = \pi$ demands l to be an integer such that $l \geq |m|$ and $l \geq |R|$.

Coming back to the full problem (86), it is clear that the functions $\tilde{Y}_{R,l,m}$ are also eigenfunctions of the angular Laplacian $4\Delta + 4\partial^2_\psi$ with

$$\hat{E}_{R,l,m} = 4l(l+1). \tag{93}$$

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6.2 Aspects of the moduli space of instantons on $\mathbf{C}P^2$ and its orbifolds

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Aspects of the moduli space of instantons on $\mathbb{C}P^2$ and its orbifolds

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We study the moduli space of (framed) self-dual instantons on $\mathbb{C}P^2$. These are described by an Atiyah-Drinfeld-Hitchin-Manin (ADHM)-like construction which allows us to compute the Hilbert series of the moduli space. The latter has been found to be blind to certain compact directions. In this paper, we probe these, finding them to correspond to a Grassmanian, upon considering appropriate ungaugings. Moreover, the ADHM-like construction can be embedded into a $3d$ gauge theory with a known gravity dual. Using this, we realize in AdS_4/CFT_3 (part of), the instanton moduli space providing at the same time further evidence supporting the AdS_4/CFT_3 duality. Moreover, upon orbifolding, we provide the ADHM-like construction of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ as well as compute its Hilbert series. As in the unorbifolded case, these turn out to coincide with those for instantons on $\mathbb{C}^2/\mathbb{Z}_n$.

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I. INTRODUCTION

In the recent past, it has become clear that studying gauge theories in diverse circumstances is of the utmost interest in order to unravel their dynamics. In particular, it is very interesting to consider their response to curvature by considering placing gauge theories on curved backgrounds. In that respect, very recently developed techniques—such as localization—allow us to compute exactly certain observables, such as partition functions and surface/line operators in certain gauge theories. In turn, these are sensible to different physical aspects. For example, while the supersymmetric partition functions of $\mathcal{N}=2$ $4d$ theories on $S^1\times S^3$ have the interpretation of an index—a weighted counting of Bogomol'nyi-Prasad-Sommereld states—the homologous computation on S^4 is interpreted as a partition function, and it is closely related to the Zamolodchikov metric [1].

In these computations, the nonperturbative sector typically plays a crucial role. In particular, it is well known that instantons are very important configurations in gauge theory. For example, the partition function of gauge theories contains contributions from saddle points of all instanton numbers. This can be made fully precise in the case of supersymmetric gauge theories with eight supercharges, when the supersymmetric partition function can be computed exactly thanks to localization (see [2] for a seminal contribution). One can then explicitly see that, in addition to purely perturbative saddle points, the partition function localizes on instantonic configurations, whose contribution one has to sum. On general grounds, such contributions are the one-loop determinants around each instanton saddle point, which can be computed by the so-called Nekrasov instanton partition function. In turn, in the case of pure gauge theories, the latter coincides with the Hilbert series of the instanton moduli space (see, e.g., [3,4]). Therefore, the construction of instanton moduli spaces, as well as the computation of their associated Hilbert series, is of the greatest importance (of course, the reasons alluded to before are just a very limited subset of those making the instanton moduli space a very interesting object).

In the case of instantons on $\mathbb{C}^2$—or its conformal compactification S^4—the problem of constructing instantons of pure gauge theories[1] with gauge group A, B, C, D was solved long ago by the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [5]. Moreover, it turns out that the ADHM construction has a natural embedding into string theory as it arises as the Higgs branch of the Dp-$Dp+4$-brane system [6–9]. In this paper, we are interested in the parallel story but for the case of $\mathbb{C}P^2$. As opposed to S^4, $\mathbb{C}P^2$ is a Kähler manifold. This naturally induces a preferred orientation which distinguishes self-dual (SD) from anti-self-dual (ASD) 2-forms. As a result, the construction of gauge connections with ASD and SD curvatures is intrinsically different. In this paper, we will concentrate on SD connections on $\mathbb{C}P^2$ (and its orbifolds). In the mathematical literature, an ADHM-like construction for such gauge bundles has been developed long ago [10–14]. Very recently, it has been shown that such construction can be embedded into a gauge field theory, which, moreover, admits a string/M theory interpretation [15]. Surprisingly, the gauge theories engineering the ADHM construction for instantons on $\mathbb{C}P^2$ are $3d$ gauge theories with $\mathcal{N}=2$ supersymmetry—that is, four supercharges. Nevertheless, as shown in [16] (see, also, [15,17,18] for a discussion in the physics context), the Hilbert series and other properties do indeed satisfy properties compatible with the expected hyper-Kähler condition of the moduli space.

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[1] We will concentrate on instantons in pure gauge theories with eight supercharges throughout the paper.

In this paper, we consider several aspects of these moduli spaces for SD instantons on $\mathbb{C}P^2$, as well as develop their construction on orbifolds of $\mathbb{C}P^2$. As introduced above, being $\mathbb{C}P^2$ a Kähler manifold, a preferred orientation is induced. In turn, this intrinsically distinguishes SD from ASD configurations. It is then natural to ask whether both types of instantons can be physically relevant. To elucidate this, we need to construct a supersymmetric gauge theory on the curved space such that its instanton sector includes SD configurations. A very useful strategy put forward by [19] is to couple the gauge theory to supergravity so that the combined system is automatically supersymmetric. Then, a suitable rigid limit freezes the gravity dynamics around the chosen background in such a way that we are left with the quantum field theory appropriately supersymmetrized on the curved space. From this perspective, the vacuum expectation values (VEVs) of the fields in the SUGRA multiplet become the supersymmetric couplings in the gauge theory. Moreover, in order to preserve supersymmetry, generically the SUGRA, the background must be nontrivial. A very natural way to supersymmetrize a gauge theory is by means of topologically twisting—perhaps including an equivariant version—with the R symmetry. Following this method, in [20] the partition function for gauge theories on Kähler spaces, in particular, $\mathbb{C}P^2$, was constructed. However, the relevant instanton sector in that case was that of ASD configurations. As we describe, this is related to the choice of topological twist: because of the Kähler property, twists based on left-handed spinors are intrinsically different from twists based on right-handed spinors. As we explicitly spell out in this paper, by choosing the appropriate twist, it is possible to construct a supersymmetric gauge theory on $\mathbb{C}P^2$ for which the relevant instanton sector contains SD configurations.

In the case of SD instantons on $\mathbb{C}P^2$, the corresponding Hilbert series was computed in [16–18] and reobtained in [15] from a physics-based approach. In particular, it was shown that these coincide with the Hilbert series of a "parent" instanton on $\mathbb{C}^2$. This immediately raises the question that, being $\mathbb{C}P^2$ a topologically nontrivial space, it is natural to expect that our instantons are described by extra topological data. In particular, given that $\mathbb{C}P^2$ contains a nontrivial $\mathbb{C}P^1$, gauge field configurations should be labeled as well by a first Chern class basically corresponding to flux on the nontrivial $\mathbb{C}P^1$. Since the Hilbert series, which coincides with the Nekrasov instanton partition function, is insensitive to this information, it follows that the partition function is independent on the choice of first Chern class for the gauge bundle. However, other observables might depend on it (in particular, surface operators). Thus, on general grounds, it is natural to explore the structure of the full moduli space. Such description has been accomplished in the mathematical literature [16–18] for the unitary case. In particular, it has been shown that the dimension of the moduli space seen by the Hilbert series is smaller than the dimension of the actual moduli space. As argued from a mathematical perspective for the unitary case, in particular, in [17], such "extra directions" are associated to (compact) Grassmanian subspaces in the full moduli space.[2] Note that these extra directions were detected by means of other methods, as being compact, the Hilbert series is blind to them. In this paper, we explore from a novel physics-based perspective, these extra directions associated to the extra topological data. Our approach applies to the unitary case as well as to orthogonal and symplectic instantons. For that matter, we consider the simplest case of a SD configuration probing these extra directions, namely, that with zero instanton number but nonzero first Chern class. Amusingly, for unitary instantons, the construction degenerates into a $3d$ version of the theory in [21], whose moduli space has been argued to be a (compact) Grassmanian manifold, thus, reassuringly recovering the expectations in the mathematical literature. This theory, which admits a brane description, provides a clear physical description of the extra directions of the moduli space not captured by the Hilbert series. Moreover, it suggests a novel way to study such extra directions by using the so-called master space [22] of the theory. The latter is an extended notion of the moduli space where one ungauges the Abelian part of the gauge symmetry. As in [23], upon appropriately ungauging $U(1)$ groups, we are effectively considering the complex cone over the compact base. In this modified scenario, we can now use the Hilbert series, which probes the extra directions finding agreement with the expectations. Moreover, we use this technique to probe the resolved moduli space for orthogonal instantons as well—symplectic instantons are trivial in this respect. Thus, our new approach provides a direct and physical method to explore in detail the moduli space of SD instantons of all classical groups on $\mathbb{C}P^2$.

Yet another very interesting aspect of the construction of SD instantons on $\mathbb{C}P^2$ is that the gauge theory containing the ADHM construction of unitary instantons admits a large N limit where it is dual to an AdS_4 geometry. It is then natural to study the instanton moduli space in the gravity dual. Similar to other examples in the literature, the gravity dual captures the subset of operators involving only bifundamental fields in the quiver corresponding to "closed string degrees of freedom" (as opposed to fundamental matter corresponding to "open string degrees of freedom"). It is possible, however, to identify this subset in the field theory for detailed comparisons. In particular, the expected hyper-Kähler structure is recovered from the AdS dual. Moreover, in order to find agreement with the field theory description, the exact R charges of the operators are required. This provides an interesting cross-check of the

[2]In [15], the full moduli space including the Grassmanian directions was called the *resolved moduli space*, as it discerns the extra directions not seen by the Hilbert series.

field theory results. At the same time, it provides very nontrivial evidence of the proposed AdS_4/CFT_3 dualities, as, in particular, it requires detailed matchings involving R charges in $\mathcal{N}=2$ theories—free to deviate largely from the free-field ones.

Starting the ADHM construction for instantons on a given space can be used to find the corresponding construction on related spaces obtained by orbifold projections. In this manner, we find the ADHM construction, as well as the Hilbert series for moduli spaces of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$, whose construction and description were not known to the best of our knowledge. As these spaces have an even richer topological structure, the identification of ADHM-like quiver data with the instanton data is more involved and not known, yet we propose some conjectures supported on the observations coming from the unorbifolded case. We stress that our approach towards exploring compact directions of the moduli space plays an important role in guessing the topological properties of instantons on the orbifolded spaces.

The structure of this paper is as follows: In Sec. II we explicitly describe the relevance of SD instantons on $\mathbb{C}P^2$ in the computation of the partition function for the topologically twisted gauge theory. In particular, we show how SD instantons on $\mathbb{C}P^2$ arise as the minima of the localization action, as well as (very briefly) review some relevant aspects of the ADHM construction in the mathematical literature. In Sec. III we study unitary instantons on $\mathbb{C}P^2$, considering, in particular, our novel approach consisting of the resolution of the extra directions upon ungauging $U(1)$'s as well as the AdS/CFT description of (part of) the instanton moduli space—this providing very nontrivial evidence of both the construction and the AdS/CFT duality, as it requires a precise matching of superconformal R charges. In Secs. IV–VI we turn to instantons on orbifolded spaces, for which we provide the first explicit description. In Sec. IV we consider the construction of unitary instantons on the orbifold space. In Sec. V we turn to the symplectic case, finding the ADHM construction of their moduli space on $\mathbb{C}P^2/\mathbb{Z}_n$. In Sec. VI we turn to orthogonal instantons, analyzing, very much like in the unitary case, the compact extra directions associated to the nontrivial topology. Moreover, we provide the construction of orthogonal instantons on the orbifolded space. We provide a short summary of the highlights as well as some conclusions in Sec. VII. Finally, we describe some exotic cases as well as compile some figures in the appendixes in order to not clutter the text.

II. SELF-DUAL INSTANTON CONTRIBUTIONS TO SUPERSYMMETRIC GAUGE THEORY ON $\mathbb{C}P^2$

We are interested in pure gauge theories on $\mathbb{C}P^2$. Hence, our first task is the construction of the supersymmetric Lagrangian for the theory on the curved manifold. For that matter, we follow the approach in [19], which amounts to considering the combined system of supergravity plus the gauge theory of interest. Then, a rigid limit freezes the gravitational dynamics so that we are automatically left with the supersymmetric gauge theory on the curved space. Since we are interested in $\mathcal{N}=2$ gauge theories, we will use conformal supergravity as in [24].

Recently, the partition function of supersymmetric gauge theories on $\mathbb{C}P^2$ was considered in [20]. However, in this paper, we are interested in a different version of the gauge theory. Recall that in order to find the supersymmetric theory, we need to solve the gravitino variation as well as the auxiliary condition in [24]. These provide both the background fields as well as the Killing spinors for the gauge theory on the curved space. A natural solution to these equations is the topological twist [25]. On general grounds, this amounts to redefining the Lorentz group—generically locally $SO(4)\sim SU(2)_{\rm left}\times SU(2)_{\rm right}$—by twisting either $SU(2)_{\rm left,right}$ with $SU(2)_R$. Nevertheless, as described in, e.g., [26], since for Kähler manifolds the holonomy is really $SU(2)_{\rm right}\times U(1)_{\rm left}$, a second version exists whereby one twists the $U(1)_{\rm left}$ by the Cartan of the $SU(2)_R$ (note that in this case, one chirality is privileged over the other by the orientation naturally induced by the Kähler form). While in [20] this latter choice was considered, in this paper we will focus on the former version of the topological twist, which can be performed both for positive and negative chiralities of the background Killing spinors.

Setting to begin with all supergravity fields other than the metric and $SU(2)_R$ gauge field to zero, the equations defining the supersymmetric backgrounds are defined by the conformal Killing spinor equation [24] (we refer to this reference for details)

$$\mathcal{D}_\mu\epsilon^i_\pm-\frac{1}{4}\gamma_\mu D\epsilon^i_\pm=0, \tag{1}$$

where the covariant derivative acting on the background Killing spinors is

$$\mathcal{D}_\mu\epsilon^i_\pm=\nabla_\mu\epsilon^i_\pm+(\mathcal{A}_\mu)^i_j\epsilon^j_\pm, \tag{2}$$

while $\mathcal{A}_\mu$ is the $SU(2)_R$ gauge field, and ∇_μ is the covariant derivative acting on spinors including the spin connection. Moreover, the metric of the $\mathbb{C}P^2$ is

$$\begin{aligned} ds_{\mathbb{C}P^2}&=d\rho^2+\frac{\sin^2\rho}{4}[d\theta^2+\sin^2\theta d\phi^2\\ &\quad+\cos^2\rho(d\psi+\cos\theta d\phi)^2],\\ \rho&\in\left[0,\frac{\pi}{2}\right],\quad \psi\in[0,4\pi],\quad \theta\in[0,\pi],\quad \phi\in[0,2\pi]. \end{aligned} \tag{3}$$

In hindsight, in this paper we are interested in keeping the positive chirality spinors. Choosing then

$$(\mathcal{A}_\mu)^i_j = -\frac{i}{4}\eta_{Iab}\omega_{\mu ab}(\sigma^I)^i_j, \tag{4}$$

where η_{Iab} is the 't Hooft symbol and σ^I are the Pauli matrices, we have that the spin connection part in the covariant derivative is canceled, so that the Killing spinors are simply[3]

$$\epsilon^1_+ = \begin{pmatrix} i\alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \epsilon^2_+ = \begin{pmatrix} 0 \\ i\alpha \\ 0 \\ 0 \end{pmatrix}, \qquad \alpha \in \mathbb{R}. \tag{5}$$

Furthermore, one can check that the remaining supergravity equation is solved upon appropriately tuning the supergravity scalar [25].

Following [24], negative chirality spinors could be included choosing a Killing vector v of $\mathbb{C}P^2$ as $\epsilon^i_- = i\not{v}\epsilon^i_+$ upon turning on $T^- = 2dv|_-$. Let us stick, however, to the topological case. Then, since the theory is invariant under the supersymmetry generated by the above ϵ^i_+, we could add to the action the Q-invariant term $-t\int \delta\mathcal{V}$, being $\delta\mathcal{V} = |\delta\Omega^i_+|^2 + |\delta\Omega^i_-|^2$. The standard argument suggests then that the action is t invariant. A straightforward calculation gives [we set $(\epsilon^i_+)^\dagger\epsilon^i_+ = 1$]

$$\delta\mathcal{V} = \frac{1}{64}(F^+)^2 + |D\bar{\phi}|^2 + \frac{1}{8}|Y^i{}_j|^2 + |[\phi,\bar{\phi}]|^2, \tag{6}$$

where we have imposed the reality condition $Y^i{}_j = (Y^j{}_i)^\star$ [20]. Since Eq. (6) is strictly positive, in the classical limit $t\to\infty$, the theory localizes on configurations such that the scalar in the vector multiplet is constant and lies along the Cartan of the gauge group while $F^+ = 0$. Note that, had we chosen to keep negative chirality spinors, we would have obtained $F^- = 0$. Being more explicit, the condition $F^+ = 0$ is, in the conventions of [24], equivalent to[4]

$$F^+ = \frac{1}{2}(F - \star F) = 0 \rightsquigarrow F = \star F. \tag{7}$$

That is, F must be SD. Since, for the standard orientation of the $\mathbb{C}P^2$, the Kähler form is also self-dual, we have that the relevant gauge configurations in this case are instantons of the same duality type of the Kähler form. This is precisely the type of instantons described in [15] using the King [13] and Bryan and Sanders [14] constructions elaborating on [10–12].

[3]We choose a chiral representation for the Dirac algebra so that $\Gamma_5 = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$.

[4]Here, $(\star F)_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$.

A. The construction of self-dual instantons on $\mathbb{C}P^2$

While we are interested in constructing self-dual instantons on $\mathbb{C}P^2$, it is, however, more convenient to regard them, upon orientation reversal of the base manifold, as ASD instantons on $\overline{\mathbb{C}P}^2$ (the opposite-oriented $\mathbb{C}P^2$). Then, we can directly borrow the construction of their moduli spaces from King [13] and Bryan and Sanders [14]. Let us give a lightning overview of the relevant ingredients of the construction and defer to [10–14] for the detailed account (see, also, [15] for more references).

On very general grounds, there is a correspondence between the moduli space of instantons on projective algebraic surfaces and the moduli space of (stable) holomorphic bundles which goes under the name of Hitchin-Kobayashi correspondence. In this context, the ADHM construction can be regarded as a device to construct holomorphic bundles over the appropriate manifold.

An alternative version of the Hitchin-Kobayashi correspondence, more useful for our purposes, was proven by Donaldson by using the so-called Ward correspondence, which associates an ASD connection—that is, a connection whose curvature is ASD—on a (not complex) manifold X to a holomorphic bundle on a related manifold X_{holo}. Roughly speaking, one regards X as a conformal compactification of some underlying complex manifold X_{cplx}. Since both the Yang-Mills equations and the self-duality constraints are conformally invariant, solutions with definite duality properties (say, ASD) on X_{cplx} can be naturally extended into solutions on X. Note that, in doing this, the behavior of the gauge field at the added point must be specified; that is, a framing must be chosen. In particular, we choose a trivial framing, where the gauge transformations become the identity at infinity.

On the other hand, it is well known that connections with an ASD curvature on a complex manifold X_{cplx} are in one-to-one correspondence with holomorphic bundles on X_{cplx}.[5] Since the moduli space of the latter is a rather sick notion, being X_{cplx} a noncompact space, we can considering a holomorphic compactification of X_{cplx} into X_{holo} whereby we add the complex line at infinity ℓ_∞ and demand the holomorphic bundle to be trivial over there. Hence, all in all, the problem of constructing trivially framed ASD connections on X is mapped to the construction of holomorphic bundles—denote them by E—over X_{holo} trivial over ℓ_∞. The ADHM construction is precisely the device constructing such bundles.

In the case at hand, we consider $X_{\mathrm{cplx}} = \widehat{\mathbb{C}}^2$, the blowup of $\mathbb{C}^2$ at a point defined as

[5]Roughly speaking, this is due to the fact that the ASD condition on a connection A is equivalent to the integrability condition $\bar{\partial}_A^2 = 0$ of $\bar{\partial}_A = \bar{\partial} + \bar{A}$, hence, defining a holomorphic bundle on X_{cplx} through the Newlander-Nirenberg theorem. See [10–14] and [15] for more references.

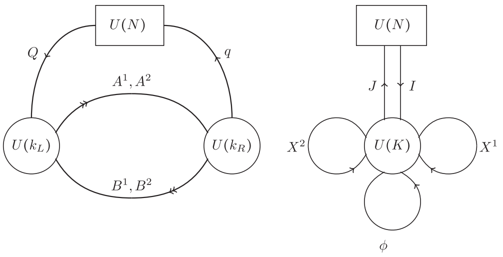

FIG. 1. Quiver diagram for $SU(N)$ instantons on $\mathbb{C}P^2$ (on the left) and for $SU(N)$ instantons on $\mathbb{C}^2$ (on the right).

$$\widehat{\mathbb{C}}^2 = \{(x_1, x_2) \times [z_1, z_2] \in \mathbb{C}^2 \times \mathbb{C}P^1 / x_1 z_1 = x_2 z_2\}. \quad (8)$$

Then, on one hand, we can find a conformal compactification of $X_{\text{cplx}} = \widehat{\mathbb{C}}^2$ into $X = \overline{\mathbb{C}P}^2$—the opposite-oriented $\mathbb{C}P^2$—as follows:

$$\widehat{\mathbb{C}}^2 \to \overline{\mathbb{C}P}^2\colon\ ((x_1, x_2) \times [z_1, z_2]) \to \begin{cases} [|x|^2, x_1, x_2], \\ [0, z_1, z_2]. \end{cases} \quad (9)$$

Note that $\widehat{\mathbb{C}P}^2$ is not really a complex manifold, as the orientation does not follow from the Kähler form.

On the other hand, we can find a holomorphic compactification by adding ℓ_∞ which compactifies $\widehat{\mathbb{C}}^2$ into $X_{\text{holo}} = \mathbb{C}P^2$ blown up at a point, that is, Hirzebruch's first surface $\mathbb{F}_1$. Hence, we have that framed ASD connections over $\mathbb{C}P^2$ are in one-to-one correspondence with holomorphic bundles over $\mathbb{F}^1$ which are trivial over ℓ_∞. Since upon orientation reversal, ASD connections on $\overline{\mathbb{C}P}^2$ become SD connections on $\mathbb{C}P^2$, it follows that the desired moduli spaces are in correspondence with holomorphic bundles over $\mathbb{F}_1$. Then, the ADHM construction is precisely the device to construct such bundles.

While here we will not dive into more details, an instrumental notion in arriving at the actual ADHM construction, from this point of view, is the associated twistor space, which takes into account the sphere bundle of compatible complex structures over X_{holo}. Instead of delving into more intricacies, here we will describe the ADHM-like description of instantons for unitary, orthogonal, and symplectic gauge groups embedded in a gauge theory as in [15], and refer to [10–14] for the details of their construction along the lines outlined here.

One word of caution is in order. Even though in the following we will loosely refer to instantons on $\mathbb{C}P^2$, the previous description of the precise construction should be borne in mind—that is, we are describing SD instantons on $\mathbb{C}P^2$ or equivalently ASD instantons on $\overline{\mathbb{C}P}^2$. Moreover, we stress that we discuss framed instantons where a particular behavior in the added line (trivial) is imposed.

III. $U(N)$ INSTANTONS ON $\mathbb{C}P^2$

As described in [15], the King [13] construction for unitary instantons on $\mathbb{C}P^2$ can be embedded into a $3d$ quiver gauge theory. The theory in question is a $3d$ $\mathcal{N} = 2$ gauge theory whose quiver is in the left panel of Fig. 1, supplemented with the superpotential

$$W = \mathrm{Tr}[A^1 B^1 A^2 B^2 - A^1 B^2 A^2 B^1 + qA^1 Q]. \quad (10)$$

Note that the chiral nature of the theory demands, because of the parity anomaly, the gauge nodes to have a nonvanishing Chern-Simons level $\frac{N}{2} + \mathbf{k}_L$ and $-\frac{N}{2} + \mathbf{k}_R$, respectively, where $\mathbf{k}_L$, $\mathbf{k}_R$ are integers including zero. In the following, we will concentrate on the case $\mathbf{k}_L = \mathbf{k}_R = 0$.

As a $3d$ gauge theory, it has been argued [27,28] that the theory flows to an IR fixed point, where the charges of the fields are listed in Table I. For the particular case $N = 1$, as argued in [28], the mesonic moduli space (excluding "Higgs-like" directions where fundamental fields take a VEV) of the theory is the direct product of a conifold times the complex line. In general, as N is increased, this geometric branch of the moduli space becomes an increasingly more involved toric manifold (see [28]).

The instanton moduli space of interest is that of $G = U(N)$ instantons on $\mathbb{C}P^2$, denoted as $M^G_{\mathbb{C}P^2}$. It arises as a Higgs-like branch of the full moduli space of the gauge theory dubbed the instanton branch where fundamental fields take a VEV. Note that the instanton gauge group appears as the flavor symmetry of the ADHM construction. Note as well that in order to specify the instanton, in

TABLE I. Transformations of the fields for the $\mathbb{C}P^2$ quiver gauge theory. Here, r is an unknown real parameter whose value, nevertheless, does not affect subsequent results.

Fields	$U(k_L)$	$U(k_R)$	$U(N)$	$SU(2)$	$U(1)_R$
A^1	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
A^2	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^1, B^2	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1]$	1/4
q	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]$	$1-1/4r$
Q	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$1/4r$
F term	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

general, a set of numbers I including the instanton number is required. We will come back to this issue below.

More precisely, as described in [15], the instanton branch of the moduli space arises when we set A^1 (as well as all monopole operators, typically denoted by T, $\tilde{T}$) to zero. It is important to note that the truncation $A^1 = T = \tilde{T} = 0$ is consistent with the quantum constraint on the moduli space introduced in [28]. Then, the only relevant F term arises from the superpotential and reads

$$\partial_{A^1} W = B^1 A^2 B^2 - B^2 A^2 B^1 + qQ. \tag{11}$$

Together with the field content and gauge groups of the $3d$ gauge theory, this constraint precisely realizes the King construction. Note that even though the flavor symmetry is $U(N)$, the $U(1)$ part is really gauged. Hence, we can think of our instantons as instantons of $SU(N)$ [even though, as we will review below, we should really think of $SU(N)/\mathbb{Z}_N$].

In the following, we are interested in the Hilbert series of the instanton moduli space. The ADHM construction just introduced (and the corresponding orthogonal and symplectic versions in addition to their orbifoldings to be described below) allows us to compute it using by now standard methods as in, e.g., [15,29–31] (see, also, [32] for the study of instantons on $\mathbb{C}^2/Z_n$). Let us pause to make a point on notation. Throughout the paper, we will denote the Hilbert series H of the instantons' moduli space as $H[I, G, M]$, being I the integers characterizing the instanton, which appears as the date of the gauge group of the ADHM construction, G those characterizing the instanton gauge group appearing as a flavor group in the ADHM construction, and M the ambient manifold of the instanton.

As anticipated, in order to specify a particular G instanton on $\mathbb{C}P^2$, a set of quantum numbers I is required. It is clear that one such integer is the instanton number. However, since $\mathbb{C}P^2$ is a topologically nontrivial manifold, it is natural to expect that instantons on $\mathbb{C}P^2$ might carry extra quantum numbers. Indeed, as reviewed in [15] following [16], we can characterize the instanton by its first Chern number $\hat{c}$ and its instanton number $\hat{k}$. Using the correspondence between ASD connections on X and holomorphic bundles E on X_{holo}, these can be written as

$$\langle c_1(E), [C] \rangle = -\hat{c}, \quad \left\langle c_2(E) - \frac{N-1}{2N} c_1(E)^2, [\mathbb{F}_1] \right\rangle = \hat{k}, \tag{12}$$

being $[C]$ the $\mathbb{C}P^1$ class inside $\mathbb{F}_1$—recall that, in this case, $X = \overline{\mathbb{C}P^2}$ and $X_{\text{holo}} = \mathbb{F}_1$. These, in turn, are related to the quiver data k_L, k_R as follows:

$$\hat{c} = k_R - k_L, \quad \hat{k} = \frac{1}{2}(k_L + k_R) - \frac{1}{2N}(k_L - k_R)^2. \tag{13}$$

As an algebraic variety, $M^{SU(N)}_{\mathbb{C}P^2}$ can be mapped into the moduli space of a related instanton on $\mathbb{C}^2$—described by the Higgs branch of the theory in the right panel of Fig. 1—in the following way,

$$\pi\colon (A^2, B^1, B^2, Q, q) \to (X^1 = A^2 B^1, X^2 = A^2 B^2, I = A^2 q, J = Q), \tag{14}$$

being X^1, X^2, I, J the fields of the quiver diagram for $\mathbb{C}^2$ theory. Indeed, if we multiply the F-term relation (11) by A^2 and we apply the map (14), we recover the F term for $SU(N)$ instantons on $\mathbb{C}^2$,

$$[X^1, X^2] + I \cdot J = 0. \tag{15}$$

In turn, the inverse map σ can also be defined as

$$\sigma\colon (X^1, X^2, I, J) \to (A^2 = \mathbf{1}_{K\times K}, B^1 = X^1, B^2 = X^2, q = I, Q = J). \tag{16}$$

Let us momentarily consider the case where $k_L = k_R$, which corresponds to $\hat{c} = 0$ and $\hat{k} = k_L$. From the construction in Eq. (14), it is clear that the integer K in the quiver in the right panel of Fig. 1 is identified with k_L. Thus, we have that as an algebraic variety, the moduli space of k_L $SU(N)$ instantons on $\mathbb{C}P^2$ is identified with the moduli space of k_L $SU(N)$ instantons on $\mathbb{C}^2$. Consistently, the Hilbert series of these instantons coincide, from which it follows that $\dim_{\mathbb{C}} M^{SU(N)}_{\mathbb{C}P^2} = 2Nk_L$.

In the general case $k_L \neq k_R$, one finds that the above construction still holds upon setting $K = \min(k_L, k_R)$. Consistently, as described in [15], the Hilbert series corresponding to the instanton branch of the quiver in the left panel of Fig. 1 coincides with the Hilbert series of the Higgs branch of the quiver in the right panel of Fig. 1, that is,

$$H[(k_L, k_R), SU(N), \mathbb{C}P^2](t, x, \mathbf{y}) = H[\min(k_L, k_R), SU(N), \mathbb{C}^2](t^3, x, \mathbf{y}), \tag{17}$$

where t is the fugacity of the R charge, x the fugacity associated with the $SU(2)$ global symmetry, and $\mathbf{y}$'s are the fugacities associated with the $U(N)$ global symmetry. Note that the fugacity associated to the R charge is rescaled from t in the $\mathbb{C}P^2$ case into t^3 in the $\mathbb{C}^2$ case.

Naively, Eq. (17) suggests that the dimension of the moduli space of unitary instantons on $\mathbb{C}P^2$ is

$$\dim_{\mathbb{C}} M^{SU(N)}_{\mathbb{C}P^2} = 2N \min(k_L, k_R). \tag{18}$$

Note that, even though the quiver is specified by three integers N, k_L, k_R, Eq. (18) is only sensitive to two of them.

However, it is possible to consider an extended notion of the moduli space where the extra directions associated to all the three quantum numbers specifying the instanton are taken into account. This is the so-called resolved (as the extra directions are discerned) moduli space denoted as $\widehat{M}^{SU(N)}_{\mathbb{C}P^2}$, whose dimension is [16–18]

$$\dim_{\mathbb{C}}\widehat{M}^{SU(N)}_{\mathbb{C}P^2} = 2\hat{k}N = \dim_{\mathbb{C}}M^{SU(N)}_{\mathbb{C}P^2} + \hat{c}(N-\hat{c}). \quad (19)$$

Note that for $\hat{c}=0$, N the dimension of $\widehat{M}^{SU(N)}_{\mathbb{C}P^2}$ is equal to the dimension of $M^{SU(N)}_{\mathbb{C}P^2}$. This suggests that $\hat{c}$ is really a modulo N quantity corresponding to an instanton gauge group which is really $SU(N)/\mathbb{Z}_N$. We warn the reader that, while in the following we will not clutter notation by suppressing the $\mathbb{Z}_N$, the global properties of the gauge group must be kept in mind.

A. The resolved moduli space and the Grassmanian

In order to explore the resolved moduli space, it is instructive to first consider the simplest case where $k_L = 0$. The theory simplifies into a one-noded quiver flavored only with fundamental fields (and not antifundamentals) shown in Fig. 2. Recall that the CS level is adjusted so as to cancel the parity anomaly, and, furthermore, there is no superpotential.

The leftover theory in this particular case corresponds to a $3d$ version of the theory considered in [21]. Then, as argued in that reference, the moduli space is a complex Grassmanian (compact) manifold, consistent with the expectations in [16–18].

We can now understand why $M^{SU(N)}_{\mathbb{C}P^2}$ is insensitive to these extra directions, as forming a compact Grassmanian manifold, the Hilbert series is blind to them. Indeed, since in the theory in Fig. 2 the gauge group is $U(k_R)$, the Higgs-like moduli space is empty, as no gauge invariant can be constructed out of fundamental fields. Consistently, formula (18) gives a zero-dimensional moduli space. However, as in [23], we can consider a version of the theory where only the non-Abelian $SU(k_R)$ part of $U(k_R)$ is gauged, while the $U(1)$ is kept as a global baryonic symmetry [alternatively, we could think of this as the

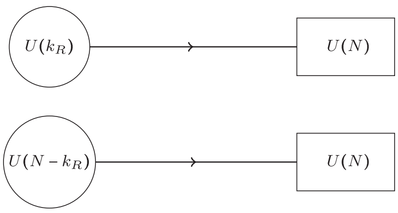

FIG. 2. Quiver diagrams for Grassmanian (we show the dual pair—see text).

master space [22] of the $U(k_R)$ theory]. In this case, we can form baryonlike gauge-invariant operators, thus, finding a nonempty moduli space which, in fact, is a complex cone over the Grassmanian. It is straightforward to compute the Hilbert series. Unrefining the flavor fugacities, we have

$$\mathrm{HS} = \int \mathrm{PE}[Nt\chi_{\square_{k_R}}], \quad (20)$$

where $\chi_{\square_{k_R}}$ is the character of the $SU(k_R)$ fundamental. Let us introduce the d-Narayana numbers

$$N_{d,n,k} = \sum_{j=0}^{k}(-1)^{k-j}\binom{dn+1}{k-j}\prod_{i=0}^{d-1}\binom{n+i+j}{n} \times \binom{n+i}{n}^{-1}. \quad (21)$$

Using them we can define the Narayana polynomial

$$\hat{P}_{d,n}(t) = \sum_{k=0}^{(d-1)(n-1)} N_{d,n,k}t^{dk}. \quad (22)$$

In terms of this polynomial, one can see that

$$\mathrm{HS} = (1-t^{k_R})^{k_R^2-1-k_RN}\hat{P}_{k_R,N-k_R}. \quad (23)$$

We can easily read off the dimension of the moduli space from the pole at $t=1$, which is simply coming from the prefactor before the Narayana polynomial, finding (this result, not known in the literature to the best of our knowledge, generalizes that in [33])

$$\dim_{\mathbb{C}}M^{SU(N)}_{\mathbb{C}P^2}|_{\mathrm{Grassmanian}} = k_R(N-k_R)+1. \quad (24)$$

Recalling that the $+1$ is due to the $U(1)$ which we are not integrating over—resulting in moduli space which is a complex cone over the Grassmanian—we find a result in accordance with Eq. (19).

Equation (24) is invariant under the exchange $k_R \leftrightarrow N-k_R$. Indeed, one can explicitly check that the Hilbert series of the theories with $SU(k_R)$ gauge group and $SU(N-k_R)$ are identical up to a trivial redefinition of t, thus, suggesting a duality among these theories. Note that this should imply nontrivial identities among Narayana polynomials, which would be interesting to explore. Such duality is also suggested by the brane construction in [21].[6] In that reference, in a IIA system consisting on an NS-brane and an NS'-N $D4$-branes intersection, k_R $D2$-branes are stretched along x^6 direction between the NS and the

[6]We should stress that the same choice of Fayet-Iliopoulos (FI) parameters as in those references related to the stability conditions in the ADHM-like construction applies.

NS'-$D4$ intersection. Then, the N $D4$'s can be broken on the NS' and, say, the lower part of them can be sent to infinity. As argued in [21], the gauge theory on the $D2$'s is precisely the $2d$ version of the gauge theory in the first panel of Fig. 2. Upon T duality along x^2, this system engineers the actual $3d$ gauge theory of interest, namely, that in the first panel of Fig. 2. Explicitly, the system contains

(i) An NS-brane along 012345.
(ii) A braneweb with an NS'-brane along 012389 meeting N $D5$-branes along 012378 and emanating a $(1,N)$ fivebrane.
(iii) k_R $D3$-branes along 0126, starting at the braneweb junction and ending on the NS.

Note that the NS'-$D4$ intersection in the IIA system becomes a braneweb in the IIB system, as $D5$-branes meeting an NS' give rise to a $(1,N)$ fivebrane. In fact, it is precisely this bending that gives the expected CS level in the $3d$ gauge theory [34,35]. In this, it is important to recall that the $D3$'s meet the fivebranes right at the junction, as this is what makes the $3d$ theory contain only fundamental (and not antifundamental) matter [21], which, in turn, generates the $\frac{N}{2}$ CS level.

We can now imagine crossing the NS to the other side. Then, due to the Hanany-Witten effect, the final configuration contains $N-k_R$ $D3$-branes but is otherwise identical, consistent with our finding that the two theories in Fig. 2 yield the same Hilbert series (for a more detailed account of the duality in the $2d$ case, we refer to [21]).

Coming back to the general discussion, in view of the $k_L=0$ case, it is natural to guess that ungauging the Abelian part of the largest gauge symmetry will allow us to resolve the extra directions in $\hat{M}$. For that matter, let us now consider the case $k_L=1$. Writing the remaining $U(k_R)$ gauge group as $U(1)\times SU(k_R)$, we can compute the Hilbert series upon integration only over the non-Abelian $SU(k_R)$ part. In this case, finding a closed analytic form seems a daunting task. Nevertheless, from explicit computations for $k_L=1$ and $k_R=2,3$ and $N=1,2,3$, we find that (the explicit forms of the Hilbert series are rather unilluminating, and we will refrain from explicitly displaying them here) reading the dimension of the moduli space from the order of the pole at $t=1$, the dimension is compatible with the formula

$$\mathrm{dim}_{\mathbb{C}}\hat{M}^{SU(N)}_{\mathbb{C}P^2}=2k_LN+\hat{c}(N-\hat{c})+1, \tag{25}$$

which is precisely the expected result (19). Unfortunately, explicitly checking higher-rank cases is technically challenging. Nevertheless, it would be very interesting to perform further checks for higher ranks.

B. Rank one and AdS/CFT

In the particular case of $k_L=k_R$, upon setting $N=1$ and for $\mathbf{k}_L=\mathbf{k}_R=0$, the theory engineering the moduli space of unitary instantons on $\mathbb{C}P^2$ becomes exactly that found in [28] to describe M2 branes probing $\mathcal{C}\times\mathbb{C}$, the direct product of a conifold times the complex line. The metric of the CY_4 cone can be written as

$$ds^2_{\mathrm{cone}}=d\rho^2+\rho^2ds^2_{\mathcal{B}}, \tag{26}$$

$$ds^2_{\mathcal{B}}=d\alpha^2+\sin^2\alpha d\gamma^2+\frac{\cos^2\alpha}{9}\left(d\psi+\sum_{i=1}^{2}\cos\theta_id\phi_i\right)^2+\sum_{i=1}^{2}\frac{\cos^2\alpha}{6}(d\theta_i^2+\sin^2\theta_id\phi_i^2). \tag{27}$$

Then, on general grounds, the near-brane geometry for a stack of k_L M2 branes probing this cone is $\mathrm{AdS}_4\times\mathcal{B}$, which, in global coordinates, can be written as

$$ds^2=-\left(1+\frac{r^2}{L^2}\right)dt^2+\frac{dr^2}{(1+\frac{r^2}{L^2})}+r^2(\sin^2\theta d\theta^2+d\phi^2)+4L^2ds^2_{\mathcal{B}}, \tag{28}$$

being L the radius of the AdS_4 space. Besides, there is a 6-form flux whose field strength integrates to k_L on $\mathcal{B}$. Hence, in the large $k_L(=k_R)$ limit, the gauge theory is holographically dual to $\mathrm{AdS}_4\times\mathcal{B}$ with k_L units of flux through $\mathcal{B}$. It is, thus, natural to wonder whether, at least partially, the moduli space of unitary instantons on $\mathbb{C}P^2$ can be geometrically realized in this context.

As discussed in [28], the gauge theory contains a mesonic branch of the moduli space which realizes the dual geometry. In general, it is natural to expect that the holographic dual captures gauge theory operators made out of bifundamental fields, while those corresponding to fundamental matter would require extra multiplets on top of the $\mathrm{AdS}_4\times\mathcal{B}$ to account for the "flavor brane open string" degrees of freedom. Hence, it is natural to expect that the sub-branch of the instanton branch involving just $\{A^2,B^i\}$ fields is visible in the geometry. This is indeed analogous to the cases discussed in [31,36], where only the "closed string fields" in the quiver are captured by the gravity dual.

More explicitly, following [31,36], it is natural to expect that this sub-branch of the instanton branch is captured by dual giant graviton branes moving in the appropriate subspace corresponding to the instanton branch. For that matter, we consider a probe M2 brane wrapping (t,Ω_2), where Ω_2 is the sphere inside the AdS_4. Moreover, we assume that $\psi=\psi(t)$ and $\phi_2=\phi_2(t)$, while

$$\gamma,\alpha,\theta_1,\phi_1,\theta_2=\mathrm{constant}. \tag{29}$$

The action for such probe brane is

$$S=-T_2\int\sqrt{-g}+T_2\int P[A^{(3)}], \tag{30}$$

which becomes

$$S = -T_2V_2 \int dt r^2 \left(\sqrt{\left(1+\frac{r^2}{L^2}\right) - \frac{4L^2\cos^2\alpha}{9}(\dot{\psi}(t)+\cos\theta_2\dot{\phi}_2(t))^2 - \frac{4L^2\cos^2\alpha\sin^2\theta_2}{6}\dot{\phi}_2(t)^2} - \frac{r^3}{L}\right).$$

It is easy to convince oneself that the equations of motion fix $\alpha = 0$ (for simplicity, from now on we set $\alpha = 0$). Then, with the Legendre transforming to the Hamiltonian $H = H(\theta_2, r, P_\psi, P_{\phi_2})$, we obtain

$$H = \frac{1}{2L}\sqrt{\frac{r^2+L^2}{L^2}}\sqrt{\frac{3(5-\cos 2\theta_2)P_\psi^2 - 24\cos\theta_2 P_\psi P_{\phi_2} + 2(6P_{\phi_2}^2 + 4L^2r^4\sin^2\theta_2 T_2^2V_2^2)}{2\sin^2\theta_2}} - \frac{V_2T_2r^3}{L}.$$

The minimum energy configurations are

$$\cos\theta_2 = \frac{P_{\phi_2}}{P_\psi}, \tag{31}$$

for which

$$r = 0 \quad \text{or} \quad r = \frac{3P_\psi}{2L^2T_2V_2}. \tag{32}$$

Both configurations are degenerated in energy, one corresponding to pointlike gravitons and the other to true dual giant gravitons. The energy is

$$H = \frac{3P_\psi}{2L}. \tag{33}$$

Coming back to the solution in Eq. (31), we can parametrize the phase space of the spinning M2 as a dynamical system by the coordinates $Q^A = \{r, \alpha, \psi, \theta_2, \phi_2\}$ and the conjugated momenta $P_A = \{P_r, P_\alpha, P_\psi, P_{\theta_2}, P_{\phi_2}\}$. Moreover, the conjugated momenta P_A must obey the following constraints:

$$f_r = P_r, \qquad f_\alpha = P_\alpha, \qquad f_{\theta_2} = P_{\theta_2},$$
$$f_\psi = P_\psi - \frac{2L^2T_2V_2r}{3},$$
$$f_{\phi_2} = P_{\phi_2} - \frac{2L^2T_2V_2r\cos\theta_2}{3}.$$

As usual, the matrix $M_{AB} = \{f_A, f_B\}_{PB}$ encodes the symplectic form associated to the phase space of our dynamical system as $\{Q^A, Q^B\}_{\rm DB} = (M_{AB})^{-1}$ (DB stands for Dirac brackets). Deleting the row and column corresponding to the trivial α coordinate, we find

$$M^{AB} = \begin{pmatrix} 0 & \frac{2L^2T_2V_2}{3} & 0 & \frac{2L^2T_2V_2\cos\theta_2}{3} \\ \frac{-2LT_2V_2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2L^2r\sin\theta_2T_2V_2}{3} \\ \frac{-2L^2\cos\theta_2T_2V_2}{3} & 0 & \frac{2L^2r\sin\theta_2T_2V_2}{3} & 0 \end{pmatrix}.$$

Therefore, the symplectic structure reads

$$\omega = \frac{2L^2T_2V_2}{3}dr\wedge d\psi + \frac{2L^2T_2V_2\cos\theta_2}{3}dr\wedge d\phi_2 - \frac{2L^2T_2V_2r\sin\theta_2}{3}d\theta_2\wedge d\phi_2.$$

Integrating, we obtain

$$\nu = \frac{2L^2T_2V_2r}{3}(d\psi + \cos\theta_2 d\phi_2) \Rightarrow \omega = d\nu. \tag{34}$$

Hence, upon introducing $\rho^2 = 4L^2T_2V_2r/3$, we just recover the data of $\mathbb{C}^2$. Following [31,36], we can do symplectic quantization of this dynamical system. On general grounds, that amounts to identifying the holomorphic functions on the phase space—in this case $\mathbb{C}^2$—with the allowed wave functions. These can easily be counted, simply obtaining the Hilbert series for $\mathbb{C}^2$.

Let us now turn to the gauge theory. As discussed, we expect our probe branes to be dual to operators on the instanton branch not containing fundamental fields. These are of the schematic form

$$\mathcal{O}_{n,m} = (A^2B^1)^n(A^2B^2)^m. \tag{35}$$

Note that the F terms imply that the B^i indices are completely symmetrized; that is, the operators $\mathcal{O}_{n,m}$ are in a spin $\frac{(n+m)}{2}$ representation of the $SU(2)$ global symmetry rotating the B^i's. Hence, for a fixed R charge $R[\mathcal{O}_{n,m}] = \frac{3}{4}(n+m)$, the number of operators is $(n+m)+1$, and the corresponding generating function is just $\sum_{j=0}^{\infty}(j+1)x^j = (1-x)^{-2}$, which is precisely the $\mathbb{C}^2$ Hilbert series; here, x is a generic fugacity.

We can explicitly compare the gauge theory operators with our probe brane configurations on the gravity side. For that matter, let us first note that exactly the same configuration on the gravity side would have been obtained fixing $\theta_2 = 0, \pi$ and having our brane orbiting $\psi \pm \phi_1$, respectively. Hence, in all our formulas, we can trade ψ for $\tilde{\psi} = \psi \pm \phi_1$. In particular, Eq. (33) becomes $HL = \frac{3}{2}P_{\tilde{\psi}}$.

In order to compare our probe branes with the gauge theory operators, we need to identify charges. It is reasonable to guess that the momentum along ψ is proportional to the R symmetry. Hence, let us identify $P_\psi = r$, being r (not to be confused with the arbitrary integer in Table I) proportional to the charge R under the $U(1)_R$ in a way which we will shortly come back to. Moreover, in order to understand the $P_{\phi_{1,2}}$ momenta, it is instructive to consider momentarily removing the quarks from the gauge theory. It then exhibits an $SU(2)_A \times SU(2)_B$ global symmetry rotating, respectively, the A^i and B^i fields. Then, the quark multiplets break the $SU(2)_A$ down to a $U(1)_A$, while the $SU(2)$ rotating the B's remains as a global symmetry. We identify the $U(1)_A$ charge denoted as Q_A, with P_{ϕ_1} as $Q_A = P_{\phi_1}$. With no loss of generality, let us assume $Q_A[A^2] = \frac{1}{2}$, which corresponds to the choice $\theta_1 = \pi$. Then $P_{\tilde{\psi}} = P_\psi - P_{\phi_1}$ translates into $P_{\tilde{\psi}} = r - Q_A$. Analogously, we identify P_{ϕ_2} with the Cartan of the $SU(2)_B$ denoted as Q_B.

Note that Eq. (31) translates into $Q_B = (r - Q_A)\cos\theta_2$, and, therefore, $Q_B \in [-(r-Q_A), (r-Q_A)]$. Let us compare this with the gauge theory operators (35). Using Table I, the charges of the operators in the expression (35) are $R[\mathcal{O}_{n,m}] = \frac{3(n+m)}{4}$ and $Q_A[\mathcal{O}_{n,m}] = \frac{n+m}{2}$. As expected, being chiral operators, they satisfy the usual relation $\Delta = R$. Moreover, it is clear that $Q_B = \frac{n-m}{2}$, so that $Q_B \in [-\frac{2R}{3}, \frac{2R}{3}]$. Comparing the ranges for Q_B in gravity and field theory, we find the identification

$$R = \frac{3}{2}(r - Q_A). \tag{36}$$

Turning now to the energy for our branes, we find $HL = \frac{3}{2}(r - Q_A)$, which, upon using Eq. (36), becomes $\Delta = R$, precisely as expected for chiral operators.

Moreover, we can explicitly fix the value of r. For that matter, let us turn to the field theory operators and consider the highest Q_B weight state, which corresponds to $m = 0$. For this one, $Q_A = Q_B = \frac{n}{2}$, while $R = \frac{3Q_A}{2}$. In turn, from the gravity side, the brane with the highest Q_B is $Q_B = r - Q_A$. Since this must correspond to $Q_B = Q_A$, we find $Q_A = 2r$. Hence, this implies $r = \frac{4R}{3}$.

We can offer an alternative test of our identifications. For that matter, let us consider metric fluctuations polarized along the internal manifold. On general grounds, these fluctuations correspond to operators of the schematic form $\mathcal{T}\mathcal{O}$, being $\mathcal{T}$ the stress-energy tensor of the theory. Note that, for the particular case when the inserted operator $\mathcal{O}$ is one of those in Eq. (35), we expect that the dimension is $3 + \Delta$. In turn, these fluctuations satisfy the Klein-Gordon equation in $\mathrm{AdS}_4 \times \mathcal{B}$. For a CY_4 of the form $\mathbb{C} \times \mathcal{C}$, this problem was considered in [37], where it was shown that the dimension of the dual operators can be written in terms of the eigenvalues of the scalar Laplacian on $\mathcal{C}$. In turn, borrowing the results from [38], the eigenvalues of the scalar Laplacian on the conifold are

$$E_{\mathcal{C}} = 6\left(\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - \frac{r^2}{8}\right), \tag{37}$$

where $\ell_{1,2}$ are, respectively, the $SU(2)_A \times SU(2)_B$ total spin and r the charge along the ψ direction. For the operators in Eq. (35), we have that $\ell_1 = \ell_2 = \ell$. In turn, the charge r must satisfy $\frac{r}{2} \in (-\ell, \ell)$. Focusing on the highest weight state, we would require $r = 2\ell$, which is nothing but $r = 2Q_A$ as seen before. Then, using [37]

$$\Delta = 3 + \frac{3}{2}\ell. \tag{38}$$

This precisely coincides with our expectations upon identifying $\Delta = \frac{3}{2}\ell$. This can be written as $\Delta = \frac{3r}{4}$, which becomes $\Delta = R$ upon using the identification $r = \frac{4R}{3}$ advocated above.

Let us stress that these tests find exact matching between the gauge theory expectations and the gravity dual computations by making explicit use of $U(1)_R$ charge assignations. Since these are not protected in $\mathcal{N} = 2$ theories, the agreement we find should be regarded as a highly nontrivial check of the duality.

So far, we have considered the case $k_L = k_R$. It is natural to expect that $k_L \neq k_R$ can be accommodated into the gravity dual by adding nonvanishing flat B_2 over a 2-cycle in the internal manifold [39]. Nevertheless, such modification of the background would not change our computation. Hence, we would find the same result even for the case $k_L \neq k_R$, in agreement with the field theory result where the Hilbert series only depends on $\min(k_L, k_R)$.

IV. $U(N)$ INSTANTONS ON $\mathbb{C}P^2/\mathbb{Z}_n$

A natural generalization of the ADHM construction of instantons on $\mathbb{C}P^2$ is to consider orbifolding the ambient manifold upon quotienting by a subgroup of its symmetries. In particular, since $\mathbb{C}P^2$ is invariant under a $U(1) \times U(1)$ action corresponding to the ϕ, ψ coordinates in Eq. (3), it is natural to consider quotienting such symmetry by some discrete subgroup of it. Note that the spinors in

Eq. (5) are constant and, moreover, annihilated by $e^{i\frac{2\pi}{k}(J_{12}-J_{34})}$ (J_{ij} are the Lorentz generators in tangent space indices $J_{ij}=\frac{i}{2}[\Gamma_i,\Gamma_j]$). Therefore, we can consider a $\mathbb{Z}_n$ orbifold of the ϕ direction whereby we restrict $\phi\sim\phi+\frac{2\pi}{n}$. In the rest of the paper, we will be interested in the ADHM construction of instantons on these orbifolded spaces. For that matter, we will take as the starting point the ADHM construction in the unorbifolded case, on which we will implement the orbifold by standard methods [9].

Let us consider the case of unitary instantons presented above. In order to find the orbifolded theory, we first need to identify the transformation properties of the fields. These read as follows:

(i) The fields A^j (with $j=1,2$) in the bifundamental representation,

$$A^j\mapsto\gamma_1A^j\gamma_2^{-1}. \tag{39}$$

(ii) The fields B_1 and B_2 in the bifundamental representation,

$$B^1\mapsto\omega_n^{-1}\gamma_2B^1\gamma_1^{-1},\qquad B^2\mapsto\omega_n\gamma_2B^2\gamma_1^{-1},\quad\text{with}\quad \omega_n=e^{2\pi i/n}. \tag{40}$$

(iii) The fields Q and q,

$$q\mapsto\gamma_2q\gamma_3^{-1},\qquad Q\mapsto\gamma_3Q\gamma_1^{-1}, \tag{41}$$

where the matrices γ_1, γ_2, and γ_3 are given by

$$\gamma_1=\mathrm{diag}(\underbrace{1,\ldots,1}_{k_1\ \text{times}},\underbrace{\omega_n,\ldots,\omega_n}_{k_3\ \text{times}},\ldots,\underbrace{\omega_n^{n-1}\ldots,\omega_n^{n-1}}_{k_{2n-1}\ \text{times}})\quad\text{with}\quad\sum_{i\,\text{odd}}^{2n-1}k_i=k_L,$$

$$\gamma_2=\mathrm{diag}(\underbrace{1,\ldots,1}_{k_2\ \text{times}},\underbrace{\omega_n,\ldots,\omega_n}_{k_4\ \text{times}},\ldots,\underbrace{\omega_n^{n-1}\ldots,\omega_n^{n-1}}_{k_{2n}\ \text{times}})\quad\text{with}\quad\sum_{i\,\text{even}}^{2n}k_i=k_R,$$

$$\gamma_3=\mathrm{diag}(\underbrace{1,\ldots,1}_{N_1\ \text{times}},\underbrace{\omega_n,\ldots,\omega_n}_{N_2\ \text{times}},\ldots,\underbrace{\omega_n^{n-1}\ldots,\omega_n^{n-1}}_{N_n\ \text{times}})\quad\text{with}\quad\sum_{i=1}^{n}N_i=N.$$

It is easy to check that the superpotential (10) is invariant under the transformations (39)–(41). In addition, the two gauge groups $U(k_L)$ and $U(k_R)$ of the initial theory and the flavor group $U(N)$ are broken into

$$U(k_L)\mapsto\bigotimes_{i\,\text{odd}}^{2n-1}U(k_i),\qquad U(k_R)\mapsto\bigotimes_{i\,\text{even}}^{2n}U(k_i),\qquad U(N)\mapsto\bigotimes_{i=1}^{n}U(N_i),$$

and after the action of the transformations (39)–(41), the various fields become

$$A^1=\begin{pmatrix}A^1_{11}&0&0&\cdots&0\\0&A^1_{22}&0&\cdots&0\\0&0&A^1_{33}&\cdots&0\\\cdots&\cdots&\cdots&\cdots&0\\0&0&0&0&A^1_{nn}\end{pmatrix},\qquad A^2=\begin{pmatrix}A^2_{11}&0&0&\cdots&0\\0&A^2_{22}&0&\cdots&0\\0&0&A^2_{33}&\cdots&0\\\cdots&\cdots&\cdots&\cdots&0\\0&0&0&0&A^2_{nn}\end{pmatrix},$$

$$B^1=\begin{pmatrix}0&0&0&\cdots&B^1_{1,n}\\B^1_{21}&0&0&\cdots&0\\0&B^1_{32}&0&\cdots&0\\\cdots&\cdots&\cdots&0&0\\0&0&0&B^1_{n,n-1}&0\end{pmatrix},\qquad B^2=\begin{pmatrix}0&B^2_{12}&0&\cdots&0\\0&0&B^2_{23}&\cdots&0\\0&0&0&\cdots&\cdots\\\cdots&\cdots&\cdots&\cdots&B^2_{n-1,n}\\B^2_{n,n-1}&0&0&0&0\end{pmatrix},$$

$$q=\begin{pmatrix}q_{11}&0&0&\cdots&0\\0&q_{22}&0&\cdots&0\\0&0&q_{33}&\cdots&0\\\cdots&\cdots&\cdots&\cdots&0\\0&0&0&0&q_{nn}\end{pmatrix},\qquad Q=\begin{pmatrix}Q_{11}&0&0&\cdots&0\\0&Q_{22}&0&\cdots&0\\0&0&Q_{33}&\cdots&0\\\cdots&\cdots&\cdots&\cdots&0\\0&0&0&0&Q_{nn}\end{pmatrix}.$$

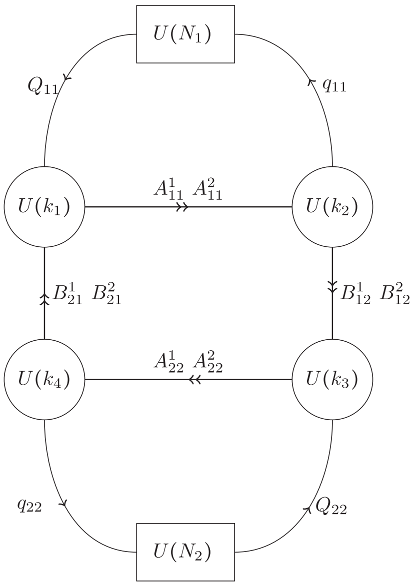

FIG. 3. Quiver diagram for the $\mathbb{C}P^2/\mathbb{Z}_2$ theory.

A. Constructing $U(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$

Let us now show the actual construction of unitary instantons on $\mathbb{C}P^2/\mathbb{Z}_n$.

1. The $\mathbb{C}P^2/\mathbb{Z}_2$ case

Let us consider the simplest case of the $\mathbb{Z}_2$ orbifold. Applying the rules above, we obtain a theory whose quiver is reported in Fig. 3 together with the superpotential (42). Note that $W_{F_0^I}$ denotes the superpotential for F_0^I (the first phase of the F_0 was studied in [40] in the case of 4d field theories and in [41] in the context of 3d field theories). Moreover, for future reference, we compile the transformation properties of the fields and the F terms under the various symmetry groups in Table II,

$$W = \mathrm{Tr}[A^i_{11}B^j_{12}A^k_{22}B^l_{21}\epsilon_{ik}\epsilon_{jl} + q_{11}A^1_{11}Q_{11} + q_{22}A^1_{22}Q_{22}] = W_{F_0^I} + \mathrm{Tr}[q_{11}A^1_{11}Q_{11} + q_{22}A^1_{22}Q_{22}]. \quad (42)$$

In the unorbifolded case, the instanton branch appeared upon setting $A^1 = 0$. Therefore, in this case, we need to impose $A^1_{11} = A^1_{22} = 0$. Then, the only relevant F terms are

$$F_1\colon\ \partial_{A^1_{11}}W = B^1_{12}A^2_{22}B^2_{21} - B^2_{12}A^2_{22}B^1_{21} + q_{11}Q_{11} = 0, \quad (43)$$

$$F_2\colon\ \partial_{A^1_{22}}W = B^1_{21}A^2_{11}B^2_{12} - B^2_{21}A^2_{11}B^1_{12} + q_{22}Q_{22} = 0. \quad (44)$$

This describes the ADHM construction for instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

As we have reviewed above, in the unorbifolded case, it is possible to map instantons on $\mathbb{C}P^2$ into instantons on $\mathbb{C}^2$. Inherited from this, we can find a mapping from the ADHM construction for instantons on the orbifolded space into that for instantons on the appropriate orbifold of $\mathbb{C}^2$. To see this, using the map π in Eq. (14), we have the following identifications between the fields of the $\mathbb{C}P^2/\mathbb{Z}_2$ theory and the fields of the $\mathbb{C}^2/\mathbb{Z}_2$ theory,

$$A_2B_2 = \begin{pmatrix} 0 & A^2_{11}B^2_{12} \\ A^2_{22}B^2_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & X^2_{12} \\ X^2_{21} & 0 \end{pmatrix} = X^2,$$
$$A^2q = \begin{pmatrix} A^2_{11}q_{11} & 0 \\ 0 & A^2_{22}q_{22} \end{pmatrix} = \begin{pmatrix} I_{11} & 0 \\ 0 & I_{22} \end{pmatrix} = I,$$
$$A_2B_1 = \begin{pmatrix} 0 & A^2_{11}B^1_{12} \\ A^2_{22}B^1_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & X^1_{12} \\ X^1_{21} & 0 \end{pmatrix} = X^1,$$
$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} = J.$$

Then, upon multiplication of the F-term relations (43) and (44) by A^1_{11} and A^2_{22}, respectively, these can be rewritten as

$$X^1_{12}X^2_{21} - X^2_{12}X^1_{21} + I_{11}J_{11} = 0, \quad (45)$$

$$X^1_{21}X^2_{12} - X^2_{21}X^1_{12} + I_{22}J_{22} = 0, \quad (46)$$

which are the F-term relations for the $\mathbb{C}^2/\mathbb{Z}_2$ theory [31]. Hence, we recover the analog to the unorbifolded case, namely, that the moduli space (at least removing possible compact directions, which we will come back to below) is biholomorphic to the moduli space of $\mathbb{C}^2/\mathbb{Z}_2$.

The Hilbert series of instantons described by the theory with flavor group $U(N_1)\times U(N_2)$ and gauge ranks $\mathbf{k} = (k_1, k_2, k_3, k_4)$[7] reads

$$H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_2](t, x, \mathbf{y}, \mathbf{d}) = \int d\mu_{U(k_1)}(\mathbf{u}) \int d\mu_{U(k_2)}(\mathbf{w}) \int d\mu_{U(k_3)}(\mathbf{z}) \times \int d\mu_{U(k_4)}(\mathbf{v})\mathrm{PE}[\chi_{A^2_{11}}t^2 + \chi_{A^2_{22}}t^2 + \chi_{B^j_{12}}t + \chi_{B^j_{21}}t + \chi_{q_{11}}t^2 + \chi_{Q_{11}}t^2 + \chi_{q_{22}}t^2 + \chi_{Q_{22}}t^2 - \chi_{F_1}t^4 - \chi_{F_2}t^4], \quad (47)$$

where we are using the following notation:

[7] We will summarize the ranks of the various gauge groups with a vector $\mathbf{k}$ and the ranks of the flavor groups with a vector $\mathbf{N}$.

TABLE II. Transformations of the fields and of the F terms for the $\mathbb{C}P^2/\mathbb{Z}_2$ theory.

Fields	$U(k_1)$	$U(k_2)$	$U(k_3)$	$U(k_4)$	$U(N_1)$	$U(N_2)$	$SU(2)$	$U(1)$
A^2_{11}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0]$	$1/2$
A^2_{22}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0]$	$1/2$
B^1_{12}, B^2_{12}	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1]$	$1/4$
B^1_{21},B^2_{21}	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1]$	$1/4$
q_{11}	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[0]$	$1-1/4r$
Q_{11}	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[0]$	$1/4r$
q_{22}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[0]$	$1-1/4r$
Q_{22}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0]$	$1/4r$
F_1	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0]$	1
F_2	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0]$	1

(i) The fugacity t is associated with the R charge and keeps track of it in units of one quarter.

(ii) The fugacities $\mathbf{u}$, $\mathbf{w}$, $\mathbf{z}$, and $\mathbf{v}$ are associated with the gauge groups $U(k_1)$, $U(k_2)$, $U(k_3)$, and $U(k_4)$, respectively.

(iii) The fugacities x, $\mathbf{y}$, and $\mathbf{d}$ are associated with the global symmetries $SU(2)$, $U(N_1)$, and $U(N_2)$, respectively.

(iv) The contribution of each field is given by

$$\chi_{A^2_{11}} = \sum_{a=1}^{k_1}\sum_{b=1}^{k_2} u_a w_b^{-1}, \qquad \chi_{A^2_{22}} = \sum_{a=1}^{k_3}\sum_{b=1}^{k_4} z_a v_b^{-1}, \qquad \chi_{B^j_{12}} = \left(x+\frac{1}{x}\right)\sum_{a=1}^{k_2}\sum_{b=1}^{k_3} w_a z_b^{-1},$$

$$\chi_{B^j_{21}} = \left(x+\frac{1}{x}\right)\sum_{a=1}^{k_4}\sum_{b=1}^{k_1} v_a u_b^{-1}, \qquad \chi_{F_1} = \sum_{a=1}^{k_1}\sum_{b=1}^{k_2} u_a^{-1} w_b, \qquad \chi_{F_2} = \sum_{a=1}^{k_3}\sum_{b=1}^{k_4} z_a^{-1} v_b,$$

$$\chi_{q_{11}} = \sum_{a=1}^{k_2}\sum_{b=1}^{N_1} w_a y_b^{-1}, \qquad \chi_{Q_{11}} = \sum_{a=1}^{N_1}\sum_{b=1}^{k_1} y_a u_b^{-1}, \qquad \chi_{q_{22}} = \sum_{a=1}^{k_4}\sum_{b=1}^{N_2} v_a d_b^{-1}, \qquad \chi_{Q_{22}} = \sum_{a=1}^{N_2}\sum_{b=1}^{k_3} d_a z_b^{-1}.$$

(v) The Haar measure of each $U(k)$ gauge group is taken equal to

$$\int d\mu_{U(k)}(\mathbf{u}) = \frac{1}{k!}\left(\prod_{j=1}^{k}\oint_{|u_j|=1}\frac{du_j}{2\pi i u_j}\right) \times \prod_{1\le i<j\le k}(u_i-u_j)(u_i^{-1}-u_j^{-1}).$$

In addition, PE stands for the plethystic exponential defined as $\mathrm{PE}[f(\cdot)] = \exp(\sum_{n=1}^{\infty}\frac{f(\cdot^n)}{n})$.

Explicit computation shows that the Hilbert series on the instanton branch for gauge group $G = U(k_1)\times U(k_2)\times U(k_3)\times U(k_4)$ with flavor group $U(N_1)\times U(N_2)$ corresponding to instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ is equal to the Hilbert series on the Higgs branch of the A_1 quiver with $U(K_1)\times U(K_2)$ gauge symmetry and global $U(N_1)\times U(N_2)$ symmetry corresponding to instantons on $\mathbb{C}^2/\mathbb{Z}_2$ [31], where

$$K_1 = \min(k_1,k_2), \qquad K_2 = \min(k_3,k_4). \tag{48}$$

In Fig. 4, we graphically summarize the relation between the theory describing instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ and that describing instantons on $\mathbb{C}^2/\mathbb{Z}_2$. Note that each flavor node flavors two adjacent nodes, which are precisely those merging into a single node in the $\mathbb{C}^2/\mathbb{Z}_2$ cousin.

Let us turn to explicit examples supporting of our claim. $U(N_1)$ instantons: $\mathbf{k} = (1,1,1,1)$ and $\mathbf{N} = (N_1, 0)$. Using Eq. (47), we have

$$\begin{aligned} H[\mathbf{k} = (1,1,1,1), \mathbf{N} = (N_1,0), \mathbb{C}P^2/\mathbb{Z}_2](t,x,\mathbf{y}) &= \frac{1}{(2\pi i)^4}\oint_{|u|=1}\frac{du}{u}\oint_{|w|=1}\frac{dw}{w}\oint_{|z|=1}\frac{dz}{z} \\ &\quad \times \oint_{|v|=1}\frac{dv}{v}\times \mathrm{PE}[\chi_{A^2_{11}}t^2 + \chi_{A^2_{22}}t^2 + \chi_{B^j_{12}}t + \chi_{B^j_{21}}t \\ &\quad + \chi_{q_{11}}t^2 + \chi_{Q_{11}}t^2 - \chi_{F_1}t^4 - \chi_{F_2}t^4], \end{aligned}$$

where the various characters are given by[8]

[8]We rewrite the flavor group $U(N_1)$ as $U(1)\times SU(N_1)$. We denote with p the fugacity of the $U(1)$ subgroup, while we denote with $\vec{y}$ the fugacities of the $SU(N_1)$ group.

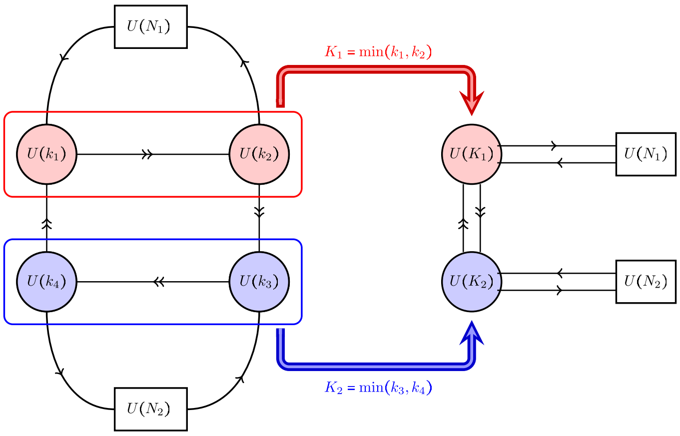

FIG. 4. Relation between the $\mathbb{C}P^2/\mathbb{Z}_2$ quiver gauge theory (on the left) and the corresponding $\mathbb{C}^2/\mathbb{Z}_2$ quiver gauge theory (on the right).

$$\chi_{A_{11}^2} = uw^{-1}, \qquad \chi_{A_{22}^2} = zv^{-1},$$
$$\chi_{B_{12}^j} = \left(x + \frac{1}{x}\right) wz^{-1}, \qquad \chi_{B_{21}^j} = \left(x + \frac{1}{x}\right) u^{-1}v,$$
$$\chi_{F_1} = u^{-1}w, \qquad \chi_{F_2} = z^{-1}v,$$
$$\chi_{q_{11}} = wp^{-1}[0, \ldots, 0, 1]_{\vec{y}}, \qquad \chi_{Q_{11}} = u^{-1}p[1, 0, \ldots, 0]_{\vec{y}}.$$

Integrating over z and v, we obtain

$$\frac{1}{(2\pi i)^2} \oint_{|u|=1} \frac{du}{u} \oint_{|w|=1} \frac{dw}{w} \frac{(1-t^6)x^2(u+t^4w)}{(t^2u-w)(t^4w-x^2u)(u-t^4x^2w)} \times \mathrm{PE}[\chi_{q_{11}} t^2 + \chi_{Q_{11}} t^2],$$

then integrating over the second gauge group, we find

$$\frac{1+t^6}{(1-t^6/x^2)(1-t^6x^2)} \times \frac{1-t^6}{(2\pi i)} \oint_{|u|=1} \frac{du}{u} \mathrm{PE}[up^{-1}t^4[0,\ldots,0,1]_{\vec{y}} + u^{-1}pt^2[1,0,\ldots 0]_{\vec{y}}].$$

We can reabsorb the fugacity p of the $U(1)$ flavor as $u' = up^{-1}$. Therefore, the previous integral becomes

$$\frac{1+t^6}{(1-t^6/x^2)(1-t^6x^2)} \times \frac{1-t^6}{(2\pi i)} \oint_{|u'|=1} \frac{du'}{u'} \times \mathrm{PE}[u't^4[0, \ldots, 0, 1]_{\vec{y}} + t^2/u'[1, 0, \ldots 0]_{\vec{y}}].$$

Finally, doing $u' = u_2/t$, the previous expression becomes

$$\frac{1+t^6}{(1-t^6/x^2)(1-t^6x^2)} \times \frac{1-t^6}{(2\pi i)} \oint_{|u_2|=1} \frac{du_2}{u_2} \mathrm{PE}[u_2t^3[0,\ldots,0,1]_{\vec{y}} + t^3u_2^{-1}[1,0,\ldots 0]_{\vec{y}}].$$

This last expression coincides with the Hilbert series for one $SU(N_1)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$ [it coincides with Eq. (2.15) of [31]].

$U(1)$ instanton: $\mathbf{k} = (2, 1, 1, 1)$ and $\mathbf{N} = (1, 0)$. Using Eq. (47), we find that

$$H[\mathbf{k} = (2, 1, 1, 1), \mathbf{N} = (1, 0), \mathbb{C}P^2/\mathbb{Z}_2](t, x) = \frac{1+t^6}{(1-t^6/x^2)(1-t^6x^2)},$$

which is the Hilbert series of one $U(1)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$.

$U(1)$ instanton: $\mathbf{k} = (2, 1, 2, 1)$ and $\mathbf{N} = (1, 0)$. Using Eq. (47), we find that

$$H[\mathbf{k} = (2, 1, 2, 1), \mathbf{N} = (1, 0), \mathbb{C}P^2/\mathbb{Z}_2](t, x) = \frac{1+t^6}{(1-t^6/x^2)(1-t^6x^2)},$$

which is again the Hilbert series of one $U(1)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$.

$U(1)$ instanton: $\mathbf{k} = (1, 2, 1, 2)$ and $\mathbf{N} = (1, 0)$. Using Eq. (47), we find that

$$H[\mathbf{k} = (1, 2, 1, 2), \mathbf{N} = (1, 0), \mathbb{C}P^2/\mathbb{Z}_2](t, x) = \frac{1+t^6}{(1-t^6/x^2)(1-t^6x^2)},$$

which is again the Hilbert series of one $U(1)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$.

$U(2)$ instanton: $\mathbf{k} = (2, 1, 1, 1)$ and $\mathbf{N} = (2, 0)$. Using Eq. (47), we find that

$$H[\mathbf{k} = (2, 1, 1, 1), \mathbf{N} = (2, 0), \mathbb{C}P^2/\mathbb{Z}_2](t, x, y_1, y_2) = \frac{(1+t^6)^2x^2y_1y_2}{(t^6-x^2)(1-t^6x^2)(t^6y_1-y_2)(y_1-t^6y_2)},$$

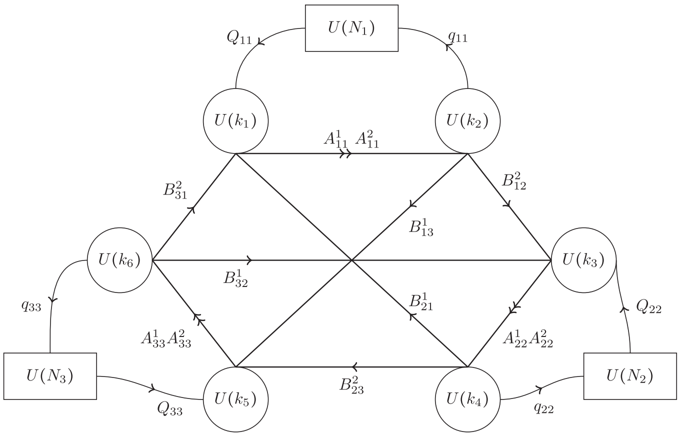

FIG. 5. The quiver diagram for the $\mathbb{C}P^2/\mathbb{Z}_3$ theory.

being y_1 and y_2 the fugacities of the flavor group. The previous expression coincides with the Hilbert series for one $U(2)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$.

$U(2)$ instanton: $\mathbf{k} = (2,2,1,1)$ and $\mathbf{N} = (2,0)$. Using Eq. (47) and unrefining for simplicity, we find

$$H[\mathbf{k} = (2,2,1,1), \mathbf{N} = (2,0), \mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1) = \frac{1+3t^6+11t^{12}+10t^{18}+11t^{24}+3t^{30}+t^{36}}{(1-t^6)^6(1+t^6)^3},$$

which is the unrefined Hilbert series for $\mathbf{K} = (2,1)$ instantons with flavor group $\mathbf{N} = (2,0)$ on $\mathbb{C}^2/\mathbb{Z}_2$.

$U(2)$ instanton: $\mathbf{k} = (2,2,1,1)$ and $\mathbf{N} = (0,2)$. Using Eq. (47), this time we find that

$$H[\mathbf{k} = (2,2,1,1), \mathbf{N} = (0,2), \mathbb{C}P^2/\mathbb{Z}_2](t,x,y_1,y_2) = \frac{(1+t^6)(x^2+t^6x^2+t^{18}x^2-t^{12}(1+x^2+x^4))y_1y_2}{(t^6-x^2)(1-t^6x^2)(t^6y_1-y_2)(y_1-t^6y_2)},$$

being y_1 and y_2 the fugacities of the $U(2)$ flavor group. The previous expression is the Hilbert series of $\mathbf{K} = (2,1)$ instantons with $\mathbf{N} = (0,2)$ on $\mathbb{C}^2/\mathbb{Z}_2$.

2. The $\mathbb{C}P^2/\mathbb{Z}_3$ case

Let us now consider the case of $\mathbb{C}P^2/\mathbb{Z}_3$. Using the rules above, we find that the quiver describing the moduli space of instantons on the $\mathbb{C}P^2/\mathbb{Z}_3$ is Fig. 5. We summarize the fields' quantum numbers in Table III.

The superpotential (10) becomes

$$W = \mathrm{Tr}[A^1_{22}B^1_{21}A^2_{11}B^2_{12} - A^1_{11}B^2_{12}A^2_{22}B^1_{21} + A^1_{33}B^1_{32}A^2_{22}B^2_{23} - A^1_{22}B^2_{23}A^2_{33}B^1_{32} - A^1_{33}B^2_{31}A^2_{11}B^1_{13} + A^1_{11}B^1_{13}A^2_{33}B^2_{31} + q_{11}A^1_{11}Q_{11} + q_{22}A^1_{22}Q_{22} + q_{33}A^1_{33}Q_{33}]. \quad (49)$$

Now the instanton branch emerges upon setting $A^1_{ii} = 0$. The relevant F terms are

$$F_1:\ \partial_{A^1_{11}}W = B^1_{13}A^2_{33}B^2_{31} - B^2_{12}A^2_{22}B^1_{21} + q_{11}Q_{11} = 0,$$
$$F_2:\ \partial_{A^1_{22}}W = B^1_{21}A^2_{11}B^2_{12} - B^2_{23}A^2_{33}B^1_{32} + q_{22}Q_{22} = 0,$$
$$F_3:\ \partial_{A^1_{33}}W = B^1_{32}A^2_{22}B^2_{23} - B^2_{31}A^2_{11}B^1_{13} + q_{33}Q_{33} = 0.$$

This defines the ADHM construction for instantons on $\mathbb{C}P^2/\mathbb{Z}_3$.

If we multiply F_1, F_2, and F_3, respectively, by A^2_{11}, A^2_{22}, and A^2_{33}, we obtain

$$A^2_{11}B^1_{13}A^2_{33}B^2_{31} - A^2_{11}B^2_{12}A^2_{22}B^1_{21} + A^2_{11}q_{11}Q_{11} = 0, \quad (50)$$

$$A^2_{22}B^1_{21}A^2_{11}B^2_{12} - A^2_{22}B^2_{23}A^2_{33}B^1_{32} + A^2_{22}q_{22}Q_{22} = 0, \quad (51)$$

$$A^2_{33}B^1_{32}A^2_{22}B^2_{23} - A^2_{33}B^2_{31}A^2_{11}B^1_{13} + A^2_{33}q_{33}Q_{33} = 0. \quad (52)$$

It is easy to check using the identification provided by the map π in Eq. (14) that the expressions (50)–(52) match the corresponding F terms of the $\mathbb{C}^2/\mathbb{Z}_3$ theory. Note that, as opposed to the unorbifolded and $\mathbb{Z}_2$ orbifold, the $SU(2)$ global symmetry rotating the B_i fields is broken due to the orbifold action. This correlates with the fact that the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ is biholomorphic to the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_n$, which exhibits a $SU(2)$ symmetry for $n = 1, 2$ but not for higher n.

The Hilbert series for $F = U(N_1)\times U(N_2)\times U(N_3)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$ with the configuration $\mathbf{k} = (k_1,k_2,k_3,k_4,k_5,k_6)$ reads

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TABLE III. Transformations of the fields and F terms for the $\mathbb{C}P^2/\mathbb{Z}_3$ theory.

Fields	$U(k_1)$	$U(k_2)$	$U(k_3)$	$U(k_4)$	$U(k_5)$	$U(k_6)$	$U(N_1)$	$U(N_2)$	$U(N_3)$	$U(1)_R$
A^2_{11}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/2$
A^2_{22}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/2$
A^2_{33}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/2$
B^1_{13}	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4$
B^1_{21}	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4$
B^1_{32}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4$
B^2_{12}	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4$
B^2_{23}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4$
B^2_{31}	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4$
q_{11}	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1-1/4r$
Q_{11}	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$1/4r$
q_{22}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$1-1/4r$
Q_{22}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$1/4r$
q_{33}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$1-1/4r$
Q_{33}	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[1,0,\ldots,0]_{+1}$	$1/4r$
F_1	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	1
F_2	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	1
F_3	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	$[\mathbf{0}]_0$	1

$$
\begin{aligned}
H[\mathbf{k},F,\mathbb{C}P^2/\mathbb{Z}_3](t,\mathbf{y},\mathbf{d},\mathbf{s}) = {} & \int d\mu_{U(k_1)}(\mathbf{u})\int d\mu_{U(k_2)}(\mathbf{w})\int d\mu_{U(k_3)}(\mathbf{z})\int d\mu_{U(k_4)}(\mathbf{v})\times\int d\mu_{U(k_5)}(\mathbf{j})\int d\mu_{U(k_6)}(\mathbf{c})\\
& \times \mathrm{PE}[\chi_{A^2_{11}}t^2+\chi_{A^2_{22}}t^2+\chi_{A^2_{33}}t^2+\chi_{B^2_{12}}t+\chi_{B^2_{23}}t+\chi_{B^2_{31}}t+\chi_{B^1_{21}}t+\chi_{B^1_{13}}t+\chi_{B^1_{32}}t+\chi_{q_{11}}t^2\\
& +\chi_{Q_{11}}t^2+\chi_{q_{22}}t^2+\chi_{Q_{22}}t^2+\chi_{q_{33}}t^2+\chi_{Q_{33}}t^2-\chi_{F_1}t^4-\chi_{F_2}t^4-\chi_{F_3}t^4], \qquad (53)
\end{aligned}
$$

where the contributions of the F terms and the various fields are given by

$$
\chi_{F_2}=\sum_{a=1}^{k_3}\sum_{b=1}^{k_4}z_a^{-1}v_b,\quad \chi_{F_3}=\sum_{a=1}^{k_5}\sum_{b=1}^{k_6}j_a^{-1}c_b,\quad \chi_{q_{11}}=\sum_{a=1}^{k_2}\sum_{b=1}^{N_1}w_a y_b^{-1},\quad \chi_{Q_{11}}=\sum_{a=1}^{N_1}\sum_{b=1}^{k_1}y_a u_b^{-1},
$$

$$
\chi_{q_{22}}=\sum_{a=1}^{k_4}\sum_{b=1}^{N_2}v_a d_b^{-1},\quad \chi_{Q_{22}}=\sum_{a=1}^{N_2}\sum_{b=1}^{k_3}d_a z_b^{-1},\quad \chi_{q_{33}}=\sum_{a=1}^{k_6}\sum_{b=1}^{N_3}c_a s_b^{-1},\quad \chi_{Q_{33}}=\sum_{a=1}^{N_3}\sum_{b=1}^{k_5}s_a j_b^{-1},
$$

$$
\chi_{A^2_{11}}=\sum_{a=1}^{k_1}\sum_{b=1}^{k_2}u_a w_b^{-1},\quad \chi_{A^2_{22}}=\sum_{a=1}^{k_3}\sum_{b=1}^{k_4}z_a v_b^{-1},\quad \chi_{A^2_{33}}=\sum_{a=1}^{k_5}\sum_{b=1}^{k_6}j_a c_b^{-1},\quad \chi_{B^2_{12}}=\sum_{a=1}^{k_2}\sum_{b=1}^{k_3}w_a z_b^{-1},\quad \chi_{B^2_{23}}=\sum_{a=1}^{k_4}\sum_{b=1}^{k_5}v_a j_b^{-1},
$$

$$
\chi_{B^2_{31}}=\sum_{a=1}^{k_6}\sum_{b=1}^{k_1}c_a u_b^{-1},\quad \chi_{B^1_{21}}=\sum_{a=1}^{k_4}\sum_{b=1}^{k_1}v_a u_b^{-1},\quad \chi_{B^1_{13}}=\sum_{a=1}^{k_2}\sum_{b=1}^{k_5}w_a j_b^{-1},\quad \chi_{B^1_{32}}=\sum_{a=1}^{k_6}\sum_{b=1}^{k_3}c_a z_b^{-1},\quad \chi_{F_1}=\sum_{a=1}^{k_1}\sum_{b=1}^{k_2}u_a^{-1}w_b.
$$

As above, the Hilbert series on the instanton branch of the quiver describing instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ with gauge group of $G = U(k_1)\times U(k_2)\times U(k_3)\times U(k_4)\times U(k_5)\times U(k_6)$ and flavor group $U(N_1)\times U(N_2)\times U(N_3)$ is equal to the Hilbert series of the Higgs branch describing the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with flavor group $U(N_1)\times U(N_2)\times U(N_3)$ instantons and gauge group $\mathbf{K} = (K_1, K_2, K_3)$ [31], where

$$
K_1=\min(k_1,k_2),\qquad K_2=\min(k_3,k_4),\quad \text{and}\quad K_3=\min(k_5,k_6). \qquad (54)
$$

We can again summarize graphically the relation between the theory describing $\mathbb{C}P^2/\mathbb{Z}_3$ instantons and its $\mathbb{C}^2/\mathbb{Z}_3$ cousin as in Fig. 6. As in the $\mathbb{Z}_2$ orbifold case, each flavor node flavors a pair of gauge nodes which "merge" into a single node in the cousin $\mathbb{C}^2/\mathbb{Z}_3$ theory.

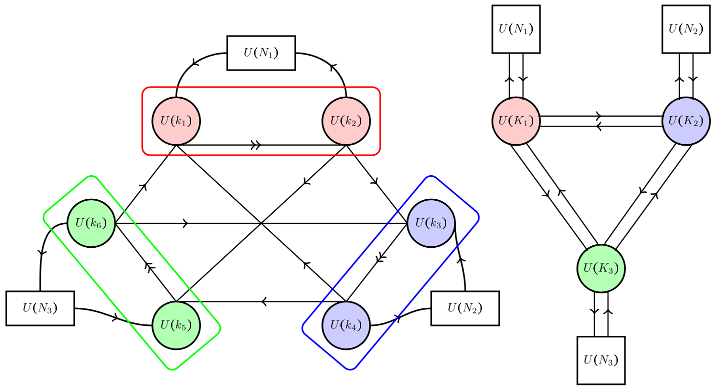

FIG. 6. Relation between the $\mathbb{C}P^2/\mathbb{Z}_3$ quiver gauge theory (on the left) and the corresponding $\mathbb{C}^2/\mathbb{Z}_3$ quiver gauge theory (on the right).

Let us support our claim with explicit examples. $U(1)$ instanton: $\mathbf{k} = (1,1,1,1,1,1)$ and $N = (1,0,0)$. Using Eq. (53), we find that

$$H[(1,1,1,1,1,1),(1,0,0),\mathbb{C}P^2/\mathbb{Z}_3](t) = \frac{1-t^3+t^6}{(1-t^3)^2(1+t^3+t^6)},$$

which is the Hilbert series for $\mathbf{N} = (1,0,0)$ instantons and $\mathbf{K} = (1,1,1)$ on $\mathbb{C}^2/\mathbb{Z}_3$. $U(2)$ instanton: $\mathbf{k} = (1,1,1,1,1,1)$ and $\mathbf{N} = (1,1,0)$. Using Eq. (53) and unrefining, we find that

$$H[(1,1,1,1,1,1),(1,1,0),\mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{1+t^6+2t^9+2t^{12}+2t^{15}+t^{18}+t^{24}}{(1-t^3)^4(1+t^3)^2(1+t^6)(1+t^3+t^6)^2},$$

which is the unrefined Hilbert series for $\mathbf{N} = (1,1,0)$ instantons and $\mathbf{K} = (1,1,1)$ on $\mathbb{C}^2/\mathbb{Z}_3$. $U(1)$ instanton: $\mathbf{k} = (2,1,1,1,1,1)$ and $\mathbf{N} = (1,0,0)$. Using Eq. (53), we find that

$$H[(2,1,1,1,1,1),(1,0,0),\mathbb{C}P^2/\mathbb{Z}_3](t) = \frac{1-t^3+t^6}{(1-t^3)^2(1+t^3+t^6)},$$

which is again the Hilbert series for $\mathbf{N} = (1,0,0)$ instantons and $\mathbf{K} = (1,1,1)$ on $\mathbb{C}^2/\mathbb{Z}_3$. $U(1)$ instanton: $\mathbf{k} = (2,1,2,1,1,1)$ and $\mathbf{N} = (1,0,0)$. Using Eq. (53), we find that

$$H[(2,1,2,1,1,1),(1,0,0),\mathbb{C}P^2/\mathbb{Z}_3](t) = \frac{1-t^3+t^6}{(1-t^3)^2(1+t^3+t^6)},$$

which is again the Hilbert series for $\mathbf{N} = (1,0,0)$ instantons and $\mathbf{K} = (1,1,1)$ on $\mathbb{C}^2/\mathbb{Z}_3$. $U(2)$ instanton: $\mathbf{k} = (2,1,1,1,1,1)$ and $\mathbf{N} = (2,0,0)$. Using Eq. (53), we find that

$$H[(2,1,1,1,1,1),(2,0,0),\mathbb{C}P^2/\mathbb{Z}_3](t,y_1,y_2) = \frac{(1-t^3+2t^6-t^9+t^{12})y_1y_2}{(1-t^3)^2(1+t^3+t^6)(t^6y_1-y_2)(t^6y_2-y_1)},$$

being y_1 and y_2 the fugacities of the flavor group $U(2)$. The previous expression is the Hilbert series for $\mathbf{N} = (2,0,0)$ instantons and $\mathbf{K} = (1,1,1)$ on $\mathbb{C}^2/\mathbb{Z}_3$. $U(2)$ instanton: $\mathbf{k} = (2,2,1,1,1,1)$ and $\mathbf{N} = (2,0,0)$. Using Eq. (53) and unrefining, we find that

$$H[(2,2,1,1,1,1),(2,0,0),\mathbb{C}P^2/\mathbb{Z}_3](t,1,1) \\ = \frac{1-t^3+2t^6-t^9+3t^{12}+2t^{15}-t^{18}-t^{21}-5t^{27}+2t^{30}-5t^{33}-t^{39}-t^{42}+2t^{45}+3t^{48}-t^{51}+2t^{54}-t^{57}+t^{60}}{(1-t^3)^4(1+t^3)^2(1+t^3+t^6)(1-t^{12})^2(1-t^{15})^2},$$

which is the Hilbert series for $\mathbf{N} = (2,0,0)$ instantons and $\mathbf{K} = (2,1,1)$ on $\mathbb{C}^2/\mathbb{Z}_3$. $U(2)$ instanton: $\mathbf{k} = (2,1,2,1,1,1)$ and $\mathbf{N} = (2,0,0)$. Using Eq. (53), we find that

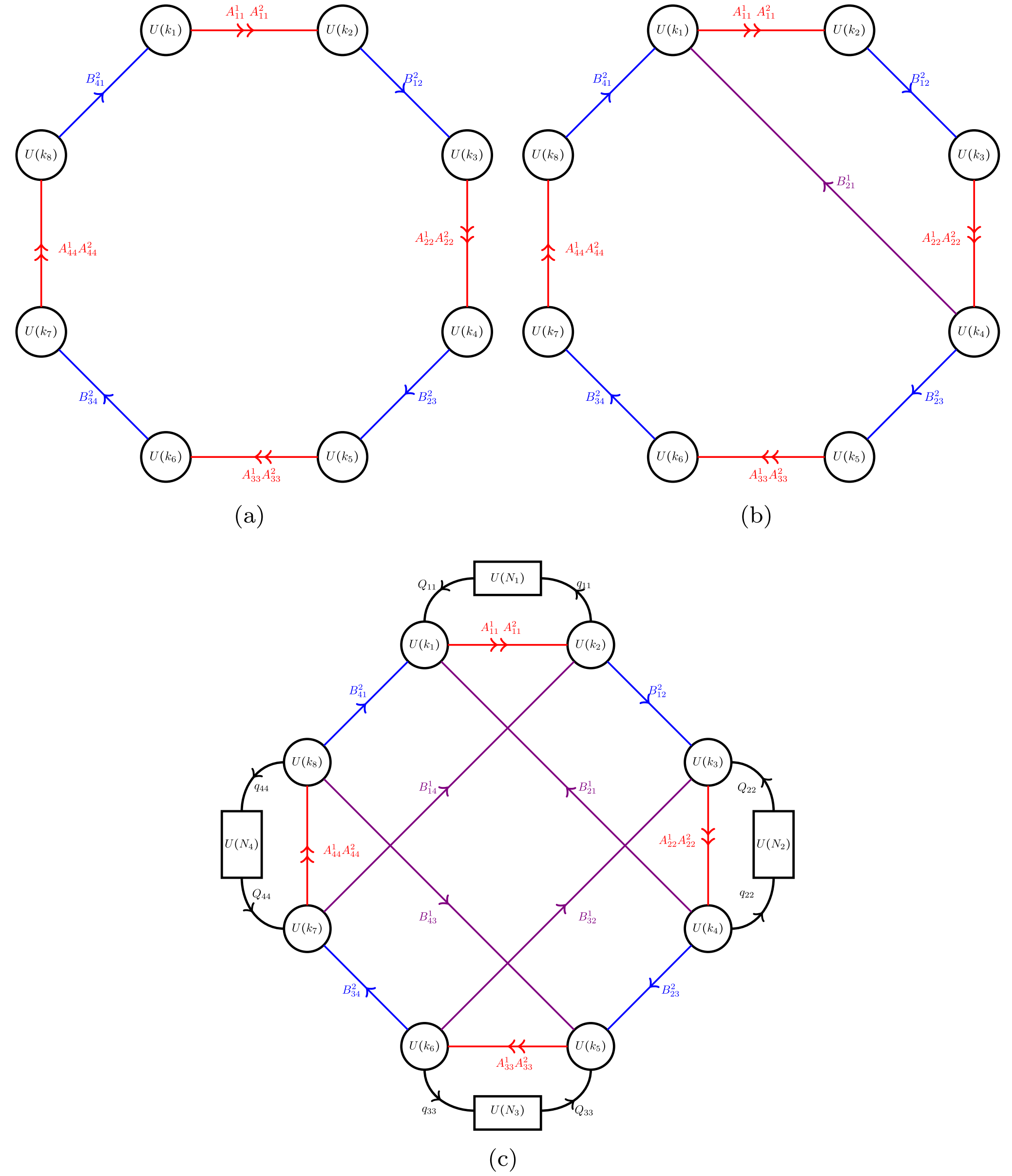

FIG. 7. (a),(b) and (c) are the steps for the construction of the quiver diagram for the $\mathbb{C}P^2/\mathbb{Z}_4$ theory.

$$H[(2,1,1,1,1,1),(2,0,0),\mathbb{C}P^2/\mathbb{Z}_3](t,y_1,y_2)=\frac{(1-t^3+2t^6-t^9+t^{12})y_1y_2}{(1-t^3)^2(1+t^3+t^6)(t^6y_1-y_2)(t^6y_2-y_1)},$$

being y_1 and y_2 the fugacities of the flavor group $U(2)$. The previous expression is the Hilbert series for $\mathbf{N} = (2, 0, 0)$ instantons and $\mathbf{K} = (1, 1, 1)$ on $\mathbb{C}^2/\mathbb{Z}_3$.

3. The $\mathbb{C}P^2/\mathbb{Z}_n$ case ($n \geq 3$)

It is now easy to generalize the previous construction of $U(N)$ instantons to higher orbifolds of $\mathbb{C}P^2$. For a general $\mathbb{Z}_n$ orbifold, the resulting procedure is as follows (see Fig. 7):

(i) The quiver has $2n$ circular nodes linked together in an alternating way; i.e., a segment with fields A^1_{ii} and A^2_{ii} is alternated with a segment with field $B^2_{i,i+1}$ [see Fig. 7(a)].

(ii) Then we add the contribution due to the fields $B^1_{i+1,i}$. In order to do this, we begin from one circular node [for example, the one in which there is the gauge group $U(k_1)$], and we move clockwise counting three segments (in this case, we will count the segment labeled by A^1_{11}, the segment labeled by B^2_{12}, and finally the segment labeled by A^1_{22}). When we reach the circular node at the end of the third segment, we draw a line between this node and the initial circular node [in this case, a line between the node $U(k_4)$ and the initial node $U(k_1)$]. This line we labeled by a $B^1_{i+1,i}$ field (in the case we are considering, by the field $B^1_{2,1}$) [see Fig. 7(b)].

(iii) We apply the same procedure starting this time from the next circular node arising from the first gauge group $U(k_L)$ [in this case, the one labeled by $U(k_3)$], and we will continue to apply this algorithm up to the end of the circular nodes arising from the decomposition of the first gauge group. Finally, we add the contributions due to the various flavor groups, and we obtain the quiver reported in Fig. 7(c).

Note that N corresponds to the sum of the ranks of the flavor nodes. In turn, the gauge ranks correspond to the instanton number as well as, together with relative flavor ranks, other quantum numbers describing the instanton (we will briefly come back to these issues below).

We can compute the Hilbert series on the instanton branch. In general, we find a correspondence between the Hilbert series for the moduli space of $\mathbf{N}=(N_1,\ldots,N_n)$ instantons with $\mathbf{k}=(k_1,k_2,\ldots,k_{2n})$ on $\mathbb{C}P^2/\mathbb{Z}_n$ and the Hilbert series for the moduli space of $\mathbf{N}=(N_1,\ldots N_n)$ instantons with $\mathbf{K}=(K_1,\ldots,K_n)$ on $\mathbb{C}^2/\mathbb{Z}_n$ upon identifying

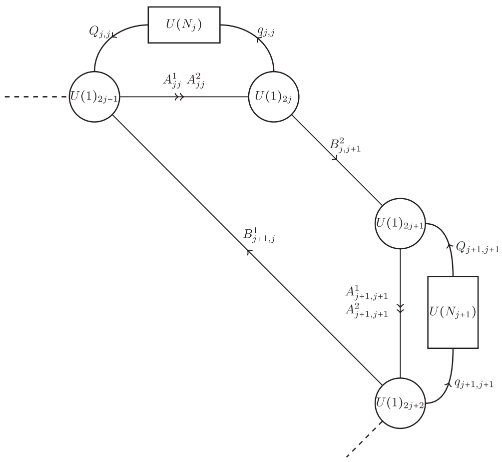

FIG. 8. Basic element of the quiver diagram for the $\mathbb{C}P^2/\mathbb{Z}_n$ theory.

$$K_1 = \min(k_1, k_2),$$
$$K_2 = \min(k_3, k_4), \ldots K_n = \min(k_{2n-1}, k_{2n}). \quad (55)$$

This can be easily proven in the particular case

$$G = \bigotimes_{i=1}^{2n} U(1)_i, \qquad F = \bigotimes_{i=1}^{n} U(N_i).$$

Moreover, we denote with z_i $i = 1, \ldots, 2n$ the fugacities of the various $U(1)_i$ gauge groups and with u_i and $\vec{y}_i$ the fugacities of each flavor group $U(N_i)$ [being u_i the fugacity of the $U(1)$ part, while $\vec{y}_i$'s are the fugacities associated with the $SU(N)$ part of the flavor group].

The Hilbert series reads

$$\begin{aligned} H[(1,1,\ldots,1),(N_1,N_2,\ldots,N_n),\mathbb{C}P^2/\mathbb{Z}_n](t,u_i,\vec{y}_i) \\ = \prod_{i=1}^{2n} \frac{1}{2\pi i} \oint_{|z_i|} \frac{dz_i}{z_i} \prod_{j=1}^{n} \chi_{A^2_{j,j}}(t, z_{2j-1}, z_{2j}) \times \chi_{B^2_{j,j+1}}(t, z_{2j}, z_{2j+1}) \chi_{B^1_{j+1,j}}(t, z_{2j}, z_{2j-1}) \chi_{F_j}(t, z_{2j-1}, z_{2j}) \\ \times \chi_{q_{j,j}}(t, z_{2j}, \vec{y}_j, u_j) \chi_{Q_{j,j}}(t, z_{2j-1}, \vec{y}_j, u_j). \end{aligned} \quad (56)$$

The contributions of the various fields are[9]

[9]See Fig. 8.

$$\chi_{A^2_{j,j}}(t,z_{2j-1},z_{2j}) = \mathrm{PE}[t^2 z_{2j-1} z_{2j}^{-1}], \qquad \chi_{B^2_{j,j+1}}(t,z_{2j},z_{2j+1}) = \mathrm{PE}[t z_{2j} z_{2j+1}^{-1}],$$
$$\chi_{B^1_{j+1,j}}(t,z_{2j+2},z_{2j-1}) = \mathrm{PE}[t z_{2j+2} z_{2j-1}^{-1}], \qquad \chi_{F_j}(t,z_{2j-1},z_{2j}) = \mathrm{PE}[-t^4 z_{2j-1}^{-1} z_{2j}],$$
$$\chi_{Q_{j,j}}(t,z_{2j-1},\vec{y}_j,u_j) = \mathrm{PE}[t^2 z_{2j-1}^{-1}[1,0,\ldots,0]_{\vec{y}_j} u_j], \qquad \chi_{q_{j,j}}(t,z_{2j},\vec{y}_j,u_j) = \mathrm{PE}[t^2 z_{2j}[0,\ldots,0,1]_{\vec{y}_j} u_j^{-1}].$$

Therefore, the Hilbert series (56) becomes

$$\prod_{i=1}^{2n} \frac{1}{2\pi i} \oint_{|z_i|} \frac{dz_i}{z_i} \prod_{j=1}^{n} \frac{\mathrm{PE}[t^2 z_{2j-1}^{-1}[1,0,\ldots,0]_{\vec{y}_j} u_j + t^2 z_{2j}[0,\ldots,0,1]_{\vec{y}_j} u_j^{-1}](z_{2j-1} - t^4 z_{2j})}{\frac{z_{2j-1}}{z_{2j}}(z_{2j} - t^2 z_{2j-1})(1 - \frac{t z_{2j}}{z_{2j+1}})(1 - \frac{t z_{2j+2}}{z_{2j-1}})}.$$

It is important to note that we can integrate over the gauge group $U(1)_i$ with an even value of the index i. This is due to the fact that the only contribution to these integrals comes from the poles located at $z_{2j} = t^2 z_{2j-1}$. Therefore, performing the integrations, we obtain

$$\prod_{i\,\mathrm{odd}}^{2n} \frac{1}{2\pi i} \oint_{|z_i|} \frac{dz_i}{z_i} \prod_{j=1}^{n} \frac{\mathrm{PE}[t^2 z_{2j-1}^{-1}[1,0,\ldots,0]_{\vec{y}_j} u_j + t^4 z_{2j-1}[0,\ldots,0,1]_{\vec{y}_j} u_j^{-1}](z_{2j-1} - t^6 z_{2j-1})}{z_{2j-1}(1 - \frac{t^3 z_{2j-1}}{z_{2j+1}})(1 - \frac{t^3 z_{2j+1}}{z_{2j-1}})},$$

then we perform the change of variables $z_{2j-1} \mapsto t z_{2j-1}$,

$$\prod_{i\,\mathrm{odd}}^{2n} \frac{1}{2\pi i} \oint_{|z_i|} \frac{dz_i}{z_i} \prod_{j=1}^{n} \frac{\mathrm{PE}[t^3 z_{2j-1}^{-1}[1,0,\ldots,0]_{\vec{y}_j} u_j + t^3 z_{2j-1}[0,\ldots,0,1]_{\vec{y}_j} u_j^{-1}](1 - t^6)}{(1 - \frac{t^3 z_{2j-1}}{z_{2j+1}})(1 - \frac{t^3 z_{2j+1}}{z_{2j-1}})}.$$

Finally, we observe that instead of considering only the odd numbers between 1 and $2n$, it is more useful to consider all the integer numbers between 1 and n. Therefore, we can make the following replacements $z_{2j-1} \mapsto z_j$ and $z_{2j+1} \mapsto z_{j+1}$, and we rewrite the previous integral as

$$\prod_{i=1}^{n} \frac{1}{2\pi i} \oint_{|z_i|} \frac{dz_i}{z_i} (1 - t^6)^n \prod_{j=1}^{n} \mathrm{PE}[t^3 z_j^{-1}[1,0,\ldots,0]_{\vec{y}_j} u_j + t^3 z_j[0,\ldots,0,1]_{\vec{y}_j} u_j^{-1}] \mathrm{PE}[t^3 z_j z_{j+1}^{-1} + t^3 z_{j+1} z_j^{-1}],$$

which is the Hilbert series for $\mathbf{N} = (N_1, N_2, \ldots, N_n)$ instantons with $\mathbf{K} = (1,1,\ldots,1)$ on $\mathbb{C}^2/\mathbb{Z}_n$ [it coincides with the expression (2.41) of [31]].

Up to now, we have deliberately postponed discussing the identification of the quantum numbers of the instanton. Recall that in the $\mathbb{C}^2/\mathbb{Z}_n$ case [31], the instanton is described by $n-1$ first Chern classes, one second Chern class, and n holonomies of the gauge field, all in all a total of $2n$ quantum numbers corresponding to the $2n$ integers specifying the A_{n-1} quiver.

In the case at hand, the quiver describing instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ is specified by a total of $3n$ integers corresponding to $2n$ gauge ranks and n flavor ranks. In turn, we expect the instanton on $\mathbb{C}P^2/\mathbb{Z}_n$ to be described by $2n-1$ first Chern classes—corresponding to n orbifold copies of the $\mathbb{C}P^2$ 2-cycle plus $n-1$ extra 2-cycles introduced by the orbifold—one second Chern class and n holonomies, hence, totaling the expected $3n$ quantum numbers. While the exact identification of integers is not known, note that, from the examples above, the mapping of the $\mathbb{C}P^2/\mathbb{Z}_n$ quiver into the $\mathbb{C}^2/\mathbb{Z}_n$ one is such that one node of the latter arises from the merging of two adjacent commonly flavored nodes of the former in such a way that the common flavor group in the $\mathbb{C}P^2/\mathbb{Z}_n$ case becomes the flavor group in the $\mathbb{C}^2/\mathbb{Z}_n$ case. Hence, it is natural to guess that the n holonomies correspond to the n flavor nodes. Moreover, the $n-1$ first Chern classes associated to the cycles arising from the orbifold are naturally associated to the differences among the minima of the ranks of each pair of "merging nodes." Obviously, there are n such nodes arising from merging, whose $n-1$ rank differences would correspond to first Chern classes. In turn, the relative rank between the merging nodes is naturally associated with the n remaining 2-cycles, orbifold copies of the original 2-cycle in $\mathbb{C}P^2$. Finally, the sum of the ranks is naturally related to the second Chern class. Note that clearly the identification of N with the sum of the ranks of the flavor nodes is consistent.

As a small consistency check, let us consider the simple case of the vanishing first Chern class associated to cycles introduced by the orbifold. This would correspond to a rank assignation of the form $(\cdots, k, q_n, k, q_{n+1}, k, \cdots)$ with $q_i > k$, so that among each "merging pair," the minimum rank is k. Then all relative rank differences among the "merged nodes" are 0 corresponding to a $\mathbb{C}^2/\mathbb{Z}_n$ instanton with zero first Chern classes. Moreover, let us consider the case of the vanishing second Chern class from the $\mathbb{C}^2/\mathbb{Z}_n$ point of view, which demands $k=0$. This is analogous to the case $k_L=0$ in Sec. III A. We are then left with a gauge rank assignation of the form $(\cdots,0,q_n,0,q_{n+1},0,\cdots)$. According to our conjecture, these integers q_i should correspond to the first Chern classes on the n 2-cycles coming from the orbifold images of the original 2-cycle. Indeed, if we consider just one of them, that is, we set all but one of the q_i's to vanish, we simply recover the Grassmanian quiver above. Note that, as expected, indeed we have n such possibilities corresponding to the n 2-cycles coming from the orbifold images of the original 2-cycle.

V. $Sp(N)$ INSTANTONS ON $\mathbb{C}P^2/\mathbb{Z}_n$

So far, we have concentrated on the case of unitary instantons. Let us now turn to the case of instantons in the symplectic gauge group. The explicit ADHM construction of such instantons was introduced in [14]. As described in [15], it can be embedded into a $3d$ gauge theory upon restricting to the appropriate instanton branch. In $3d\,\mathcal{N}=2$ notation, such theory contains one $U(k)$ vector multiplet coupled to one chiral multiplet $\tilde{A}$ in the second rank antisymmetric tensor representation of the gauge group and three chiral multiplets S_1, S_2, $\tilde{S}$ in the second rank symmetric tensor representation. In addition, there is a number of chiral multiplets in the fundamental representation with an $Sp(N)$ global symmetry. The corresponding quiver is reported in Fig. 9.

In turn, the superpotential is

$$W=\epsilon^{\alpha\beta}(S_\alpha)_{ab}\tilde{S}^{bc}(S_\beta)_{cd}\tilde{A}^{da}+\tilde{A}^{ab}Q^i{}_aQ^j{}_bJ_{ij}, \qquad (57)$$

being J the $Sp(N)$ symplectic matrix. As shown in [15], the instanton branch emerges upon setting $\tilde{A}$—as well as the monopole operators—to zero.

As in the unitary case, it is possible to embed the $\mathbb{C}P^2$ symplectic instantons ADHM construction into the $\mathbb{C}^2$ symplectic ADHM construction and vice versa [15]. It should be noted though that now the equivalent to the map π in Eq. (14) is quadratic and, hence, does not define a proper mapping. Nevertheless, as a consequence, the Hilbert series for symplectic instantons on $\mathbb{C}P^2$ coincides with that of symplectic instantons on $\mathbb{C}^2$. We refer to [15] for further details.

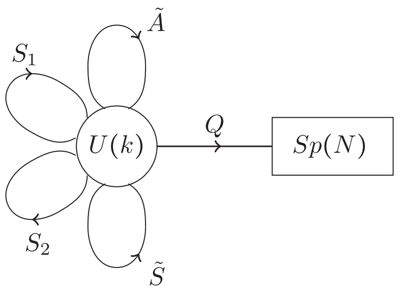

FIG. 9. Quiver diagram for $Sp(N)$ instantons on $\mathbb{C}P^2$.

A. Constructing $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$

Just as in the case of unitary instantons, we can consider orbifolding the base $\mathbb{C}P^2$ manifold and study $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$. It is then natural to engineer the ADHM-like construction by orbifolding the $\mathbb{C}P^2$ case, just as for unitary instantons. As a guideline, let us compare with the case of instantons on $\mathbb{C}^2$ and its orbifolds [31]. The gauge theory realizing the ADHM construction for unitary instantons on $\mathbb{C}^2/\mathbb{Z}_n$ can be thought of as the world volume theory on a $D3$-$D7$ system, where the transverse directions to the $D3$'s inside the $D7$'s wrap $\mathbb{C}^2/\mathbb{Z}_n$. Then, symplectic (and orthogonal) instantons can be constructed upon adding $O7$ planes of the appropriate charge. A comprehensive picture appears upon T duality along the asymptotically locally Euclidean (ALE) space. Then, the $D3$-branes are mapped to $D4$-branes wrapping a circle. In turn, the $D7$'s are mapped into $D6$ at fixed positions in the circle. Finally, n NS5-branes on the circle arise from T dualizing the ALE space. In this context, the construction of symplectic (alternatively, orthogonal) instantons boils down to adding two identical— because they come from T duality of a single $O7$-$O6$ plane of the appropriate charge at opposite points in the circle such that each side of the circle mirrors —due to the orientifold projection—the other side. This procedure highlights an obvious difference between the cases of even and odd orbifolds. As the distribution of NS5-branes must be symmetric on the circle, for an odd n, it is clear that one such NS5 must be stuck in an orientifold plane. In turn, in the case of even n, we can have a symmetric distribution by either sticking one NS5 at each O plane or not sticking any NS5's on the O planes. These possibilities lead, respectively, to the so-called no-vector structure (NVS) and vector structure (VS). We refer to [31] and references therein for further explanations. Note that the T-duality construction suggests that the two O planes are of the same type. Nevertheless, once in the IIA setup, one might imagine other versions whereby the O planes are of different type. These configurations were dubbed hybrid in [31]. We will briefly touch on the equivalent to these in the case at hand below, showing an explicit example in Appendix A.

In view of the $\mathbb{C}^2/\mathbb{Z}_n$ case, it is natural to proceed in a similar way in the case of instantons on the orbifolded $\mathbb{C}P^2$, that is, first consider orbifolding unitary instantons and then considering orientifolding. Note, however, that in this case, the brane picture is much less clear. Nevertheless, as we will see, the results are qualitatively similar. Since we will set monopole operators to zero, formally the procedure is identical to the case of 4*d* gauge theories. Hence, we can borrow the technology developed [42,43] to construct the relevant theories.

As illustrated in [42], the orientifold field theory is obtained from the parent field theory performing a $\mathbb{Z}_2$ identification of the gauge groups, chiral multiplets, and superpotential couplings. As explained in [43], this means that the O-plane involution defines a $\mathbb{Z}_2$ automorphism of the quiver diagram that reverses the directions of the arrows. Therefore, the quiver of the parent theory has a $\mathbb{Z}_2$ symmetry that can be visualized as a reflection through a fixed line once we embed the quiver diagram in $\mathbb{R}^2$. In the following, we will follow the method used in [43] that allows us to obtain the orientifold theory starting directly from its quiver diagram. Of course, as can be verified, the application of the method of [42] that acts on the dimer diagram of the theory leads to the same results.

In order to explain how this procedure works, we apply it to the case of the $\mathbb{C}P^2/\mathbb{Z}_2$ theory, and we refer to [43] for the analysis of the general case. An inspection of the corresponding quiver diagram shows that there are two inequivalent ways to cut it with a line, such that the quiver displays arrows reversing the symmetry with respect to this line (see Fig. 10).

In order to obtain the corresponding orientifold theory, we label each node and each line intersecting perpendicularly to the cutting line with a sign (denoted with a roman number in the figure) that can be positive or negative. Then, the orientifold theory is constructed as follows. Each node untouched by the cutting line corresponds to a $U(k)$ group, while each node touched by the line corresponds to an $SO(N)$ or $Sp(N)$ (for a positive or negative sign, respectively) in the orientifold field theory. In the same way, each edge of the quiver diagram away from the cutting line corresponds to bifundamental matter, while each edge crossing the cutting line perpendicularly corresponds to symmetric matter (positive sign) or antisymmetric matter (negative sign) in the orientifold field theory. The values of the signs must be fixed requiring that the superpotential of the parent theory is invariant under the involution. Note that, in general, more than one choice is allowed. For example, in the case of the quiver diagram in Fig. 10(b), we can choose the following values of the signs $(+,+,+,+)$, $(-,+,+,-)$, $(+,-,+,-)$, $(+,+,-,-)$. In the following, we will always fix the signs in order to obtain the theory whose Higgs branch describes the moduli space for $Sp(N)$ instantons (respectively, SO) on $\mathbb{C}P^2/\mathbb{Z}_n$, which, in the case at the hand, means to select the $(+,+,+,+)$ configuration. The remaining allowed choices correspond to the "hybrid configurations" discussed in [31]. Even though we will not touch upon these further in this paper, we present an explicit example in Appendix A.

Therefore, as in [31], we have two different situations depending on whether the degree of the orbifold is even or odd.

(i) If n is odd, we have only one type of quiver diagram corresponding to the fact that we have only one inequivalent way to cut it with a line.

(ii) If n is even, we have two types of quiver gauge theories corresponding to the two possible inequivalent ways to cut it with a fixed line. These two cases are just the equivalent of the vector-structure and no-vector-structure cases for $\mathbb{C}^2/\mathbb{Z}_n$ symplectic instantons. By analogy, in the following we will refer to them as the VS and the NVS, respectively.

Note that N corresponds to the sum of the ranks of the flavor groups in the ADHM quiver. In turn, gauge group ranks correspond to the instanton number (as well as to other possible quantum numbers labeling the instanton).

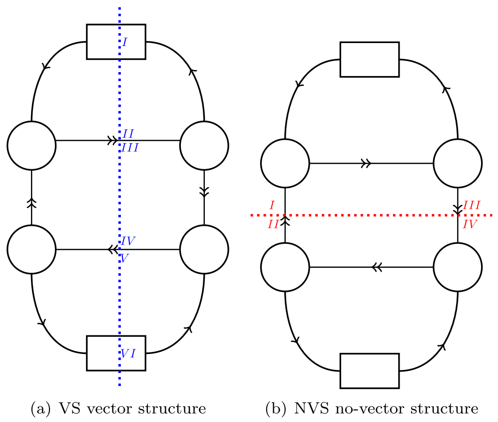

FIG. 10. The two inequivalent ways to obtain the $\mathbb{C}P^2/\mathbb{Z}_2$ orientifold theory. The VS case (a) and the NVS case (b).

1. $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$: VS

Starting from the $\mathbb{C}P^2/\mathbb{Z}_2$ and applying the rules above, we can obtain the VS theory for $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$. The corresponding quiver diagram is reported in Fig. 11, while we summarize the transformations of the

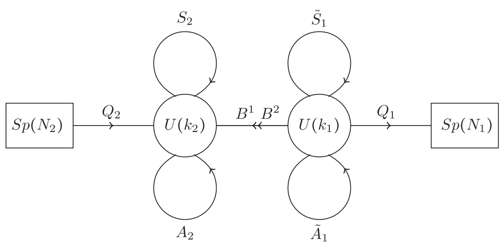

FIG. 11. Quiver diagram for VS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

TABLE IV. Transformations of the fields for VS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

Fields	$U(k_1)$	$U(k_2)$	$Sp(N_1)$	$Sp(N_2)$	$SU(2)$	$U(1)$
$\tilde{A}_1$	$[0,1,0\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	$1/2$
$\tilde{S}_1$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	$1/2$
A_2	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	$1/2$
S_2	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	$1/2$
B^1, B^2	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1]$	$1/4$
Q_1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[\mathbf{0}]$	$[0]$	$1/2$
Q_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[0]$	$1/2$
F_1	$[0,1,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	1
F_2	$[\mathbf{0}]$	$[0,1,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	1

fields under the different groups in Table IV. Note that $N = N_1 + N_2$.

The branch of the moduli space that can be identified with $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ is the one on which $\tilde{A}_1 = 0$ and $A_2 = 0$. Then, the Hilbert series of the instanton branch corresponding to the VS theory with flavor symmetry $Sp(N_1)\times Sp(N_2)$ and gauge ranks $\mathbf{k} = (k_1, k_2)$ is

$$H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_2](t, x, \mathbf{y}, \mathbf{d}) = \int d\mu_{U(k_1)}(\mathbf{z}) \int d\mu_{U(k_2)}(\mathbf{p}) \mathrm{PE}[\chi_{S^2} t^2 + \chi_{\tilde{S}^1} t^2 + \chi_{B^j} t + \chi_{Q_1} t^2 + \chi_{Q_2} t^2 - \chi_{F_1} t^4 - \chi_{F_2} t^4], \quad (58)$$

where $\mathbf{z}$ and $\mathbf{p}$ are the fugacities of the $U(k_1)$ and $U(k_2)$ gauge groups, respectively, while $\mathbf{y}$ and $\mathbf{d}$ denote the fugacities of the $Sp(N_1)$ and $Sp(N_2)$ flavor groups, respectively. Finally, x denotes the fugacity of the global $SU(2)$ symmetry rotating the B_1 and B_2 fields. The contribution of each field is given by

$$\chi_{Q_1} = \sum_{i=1}^{N_1}\left(y_i + \frac{1}{y_i}\right)\sum_{a=1}^{k_1} z_a,$$

$$\chi_{Q_2} = \sum_{j=1}^{N_2}\left(d_j + \frac{1}{d_j}\right)\sum_{b=1}^{k_2} p_b^{-1}, \qquad \chi_{F_1} = \sum_{1\le a<b\le k_1} z_a z_b,$$

$$\chi_{S_2} = \sum_{1\le a\le b\le k_2} p_a p_b, \qquad \chi_{\tilde{S}_1} = \sum_{1\le a\le b\le k_1} z_a^{-1} z_b^{-1},$$

$$\chi_{B^j} = \left(x + \frac{1}{x}\right)\sum_{a=1}^{k_1}\sum_{b=1}^{k_2} z_a p_b^{-1}, \qquad \chi_{F_2} = \sum_{1\le a<b\le k_2} p_a^{-1} p_b^{-1}.$$

Explicit computation shows that the Hilbert series for the instanton branch of the VS theory with gauge group $G = U(k_1)\times U(k_2)$ and flavor group $Sp(N_1)\times Sp(N_2)$ corresponding to the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ turns out to be equal to the Hilbert series for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with gauge group $G = O(K_1)\times O(K_2)$ (see [31] for more details). The two theories share the same flavor groups, and the gauge groups are related as

$$K_1 = k_1, \qquad K_2 = k_2. \quad (59)$$

Let us show some explicit examples supporting our claim. $Sp(2)$ instanton: $\mathbf{k} = (1,1)$ and $\mathbf{N} = (1,1)$. Using Eq. (58) and unrefining, we find that

$$H[\mathbf{k} = (1,1), Sp(1)\times Sp(1), \mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1) = \frac{1 - 2t^3 + 6t^6 - 2t^9 + t^{12}}{(1-t^3)^6(1+t^3)^4},$$

which is the unrefined Hilbert series for $Sp(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbf{K} = (1,1)$ and $\mathbf{N} = (1,1)$. $Sp(3)$ instanton: $\mathbf{k} = (1,1)$ and $\mathbf{N} = (1,2)$. Using Eq. (58) and unrefining, we find that

$$H[\mathbf{k} = (1,1), Sp(1)\times Sp(2), \mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1,1) = \frac{(1+t^6)(1 - 2t^3 + 10t^6 - 2t^9 + t^{12})}{(1-t^3)^8(1+t^3)^6},$$

which is the unrefined Hilbert series for $Sp(3)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbf{K} = (1,1)$ and $\mathbf{N} = (1,2)$.

2. *Sp(N) instantons on $\mathbb{C}P^2/\mathbb{Z}_2$: NVS*

Let us now consider the second possible configuration corresponding to the NVS case. The quiver diagram of the corresponding theory is reported in Fig. 12, while the

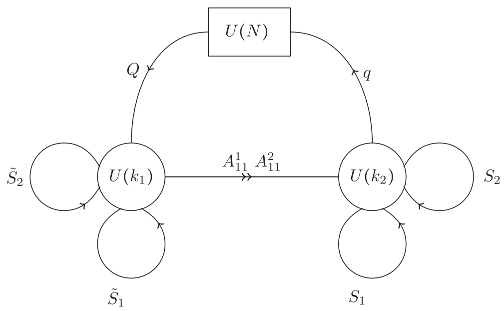

FIG. 12. Quiver diagram for NVS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

TABLE V. Transformations of the fields for NVS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

Fields	$U(k_1)$	$U(k_2)$	$U(N)$	$SU(2)$	$U(1)$
$\tilde{S}_1$, $\tilde{S}_2$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1]$	$1/4$
S_1, S_2	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[1]$	$1/4$
A^2_{11}	$[1,0,\ldots,0]_{+1}$	$[0,0,\ldots,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
q	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$1/2$
Q	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$1/2$
F	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

transformations of the fields and of the F term are summarized in Table V.

The branch of the moduli space that can be identified with $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ is the one on which $A^1_{11}=0$. Then, the Hilbert series of the instanton branch corresponding to the NVS theory with flavor symmetry $U(N)$ and gauge ranks $\mathbf{k}=(k_1,k_2)$ is

$$H[\mathbf{k},F,\mathbb{C}P^2/\mathbb{Z}_2](t,x,\mathbf{y}) = \int d\mu_{U(k_1)}(\mathbf{z})\int d\mu_{U(k_2)}(\mathbf{p})\times \mathrm{PE}[\chi_{S_i}t+\chi_{\tilde{S}_j}t + \chi_{A^2_{11}}t^2+\chi_Q t^2+\chi_q t^2-\chi_F t^4], \tag{60}$$

where $\mathbf{z}$ and $\mathbf{p}$ are the fugacities of the $U(k_1)$ and $U(k_2)$ gauge groups, respectively, while $\mathbf{y}$ denotes the fugacity of the $U(N)$ flavor group, and x denotes the fugacity of the global $SU(2)$ symmetry acting separately on the two doublets $\tilde{S}_\alpha$ and S_β. The contribution of each field is given by

$$\chi_{S_j}=\left(x+\frac{1}{x}\right)\sum_{1\le a\le b\le k_2}p_a p_b,\quad \chi_{\tilde{S}_i}=\left(x+\frac{1}{x}\right)\sum_{1\le a\le b\le k_1}z_a^{-1}z_b^{-1},$$

$$\chi_{A^2_{11}}=\sum_{a=1}^{k_1}\sum_{b=1}^{k_2}z_a p_b^{-1},\quad \chi_Q=\sum_{i=1}^{N}\sum_{a=1}^{k_1}z_a^{-1}y_i,$$

$$\chi_q=\sum_{j=1}^{N}\sum_{b=1}^{k_2}p_b y_j^{-1},\quad \chi_F=\sum_{a=1}^{k_1}\sum_{b=1}^{k_2}z_a^{-1}p_b.$$

In this case, by explicit computation of the Hilbert series of the instanton branch of the NVS theory with gauge group $G=U(k_1)\times U(k_2)$ and flavor group $U(N)$ for the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_2$, we find that it turns out to be equal to the Hilbert series for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with gauge group $G=U(K_1)$ (see [31] for more details). The two theories share the same flavor group, and the gauge groups are related in the following way:

$$K_1=\min(k_1,k_2). \tag{61}$$

Let us explicitly show a few examples supporting our claim. $Sp(1)$ instanton: $\mathbf{k}=(1,1)$ and $N=1$. Using Eq. (60) and unrefining, we find that

$$H[\mathbf{k}=(1,1),U(1),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1) = \frac{1+2t^6+2t^9+2t^{12}+t^{18}}{(1-t^3)^4(1+2t^3+2t^6+t^9)^2},$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=1$ and $K_1=1$. $Sp(2)$ instanton: $\mathbf{k}=(1,1)$ and $N=2$. Using Eq. (60) and unrefining, we find that

$$H[\mathbf{k}=(1,1),U(2),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1) = \frac{1-t^3+5t^6+4t^9+4t^{12}+4t^{15}+5t^{18}-t^{21}+t^{24}}{(1-t^3)^6(1+t^3)^2(1+t^3+t^6)^3},$$

which is the Hilbert series for $Sp(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=2$ and $K_1=1$. $Sp(1)$ instanton: $\mathbf{k}=(2,1)$ and $N=1$. Using Eq. (60) and unrefining, we find that

$$H[\mathbf{k}=(2,1),U(1),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1) = \frac{1+2t^6+2t^9+2t^{12}+t^{18}}{(1-t^3)^4(1+2t^3+2t^6+t^9)^2},$$

which is again the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=1$ and $K_1=1$. $Sp(1)$ instanton: $\mathbf{k}=(2,2)$ and $N=1$. Using Eq. (60) and unrefining, we obtain

$$H[\mathbf{k}=(2,2),U(1),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1) = \frac{1}{(1-t^3)^8(1+t^3)^4(1+t^6)^2(1+t^3+t^6)^2(1+t^3+t^6+t^9+t^{12})^2}(1+2t^6+2t^9+9t^{12}+10t^{15}+15t^{18}+18t^{21} + 28t^{24}+26t^{27}+34t^{30}+26t^{33}+\text{palindrome}+t^{60}),$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=1$ and $K_1=2$. In the NVS case, we can graphically summarize the relation between the parent $\mathbb{C}^2/\mathbb{Z}_2$ instanton and the $\mathbb{C}P^2/\mathbb{Z}_2$ one as in Fig. 13. Note that, as in the unitary instanton case, we again have a merging of the flavored pair of gauge nodes into a single node with the rank the minimum of the "merged ones."

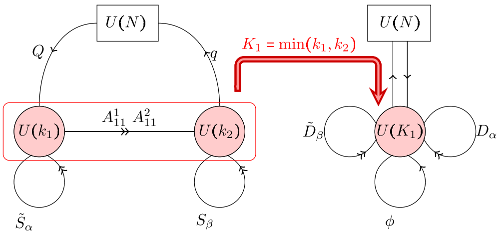

FIG. 13. Relation between the $\mathbb{C}P^2/\mathbb{Z}_2$ quiver gauge theory in the NVS case (on the left) and the $\mathbb{C}^2/\mathbb{Z}_2$ quiver gauge theory (on the right). $\tilde{D}_\beta$ are two fields in the symmetric conjugate representation of the gauge group $U(K_1)$, while D_α are two fields in the symmetric representation of the gauge group $U(K_1)$ (see [31] for more details).

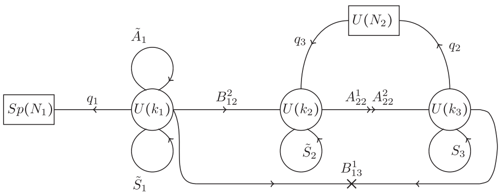

FIG. 14. Quiver diagram for symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_3$.

3. $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$

For the case of odd orbifolds, there is only one inequivalent choice. We report in Fig. 14 the quiver diagram of the corresponding field theory, while we summarize the fields and F-term transformations in Table VI. Note that $N = N_1 + N_2$.

The branch of the moduli space that can be identified with $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$ is the one on which $A^1_{22} = 0$ and $\tilde{A}_1 = 0$. The Hilbert series of the instanton branch corresponding to the theory with flavor symmetry $Sp(N_1) \times U(N_2)$ and gauge ranks $\mathbf{k} = (k_1, k_2, k_3)$ is

$$H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_3](t, x, \mathbf{y}, \mathbf{d}) = \int d\mu_{U(k_1)}(\mathbf{z}) \int d\mu_{U(k_2)}(\mathbf{p}) \int d\mu_{U(k_3)}(\mathbf{w}) \times \mathrm{PE}[\chi_{q_1} t^2 + \chi_{q_2} t^2 + \chi_{q_3} t^2 + \chi_{B^2_{12}} t + \chi_{A^2_{22}} t^2 + \chi_{B^1_{13}} t + \chi_{\tilde{S}_1} t^2 + \chi_{\tilde{S}_2} t + \chi_{S_3} t - \chi_{F_1} t^4 - \chi_{F_2} t^4], \quad (62)$$

where $\mathbf{z}$, $\mathbf{p}$, and $\mathbf{w}$ are the fugacities of the $U(k_1)$, $U(k_2)$, and $U(k_3)$ gauge groups, respectively, while $\mathbf{y}$ denotes the fugacity of the $Sp(N_1)$ flavor group and $\mathbf{d}$ the fugacity of the $U(N_2)$ flavor group. Finally, x is the fugacity of the $U(1)$ symmetry acting on the $\tilde{S}_2$ and S_3 fields. The contribution of each field and of the F terms are

$$\chi_{\tilde{S}_1} = \sum_{1 \le a \le b \le k_1} z_a^{-1} z_b^{-1}, \qquad \chi_{\tilde{S}_2} = \sum_{1 \le a \le b \le k_2} p_a^{-1} p_b^{-1} x^{-1},$$

$$\chi_{S_3} = \sum_{1 \le a \le b \le k_3} w_a w_b x,$$

$$\chi_{q_1} = \sum_{a=1}^{k_1} \sum_{i=1}^{N_1} z_a \left(y_i + \frac{1}{y_i} \right), \qquad \chi_{q_2} = \sum_{a=1}^{k_3} \sum_{j=1}^{N_2} w_a d_j^{-1},$$

$$\chi_{q_3} = \sum_{a=1}^{k_2} \sum_{j=1}^{N_2} p_a^{-1} d_j, \qquad \chi_{F_2} = \sum_{a=1}^{k_2} \sum_{b=1}^{k_3} p_a^{-1} w_b,$$

$$\chi_{B^2_{12}} = \sum_{a=1}^{k_1} \sum_{b=1}^{k_2} z_a p_b^{-1}, \qquad \chi_{A^2_{22}} = \sum_{a=1}^{k_2} \sum_{b=1}^{k_3} p_a w_b^{-1},$$

$$\chi_{B^1_{13}} = \sum_{a=1}^{k_1} \sum_{b=1}^{k_3} z_a w_b, \qquad \chi_{F_1} = \sum_{1 \le a < b \le k_1} z_a z_b.$$

By explicit computation, we find that the Hilbert series of the theory with gauge group $G = U(k_1) \times U(k_2) \times U(k_3)$ and flavor group $Sp(N_1) \times U(N_2)$ for the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_3$ coincides with the Hilbert series for the moduli space of $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with gauge group $G = O(K_1) \times U(K_2)$ and flavor

TABLE VI. Transformations of the fields for symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_3$.

Fields	$U(k_1)$	$U(k_2)$	$U(k_3)$	$Sp(N_1)$	$U(N_2)$	$U(1)$	$U(1)$
q_1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
q_2	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots 0]_{+1}$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$1/2$
q_3	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$1/2$
B^2_{12}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/4$
A^2_{22}	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
B^1_{13}	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/4$
$\tilde{S}_1$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
$\tilde{S}_2$	$[\mathbf{0}]$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/x$	$1/4$
S_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	x	$1/4$
F_1	$[0,1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

group $Sp(N_1) \times U(N_2)$ (see [31] for more details) upon identifying

$$K_1 = k_1, \qquad K_2 = \min(k_2, k_3). \tag{63}$$

Let us turn to explicit examples supporting our claim. $Sp(1)$ instanton: $\mathbf{k} = (1,1,1)$ and $\mathbf{N} = (1,0)$. Using Eq. (62) and unrefining, we find that

$$H[\mathbf{k} = (1,1,1), Sp(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{(1+t^6)(1-t^3+t^6)}{(1-t^3)^4(1+t^3)^2(1+t^3+t^6)},$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N}$=(1,0) and $\mathbf{K}$=(1,1). $Sp(1)$ instanton: $\mathbf{k} = (1,1,1)$ and $\mathbf{N} = (0,1)$. Using Eq. (62) and unrefining, we find that

$$H[\mathbf{k} = (1,1,1), U(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{1+t^6+2t^9+2t^{12}+2t^{15}+t^{18}+t^{24}}{(1-t^3)^4(1+t^3)^2(1+t^6)(1+t^3+t^6)^2},$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (0,1)$ and $\mathbf{K} = (1,1)$. $Sp(2)$ instanton: $\mathbf{k} = (1,1,1)$ and $\mathbf{N} = (1,1)$. Using Eq. (62) and unrefining, we find that

$$H[\mathbf{k} = (1,1,1), Sp(1) \times U(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1,1) = \frac{1-2t^3+5t^6-2t^9+6t^{12}-2t^{15}+5t^{18}-2t^{21}+t^{24}}{(1-t^3)^6(1+t^6)(1+2t^3+2t^6+t^9)^2},$$

which is the Hilbert series for $Sp(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (1,1)$ and $\mathbf{K} = (1,1)$. $Sp(1)$ instanton: $\mathbf{k} = (1,2,1)$ and $\mathbf{N} = (1,0)$. Using Eq. (62) and unrefining, we find that

$$H[\mathbf{k} = (1,2,1), Sp(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{(1+t^6)(1-t^3+t^6)}{(1-t^3)^4(1+t^3)^2(1+t^3+t^6)},$$

which is again the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (1,0)$ and $\mathbf{K} = (1,1)$. $Sp(1)$ instanton: $\mathbf{k} = (1,1,2)$ and $\mathbf{N} = (1,0)$. Using Eq. (62) and unrefining, we find that

$$H[\mathbf{k} = (1,2,1), Sp(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{(1+t^6)(1-t^3+t^6)}{(1-t^3)^4(1+t^3)^2(1+t^3+t^6)},$$

which is again the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (1,0)$ and $\mathbf{K} = (1,1)$. $Sp(1)$ instanton: $\mathbf{k} = (1,1,2)$ and $\mathbf{N} = (0,1)$. Using Eq. (62) and unrefining, we find that

$$H[\mathbf{k} = (1,1,2), U(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{1+t^6+2t^9+2t^{12}+2t^{15}+t^{18}+t^{24}}{(1-t^3)^4(1+t^3)^2(1+t^6)(1+t^3+t^6)^2},$$

which is again the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (0,1)$ and $\mathbf{K} = (1,1)$. $Sp(1)$ instanton: $\mathbf{k} = (1,2,1)$ and $\mathbf{N} = (0,1)$. Using Eq. (62) and unrefining, we find

$$H[\mathbf{k} = (1,2,1), U(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) = \frac{1+t^6+2t^9+2t^{12}+2t^{15}+t^{18}+t^{24}}{(1-t^3)^4(1+t^3)^2(1+t^6)(1+t^3+t^6)^2},$$

which is again the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (0,1)$ and $\mathbf{K} = (1,1)$. $Sp(1)$ instanton: $\mathbf{k} = (2,1,1)$ and $\mathbf{N} = (1,0)$. Using Eq. (62) and unrefining, we find that

$$\begin{aligned} H[\mathbf{k} = (2,1,1), Sp(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1) &= \frac{1}{(1-t^3)^6(1+t^3)^4(1+t^3+t^6)(1+t^3+2t^6+2t^9+2t^{12}+t^{15}+t^{18})^2}(1+t^3 \\ &\quad + 3t^6+4t^9+8t^{12}+14t^{15}+19t^{18}+23t^{21}+27t^{24}+26t^{27}+27t^{30}+\text{palindrome}+t^{54}) \\ &= 1+4t^6+2t^9+13t^{12}+14t^{15}+33t^{18}+42t^{21}+80t^{24}+104t^{27}+o(t^{27}), \end{aligned}$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (1,0)$ and $\mathbf{K} = (2,1)$.

As shown in Fig. 15, we can graphically summarize the relation between the symplectic $\mathbb{C}P^2/\mathbb{Z}_3$ instanton and its cousin on $\mathbb{C}^2/\mathbb{Z}_3$ as the merging of the flavored pair of gauge nodes into a single node whose rank is the minimum among the "merging ones."

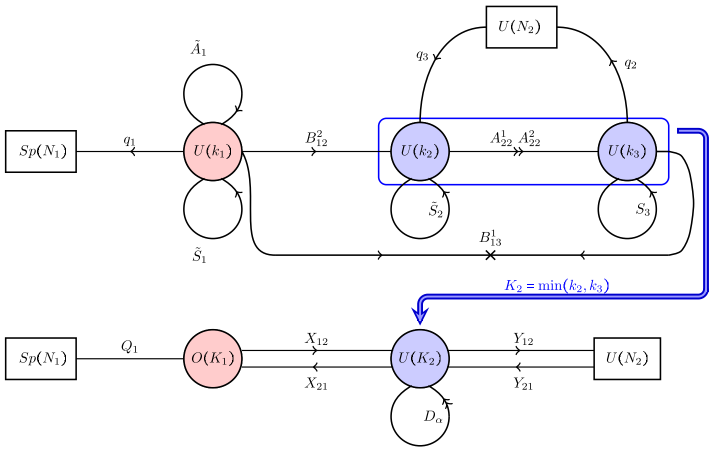

FIG. 15. Relation between the quiver diagram for $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$ and the quiver diagram for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$. In the figure, the symbol D_α denotes two fields transforming in the symmetric representation of the gauge group $U(K_2)$ (however, see [31] for more details).

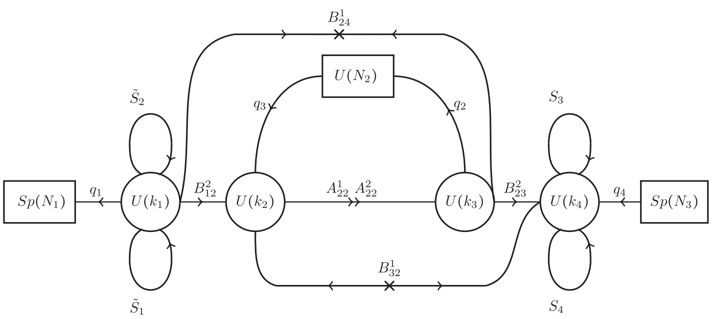

FIG. 16. Quiver diagram for VS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

4. $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$: VS

Starting from the theory whose instanton branch describes instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ and applying the rules in [42], we obtain the theory for $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ in the VS case. The corresponding quiver diagram is reported in Fig. 16, while we summarize the transformations of the fields under the different groups in Table VII.

TABLE VII. Transformation of the fields for VS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

Fields	$U(k_1)$	$U(k_2)$	$U(k_3)$	$U(k_4)$	$Sp(N_1)$	$U(N_2)$	$Sp(N_3)$	$U(1)$
B^2_{12}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
A^2_{22}	$[\mathbf{0}]$	$[1,0\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^2_{23}	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
$\tilde{S}_2$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
S_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[2,0\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^1_{24}	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
B^1_{32}	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
q_1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots.,0]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
q_3	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	1/2
q_2	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	1/2
q_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	1/2
F_1	$[0,1,0\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,1,0,\ldots.,0]_{-1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

The branch of the moduli space that can be identified with $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ is the one on which $A^1_{22}=0$, $\tilde{S}_1=0$, and $S_3=0$. The Hilbert series of the instanton branch corresponding to the VS theory with flavor symmetry $Sp(N_1)\times U(N_2)\times Sp(N_3)$ and gauge ranks $\mathbf{k}=(k_1,k_2,k_3,k_4)$ is

$$H[\mathbf{k},F,\mathbb{C}P^2/\mathbb{Z}_4](t,x,\mathbf{y},\mathbf{d},\mathbf{u}) = \int d\mu_{U(k_1)}(\mathbf{z})\int d\mu_{U(k_2)}(\mathbf{p})\int d\mu_{U(k_3)}(\mathbf{w})\int d\mu_{U(k_4)}(\mathbf{v}) \times \mathrm{PE}[\chi_{q_1}t^2+\chi_{q_2}t^2+\chi_{q_3}t^2+\chi_{q_4}t^2+\chi_{B^2_{12}}t+\chi_{A^2_{22}}t^2+\chi_{B^2_{23}}t+\chi_{B^1_{24}}t+\chi_{B^1_{32}}t + \chi_{\tilde{S}_2}t^2+\chi_{S_4}t^2-\chi_{F_1}t^4-\chi_{F_2}t^4-\chi_{F_3}t^4], \tag{64}$$

where $\mathbf{z}$, $\mathbf{p}$, $\mathbf{w}$, and $\mathbf{v}$ are the fugacities of the $U(k_1)$, $U(k_2)$, $U(k_3)$, and $U(k_4)$ gauge groups, respectively, while $\mathbf{y}$, $\mathbf{d}$, and $\mathbf{u}$ denote the fugacities of the $Sp(N_1)$ flavor group, the $U(N_2)$ flavor group, and the $Sp(N_3)$, respectively. The contributions of the various fields are

$$\chi_{B^2_{12}}=\sum_{a=1}^{k_1}\sum_{b=1}^{k_2}z_a p_b^{-1},\qquad \chi_{A^2_{22}}=\sum_{a=1}^{k_2}\sum_{b=1}^{k_3}p_a w_b^{-1},\qquad \chi_{B^2_{23}}=\sum_{a=1}^{k_3}\sum_{b=1}^{k_4}w_a v_b^{-1},$$

$$\chi_{S_4}=\sum_{1\le a\le b\le k_4}v_a v_b,\qquad \chi_{F_1}=\sum_{1\le a<b\le k_1}z_a z_b,\qquad \chi_{F_3}=\sum_{1\le a<b\le k_4}v_a^{-1}v_b^{-1},$$

$$\chi_{B^1_{24}}=\sum_{a=1}^{k_1}\sum_{b=1}^{k_3}z_a w_b,\qquad \chi_{B^1_{32}}=\sum_{a=1}^{k_2}\sum_{b=1}^{k_4}p_a^{-1}v_b^{-1},\qquad \chi_{\tilde{S}_2}=\sum_{1\le a\le b\le k_1}z_a^{-1}z_b^{-1},\qquad \chi_{F_2}=\sum_{a=1}^{k_2}\sum_{b=1}^{k_3}p_a^{-1}w_b,$$

$$\chi_{q_1}=\sum_{a=1}^{k_1}\sum_{j=1}^{N_1}z_a\left(y_j+\frac{1}{y_j}\right),\qquad \chi_{q_3}=\sum_{j=1}^{N_2}\sum_{b=1}^{k_2}d_j p_b^{-1},\qquad \chi_{q_2}=\sum_{a=1}^{k_3}\sum_{i=1}^{N_2}w_a d_i^{-1},\qquad \chi_{q_4}=\sum_{a=1}^{k_4}\sum_{i=1}^{N_3}v_a^{-1}\left(u_i+\frac{1}{u_i}\right).$$

By explicit computation of the instanton branch Hilbert series for the theory with gauge group $G=U(k_1)\times U(k_2)\times U(k_3)\times U(k_4)$ and flavor group $Sp(N_1)\times U(N_2)\times Sp(N_3)$, we find that it is equal to the Hilbert series for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with gauge group $G=O(K_1)\times U(K_2)\times O(K_3)$ and flavor group $Sp(N_1)\times U(N_2)\times Sp(N_3)$ (see [31] for more details) upon identifying

$$K_1=k_1,\qquad K_2=\min(k_2,k_3),\qquad K_3=k_3. \tag{65}$$

Let us show some explicit examples supporting our claim. $Sp(1)$ instanton: $\mathbf{k}=(1,1,1,1)$ and $\mathbf{N}=(1,0,0)$. Using Eq. (64) and unrefining, we find that

$$H[\mathbf{k}=(1,1,1,1),Sp(1),\mathbb{C}P^2/\mathbb{Z}_4](t,1)=\frac{1+t^{12}}{(1-t^6)^4}, \tag{66}$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N}=(1,0,0)$ and $\mathbf{K}=(1,1,1)$. $Sp(1)$ instanton: $\mathbf{k}=(1,1,1,1)$ and $\mathbf{N}=(0,1,0)$. Using Eq. (64) and unrefining, we find that

$$H[\mathbf{k}=(1,1,1,1),U(1),\mathbb{C}P^2/\mathbb{Z}_4](t,1)=\frac{1+4t^{12}+t^{24}}{(1-t^6)^4(1+t^6)^2},$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N}=(0,1,0)$ and $\mathbf{K}=(1,1,1)$. $Sp(1)$ instanton: $\mathbf{k}=(1,2,1,1)$ and $\mathbf{N}=(1,0,0)$. Using Eq. (64) and unrefining, we find again the expression (66). $Sp(1)$ instanton: $\mathbf{k}=(1,1,2,1)$ and $\mathbf{N}=(1,0,0)$. Using Eq. (64) and unrefining, we find again the expression (66). $Sp(1)$ instanton: $\mathbf{k}=(2,1,1,1)$ and $\mathbf{N}=(1,0,0)$. Using Eq. (64) and unrefining, we find that

$$H[\mathbf{k}=(2,1,1,1),Sp(1),\mathbb{C}P^2/\mathbb{Z}_4](t,1) =\frac{1+t^6+5t^{12}+8t^{18}+8t^{24}+8t^{30}+5t^{36}+t^{42}+t^{48}}{(1-t^6)^6(1+t^6)(1+t^6+t^{12})^2},$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N}=(1,0,0)$ and $\mathbf{K}=(2,1,1)$.

We can graphically relate the symplectic VS $\mathbb{C}P^2/\mathbb{Z}_4$ instantons with their cousin on $\mathbb{C}^2/\mathbb{Z}_4$ as in Fig. 17.

5. *Sp*(*N*) *instantons on* $\mathbb{C}P^2/\mathbb{Z}_4$*: NVS*

Let us now consider the second configuration leading to the NVS case. The quiver diagram of the corresponding theory is reported in Fig. 18, while the transformations of the fields and of the F terms are summarized in Table VIII.

The branch of the moduli space that can be identified with $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ in the NVS case is the one on which $A^1_{11}=0$ and $A^1_{33}=0$. The Hilbert series of the instanton branch corresponding to the NVS theory with flavor symmetry $U(N_1)\times U(N_2)$ and gauge ranks $\mathbf{k}=(k_1,k_2,k_3,k_4)$ is

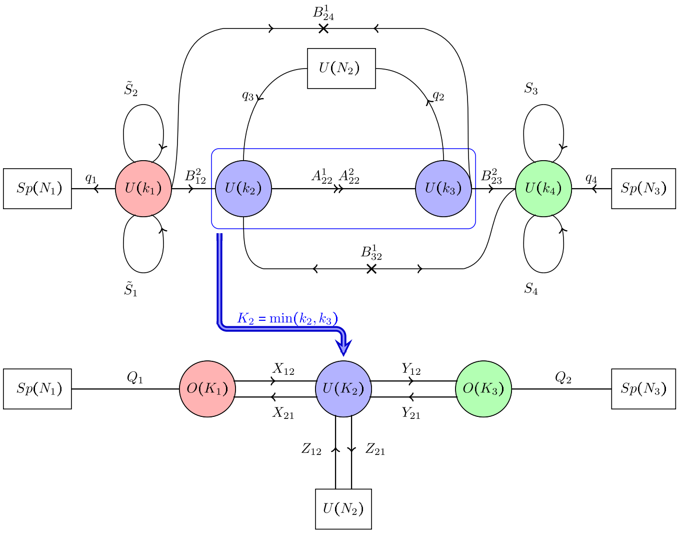

FIG. 17. Relation between the $\mathbb{C}P^2/\mathbb{Z}_4$ quiver gauge theory in the VS case and the corresponding $\mathbb{C}^2/\mathbb{Z}_4$ quiver gauge theory.

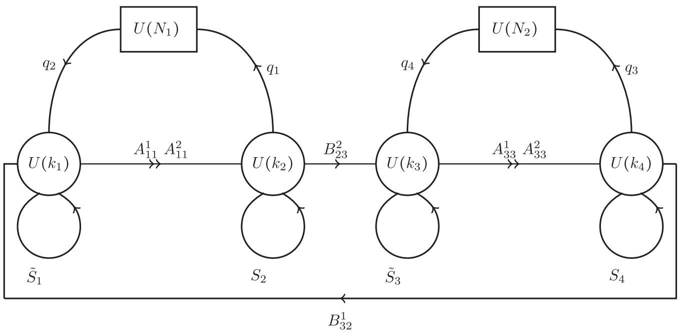

FIG. 18. Quiver diagram for NVS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

$$\begin{aligned} H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_4](t, x, \mathbf{y}, \mathbf{d}) = &\int d\mu_{U(k_1)}(\mathbf{z}) \int d\mu_{U(k_2)}(\mathbf{p}) \int d\mu_{U(k_3)}(\mathbf{w}) \\ &\times \int d\mu_{U(k_4)}(\mathbf{v}) \times \mathrm{PE}[\chi_{q_1} t^2 + \chi_{q_2} t^2 + \chi_{q_3} t^2 + \chi_{q_4} t^2 + \chi_{B^2_{23}} t + \chi_{A^2_{11}} t^2 + \chi_{A^2_{33}} t^2 \\ &+ \chi_{B^1_{32}} t + \chi_{\tilde{S}_1} t + \chi_{S_2} t + \chi_{\tilde{S}_3} t + \chi_{S_4} t - \chi_{F_1} t^4 - \chi_{F_2} t^4], \end{aligned} \tag{67}$$

where $\mathbf{z}$, $\mathbf{p}$, $\mathbf{w}$, and $\mathbf{v}$ are the fugacities of the $U(k_1)$, $U(k_2)$, $U(k_3)$, and $U(k_4)$ gauge groups, respectively, while $\mathbf{y}$ and $\mathbf{d}$ denote the fugacities of the $U(N_1)$ flavor group and the $U(N_2)$ flavor group, respectively. The contributions of the various fields are given by

TABLE VIII. Transformation of the fields for NVS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

Fields	$U(k_1)$	$U(k_2)$	$U(k_3)$	$U(k_4)$	$U(N_1)$	$U(N_2)$	$U(1)$
A^2_{11}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^2_{23}	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
A^2_{33}	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^1_{32}	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
$\tilde{S}_1$	$[2,0\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
S_2	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
$\tilde{S}_3$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
S_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
q_1	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	1/2
q_2	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	1/2
q_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	1/2
q_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	1/2
F_1	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

$$\chi_{S_4} = \sum_{1\le a\le b\le k_4} v_a v_b, \qquad \chi_{F_1} = \sum_{a=1}^{k_1}\sum_{b=1}^{k_2} p_b z_a^{-1}, \qquad \chi_{F_2} = \sum_{a=1}^{k_3}\sum_{b=1}^{k_4} w_a^{-1} v_b,$$

$$\chi_{\tilde{S}_1} = \sum_{1\le a\le b\le k_1} z_a^{-1} z_b^{-1}, \qquad \chi_{S_2} = \sum_{1\le a\le b\le k_2} p_a p_b, \qquad \chi_{\tilde{S}_3} = \sum_{1\le a\le b\le k_3} w_a^{-1} w_b^{-1},$$

$$\chi_{q_1} = \sum_{a=1}^{k_2}\sum_{i=1}^{N_1} p_a y_i^{-1}, \qquad \chi_{q_2} = \sum_{a=1}^{k_1}\sum_{i=1}^{N_1} z_a^{-1} y_i, \qquad \chi_{q_3} = \sum_{a=1}^{k_4}\sum_{j=1}^{N_2} v_a d_j^{-1}, \qquad \chi_{q_4} = \sum_{a=1}^{k_3}\sum_{j=1}^{N_2} w_a^{-1} d_j,$$

$$\chi_{A^2_{11}} = \sum_{a=1}^{k_1}\sum_{b=1}^{k_2} z_a p_b^{-1}, \qquad \chi_{B^2_{23}} = \sum_{a=1}^{k_2}\sum_{b=1}^{k_3} p_a w_b^{-1}, \qquad \chi_{A^2_{33}} = \sum_{a=1}^{k_3}\sum_{b=1}^{k_4} w_a v_b^{-1}, \qquad \chi_{B^1_{32}} = \sum_{a=1}^{k_1}\sum_{b=1}^{k_4} v_b z_a^{-1}.$$

Explicit computation of the instanton branch Hilbert series with gauge group $G = U(k_1)\times U(k_2)\times U(k_3)\times U(k_4)$ and flavor group $U(N_1)\times U(N_2)$ shows that it coincides with the Hilbert series for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with gauge group $G = U(K_1)\times U(K_2)$ and flavor group $U(N_1)\times U(N_2)$ (see [31] for more details) upon the identification

$$K_1 = \min(k_1, k_2), \qquad K_2 = \min(k_3, k_4). \tag{68}$$

Let us show a few explicit examples. $Sp(1)$ instanton: $\mathbf{k} = (1,1,1,1)$ and $\mathbf{N} = (1,0)$. Using Eq. (67) and unrefining, we find that

$$H[\mathbf{k} = (1,1,1,1), U(1), \mathbb{C}P^2/\mathbb{Z}_4](t,1) = \frac{1 - t^3 + 2t^9 - t^{15} + t^{18}}{(1-t^3)^4(1+t^3)^2(1+t^3+t^6+t^9+t^{12})}, \tag{69}$$

which is the Hilbert series for $Sp(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N} = (1,0)$ and $\mathbf{K} = (1,1)$. $Sp(2)$ instanton: $\mathbf{k} = (1,1,1,1)$ and $\mathbf{N} = (1,1)$. Using Eq. (67) and unrefining, we find that

$$\begin{aligned} &H[\mathbf{k} = (1,1,1,1), U(1)\times U(1), \mathbb{C}P^2/\mathbb{Z}_4](t,1,1) \\ &= \frac{1 + 2t^6 + 3t^9 + 8t^{12} + 11t^{15} + 13t^{18} + 12t^{21} + 13t^{24} + 11t^{27} + 8t^{30} + 3t^{33} + 2t^{36} + t^{42}}{(1-t^3)^6(1+t^3)^2(1+t^3+t^6)^3(1+t^3+2t^6+2t^9+2t^{12}+t^{15}+t^{18})}, \end{aligned}$$

which is the Hilbert series for $Sp(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N} = (1,1)$ and $\mathbf{K} = (1,1)$. $Sp(1)$ instantons: $\mathbf{k} = (1,2,1,1)$ and $\mathbf{N} = (1,0)$. Using Eq. (67), we find again the expression (69).

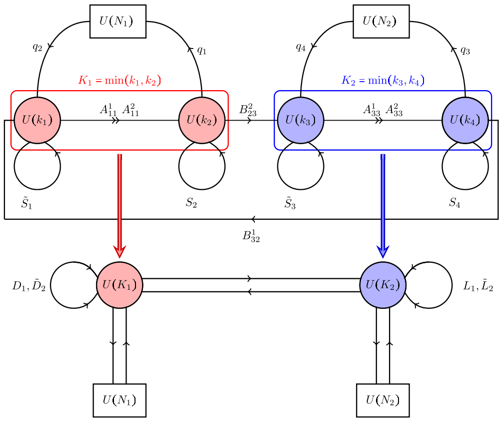

FIG. 19. Relation between the $\mathbb{C}P^2/\mathbb{Z}_4$ quiver gauge theory in the NVS case and the corresponding $\mathbb{C}^2/\mathbb{Z}_4$ quiver gauge theory, where D_1, $\tilde{D}_2$ are two fields in the symmetric representation of the gauge group $U(K_1)$, while L_1, $\tilde{L}_2$ are two fields in the symmetric representation of the gauge group $U(K_2)$ (however, see [31] for more details regarding the $\mathbb{C}^2/\mathbb{Z}_4$ theory).

Finally, in Fig. 19 we graphically show the relation between symplectic NVS instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ and their cousins on $\mathbb{C}^2/\mathbb{Z}_4$.

6. $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ with $n > 4$

Let us now consider the generic case of instantons on $\mathbb{Z}_n$ orbifolds of $\mathbb{C}P^2$ with $n > 4$. Based on the previous examples, we can extract the generic pattern of both the quiver as well as the relation between the symplectic instanton on $\mathbb{C}P^2/\mathbb{Z}_n$ with its relative on $\mathbb{C}^2/\mathbb{Z}_n$.

Recall that N is the sum of the ranks of the flavor groups in the ADHM quiver, while the ranks of the gauge groups are related to instanton number and, together with the relative flavor ranks, to other possible quantum numbers labeling the instanton. Unfortunately, the precise identification between quiver data and instanton data is not known. $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$. Elaborating on the previous examples, we conjecture that the theory describing symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$ is related to its counterpart on $\mathbb{C}^2/\mathbb{Z}_{2n+1}$ as in Fig. 31. Moreover, while the flavor groups continue to be the same, the ranks of the gauge groups are related in the following way:

$$K_1 = k_1, \qquad K_2 = \min(k_2, k_3),$$
$$K_3 = \min(k_4, k_5), \ldots K_{n+1} = \min(k_{2n}, k_{2n+1}). \qquad (70)$$

$Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$: VS. Elaborating on the lowest n cases, we can extrapolate both the quiver for VS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ and their relation to their cousins (of course, VS) on $\mathbb{C}^2/\mathbb{Z}_{2n}$ as shown in Fig. 32. Moreover, while flavor nodes remain the same, the gauge rank identification is as follows:

$$K_1 = k_1, \qquad K_2 = \min(k_2, k_3), \ldots$$
$$K_{n-1} = \min(k_{2n-2}, k_{2n-1}), \qquad K_n = k_{2n}. \qquad (71)$$

$Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$: NVS. Elaborating on the lowest n cases, in this case, we can extrapolate both the quiver for NVS symplectic instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ and their relation to their cousins (of course, NVS) on $\mathbb{C}^2/\mathbb{Z}_{2n}$ as shown in Fig. 33. Moreover, while the flavor nodes remain the same, the gauge rank identification is as follows:

$$K_1 = \min(k_1, k_2),$$
$$K_2 = \min(k_3, k_4), \ldots K_n = \min(k_{2n-1}, k_{2n}). \qquad (72)$$

It is interesting to note that the merging nodes are those going over, in the $\mathbb{C}^2/\mathbb{Z}_n$ parent, to unitary gauge groups. In turn, in the parent $\mathbb{C}^2/\mathbb{Z}_n$, these are the nodes admitting a blowup mode through the FI parameter. It would be interesting to have a deeper understanding of these facts, as well as the topological data characterizing Sp instantons on $\mathbb{C}P^2/\mathbb{Z}_n$.

VI. $SO(N)$ INSTANTONS ON $\mathbb{C}P^2$ AND ITS ORBIFOLDS

We now turn to the case of orthogonal instantons on $\mathbb{C}P^2$ and its orbifolds. As described in [15], the ADHM construction for orthogonal instantons can be embedded into a $3d$ gauge theory which, in $3d$ $\mathcal{N} = 2$ language, contains a $U(2k)$ vector multiplet as well as one chiral multiplet $\tilde{S}$ in the symmetric two-index tensor representation of the gauge group and three chiral multiplets $A_1, A_2, \tilde{A}$ in the antisymmetric two-index tensor representation of the gauge group. The corresponding quiver is shown in Fig. 20. Note that the total flavor rank corresponds to N, while the gauge ranks—as well as the relative configurations of the

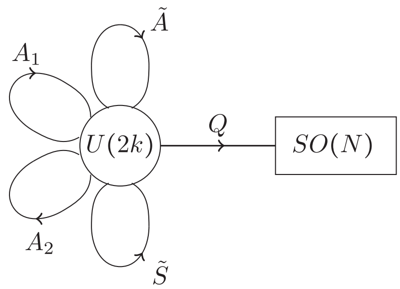

FIG. 20. Quiver diagram for $SO(N)$ instantons on $\mathbb{C}P^2$.

flavor ranks—correspond to instanton number and other data specifying the instanton.

In turn, the superpotential reads

$$W = \epsilon^{\alpha\beta}(A_\alpha)_{ab}\tilde{A}^{bc}(A_\beta)_{cd}\tilde{S}^{da} + \tilde{S}^{ab}Q^i{}_a Q^j{}_b M_{ij}, \qquad (73)$$

being M given by

$$M^{SO(2N)} = \begin{pmatrix} 0 & \mathbf{1}_{N\times N} \\ \mathbf{1}_{N\times N} & 0 \end{pmatrix},$$
$$M^{SO(2N+1)} = \begin{pmatrix} 0 & \mathbf{1}_{N\times N} & 0 \\ \mathbf{1}_{N\times N} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (74)$$

As shown in [15], the construction of orthogonal instantons on $\mathbb{C}P^2$ can be embedded into that of a parent orthogonal instanton on $\mathbb{C}^2$. As a consequence, the Hilbert series of the instanton on $\mathbb{C}P^2/\mathbb{Z}_n$ matches that of its counterpart on $\mathbb{C}^2$.

A. Resolved moduli space for orthogonal instantons

The gauge group in the ADHM construction of orthogonal instantons on $\mathbb{C}P^2$ is $U(2k)$. However, as shown in [15], k can be a half-integer while the Hilbert series is only sensitive to $\lfloor k \rfloor$, that is, the largest integer which is smaller or equal to k. In fact, it was conjectured that the instantons are distinguished by their second Stiefel-Whitney class written as $2(k - \lfloor k \rfloor)$. From this perspective, it is also natural to expect a notion of "resolved moduli space"—resolved, as in the unitary case, in the sense that these extra directions associate to other quantum numbers are discerned.

In order to explore the possibility of such resolved moduli space, following the example set by the unitary case, let us consider the simplest case where such extra directions are present. The instanton number was conjectured to be $\lfloor k \rfloor$. Then the analogous, for orthogonal instantons, to the case of a unitary instanton with $k_L = 0$ (as discussed in Sec. III A) is $k = \frac{1}{2}$, corresponding to a $U(1)$ gauge theory. Such theory does not have the antisymmetric matter, and on the instanton branch, $\tilde{S} = 0$. Therefore, the theory only contains the Q's out of which no gauge invariant can be constructed. Hence, very much like the Grassmanian, we find a extra compact manifold associated to the extra directions labeled in this case by the Stiefel-Whitney class. Just like in the unitary case, we can imagine resolving these directions by ungauging the $U(1)$ global symmetry. It is then straightforward to compute the instanton branch Hilbert series, which, upon unrefining the $SO(N)$ labels, reads

$$\mathrm{HS} = \frac{1+t}{(1-t)^{N-1}}. \qquad (75)$$

Interestingly, this can be written as

$$\mathrm{HS} = \frac{2}{(1-t)^{N-1}} - \frac{1}{(1-t)^{N-2}}, \qquad (76)$$

which is the Hilbert series for two $\mathbb{C}^{N-1}$ meeting at a $\mathbb{C}^{N-2}$. This is a dimension $N-1$ manifold analogous to the cone over the Grassmanian in the unitary case. Note that the dimension of the resolved moduli space is $2k(N-2)$, while that seen by the Hilbert series is $2\lfloor k \rfloor(N-2)$ [15]. Hence, the difference is $2(N-2)(k - \lfloor k \rfloor)$. Particularizing to the case $k = \frac{1}{2}$, this is an $(N-2)$-dimensional compact manifold. Then, the complex cone over it is a $N-1$ complex dimensional manifold, just as we have found.

Note that the case of symplectic instantons does not admit a similar construction. For example, in the quiver in Fig. 9, the instanton branch appears upon setting to zero an antisymmetric field while keeping the symmetric fields. Hence, the theory is never empty of gauge-invariant operators, as it happens in the case of unitary and orthogonal instantons, therefore, suggesting that no compact directions exist in that case.

B. Constructing $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$

Let us now turn to the construction of orthogonal instantons upon orbifolding the base space. In view of the ALE case, and following the symplectic instanton case in Sec. V, we construct the theories whose instanton branch describes orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ by first orbifolding and then orientifolding the unitary instanton case following the rules in [42,43]. As for symplectic instantons, we have qualitatively different situations depending on whether n is even or odd:

(i) If n is odd, we have only one type of quiver diagram corresponding to the fact that we have only one inequivalent way to cut the quiver diagram with a line.

(ii) If n is even, we have two types of quiver gauge theories corresponding to two possible inequivalent ways in which we can cut the quiver diagram with a line. Inspired by the ALE case, we will refer to them as the VS case and the NVS case, respectively.

Also, in this case, there can be hybrid configurations associated with one choice for the values of the signs implementing the orientifold prescription. As above, we restrict our analysis to the configuration of signs corresponding to the quantum field theory whose instanton branch describes orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ which, for the case of even n, are either VS or NVS. Just as in the other cases, the rank of the $SO(N)$ bundle corresponds to the sum of flavor ranks in the ADHM quiver. The rest of the ADHM data correspond to other data specifying the instanton.

1. $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$: VS

Starting from the $\mathbb{C}P^2/\mathbb{Z}_2$ and applying the rules in [42], we obtain the theory for $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$. The corresponding quiver diagram is reported in Fig. 21, while

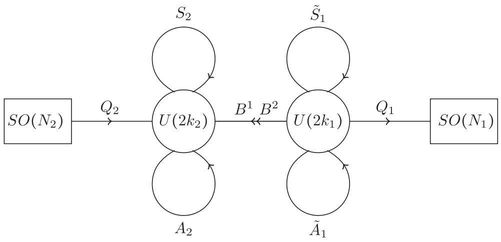

FIG. 21. Quiver diagram for VS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

we summarize the transformations of the fields under the different groups in Table IX.

The branch of the moduli space that can be identified with $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ is the one on which $\tilde{S}_1 = 0$ and $S_2 = 0$. The Hilbert series of the instanton branch corresponding to the VS theory with flavor symmetry $SO(N_1) \times SO(N_2)$ and gauge ranks $\mathbf{k} = (k_1, k_2)$ is

$$H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_2](t, x, \mathbf{y}, \mathbf{d}) = \int d\mu_{U(2k_1)}(\mathbf{z}) \int d\mu_{U(2k_2)}(\mathbf{p}) \mathrm{PE}[\chi_{A_2} t^2 + \chi_{\tilde{A}_1} t^2 + \chi_{B_j} t + \chi_{Q_1} t^2 + \chi_{Q_2} t^2 - \chi_{F_1} t^4 - \chi_{F_2} t^4], \tag{77}$$

where $\mathbf{z}$ and $\mathbf{p}$ are the fugacities of the $U(2k_1)$ and $U(2k_2)$ gauge groups, respectively, while $\mathbf{y}$ and $\mathbf{d}$ denote the fugacities of the $SO(N_1)$ and $SO(N_2)$ flavor groups. Finally, x is the fugacity of the $SU(2)$ symmetry acting on the B_j doublet. The contribution of each field is given by

$$\chi_{F_1} = \sum_{1 \le a \le b \le 2k_1} z_a z_b, \qquad \chi_{F_2} = \sum_{1 \le a \le b \le 2k_2} p_a^{-1} p_b^{-1},$$

$$\chi_{A_2} = \sum_{1 \le a < b \le 2k_2} p_a p_b, \qquad \chi_{\tilde{A}_1} = \sum_{1 \le a < b \le 2k_1} z_a^{-1} z_b^{-1}, \qquad \chi_{B^j} = \left(x + \frac{1}{x}\right) \sum_{a=1}^{2k_1} \sum_{b=1}^{2k_2} z_a p_b^{-1},$$

$$\chi_{Q_1} = \left(\sum_{a=1}^{2k_1} z_a\right) \times \begin{cases} \sum_{i=1}^{N_1/2} \left(y_i + \frac{1}{y_i}\right) & N_1 \text{ even}, \\ 1 + \sum_{i=1}^{(N_1-1)/2} \left(y_i + \frac{1}{y_i}\right) & N_1 \text{ odd}, \end{cases} \qquad \chi_{Q_2} = \left(\sum_{b=1}^{2k_2} p_b^{-1}\right) \times \begin{cases} \sum_{i=1}^{N_2/2} \left(d_i + \frac{1}{d_i}\right) & N_2 \text{ even}, \\ 1 + \sum_{i=1}^{(N_2-1)/2} \left(d_i + \frac{1}{d_i}\right) & N_2 \text{ odd}. \end{cases}$$

Explicitly computing the Hilbert series with gauge group $G = U(2k_1) \times U(2k_2)$ and flavor group $SO(N_1) \times SO(N_2)$ for the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ shows that it is equal to the Hilbert series for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with gauge group $G = Sp(K_1) \times Sp(K_2)$ (see [31] for more details) upon identifying

$$K_1 = k_1, \qquad K_2 = k_2. \tag{78}$$

Let us show a few explicit examples. $SO(5)$ instanton: $\mathbf{k} = (1, 1)$ and $\mathbf{N} = (2, 3)$. Using Eq. (77) and unrefining, we find that

$$H[\mathbf{k} = (1,1), SO(2) \times SO(3), \mathbb{C}P^2/\mathbb{Z}_2](t, 1, 1, 1) = \frac{1 - t^3 + 5t^6 + 4t^9 + 4t^{12} + 4t^{15} + 5t^{18} - t^{21} + t^{24}}{(1 - t^3)^6 (1 + t^3)^2 (1 + t^3 + t^6)^3},$$

which is the Hilbert series for the $SO(5)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbf{K} = (1, 1)$ and $\mathbf{N} = (2, 3)$. $SO(6)$ instanton:

TABLE IX. Transformations of the fields for VS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

Fields	$U(2k_1)$	$U(2k_2)$	$SO(N_1)$	$SO(N_2)$	$SU(2)$	$U(1)$
$\tilde{A}_1$	$[0,1,0,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
$\tilde{S}_1$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
A_2	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
S_2	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^j	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1]$	1/4
Q_1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
Q_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[\mathbf{0}]$	1/2
F_1	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

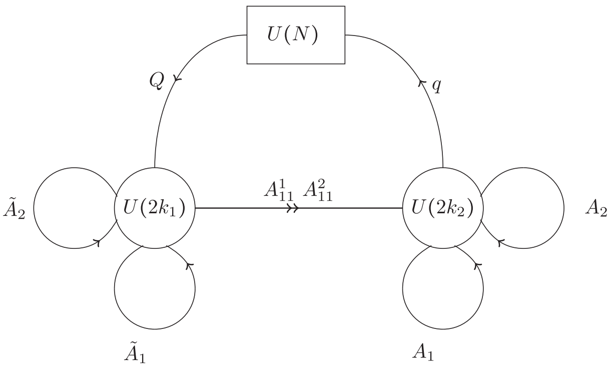

FIG. 22. Quiver diagram for NVS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

$\mathbf{k} = (1,1)$ and $\mathbf{N} = (3,3)$. Using Eq. (77) and unrefining, we find that

$$H[\mathbf{k}=(1,1), SO(3)\times SO(3), \mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1) = \frac{1-2t^3+8t^6+5t^{12}+12t^{15}+5t^{18}+8t^{24}-2t^{27}+t^{30}}{(1-t^3)^8(1+t^3)^2(1+t^3+t^6)^4},$$

which is the Hilbert series for the $SO(6)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbf{K} = (1,1)$ and $\mathbf{N} = (3,3)$.

2. $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$: NVS

Let us now consider the case of orthogonal NVS instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ upon choosing the other nonequivalent way to cut the quiver diagram. The quiver diagram of the corresponding theory is reported in Fig. 22, while the transformations of the fields and of the F term are summarized in Table X.

The branch of the moduli space that can be identified with $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_2$ is the one on which $A_{11}^1 = 0$. The Hilbert series of the instanton branch corresponding to the NVS theory with flavor symmetry $U(N)$ and ranks $\mathbf{k} = (k_1, k_2)$ is

$$H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_2](t,x,\mathbf{y}) = \int d\mu_{U(2k_1)}(\mathbf{z}) \int d\mu_{U(2k_2)}(\mathbf{p}) \times \mathrm{PE}[\chi_{A_i} t + \chi_{\tilde{A}_j} t + \chi_{A_{11}^2} t^2 + \chi_Q t^2 + \chi_q t^2 - \chi_F t^4], \quad (79)$$

where $\mathbf{z}$ and $\mathbf{p}$ are the fugacities of the $U(2k_1)$ and $U(2k_2)$ gauge groups, respectively, while $\mathbf{y}$ denotes the fugacity of the $U(N)$ flavor group. Finally, x is the fugacity of the $SU(2)$ acting on the A_β and $\tilde{A}_\alpha$ doublets. The contribution of each field is given by

$$\chi_{A_j} = \left(x+\frac{1}{x}\right)\sum_{1\le a<b\le 2k_2} p_a p_b, \qquad \chi_{\tilde{A}_i} = \left(x+\frac{1}{x}\right)\sum_{1\le a<b\le 2k_1} z_a^{-1} z_b^{-1},$$

$$\chi_{A_{11}^2} = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_2} z_a p_b^{-1}, \qquad \chi_Q = \sum_{i=1}^{N}\sum_{a=1}^{2k_1} z_a^{-1} y_i, \qquad \chi_q = \sum_{j=1}^{N}\sum_{b=1}^{2k_2} p_b y_j^{-1}, \qquad \chi_F = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_2} z_a^{-1} p_b.$$

The explicit computation of the instanton branch Hilbert series with gauge group $G = U(2k_1)\times U(2k_2)$ and flavor group $U(N)$ shows that it coincides with the Hilbert series for $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ with gauge group $G = U(2K_1)$ (see [31] for more details regarding the $\mathbb{C}^2/\mathbb{Z}_2$ Hilbert series) upon setting

TABLE X. Transformations of the fields for NVS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_2$.

Fields	$U(2k_1)$	$U(2k_2)$	$U(N)$	$SU(2)$	$U(1)$
$\tilde{A}_1, \tilde{A}_2$	$[0,1,0\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1]$	1/4
A_1, A_2	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[1]$	1/4
A_{11}^2	$[1,0,\ldots,0]_{+1}$	$[0,0,\ldots,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
q	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,0,\ldots,1]$	$[\mathbf{0}]$	1/2
Q	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[\mathbf{0}]$	1/2
F	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

$$K_1 = \min(k_1, k_2). \tag{80}$$

Let us show explicit examples supporting our claim. $SO(6)$ instanton: $\mathbf{k} = (1,1)$ and $N = 3$. Using Eq. (79) and unrefining, we find that

$$H[\mathbf{k} = (1,1), U(3), \mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1,1) = \frac{1 + 2t^3 + 9t^6 + 24t^9 + 50t^{12} + 76t^{15} + 108t^{18} + 120t^{21} + 108t^{24} + \text{palindrome} + \cdots + t^{42}}{(1-t^3)^8(1+t^3)^6(1+t^3+t^6)^{12}},$$

which is the Hilbert series for the $SO(6)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbf{K} = (1,1)$ and $N = 3$. $SO(8)$ instanton: $\mathbf{k} = (1,1)$ and $N = 4$. Using Eq. (79) and unrefining, we find that

$$\begin{aligned} H[\mathbf{k} = (1,1), U(4), \mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1,1,1) &= \frac{1}{(1-t^3)^{12}(1+t^3)^8(1+t^3+t^6)^{18}}(1 + 2t^3 + 14t^6 \\ &\quad + 44t^9 + 123t^{12} + 272t^{15} + 546t^{18} + 886t^{21} \\ &\quad + 1259t^{24} + 1544t^{27} + 1678t^{30} \\ &\quad + \text{palindrome} + \cdots + t^{60}), \end{aligned}$$

which is the Hilbert series for the $SO(8)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbf{K} = (1,1)$ and $N = 4$.

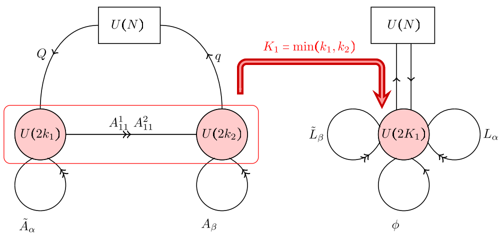

FIG. 23. Relation between the $\mathbb{C}P^2/\mathbb{Z}_2$ quiver gauge theory in the NVS case (on the left) and the corresponding $\mathbb{C}^2/\mathbb{Z}_2$ quiver gauge theory (on the right), where $\tilde{L}_\beta$ are two fields in the antisymmetric conjugate representation of the gauge group $U(2K_1)$, while L_α are two fields in the antisymmetric representation of the gauge group $U(2K_1)$ (see [31] for more details).

We graphically summarize in Fig. 23 the relation between the NVS orthogonal instanton on $\mathbb{C}P^2/\mathbb{Z}_2$ and its cousin on $\mathbb{C}^2/\mathbb{Z}_2$.

3. $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$

In this case, there is only one inequivalent choice of the orientifold action. We report in Fig. 24 the quiver diagram of the corresponding field theory, while we summarize the fields and F-term transformations in Table XI.

The branch of the moduli space that can be identified with $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$ is the one on which $A^1_{22} = 0$ and $\tilde{S}_1 = 0$. The Hilbert series of the instanton branch corresponding to a theory with flavor symmetry $SO(N_1) \times U(N_2)$ and gauge ranks $\mathbf{k} = (k_1, k_2, k_3)$ is

$$\begin{aligned} H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_3](t, x, \mathbf{y}, \mathbf{d}) &= \int d\mu_{U(2k_1)}(\mathbf{z}) \int d\mu_{U(2k_2)}(\mathbf{p}) \int d\mu_{U(2k_3)}(\mathbf{w}) \\ &\quad \times \text{PE}[\chi_{q_1} t^2 + \chi_{q_2} t^2 + \chi_{q_3} t^2 + \chi_{B^2_{12}} t + \chi_{A^2_{22}} t^2 \\ &\quad + \chi_{B^1_{13}} t + \chi_{\tilde{A}_1} t^2 + \chi_{\tilde{A}_2} t + \chi_{A_3} t - \chi_{F_1} t^4 - \chi_{F_2} t^4], \end{aligned} \tag{81}$$

where $\mathbf{z}$, $\mathbf{p}$, and $\mathbf{w}$ are the fugacities of the $U(2k_1)$, $U(2k_2)$, and $U(2k_3)$ gauge groups, respectively, while $\mathbf{y}$ denotes the fugacity of the $SO(N_1)$ flavor group and $\mathbf{d}$ the fugacity of the $U(N_2)$ gauge group. Finally, x is the fugacity of the $U(1)_x$ symmetry acting on $\tilde{A}_2$ and A_3. The contribution of each field and of the F terms are

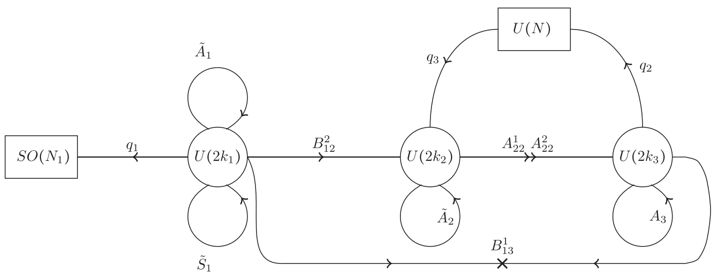

FIG. 24. Quiver diagram for $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$.

TABLE XI. Transformations of the fields for $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$.

Fields	$U(2k_1)$	$U(2k_2)$	$U(2k_3)$	$SO(N_1)$	$U(N_2)$	$U(1)_x$	$U(1)$
q_1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
q_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots 0]_{+1}$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$1/2$
q_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$1/2$
B_{12}^2	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/4$
A_{22}^2	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
B_{13}^1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/4$
$\tilde{A}_1$	$[0,1,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
$\tilde{A}_2$	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/x$	$1/4$
A_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	x	$1/4$
F_1	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

$$\chi_{B_{12}^2}=\sum_{a=1}^{2k_1}\sum_{b=1}^{2k_2} z_a p_b^{-1},\qquad \chi_{A_{22}^2}=\sum_{a=1}^{2k_2}\sum_{b=1}^{2k_3} p_a w_b^{-1},\qquad \chi_{B_{13}^1}=\sum_{a=1}^{2k_1}\sum_{b=1}^{2k_3} z_a w_b,\qquad \chi_{F_1}=\sum_{1\le a\le b\le 2k_1} z_a z_b,$$

$$\chi_{q_1}=\sum_{a=1}^{2k_1} z_a\times\begin{cases}\sum_{i=1}^{N_1/2}\left(y_i+\frac{1}{y_i}\right) & N_1\text{ even},\\ 1+\sum_{i=1}^{(N_1-1)/2}\left(y_i+\frac{1}{y_i}\right) & N_1\text{ odd},\end{cases}\qquad \chi_{q_2}=\sum_{b=1}^{2k_3}\sum_{j=1}^{N_2} w_b d_j^{-1},\qquad \chi_{q_3}=\sum_{a=1}^{2k_2}\sum_{j=1}^{N_2} p_a^{-1} d_j,$$

$$\chi_{\tilde{A}_1}=\sum_{1\le a<b\le 2k_1} z_a^{-1} z_b^{-1},\qquad \chi_{\tilde{A}_2}=\sum_{1\le a<b\le 2k_2} p_a^{-1} p_b^{-1} x^{-1},\qquad \chi_{A_3}=\sum_{1\le a<b\le 2k_3} w_a w_b x,\qquad \chi_{F_2}=\sum_{a=1}^{2k_2}\sum_{b=1}^{2k_3} p_a^{-1} w_b.$$

By explicitly evaluating the Hilbert series with gauge group $G=U(2k_1)\times U(2k_2)\times U(2k_3)$ and flavor group $SO(N_1)\times U(N_2)$ for the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_3$, we find it to be equal to the Hilbert series for $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with gauge group $G=Sp(K_1)\times U(2K_2)$ and flavor group $SO(N_1)\times U(N_2)$ (see [31] for more details) with the identification

$$K_1=k_1,\qquad K_2=\min(k_2,k_3).\qquad (82)$$

Supporting our claim, we show a few explicit examples. $SO(5)$ instanton: $\mathbf{k}=(1,1,1)$ and $\mathbf{N}=(3,1)$. Using Eq. (81) and unrefining, we find that

$$H[\mathbf{k}=(1,1,1),SO(3)\times U(1),\mathbb{C}P^2/\mathbb{Z}_3](t,1,1,1)$$
$$=\frac{1}{(1-t^3)^6(1+t^3)^4(1+t^6)^2(1+t^3+t^6)^3}(1+t^3+4t^6$$
$$+9t^9+18t^{12}+25t^{15}+33t^{18}$$
$$+30t^{21}+33t^{24}+\text{palindrome}+\cdots+t^{42}),$$

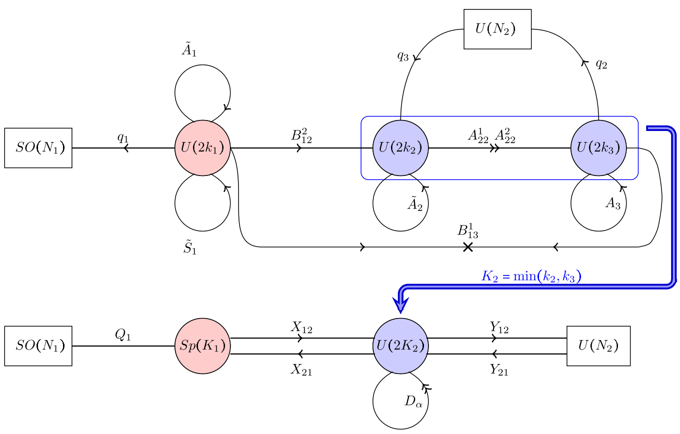

FIG. 25. Relation between the quiver diagram for $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_3$ and the quiver diagram for $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$, being D_α two fields transforming in the antisymmetric representation of $U(2K_2)$ gauge group (see [31] for more details).

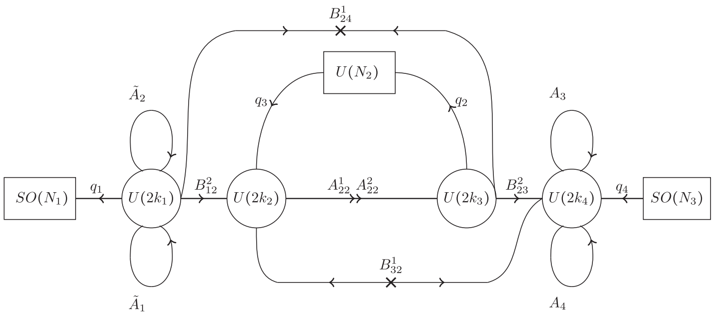

FIG. 26. Quiver diagram for VS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

which is the Hilbert series for the $SO(5)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (3,1)$ and $\mathbf{K} = (1,1)$. $SO(5)$ instanton: $\mathbf{k} = (1,1,1)$ and $\mathbf{N} = (1,2)$. Using Eq. (81) and unrefining, we find that

$$H[\mathbf{k} = (t,1,1,1), SO(2)\times U(1), \mathbb{C}P^2/\mathbb{Z}_3](t,1,1,1) = \frac{1-2t^3+5t^6-2t^9+6t^{12}-2t^{15}+5t^{18}-2t^{21}+t^{24}}{(1-t^3)^6(1+t^6)(1+2t^3+2t^6+t^9)^2},$$

which is the Hilbert series for the $SO(5)$ instantons on $\mathbb{C}^2/\mathbb{Z}_3$ with $\mathbf{N} = (1,2)$ and $\mathbf{K} = (1,1)$. We can, as well, graphically summarize the relation between the orthogonal instanton on $\mathbb{C}P^2/\mathbb{Z}_3$ and its cousin on $\mathbb{C}^2/\mathbb{Z}_3$ as in Fig. 25.

4. $SO(N)$ *instantons on* $\mathbb{C}P^2/\mathbb{Z}_4$*: VS*

Starting from the theory for unitary instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ and applying the rules in [42,43], we obtain the theory for $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ in the VS case. The corresponding quiver diagram is reported in Fig. 26, while we summarize the transformations of the fields under the different groups in Table XII.

The branch of the moduli space that can be identified with $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ is the one on which $A^1_{22} = 0$, $\tilde{A}_1 = 0$, and $A_3 = 0$. The Hilbert series of the instanton branch corresponding to the VS theory with flavor symmetry $SO(N_1)\times U(N_2)\times SO(N_3)$ and gauge ranks $\mathbf{k} = (k_1,k_2,k_3,k_4)$ is

$$\begin{aligned} H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_4](t,x,\mathbf{y},\mathbf{d},\mathbf{u}) &= \int d\mu_{U(2k_1)}(\mathbf{z}) \int d\mu_{U(2k_2)}(\mathbf{p}) \int d\mu_{U(2k_3)}(\mathbf{w}) \\ &\times \int d\mu_{U(2k_4)}(\mathbf{v}) \times \mathrm{PE}[\chi_{q_1}t^2 + \chi_{q_2}t^2 + \chi_{q_3}t^2 \\ &+ \chi_{q_4}t^2 + \chi_{B^2_{12}}t + \chi_{A^2_{22}}t^2 + \chi_{B^2_{23}}t + \chi_{B^1_{24}}t + \chi_{B^1_{32}}t \\ &+ \chi_{\tilde{A}_2}t^2 + \chi_{A_4}t^2 - \chi_{F_1}t^4 - \chi_{F_2}t^4 - \chi_{F_3}t^4], \end{aligned} \tag{83}$$

where $\mathbf{z}$, $\mathbf{p}$, $\mathbf{w}$, and $\mathbf{v}$ are the fugacities of the $U(2k_1)$, $U(2k_2)$, $U(2k_3)$, and $U(2k_4)$ gauge groups, respectively, while $\mathbf{y}$ and $\mathbf{d}$ denote the fugacities of the $SO(N_1)$ flavor group of the $U(N_2)$ flavor group and of the $SO(N_3)$ flavor group, respectively. The contributions of the various fields are

TABLE XII. Transformation of the fields for VS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

Fields	$U(2k_1)$	$U(2k_2)$	$U(2k_3)$	$U(2k_4)$	$SO(N_1)$	$U(N_2)$	$SO(N_3)$	$U(1)$
B^2_{12}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
A^2_{22}	$[\mathbf{0}]$	$[1,0,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^2_{23}	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
$\tilde{A}_2$	$[0,1,0\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
A_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,1,0\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^1_{24}	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
B^1_{32}	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
q_1	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots.,0]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
q_3	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	1/2
q_2	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	1/2
q_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]$	1/2
F_1	$[2,0\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[2,0,\ldots,,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

$$\chi_{B^2_{12}} = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_2} z_a p_b^{-1}, \qquad \chi_{A^2_{22}} = \sum_{a=1}^{2k_2}\sum_{b=1}^{2k_3} p_a w_b^{-1}, \qquad \chi_{B^2_{23}} = \sum_{a=1}^{2k_3}\sum_{b=1}^{2k_4} w_a v_b^{-1}, \qquad \chi_{\tilde{A}_2} = \sum_{1\le a<b\le 2k_1} z_a^{-1} z_b^{-1},$$

$$\chi_{q_1} = \sum_{a=1}^{2k_1} z_a \times \begin{cases} \sum_{i=1}^{N_1/2}(y_i + \frac{1}{y_i}) & N_1 \text{ even}, \\ 1 + \sum_{i=1}^{(N_1-1)/2}(y_i+\frac{1}{y_i}) & N_1 \text{ odd}, \end{cases} \qquad \chi_{q_3} = \sum_{j=1}^{N_2}\sum_{b=1}^{2k_2} d_j p_b^{-1}, \qquad \chi_{B^1_{32}} = \sum_{a=1}^{2k_2}\sum_{b=1}^{2k_4} p_a^{-1} v_b^{-1},$$

$$\chi_{B^1_{24}} = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_3} z_a w_b, \qquad \chi_{q_2} = \sum_{a=1}^{2k_3}\sum_{i=1}^{N_2} w_a d_i^{-1}, \qquad \chi_{q_4} = \sum_{a=1}^{2k_4} v_a^{-1} \times \begin{cases} \sum_{i=1}^{N_3/2}(y_i + \frac{1}{y_i}) & N_3 \text{ even}, \\ 1 + \sum_{i=1}^{(N_3-1)/2}(y_i+\frac{1}{y_i}) & N_3 \text{ odd}, \end{cases}$$

$$\chi_{A_4} = \sum_{1\le a<b\le 2k_4} v_a v_b, \qquad \chi_{F_1} = \sum_{1\le a\le b\le 2k_1} z_a z_b, \qquad \chi_{F_2} = \sum_{a=1}^{2k_2}\sum_{b=1}^{2k_3} p_a^{-1} w_b, \qquad \chi_{F_3} = \sum_{1\le a\le b\le 2k_4} v_a^{-1} v_b^{-1}.$$

By computing the Hilbert series with gauge group $G = U(2k_1)\times U(2k_2)\times U(2k_3)\times U(2k_4)$ and flavor group $SO(N_1)\times U(N_2)\times SO(N_3)$, we find that it turns out to be equal to the Hilbert series for $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with gauge group $G = Sp(K_1)\times U(2K_2)\times Sp(K_3)$ and flavor group $SO(N_1)\times U(N_2)\times SO(N_3)$ (see [31] for more details) with the identification

$$K_1 = k_1, \qquad K_2 = \min(k_2,k_3), \qquad K_3 = k_4. \tag{84}$$

Let us now show a few explicit examples. $SO(6)$ instanton: $\mathbf{k} = (1,1,1,1)$ and $\mathbf{N} = (2,0,4)$. Using Eq. (83) and unrefining, we find that

$$H[\mathbf{k}=(1,1,1,1), SO(2)\times SO(4), \mathbb{C}P^2/\mathbb{Z}_4](t,1,1) = \frac{1+4t^6+22t^{12}+36t^{18}+54t^{24}+36t^{30}+22t^{36}+4t^{42}+t^{48}}{(1-t^3)^8(1+t^3)^8(1+t^6)^4},$$

which is the Hilbert series for the $SO(6)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N} = (2,0,4)$ and $\mathbf{K} = (1,1,1)$. $SO(6)$ instanton: $\mathbf{k} = (1,1,1,1)$ and $\mathbf{N} = (2,1,2)$. Using Eq. (83) and unrefining, we find that

$$\begin{aligned} &H[\mathbf{k}=(1,1,1,1), SO(2)\times U(1)\times SO(2), \mathbb{C}P^2/\mathbb{Z}_4](t,1,1,1) \\ &= \frac{1}{(1-t^3)^8(1+t^3)^4(1+t^6)^2(1+t^3+t^6)^{12}(1+t^3+t^6+t^9+t^{12})}(1+t^3+3t^6+7t^9+18t^{12}+33t^{15} \\ &\quad + 51t^{18}+69t^{21}+93t^{24}+110t^{27}+120t^{30}+110t^{33}+\text{palindrome}+\cdots+t^{60}), \end{aligned}$$

which is the Hilbert series for $SO(6)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N} = (2,1,2)$ and $\mathbf{K} = (1,1,1)$. Finally, we summarize in Fig. 27 the relation between the theory describing VS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ and its cousin on $\mathbb{C}^2/\mathbb{Z}_4$.

5. $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$: NVS

Let us now consider the second possibility leading to the NVS case. The quiver diagram of the corresponding theory is reported in Fig. 28, while the transformations of the fields and of the F terms are summarized in Table XIII.

The branch of the moduli space that can be identified with $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ is the one on which $A^1_{11} = 0$ and $A^1_{33} = 0$. The Hilbert series of the instanton branch corresponding to the NVS theory with flavor symmetry $U(N_1)\times U(N_2)$ and gauge ranks $\mathbf{k} = (k_1,k_2,k_3,k_4)$ is

$$\begin{aligned} &H[\mathbf{k}, F, \mathbb{C}P^2/\mathbb{Z}_4](t,x,\mathbf{y},\mathbf{d}) \\ &= \int d\mu_{U(2k_1)}(\mathbf{z}) \int d\mu_{U(2k_2)}(\mathbf{p}) \int d\mu_{U(2k_3)}(\mathbf{w}) \\ &\quad \times \int d\mu_{U(2k_4)}(\mathbf{v}) \times \mathrm{PE}[\chi_{q_1}t^2 + \chi_{q_2}t^2 + \chi_{q_3}t^2 \\ &\quad + \chi_{q_4}t^2 + \chi_{B^2_{23}}t + \chi_{A^2_{11}}t^2 + \chi_{A^2_{33}}t^2 + \chi_{B^1_{32}}t + \chi_{\tilde{A}_1}t \\ &\quad + \chi_{A_2}t + +\chi_{\tilde{A}_3}t + \chi_{A_4}t - \chi_{F_1}t^4 - \chi_{F_2}t^4], \end{aligned} \tag{85}$$

where $\mathbf{z}$, $\mathbf{p}$, $\mathbf{w}$, and $\mathbf{v}$ are the fugacities of the $U(2k_1)$, $U(2k_2)$, $U(2k_3)$, and $U(2k_4)$ gauge groups, respectively, while $\mathbf{y}$ and $\mathbf{d}$ denote the fugacities of the $U(N_1)$ flavor group and the $U(N_2)$ flavor group, respectively. The contributions of the various fields are given by

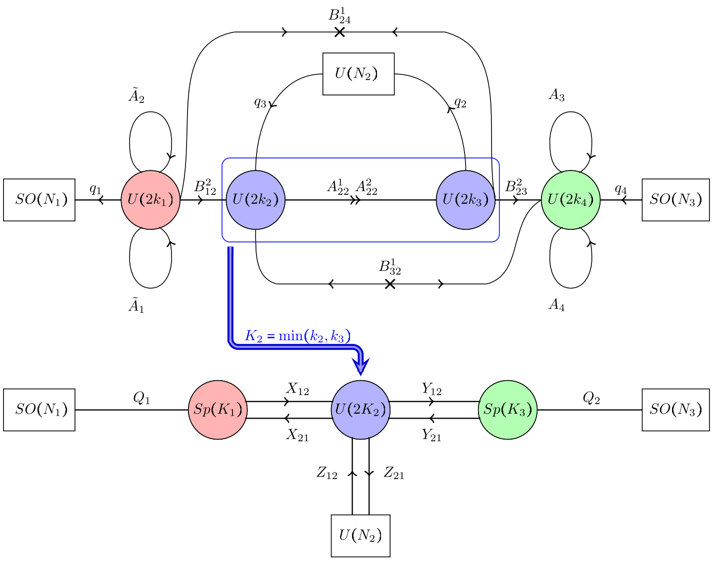

FIG. 27. Relation between the $\mathbb{C}P^2/\mathbb{Z}_4$ quiver gauge theory in the VS case and its relation with the corresponding $\mathbb{C}^2/\mathbb{Z}_4$ quiver gauge theory.

$$\chi_{\tilde{A}_3} = \sum_{1\le a<b\le 2k_3} w_a^{-1} w_b^{-1}, \qquad \chi_{A_4} = \sum_{1\le a<b\le 2k_4} v_a v_b, \qquad \chi_{q_1} = \sum_{a=1}^{2k_2}\sum_{i=1}^{N_1} p_a y_i^{-1},$$

$$\chi_{B^1_{32}} = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_4} v_b z_a^{-1}, \qquad \chi_{\tilde{A}_1} = \sum_{1\le a<b\le 2k_1} z_a^{-1} z_b^{-1}, \qquad \chi_{A_2} = \sum_{1\le a<b\le 2k_2} p_a p_b,$$

$$\chi_{q_2} = \sum_{a=1}^{2k_1}\sum_{i=1}^{N_1} z_a^{-1} y_i, \qquad \chi_{q_3} = \sum_{a=1}^{2k_4}\sum_{j=1}^{N_2} v_a d_j^{-1}, \qquad \chi_{q_4} = \sum_{a=1}^{2k_3}\sum_{j=1}^{N_2} w_a^{-1} d_j, \qquad \chi_{F_1} = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_2} p_b z_a^{-1},$$

$$\chi_{A^2_{11}} = \sum_{a=1}^{2k_1}\sum_{b=1}^{2k_2} z_a p_b^{-1}, \qquad \chi_{B^2_{23}} = \sum_{a=1}^{2k_2}\sum_{b=1}^{2k_3} p_a w_b^{-1}, \qquad \chi_{A^2_{33}} = \sum_{a=1}^{2k_3}\sum_{b=1}^{2k_4} w_a v_b^{-1}, \qquad \chi_{F_2} = \sum_{a=1}^{2k_3}\sum_{b=1}^{2k_4} w_a^{-1} v_b.$$

Performing the computation of the Hilbert series with gauge group $G = U(2k_1) \times U(2k_2) \times U(2k_3) \times U(2k_4)$ and flavor group $U(N_1) \times U(N_2)$, we find that it coincides with the Hilbert series for $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with gauge group $G = U(2K_1) \times U(2K_2)$ and flavor group $U(N_1) \times U(N_2)$ (see [31] for more details) with the identification

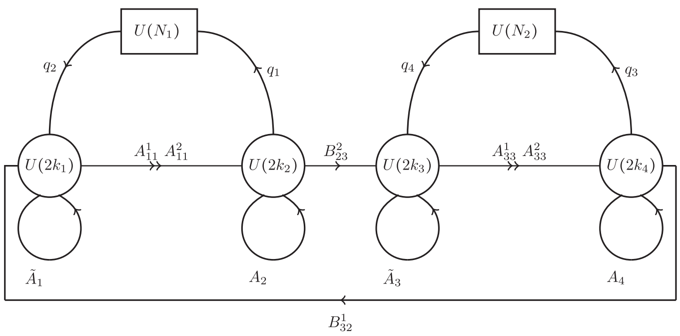

FIG. 28. Quiver diagram for NVS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

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TABLE XIII. Transformation of the fields for NVS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_4$.

Fields	$U(2k_1)$	$U(2k_2)$	$U(2k_3)$	$U(2k_4)$	$U(N_1)$	$U(N_2)$	$U(1)$
A^2_{11}	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^2_{23}	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
A^2_{33}	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/2
B^1_{32}	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
$\tilde{A}_1$	$[0,1,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
A_2	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
$\tilde{A}_3$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
A_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1/4
q_1	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	1/2
q_2	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	1/2
q_3	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	1/2
q_4	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	1/2
F_1	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	1
F_2	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	1

$$K_1 = \min(k_1, k_2), \qquad K_2 = \min(k_3, k_4). \tag{86}$$

Let us show an explicit example of our claim. $SO(6)$ instanton: $\mathbf{k} = (1,1,1,1)$ and $\mathbf{N} = (2,1)$. Using Eq. (85) and unrefining, we obtain

$$\begin{aligned} H[\mathbf{k} = (1,1,1,1), U(2)\times U(1), \mathbb{C}P^2/\mathbb{Z}_4](t,1,1,1) = {} & \frac{1}{(1-t^3)^8(1+t^3)^6(1+t^6)^3(1+t^3+t^6)^3(1+t^3+t^6+t^9+t^{12})^2} \\ & \times (1 + 3t^3 + 9t^6 + 22t^9 + 54t^{12} + 114t^{15} + 219t^{18} + 371t^{21} \\ & + 582t^{24} + 827t^{27} + 1092t^{30} + 1323t^{33} + 1493t^{36} \\ & + 1548t^{39} + 1493t^{42} + \text{palindrome} + t^{72}), \end{aligned}$$

which is the Hilbert series for $SO(6)$ instantons on $\mathbb{C}^2/\mathbb{Z}_4$ with $\mathbf{N} = (2,1)$ and $\mathbf{K} = (1,1)$. Finally, we graphically summarize the relation between the theory describing the NVS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_4$ and its cousin on $\mathbb{C}^2/\mathbb{Z}_4$ in Fig. 29.

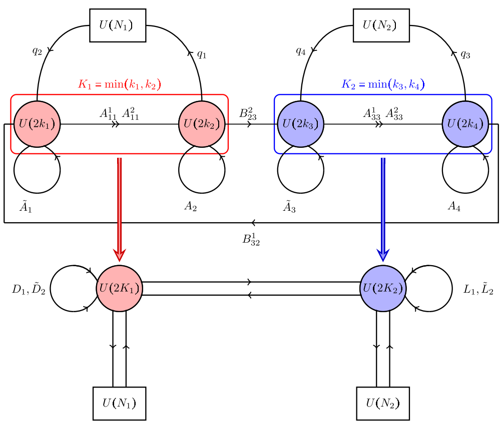

FIG. 29. Relation between the $\mathbb{C}P^2/\mathbb{Z}_4$ quiver gauge theory in the NVS case and the corresponding $\mathbb{C}^2/\mathbb{Z}_4$ quiver gauge theory, where D_1, $\tilde{D}_2$ are two fields in the antisymmetric representation of the gauge group $U(2K_1)$, while L_1, $\tilde{L}_2$ are two fields in the antisymmetric representation of the gauge group $U(2K_2)$.

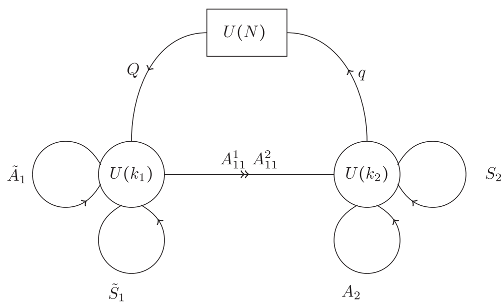

FIG. 30. Quiver diagram for instantons of the hybrid configuration on $\mathbb{C}P^2/\mathbb{Z}_2$.

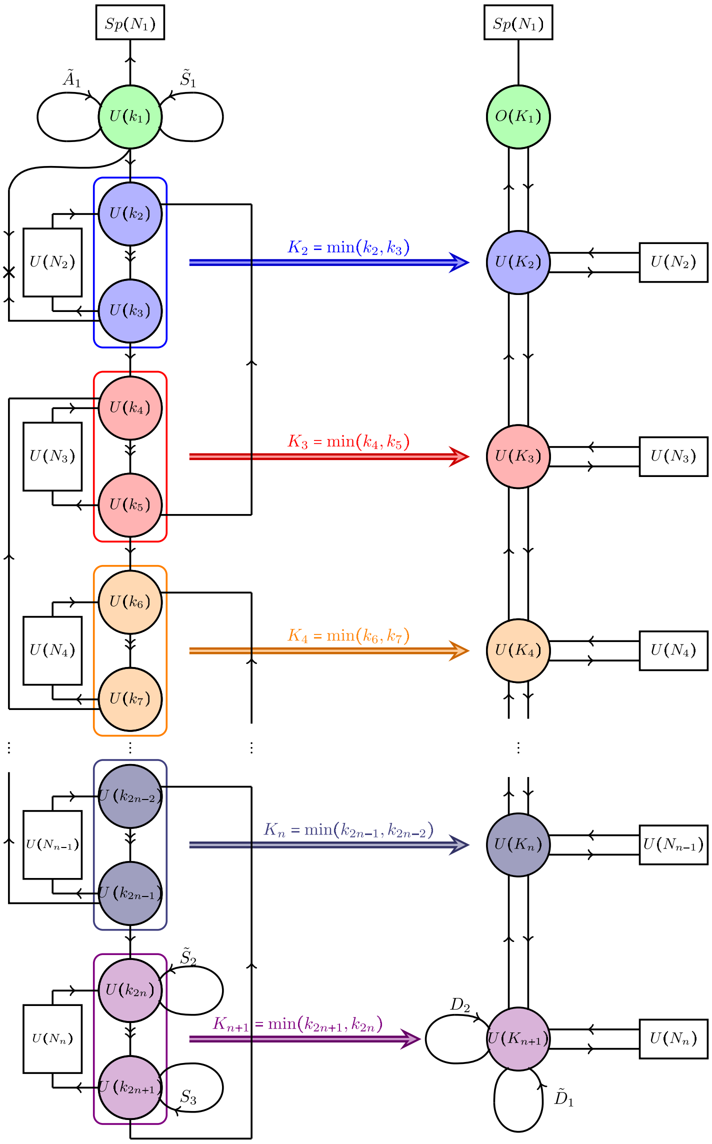

FIG. 31. Relation between the quiver diagram for $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$ (on the left) and the quiver diagram for $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_{2n+1}$ (on the right), where $\tilde{D}_1$ and D_2 are two fields in the symmetric representation of the gauge group $U(K_{n+1})$.

6. $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ with $n > 4$

Let us now consider the generic case of instantons on $\mathbb{Z}_n$ orbifolds of $\mathbb{C}P^2$ with $n > 4$. Based on the previous examples above, we can extract the generic pattern of both the quiver as well as the relation between the orthogonal instanton on $\mathbb{C}P^2/\mathbb{Z}_n$ with its relative on $\mathbb{C}^2/\mathbb{Z}_n$.

Recall that N is the sum of the ranks of the flavor groups in the ADHM quiver, while the ranks of the gauge groups

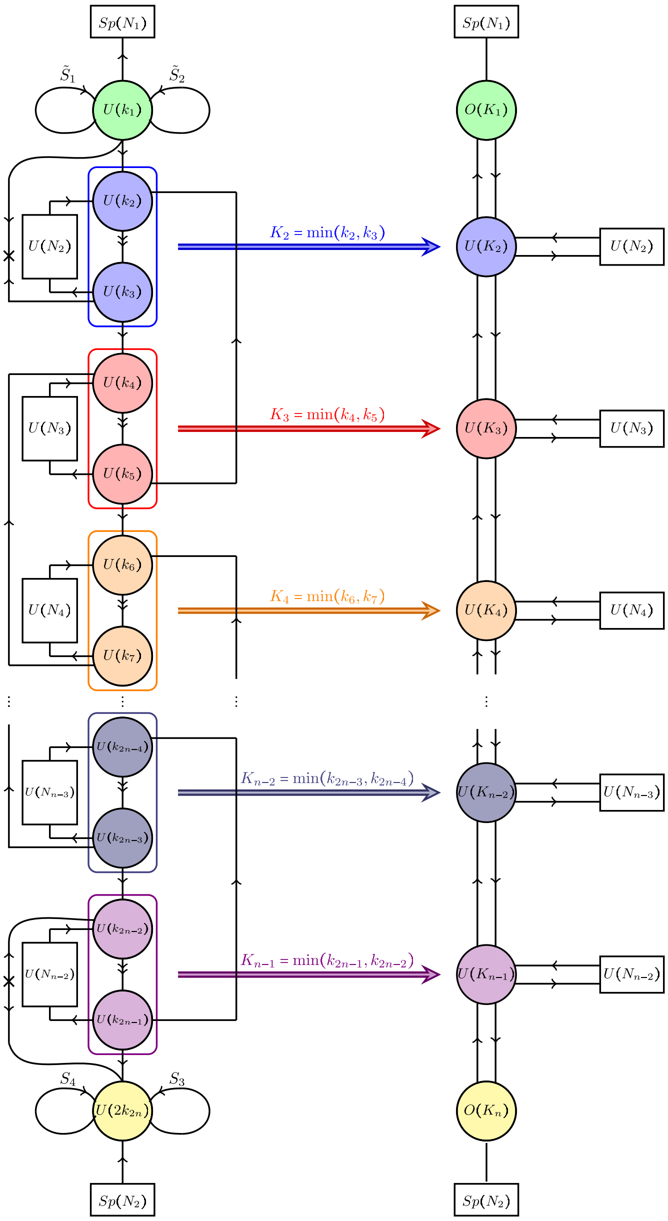

FIG. 32. Relation between the quiver diagram for VS $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ (on the left) and the quiver diagram for VS $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_{2n}$ (on the right).

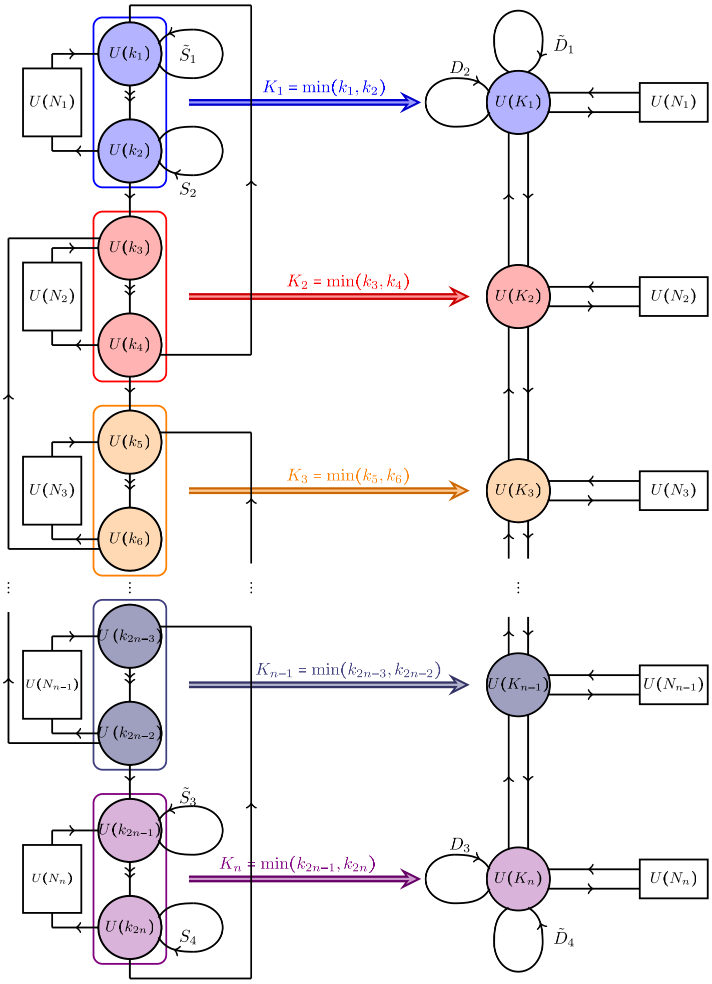

FIG. 33. Relation between the quiver diagram for NVS $Sp(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ (on the left) and the quiver diagram for NVS $Sp(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_{2n}$ (on the right), where $\tilde{D}_1$ and D_2 are two fields in the symmetric representation of the gauge group $U(K_1)$, while D_3 and $\tilde{D}_4$ are two fields in the symmetric representation of the gauge group $U(K_n)$.

are related to instanton number and, together with the relative flavor ranks, to other possible quantum numbers labeling the instanton. Unfortunately, also in this case, the precise identification between quiver data and instanton data is not known. $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$. Elaborating on the previous examples, we conjecture that the theory describing orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$ is related to its counterpart on $\mathbb{C}^2/\mathbb{Z}_{2n+1}$ as in Fig. 34. Moreover, the gauge ranks are related by

$$K_1 = k_1, K_2 = \min(k_2, k_3),$$
$$K_3 = \min(k_4, k_5), \ldots K_{n+1} = \min(k_{2n}, k_{2n+1}). \qquad (87)$$

$SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$: VS. In this case, based on the lowest n examples, the relation between the theory describing VS instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ and their VS counterparts on $\mathbb{C}^2/\mathbb{Z}_{2n}$ is summarized in Fig. 35. In addition, we find the gauge rank identification

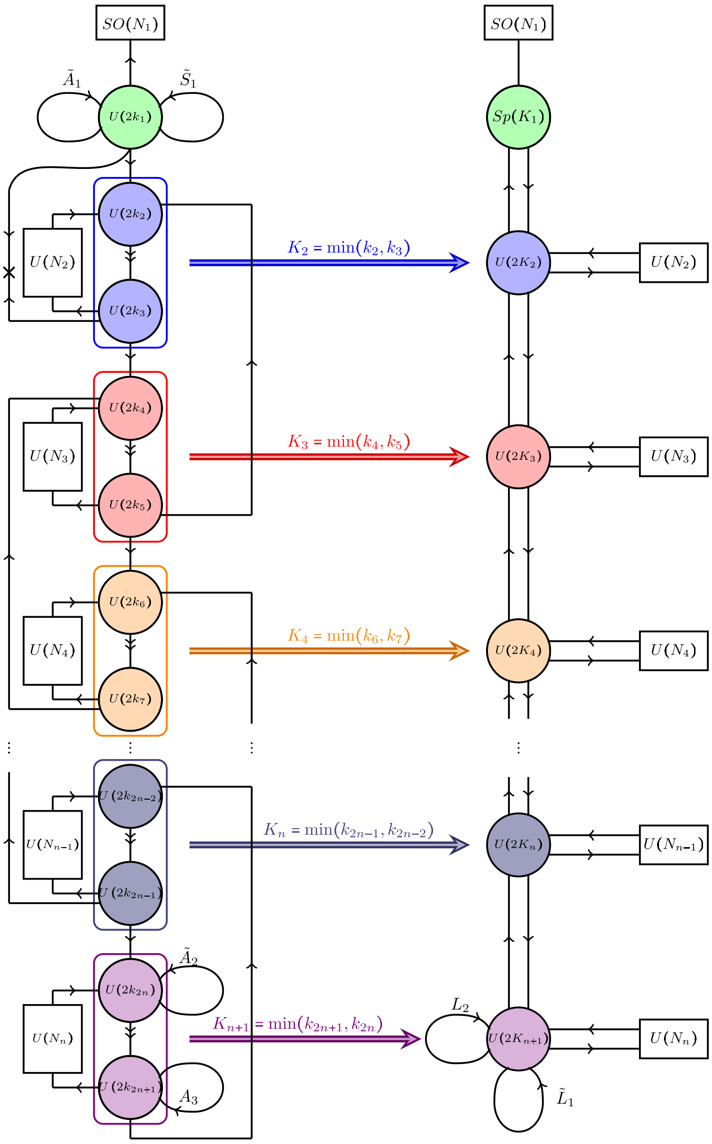

FIG. 34. Relation between the quiver diagram for $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$ (on the left) and the quiver diagram for $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_{2n+1}$ (on the right), where $\tilde{L}_1$ and L_2 are two fields in the antisymmetric representation of the gauge group $U(2K_{n+1})$.

$$K_1 = k_1, K_2 = \min(k_2, k_3), \ldots K_{n-1} = \min(k_{2n-2}, k_{2n-1}), \quad K_n = k_{2n}. \tag{88}$$

$SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$: NVS. Elaborating on the previous examples, we conjecture that the theory describing NVS orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n+1}$ is related to its NVS counterpart on $\mathbb{C}^2/\mathbb{Z}_{2n+1}$ as in Fig. 36. In addition, the gauge rank assignation is

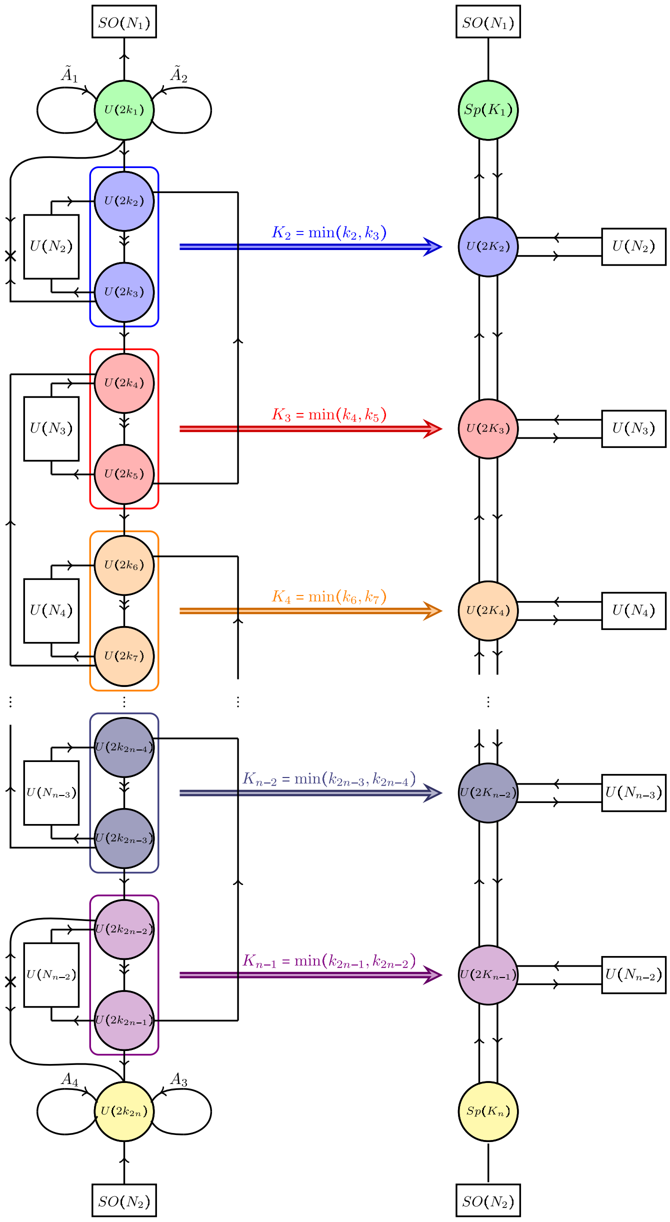

FIG. 35. Relation between the quiver diagram for VS $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ (on the left) and the quiver diagram for VS $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_{2n}$ (on the right).

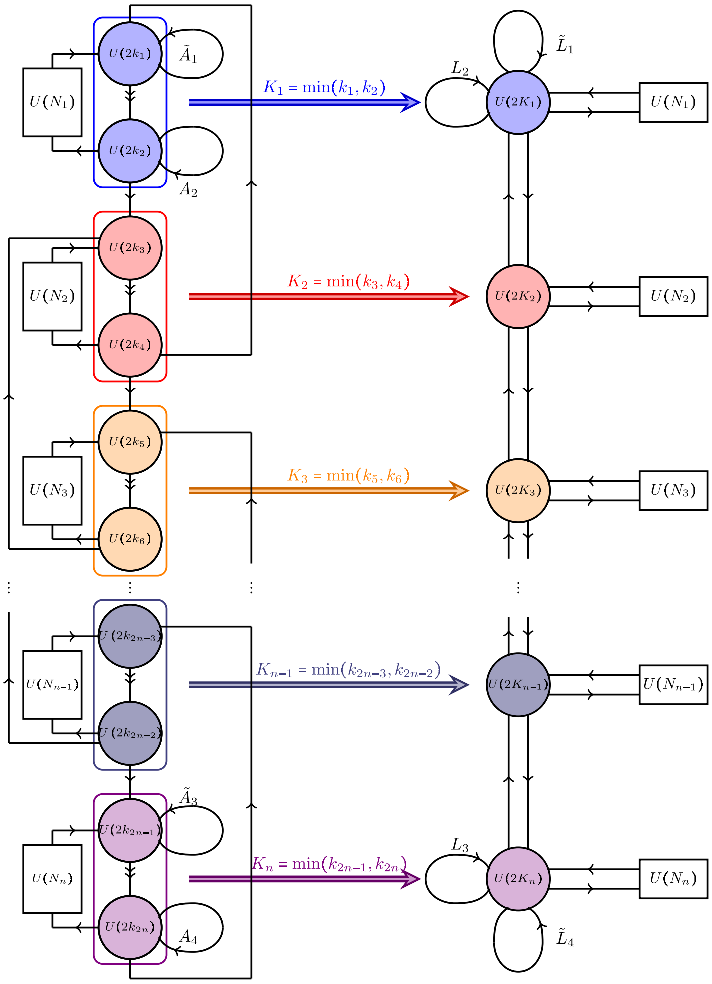

FIG. 36. Relation between the quiver diagram for NVS $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_{2n}$ (on the left) and the quiver diagram for NVS $SO(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_{2n}$ (on the right), where $\tilde{L}_1$ and L_2 are two fields in the antisymmetric representation of the gauge group $U(2K_1)$, while L_3 and $\tilde{L}_4$ are two fields in the antisymmetric representation of the gauge group $U(2K_n)$.

$$K_1 = \min(k_1, k_2), K_2 = \min(k_3, k_4), \ldots$$
$$K_n = \min(k_{2n-1}, k_{2n}). \tag{89}$$

Note that, as in the symplectic case, the merging nodes are those going over to unitary nodes in the parent $\mathbb{C}^2/\mathbb{Z}_n$ theory. It would be very interesting to understand this feature deeper, as well as the topological data classifying orthogonal instantons.

VII. CONCLUSIONS

In this paper, we analyzed and clarified several aspects of the moduli space of instantons on $\mathbb{C}P^2$. First, we explicitly spelled in which context the instanton configurations arising from the ADHM-like construction on $\mathbb{C}P^2$ are relevant. Then, by using master space techniques, we explored from a physical perspective the topological properties of the instanton moduli space to which the Hilbert series alone is blind. In the particular case of unitary instantons, an AdS/CFT approach is feasible, finding perfect agreement between gauge theory and gravity computations. Moreover, this can be regarded as a nontrivial check of the alluded AdS/CFT pair, as it is sensible, in particular, to nonprotected scaling dimensions of operators in $\mathcal{N} = 2$ theories. We also provided the construction of instantons on orbifolds of $\mathbb{C}P^2$. While their topological classification is not fully understood, by using our master space approach, we are able to provide conjectures on the identification of quantum numbers and quiver data.

Since $\mathbb{C}P^2$ is a Kähler manifold, its Kähler form naturally induces an orientation which, in particular, intrinsically distinguishes ASD and SD 2-forms. This is very relevant for the construction of gauge bundles whose curvature has definite duality properties, as such construction will be different depending on whether we are interested in the SD or ASD case. In this paper, we were interested in SD connections, whose physical relevance in a suitably constructed gauge theory we have shown. In turn, these are the ones which admit an ADHM-like construction recently embedded into a $3d$ $\mathcal{N} = 2$ gauge theory arising from a brane construction in [15].

Since $\mathbb{C}P^2$ is a topologically nontrivial manifold, the gauge bundles of interest are classified by more than simply the instanton number. Indeed, they admit a nonzero first Chern class. As a consequence, the moduli space of instantons on $\mathbb{C}P^2$ typically has compact submanifolds associated to these extra directions. In turn, the Hilbert series of the moduli space—that is, the generating function of holomorphic functions on the instanton moduli space or, equivalently, the generating function of gauge-invariant operators in the ADHM description of the instanton moduli space—which coincides with the Nekrasov instanton partition function, and it is, therefore, a very interesting quantity, is not sensible to these compact directions. Hence, in retrospect, it is natural to expect that it would coincide with the Hilbert series for a parent instanton on $\mathbb{C}^2$, as it was explicitly shown in [15]. In this paper, we provided evidence of this picture by probing the compact directions upon using a novel approach. Focusing on the simplest case admitting such directions, and following [23], we considered the master space of the gauge theory describing these instantons. This amounts to ungauging a $U(1)$, which allows us to construct extra gauge invariants otherwise not present. These precisely reproduce a moduli space, which is a complex cone over the noncompact directions. By using this strategy, we were able to understand the extra directions in the unitary and orthogonal cases. In turn, the case of symplectic instantons does not admit a similar construction, consistent with the observation in [15] that it does not involve quantum numbers other than the instanton number. Note, however, that we explicitly checked this picture for the lowest instanton numbers. It would be worth exploring this new approach further to all instanton numbers, including studying the geometry of the moduli space with extra directions, which is not simply a direct product of the noncompact times the compact directions (this can be easily checked already in the simplest cases by studying the relations among operators in the moduli space).

The case of unitary instantons is particularly interesting, as its AHDM construction is in terms of the gauge theory dual to M2 branes probing a certain CY_4 cone [28]. Hence, it is natural to guess that, at least partially, the instanton moduli space can be read from the AdS/CFT duality. Typically, fundamental degrees of freedom—that is, open stringlike—are not captured by the geometry alone in AdS/CFT. Hence, it is natural to expect that the backgrounds in [28] can capture only the part of the instanton moduli space which does not involve fundamental fields. We explicitly checked this proposal, finding complete agreement between field theory results and gravity computations. Turning things around, we can think of our results as a nontrivial check of the proposed $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality in [28], where we explicitly match charges in field theory with geometrical data in AdS.

The ambient manifold where our instantons live is $\mathbb{C}P^2$, which is, in particular, a toric manifold. Being acted by a $\mathbb{T}^2$, it is natural to consider quotienting by a discrete subgroup—that is, orbifolding. In turn, by means of the standard methods, we can orbifold the $\mathbb{C}P^2$ ADHM construction as a field theory to find the ADHM construction of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$. This way, we constructed the ADHM construction for unitary, symplectic, and orthogonal instantons on $\mathbb{C}P^2/\mathbb{Z}_n$. Note that the orbifolded space has a nontrivial topology containing 2-cycles of a somewhat different origin. On one hand, we originally had a 2-cycle in the $\mathbb{C}P^2$ which gets mirrored by the orbifold. On the other hand, the orbifold introduces extra (vanishing) 2-cycles at the orbifold fixed point. It is natural to expect that the cycles originating from the original one in $\mathbb{C}P^2$ are invisible to the Hilbert series—just as the original one was—while the others introduced by the orbifold are, indeed, visible. In fact, it is natural to guess that the Hilbert series for instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ coincides with the Hilbert series of a parent instanton on $\mathbb{C}^2/\mathbb{Z}_n$ just as in the unorbifolded case. Note that, consistently, the Hilbert series of instantons on $\mathbb{C}^2/\mathbb{Z}_n$ is, indeed, sensible to the 2-cycles associated to the orbifold fixed point [31].[10] In this paper, we, indeed, confirmed this picture, in particular, by explicitly showing the matching of the $\mathbb{C}P^2/\mathbb{Z}_n$ Hilbert series with that of a parent $\mathbb{C}^2/\mathbb{Z}_n$ one. As shown in the text, the process suggests a certain "folding" of the $\mathbb{C}P^2/\mathbb{Z}_n$ quiver by "node merging" into that of $\mathbb{C}^2/\mathbb{Z}_n$. In fact, since at least for unitary instantons on $\mathbb{C}^2/\mathbb{Z}_n$, the matching

[10] Strictly speaking, this applies to unitary instantons. The case of orthogonal and symplectic instantons is more involved, as the ADHM construction does not allow for enough FI parameters so as to blow up all cycles (see [44] for related discussions).

between quiver data and instanton data is known, this naturally suggests, at least partially, an identification of the quiver data with the instanton data in the $\mathbb{C}P^2/\mathbb{Z}_n$ case. Unfortunately, the full identification with the ADHM quiver data of the relevant quantum numbers specifying instantons on the orbifolded $\mathbb{C}P^2$ space is not known. Nevertheless, we provided—at least for the case of unitary instantons—certain conjectures based on the mapping into $\mathbb{C}^2/\mathbb{Z}_n$ based, in particular, on our approach via the master space to all directions in the moduli space. As a check, the expected compact directions can be recovered upon appropriate ungaugings of $U(1)$'s. Of course, a more comprehensive study of these aspects would be very interesting.

ACKNOWLEDGMENTS

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APPENDIX A: HYBRID CONFIGURATION (AN EXAMPLE)

In this appendix, we study an example of a hybrid configuration, making the following choice for the charges of the orientifolds plane in Fig. 10 $(I, II, III, IV) = (+,-,+,-)$. The corresponding quiver is reported in Fig. 30, while the transformations of the fields are summarized in Table XIV.

The Hilbert series of the hybrid configuration is given by

$$\begin{aligned} &H[\mathbf{k}, F, \mathbb{C}P^2/Z_2](t, a, s, \mathbf{y}) \\ &\quad = \int d\mu_{U(k_1)}(\mathbf{z}) \int d\mu_{U(k_2)}(\mathbf{p}) \mathrm{PE}[\chi_{\tilde{S}_1} t + \chi_{S_2} t + \chi_{\tilde{A}_1} t \\ &\qquad + \chi_{A_2} t + \chi_{A_{11}^2} t^2 + \chi_Q t^2 + \chi_q t^2 - \chi_F t^4], \end{aligned} \tag{A1}$$

where $\mathbf{z}$ and $\mathbf{p}$ are the fugacities of the $U(k_1)$ and $U(k_2)$ gauge groups, respectively, $\mathbf{y}$ denotes the fugacity of the $U(N)$ flavor group, s denotes the fugacity of the global $U(1)_s$ symmetry acting $\tilde{S}_1$ and S_2, while a denotes the fugacity of the global $U(1)_a$ symmetry acting on $\tilde{A}_1$ and A_2. The contribution of each field is given by

$$\chi_{A_{11}^2} = \sum_{a=1}^{k_1} \sum_{b=1}^{k_2} z_a p_b^{-1}, \qquad \chi_Q = \sum_{i=1}^{N} \sum_{a=1}^{k_1} z_a^{-1} y_i,$$

$$\chi_q = \sum_{j=1}^{N} \sum_{b=1}^{k_2} p_b y_j^{-1}, \qquad \chi_F = \sum_{a=1}^{k_1} \sum_{b=1}^{k_2} z_a^{-1} p_b,$$

$$\chi_{S_2} = s \sum_{1\leq a\leq b\leq k_2} p_a p_b, \qquad \chi_{\tilde{S}_1} = \frac{1}{s} \sum_{1\leq a\leq b\leq k_1} z_a^{-1} z_b^{-1},$$

$$\chi_{A_2} = a \sum_{1\leq a<b\leq k_2} p_a p_b, \qquad \chi_{\tilde{A}_1} = \frac{1}{a} \sum_{1<a<b\leq k_1} z_a^{-1} z_b^{-1}.$$

In this case, by explicit computation of the Hilbert series for the hybrid configuration with gauge group $G = U(k_1) \times U(k_2)$ and flavor group $U(N)$, we find it to be equal to the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with gauge group $G = U(K_1)$ (see [31] for more details). The two theories share the same flavor group, and the gauge groups are related in the following way:

$$K_1 = \min(k_1, k_2). \tag{A2}$$

Let us explicitly show a few examples supporting our claim. $\mathbf{k} = (1, 1)$ and $N = 1$. Using Eq. (A1) and unrefining, we find that

$$H[\mathbf{k} = (1,1), U(1), \mathbb{C}P^2/\mathbb{Z}_2](t, 1, 1) = \frac{1 - t^{18}}{(1 - t^6)(1 - t^9)^2},$$

which is the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N = 1$ and $K_1 = 1$. $\mathbf{k} = (1, 1)$ and $N = 2$. Using Eq. (A1) and unrefining, we find that

TABLE XIV. Transformations of the fields for instantons of the hybrid configuration on $\mathbb{C}P^2/\mathbb{Z}_2$.

Fields	$U(k_1)$	$U(k_2)$	$U(N)$	$U(1)_s$	$U(1)_a$	$U(1)$
$\tilde{S}_1$	$[2,0,\ldots,0]_{-2}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/s$	$[0]$	$1/4$
S_2	$[\mathbf{0}]$	$[2,0,\ldots,0]_{+2}$	$[\mathbf{0}]$	s	$[\mathbf{0}]$	$1/4$
$\tilde{A}_1$	$[0,1,0\ldots,0]_{-1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/a$	$1/4$
A_2	$[\mathbf{0}]$	$[0,1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	a	$1/4$
A_{11}^2	$[1,0,\ldots,0]_{+1}$	$[0,0,\ldots,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
q	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
Q	$[0,\ldots,0,1]_{+1}$	$[\mathbf{0}]$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$1/2$
F	$[0,\ldots,0,1]_{+1}$	$[1,0,\ldots,0]_{+1}$	$[\mathbf{0}]$	$[\mathbf{0}]$	$[0]$	1

$$H[\mathbf{k}=(1,1),U(2),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1)=\frac{1+2t^6+4t^9+2t^{12}+t^{18}}{(1-t^3)^4(1+2t^3+2t^6+t^9)^2},$$

which is the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=2$ and $K_1=1$. $\mathbf{k}=(1,2)$ and $N=2$. Using Eq. (A1) and unrefining, we find that

$$H[\mathbf{k}=(1,2),U(2),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1)=\frac{1+2t^6+4t^9+2t^{12}+t^{18}}{(1-t^3)^4(1+2t^3+2t^6+t^9)^2},$$

which is again the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=2$ and $K_1=1$. $\mathbf{k}=(1,1)$ and $N=3$. Using Eq. (A1) and unrefining, we find that

$$H[\mathbf{k}=(1,1),U(3),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1)=\frac{1+t^3+6t^6+15t^9+21t^{12}+18t^{15}+21t^{18}+15t^{21}+6t^{24}+t^{27}+t^{30}}{(1-t^3)^6(1+t^3)^4(1+t^3+t^6)^3},$$

which is the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=3$ and $K_1=1$. $\mathbf{k}=(1,2)$ and $N=3$. Using Eq. (A1) and unrefining, we find that

$$H[\mathbf{k}=(1,1),U(3),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1)=\frac{1+t^3+6t^6+15t^9+21t^{12}+18t^{15}+21t^{18}+15t^{21}+6t^{24}+t^{27}+t^{30}}{(1-t^3)^6(1+t^3)^4(1+t^3+t^6)^3},$$

which is again the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=3$ and $K_1=1$. $\mathbf{k}=(1,1)$ and $N=4$. Using Eq. (A1) and unrefining, we find that

$$H[\mathbf{k}=(1,1),U(4),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1)=\frac{1+2t^3+13t^6+40t^9+86t^{12}+132t^{15}+194t^{18}+220t^{21}+194t^{24}+\text{palindrome}+t^{42}}{(1-t^3)^8(1+t^3)^6(1+t^3+t^6)^4},$$

which is the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=4$ and $K_1=1$. $\mathbf{k}=(2,2)$ and $N=1$. Using Eq. (A1) and unrefining, we find that

$$H[\mathbf{k}=(2,2),U(1),\mathbb{C}P^2/\mathbb{Z}_2](t,1,1,1)=\frac{1-t^3+2t^9-t^{15}+t^{18}}{(1-t^3)^4(1+t^3)^2(1+t^3+t^6+t^9+t^{12})},$$

which is the Hilbert series for the *SA* hybrid configuration on $\mathbb{C}^2/\mathbb{Z}_2$ with $N=1$ and $K_1=2$.

APPENDIX B: QUIVERS AND RELATIONS FOR $Sp(N)$ AND $SO(N)$ INSTANTONS ON $\mathbb{C}P^2/\mathbb{Z}_n$ WITH $n>4$

In this appendix, we collect the quiver diagrams for $Sp(N)$ and $SO(N)$ instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ (with $n>4$) showing their relations with the corresponding quiver diagrams of the corresponding $\mathbb{C}^2/\mathbb{Z}_n$ theory.

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6.3 Rigid supersymmetry from conformal supergravity in $5d$

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Rigid supersymmetry from conformal supergravity in five dimensions

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Abstract: We study the rigid limit of 5d conformal supergravity with minimal supersymmetry on Riemannian manifolds. The necessary and sufficient condition for the existence of a solution is the existence of a conformal Killing vector. Whenever a certain SU(2) curvature becomes abelian the backgrounds define a transversally holomorphic foliation. Subsequently we turn to the question under which circumstances these backgrounds admit a kinetic Yang-Mills term in the action of a vector multiplet. Here we find that the conformal Killing vector has to be Killing. We supplement the discussion with various appendices.

Keywords: Supersymmetric gauge theory, Field Theories in Higher Dimensions, Supergravity Models

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1 Introduction

Over the very recent past much effort has been devoted to the study of supersymmetric gauge theories on general spaces. Part of this interest has been triggered by the development of computational methods allowing to exactly compute certain (supersymmetric) observables, such as the supersymmetric partition function (starting with the seminal paper [1]), indices or Wilson loops. This program has been very successfully applied to the cases of 4d and 3d gauge theories, and it is only very recently that the 5d case has been considered (e.g. [2–7]). On the other hand, it has become clear that the dynamics of 5d gauge theories is in fact very interesting, as, contrary to the naive intuition, at least for the case of supersymmetric theories, they can be at fixed points exhibiting rather amusing behavior as pioneered in [8]. In particular, these theories often show enhanced global symmetries which can be both flavor-like or spacetime-like. The key observation is that vector multiplets in 5d come with an automatically conserved topological current $j \sim \star F \wedge F$ under which instanton particles are electrically charged. These particles provide extra states needed to enhance perturbative symmetries, both flavor or spacetime — such as what it is expected to happen in the maximally supersymmetric case, where the theory grows one extra dimension and becomes the (2, 0) 6d theory. In fact, very recently the underlying mechanism for these enhancements has been considered from various points of view [9–12].

Five-dimensional gauge theories have a dimensionful Yang-Mills coupling constant which is irrelevant in the IR. Hence they are non-renormalizable and thus *a priori* naively uninteresting. However, as raised above, at least for supersymmetric theories the situation is, on the contrary, very interesting as, by appropriately choosing gauge group and matter content, the $g_{\rm YM}$ coulpling which plays the role of UV cut-off can be removed in such a way that one is left with an isolated fixed point theory [8]. From this perspective, it is natural to start with the fixed point theory and think of the standard gauge theory as a deformation whereby one adds a $g_{\rm YM}^{-2}F^2$ term. In fact, the $g_{\rm YM}^{-2}$ can be thought as the VEV for a scalar in a background vector multiplet. Hence, for any gauge theory arising from a UV fixed point[1] we can imagine starting with the conformal theory including a background vector multiplet such that, upon giving a non-zero VEV to the background scalar, it flows to the desired 5d gauge theory. This approach singles out 5d conformally coupled multiplets as the interesting objects to construct.

As described above, on general grounds considering the theory on arbitrary manifolds is very useful, as for example, new techniques allow for exact computation of supersymmetric observables. The first step in this program is of course the construction of the supersymmetric theory on the given (generically curved) space, which is *per se* quite nontrivial. However, the approach put forward by [13] greatly simplifies the task. The key idea is to consider the combined system of the field theory of interest coupled to a suitable supergravity, which, by definition, preserves supersymmetry in curved space. Then, upon taking a suitable rigid limit freezing the gravity dynamics, we can think of the solutions to the gravity sector as providing the background for the dynamical field theory of interest. Note that, since the combined supergravity+field theory is considered off-shell, both sectors can be analyzed as independent blocks in the rigid limit, that is, one can first solve for the supergravity multiplet and then regard such solution as a frozen background for the field theory, where the supergravity background fields act as supersymmetric couplings. Of course, the supergravity theory to use must preserve the symmetries of the field theory which, at the end of the day, we are interested in. Hence, in the case of 5d theories, it is natural to consider conformal supergravity coupled to the conformal matter multiplets described above.

Following this approach, in this paper we will consider 5d conformal supergravity [14–16] coupled to 5d conformal matter consisting of both vector and hyper multiplets.

[1]Note that the theories outside of this class do require a (presumably stringy) UV completion. Hence the class of theories which we are considering is in fact the most generic class of 5d supersymmetric quantum field theories.

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As remarked above, the Yang-Mills coupling constant is dimensionful. Hence, the action for the vector multiplets is not the standard quadratic one with a Maxwell kinetic term but rather a cubic action which can be thought as the supersymmetric completion of 5d Chern-Simons. As anticipated, in the rigid limit we can separate the analysis of the gravity multiplet as providing the supersymmetric background for the field theory. One is thus prompted to study the most generic backgrounds where 5d gauge theories with $\mathcal{N} = 1$ supersymmetry can be constructed by analyzing generic solutions of the 5d $\mathcal{N} = 2$ conformal supergravity. Solutions to various 5d supergravities on (pseudo-) Riemannian manifolds have been studied in different approaches in [17–23]. For $\mathcal{N} = 1$ Poincaré supergravity, the necessary and sufficient condition for the existence of a global solution is the existence of a non-vanishing Killing vector. If one considers conformal supergravity this condition becomes the existence of a conformal Killing vector (CKV).[2] In this paper we analyze Euclidean solutions of 5d conformal supergravity in terms of component fields. Our analysis proceeds along the lines of [24]. Interestingly, by studying the conditions under which a VEV for the scalar in the background vector multiplets paying the role of $g_{\rm YM}^{-2}$ can be given in a supersymmetric way, we find that such vector must be in fact Killing. Hence in this case we simply recover the results obtained using Poincaré supergravity.

Our results rely on some reality conditions satisfied by the supersymmetry spinors. In the Lorentzian theory, the spinors generally satisfy a symplectic Majorana condition (A.3). If one imposes the same condition in the Euclidean case, there are immediate implication for the spinor bilinears (3.1) that play an important role in the analysis. Namely, the scalar bilinear s is real and vanishes if and only if the spinor vanishes, while the vector bilinear v — the aforementioned CKV — is real. One should note however that the symplectic Majorana condition is not equivalent to these conditions for s and v. Instead, (A.3) is slightly stronger, while our results only depend on the milder assumptions on the bilinears.

While the existence of the CKV is a necessary and sufficient condition many of the backgrounds exhibit a more interesting geometric structure — that of a transversally holomorphic foliation (THF). These appeared already in the context of rigid supersymmetry in three dimensions [25] and one can think of it as an almost complex structure on the space transverse to the CKV that satisfies a certain integrability condition. A simple example of a five manifold endowed with a THF is given by Sasakian manifolds. Here, the existence of the THF was exploited in [26] in order to show that the perturbative partition function can be calculated by counting holomorphic functions on the associated Kähler cone. Similar considerations were used in [21] to solve the BPS equations on the Higgs branch. This gives rise to the question whether such simplifications occur in localization calculations on more generic five manifolds admitting rigid supersymmetry. This was addressed in [23] in

[2]This statement assumes the spinor — and thus the vector — to be non-vanishing. In the case of Poincaré supergravity, this is always the case if the manifold is connected. After all, the relevant KSE is of the form $\partial_\mu \epsilon^i = \mathcal{O}(\epsilon^i)$. If the spinor vanishes at a point, it vanishes on the whole manifold. For conformal supergravity however, the KSE takes the form of a twistor equation, $\partial_\mu \epsilon^i - \frac{1}{4}\gamma_{\mu\nu}\partial^\nu \epsilon^i = \mathcal{O}(\epsilon^i)$, which has non-trivial solutions even if the right hand side vanishes. The simplest example of this is given by the superconformal supersymmetry in $\mathbb{R}^5$. See section 6.1. Here $\epsilon^i|_{x^\mu=0} = v|_{x^\mu=0} = 0$, yet the global solution is non-trivial. In such cases a more careful analysis is necessary.

the context of 5d $\mathcal{N}=1$ Poincaré supergravity. Here it was shown that a necessary and sufficient condition for such manifolds to admit a supersymmetric background is the existence of a Killing vector. If an su(2)-valued scalar in the Weyl multiplet is non-vanishing and covarinatly constant along the four-dimensional leaves of the foliation it follows furthermore that the solution defines a THF. Subsequently it was argued that the existence of a THF (or that of an integrable Cauchy-Riemann structure) is sufficient to lead to similar simplifications in the context of localization as in [21, 26].

With this motivation in mind we will address the question under which circumstances generic backgrounds of the conformal supergravity in question admit THFs. Our results are to be seen in the context of the very recent paper [22]. We will find that the necessary and sufficient condition for the solution to support a THF is the existence of a global section of an su(2)/$\mathbb{R}$ bundle that is covariantly constant with respect to a connection $\mathcal{D}^Q$ that arises from the intrinsic torsions parametrizing the spinor.

The outline of the rest of the paper is as follows. In section 2 we offer a lightning review of the relevant aspects of superconformal 5d supergravity, with our conventions compiled in appendix A and further details described in appendix B. In section 3 we turn to the analysis of the general solutions of the supergravity, showing that the necessary condition for supersymmetry is the presence of a conformal Killing vector. Moreover, we will see that given a Killing spinor and the related CKV the general solution depends only on an su(2)-valued Δ^{ij} and a vector W that is orthogonal to the CKV. Both are determined by solving simple ODEs that become trivial if one goes to a frame where the CKV is Killing. In section 4 we study under which conditions it is possible to turn on a VEV for scalars in background vector multiplets thus flowing to a standard gauge theory, finding that the requirement is that the vector is not only conformal Killing but actually Killing. In section 5 we derive the conditions for the existence of a THF. In section 6 we show how some particular examples fit into our general structure, describing in particular the cases of $\mathbb{R}\times S^4$ relevant for the index computation of [5] and the S^5 relevant for the partition function computation of [2, 3]. We finish with some conclusions in section 7.

Note added. While this work was in its final stages we received [22], which has a substantial overlap with our results.

2 Five-dimensional conformal supergravity

Let us begin by reviewing the five-dimensional, $\mathcal{N}=2$ conformal supergravity of [15, 16].[3] The theory has $\mathrm{SU}(2)_R$ R-symmetry. The Weyl multiplet contains the vielbein e_μ^a, the $\mathrm{SU}(2)_R$ connection $V_\mu^{(ij)}$, an antisymmetric tensor $T_{\mu\nu}$, a scalar D, the gravitino ψ_μ^i and the dilatino χ^i. Our conventions are summarised in appendix A.

[3] A word on notation is in order here. We stress that we are discussing minimal supersymmetry in five dimensions.

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The supersymmetry variations of the gravitino and dilatino are

$$\delta\psi^i_\mu = \mathcal{D}_\mu\epsilon^i + \imath\gamma\cdot T\gamma_\mu\epsilon^i - \imath\gamma_\mu\eta^i, \tag{2.1}$$

$$\delta\chi^i = \frac{1}{4}\epsilon^i D - \frac{1}{64}\gamma\cdot\hat{R}^{ij}(V)\epsilon_j + \frac{\imath}{8}\gamma^{\mu\nu}\not{\nabla}T_{\mu\nu}\epsilon^i - \frac{\imath}{8}\gamma^\mu\nabla^\nu T_{\mu\nu}\epsilon^i - \frac{1}{4}\gamma^{\kappa\lambda\mu\nu}T_{\kappa\lambda}T_{\mu\nu}\epsilon^i + \frac{1}{6}T^2\epsilon^i + \frac{1}{4}\gamma\cdot T\eta^i. \tag{2.2}$$

Up to terms $\mathcal{O}(\psi_\mu, \chi^i)$,

$$\mathcal{D}_\mu\epsilon^i = \partial_\mu\epsilon^i + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon^i + \frac{1}{2}b_\mu\epsilon^i - V^{ij}_\mu\epsilon_j, \tag{2.3}$$

$$\hat{R}^{ij}_{\mu\nu}(V) = dV^{ij}_{\mu\nu} - 2V^{k(i}_{[\mu}V_{\nu]k}{}^{j)}. \tag{2.4}$$

In what follows we will set the Dilation gauge field b_μ to zero.

As usual, taking the γ-trace of the gravitino equation allows to solve for the superconformal parameters as $\eta^i = -\frac{\imath}{5}\not{\mathcal{D}}\epsilon^i + \frac{1}{5}T\cdot\gamma\epsilon^i$. Hence, we can rewrite the equations arising from the gravitino and dilatino as

$$0 = \mathcal{D}_\mu\epsilon^i - \frac{1}{4}\gamma_{\mu\nu}\mathcal{D}^\nu\epsilon^i + \imath\gamma_{\mu\kappa\lambda}T^{\kappa\lambda}\epsilon^i - 3\imath T_{\mu\nu}\gamma^\nu\epsilon^i, \tag{2.5}$$

$$0 = \frac{1}{128}\epsilon^i(32D + R) + \frac{1}{15}T_{\mu\nu}T^{\mu\nu}\epsilon^i + \frac{1}{8}\mathcal{D}^\mu\mathcal{D}_\mu\epsilon^i + \frac{3\imath}{40}\gamma_{\kappa\lambda\mu}T^{\kappa\lambda}\mathcal{D}^\mu\epsilon^i + \frac{11\imath}{40}\gamma^\mu T_{\mu\nu}\mathcal{D}^\nu\epsilon^i + \frac{\imath}{4}\gamma_{\mu\kappa\lambda}\nabla^\mu T^{\kappa\lambda}\epsilon^i + \frac{\imath}{2}\gamma^\mu\nabla^\nu T_{\mu\nu}\epsilon^i - \frac{1}{5}\gamma^{\kappa\lambda\mu\nu}T_{\kappa\lambda}T_{\mu\nu}\epsilon^i. \tag{2.6}$$

Here, R is the Ricci scalar and the rewriting of the dilatino equation uses the gravitino equation. One could also rewrite the latter using $\not{\mathcal{D}}^2$ as in [24], yet we found the above formulation to be more economical in this case.

3 General solutions of $\mathcal{N} = 2$ conformal supergravity

General solutions to five-dimensional conformal supergravity have been constructed in [20] using superspace techniques. In this section we will provide an alternative derivation of the most general solutions to $\mathcal{N} = 2$ conformal supergravity in euclidean signature using component field considerations along the lines of [24]. Before turning to the details, let us recall a counting argument from [24] regarding these solutions: in general the gravitino yields 40 scalar equations. Eliminating the superconformal spinor η^i removes 8. As we will see, the gravitino equation then also fixes the 10 components of the antisymmetric tensor and 8 of the components of the SU(2) connection. This leaves us with 14, which is exactly enough to remove the traceless, symmetric part of a two-tensor P which will appear in the intrinsic torsion. Since the trace is undetermined we will find a CKV; the vector is Killing if the trace vanishes. The remaining 7 components of the SU(2) connection and the scalar in the Weyl multiplet will then be determined by the eight equations arising from the dilatino variation.

In order to study solutions of (2.5) and (2.6), we introduce the bispinors

$$\begin{aligned} s &= \epsilon^i C \epsilon_i, \\ v &= (\epsilon^i C \gamma_\mu \epsilon_i) dx^\mu, \\ \Theta^{ij} &= (\epsilon^i C \gamma_{\mu\nu} \epsilon^j) dx^\mu \otimes dx^\nu, \end{aligned} \tag{3.1}$$

In what follows, we will assume the scalar s to be non-zero and the one-form v to be real. These assumptions are implied if one imposes a symplectic Majorana condition such as (A.3). Furthermore, note that $v^2 = s^2$.

The one-form then decomposes the tangent bundle into a horizontal and a vertical part, with the former being defined as $TM_H = \{X \in TM | v(X) = 0\}$ and TM_V as its orthogonal complement. Due to the existence of a metric we use v to refer both to the one-form and the correspoding vector and an analogous decomposition into horizontal and vertical forms extends to the entire exterior algebra. In turn, the two-forms Θ^{ij} are fully horizontal and anti self-dual[4] with respect to the automorphism $\iota_{s^{-1}v} \star : \Lambda^2_H \to \Lambda^2_H$:

$$\iota_v \Theta^{ij} = 0, \qquad \iota_{s^{-1}v} \star \Theta^{ij} = -\Theta^{ij}. \tag{3.2}$$

One finds that the spinor is chiral with respect to the vector $s^{-1}v$,

$$s^{-1} v^\mu \gamma_\mu \epsilon^i = \epsilon^i. \tag{3.3}$$

Note that the sign here is mainly a question of convention. Had we defined v with an additional minus sign, we would find the spinor to be anti-chiral and Θ^{ij} to be self-dual. One can see this by considering the transformation $v \mapsto -v$. In addition, we define the operator $\Pi^\mu_\nu = \delta^\mu_\nu - s^{-2} v^\mu v_\nu$ which projects onto the horizontal space. A number of additional useful identities involving Θ^{ij} are given in appendix A.

Next, we parametrize the covariant derivative of the supersymmetry spinor using intrinsic torsions as in [19],

$$\nabla_\mu \epsilon^i \equiv P_{\mu\nu} \gamma^\nu \epsilon^i + Q^{ij}_\mu \epsilon_j. \tag{3.4}$$

Here, $P_{\mu\nu}$ is a two-tensor while Q^{ij}_μ is symmetric in its $\mathrm{SU}(2)_R$ indices. Rewriting the torsions in terms of the supersymmetry spinor one finds

$$s P_{\mu\nu} = \epsilon^i \gamma_\nu \nabla_\mu \epsilon_i = \frac{1}{2} \nabla_\mu v_\nu, \qquad s Q^{ij}_\mu = 2 \epsilon^{(i} \nabla_\mu \epsilon^{j)}. \tag{3.5}$$

3.1 The gravitino equation

We now turn to the study of generic solutions of (2.5) and (2.6) using the intrinsic torsions. The reader interested in intermediate results and some technical details might want to

[4] Explicitly, the self-duality condition is

$$\Theta^{ij}_{\mu\nu} = -\frac{1}{2} s^{-1} \epsilon_{\mu\nu\kappa\lambda\rho} \Theta^{ij\kappa\lambda} v^\rho, \qquad \epsilon_{\lambda\mu\nu\sigma\tau} \Theta^{ij\sigma\tau} = -3! s^{-1} \Theta^{ij}_{[\lambda\mu} v_{\nu]}.$$

consult appendix B. To begin, substituting (3.4) and contracting with $\epsilon_i\gamma_\kappa$ as well as ϵ^j and symmetrizing in i, j one finds that (2.5) is equivalent to

$$0 = \frac{5}{4}s\left(P_{(\mu\nu)} - \frac{1}{5}g_{\mu\nu}P^\lambda{}_\lambda\right) + \frac{3}{4}s\left(P_{[\mu\nu]} - 4\imath T_{\mu\nu}\right) + \frac{1}{4}\epsilon_{\mu\nu\kappa\lambda\rho}(P^{[\kappa\lambda]} - 4\imath T^{\kappa\lambda})v^\rho$$
$$+\frac{1}{8}\epsilon_{\mu\nu\rho\sigma\tau}(Q-V)^{\rho ij}\Theta_{ij}^{\sigma\tau}, \tag{3.6}$$
$$0 = \frac{1}{2}s(Q-V)^{ij}_\mu + \frac{1}{4}(Q-V)^{\nu(j}{}_k\Theta^{i)k}_{\mu\nu} + \frac{1}{8}\epsilon_{\mu\kappa\lambda\sigma\tau}(P-4\imath T)^{\kappa\lambda}\Theta^{ij\sigma\tau}. \tag{3.7}$$

Clearly, the symmetric part in (3.6) has to vanish independently; so we find

$$P_{(\mu\nu)} = \frac{1}{5}g_{\mu\nu}P^\lambda{}_\lambda. \tag{3.8}$$

This implies that v is a conformal Killing vector as can be seen using (3.5).

By contracting the two remaining equations with v^μ, one finds

$$0 = 3sv^\mu(P-4\imath T)_{[\mu\nu]} - s\Theta^{ij}_{\nu\mu}(Q-V)^\mu_{ij}, \tag{3.9}$$
$$0 = 2sv^\mu(Q-V)^{ij}_\mu - s\Theta^{ij}_{\mu\nu}(P-4\imath T)^{\mu\nu}. \tag{3.10}$$

Projecting (3.6) on the horizontal space, we find that $\Pi(P-4\imath T)$ is anti self-dual.

$$0 = (P-4\imath T)^+. \tag{3.11}$$

Contracting (3.10) with $\Theta_{ij\kappa\lambda}$ and using (A.5) gives us the horizontal, self-dual part.

$$(P-4\imath T)^- = s^{-2}\Theta^{ij}\imath_v(Q-V)_{ij}. \tag{3.12}$$

By now we have equations for the self-dual, anti self-dual and vertical components of $(P-4\imath T)_{[\mu\nu]}$, which means that all components of this two-form are determined. Putting everything together, we find

$$s^2(P-4\imath T)_{[\mu\nu]} = \frac{1}{3}\left[(v\wedge\Theta^{ij})_{\mu\nu\rho} + 2\Theta^{ij}_{\mu\nu}v_\rho\right](Q-V)^\rho_{ij}. \tag{3.13}$$

The only equation we have not considered so far is the horizontal projection of (3.7). After using (A.6), (3.9) and (3.10) this simplifies to

$$s\Pi_\mu{}^\nu(Q-V)_\nu{}^i{}_j = -\frac{1}{2}[(Q-V)^\nu, \Theta_{\mu\nu}]^i{}_j. \tag{3.14}$$

In summary, the gravitino is solved by (3.13) and (3.14).

Note that one can solve (3.14) by brute force after picking explicit Dirac matrices. One finds that the equation leaves seven components of $(Q-V)$ unconstrained. Three of these have to be parallel to v as they do not enter in (3.14). This suggests that it is possible to package the seven missing components into a triplet Δ^{ij} (three components) and a horizontal vector W^μ (four) and parametrize a generic solution of the gravitino equation as

$$(Q-V)^{ij}_\mu = s^{-1}\left(v_\mu\Delta^{ij} + W^\lambda\Theta^{ij}_{\lambda\mu}\right) \qquad \text{s.t.} \qquad v(W) = 0, \Delta^{ij} = \Delta^{ji}. \tag{3.15}$$

Using (A.6) one can verify that (3.15) satisfies (3.14). The above implies that

$$T_{\mu\nu} = \frac{\imath}{4}\left(s^{-1}\Theta^{ij}_{\mu\nu}\Delta^{ij} + s^{-1}v_{[\mu}W_{\nu]} - P_{[\mu\nu]}\right). \tag{3.16}$$

3.2 The dilatino equation

We finally turn to the dilatino equation (2.6). To begin, we note that between Δ^{ij}, W_μ, D there are eight unconstrained functions remaining while the dilatino equation provides eight constraints. We can thus expect that there will be no further constraints on the geometry. In this respect, similarly to [24], supersymmetry is preserved as long as the manifold supports a conformal Killing vector v.

In what follows we will need to deal with terms involving derivatives of the spinor bilinears (3.1). To do so we use the identities

$$\nabla_\mu s = 2P_{\mu\nu}v^\nu, \tag{3.17}$$

$$\nabla_\mu v_\nu = 2sP_{\mu\nu}, \tag{3.18}$$

$$\nabla_\mu \Theta^{ij}_{\kappa\lambda} = 3!s^{-1}\Theta^{ij}_{[\kappa\lambda}v_{\rho]}P_\mu{}^\rho - 2\Theta^{k(i}_{\kappa\lambda}Q^{j)}_{\mu k}, \tag{3.19}$$

$$\nabla_{[\lambda}P_{\mu\nu]} = -s^{-1}P_{[\mu\nu}\nabla_{\lambda]}s. \tag{3.20}$$

Contracting (2.6) with ϵ^j and symmetrizing over the $\mathrm{SU}(2)_R$ indices i, j one finds

$$0 = \frac{1}{8}\epsilon^{(i}\mathcal{D}^\mu\mathcal{D}_\mu\epsilon^{j)} + \frac{3\imath}{40}\epsilon^{(i}\gamma_{\kappa\lambda\mu}T^{\kappa\lambda}\mathcal{D}^\mu\epsilon^{j)} + \frac{11\imath}{40}\epsilon^{(i}\gamma^\mu T_{\mu\nu}\mathcal{D}^\nu\epsilon^{j)} + \frac{\imath}{4}\epsilon^{(i}\gamma_{\kappa\lambda\mu}\epsilon^{j)}\nabla^\mu T^{\kappa\lambda}. \tag{3.21}$$

Substituting (3.15) and (3.16) one finds after a lengthy calculation[5]

$$\pounds_v\Delta^i{}_j = -\frac{2}{5}sP^\mu{}_\mu\Delta^i{}_j - [\iota_v Q + P^{[\mu\nu]}\Theta_{\mu\nu}, \Delta]^i{}_j. \tag{3.22}$$

Contracting (2.6) with $-\epsilon_i\gamma_\mu$ one obtains

$$0 = v_\mu\left(\frac{32D+R}{128} + \frac{1}{15}T_{\mu\nu}T^{\mu\nu}\right) + \frac{1}{8}\epsilon^i\gamma_\mu\mathcal{D}^\nu\mathcal{D}_\nu\epsilon_i + \frac{3\imath}{40}\epsilon^i\gamma_\mu\gamma_{\kappa\lambda\nu}T^{\kappa\lambda}\mathcal{D}^\nu\epsilon_i + \frac{11\imath}{40}\epsilon^i\gamma_\mu\gamma^\kappa T_{\kappa\lambda}\mathcal{D}^\lambda\epsilon_i + \frac{\imath}{4}\epsilon_\mu{}^{\nu\kappa\lambda\sigma}v_\sigma\nabla_\nu T_{\kappa\lambda} + \frac{\imath s}{2}\nabla^\nu T_{\mu\nu} - \frac{s}{5}\epsilon_\mu{}^{\kappa\lambda\sigma\tau}T_{\kappa\lambda}T_{\sigma\tau}. \tag{3.23}$$

The vertical component of this fixes the scalar D.

$$0 = 480sD + 15sR + 48s(P^\mu{}_\mu)^2 - 130sW^2 + 60\epsilon_{\kappa\lambda\mu\nu\rho}P^{[\kappa\lambda]}P^{[\mu\nu]}v^\rho - 160s\Delta^{ij}\Delta_{ij} + 100P_{[\mu\nu]}(sP^{[\mu\nu]} - 2v^\mu W^\nu) - 200P^{[\mu\nu]}\Theta^{ij}_{\mu\nu}\Delta_{ij} + 48v^\mu\nabla_\mu P^\rho{}_\rho - 120s\nabla^\mu W_\mu. \tag{3.24}$$

The horizontal part of (3.23) yields a differential equation for W

$$\pounds_v W_\kappa = \frac{1}{50}\Pi_\kappa{}^\lambda\left(3s^2P^\mu{}_\mu W_\lambda - 34P^\rho{}_\rho P_{[\lambda\mu]}v^\mu - 20s\nabla_\lambda P^\rho{}_\rho\right). \tag{3.25}$$

Note that the left hand side is horizontal since $\iota_v\pounds_v W = \iota_v\iota_v dW = 0$.

Similar to the discussion in [24], we note that one can always solve (3.22) and (3.25) locally. Moreover, after a Weyl transformation to a frame where v is not only conformal Killing yet actually Killing, that is, setting $P^\mu{}_\mu = 0$, both equations simplify considerably.

[5] We found the Mathematica package `xAct` [27, 28] very useful.

All the source terms in the latter vanish which is now solved by $W = 0$ while the former becomes purely algebraic,

$$0 = [\iota_v Q + P^{[\mu\nu]}\Theta_{\mu\nu}, \Delta]^i{}_j, \tag{3.26}$$

and is solved by $\Delta = s^{-1} f(\iota_v Q + P^{[\mu\nu]}\Theta_{\mu\nu})$ for a generic, possibly vanishing, function f as long as $\pounds_v f = 0$. The factor s^{-1} is simply included here to render Δ invariant under $\epsilon^i \to \lambda\epsilon^i$ for $\lambda \in \mathbb{C}$.

An alternative way to see that (3.22) and (3.25) can be solved globally is by direct construction of the solution following the approach of section 5 in [23]. Thus, the existence of a non-vanishing CKV is not only necessary, but also sufficient. See also footnote 2.

4 Yang-Mills theories from conformal supergravity

The solutions described above provide the most general backgrounds admitting a five-dimensional, minimally supersymmetric quantum field theory arising in the rigid limit of conformal supergravity. In the maximally supersymmetric case a more general class of solutions is possible, since the R-symmetry of maximal supergravity is SO(5), one can define supersymmetric field theories on generic five manifolds by twisting with the whole SO(5). Such field theories were considered in [29]. An embedding in supergravity should be possible starting from [30].

Of course, in the case at hand our starting point is conformal supergravity, so only conformal multiplets can be consistently coupled to the theory. While the hypermultiplet is conformally invariant *per se*, the vector multiplet with the standard Maxwell kinetic term breaks conformal invariance as the Yang-Mills coupling has negative mass dimensions. Therefore the action for the conformally coupled vector multiplet is a non-standard cubic action which can be thought as the supersymmetric completion of 5d Chern-Simons. Such action contains in particular a coupling of the form $C_{IJK}\,\sigma^I\,F^J\,F^K$, where F^I is the field strength of the I-th vector multiplet, σ^I its corresponding real scalar and C_{IJK} a suitable matrix encoding the couplings among all vector multiplets (we refer to [15, 16] for further explanations). Thus we can imagine constructing a standard gauge theory by starting with a conformal theory and giving suitable VEVs to scalars in background abelian vector multiplets. Of course, such VEVs must preserve supersymmetry. To that end, let us consider the SUSY variation of a background vector multiplet. As usual, only the gaugino variation is relevant, which, in the conventions of [16], reads

$$\delta\Omega^i_B = -\frac{\imath}{2}\,\nabla\!\!\!/\sigma_B\,\epsilon^i + Y^i_{B\,j}\,\epsilon^j + \sigma_B\,\gamma\cdot T\epsilon^i + \sigma_B\,\eta^i\,, \tag{4.1}$$

where we have set to zero the background gauge field. The $Y^i_{B\,j}$ are a triplet of auxiliary scalars in the vector multiplet. Contracting with ϵ_i it is straightforward to see that, in order to have a supersymmetric VEV, we must have

$$\pounds_v\sigma_B + \frac{2\,s}{5}\,P^\mu{}_\mu\sigma_B = 0\,, \tag{4.2}$$

while the other contractions fix the value of $Y^{i}_{B\,j}$. The VEV of σ_B is $g_{\rm YM}^{-2}$, and as such one would like it to be a constant. Therefore, equation (4.2) gives us an obstruction for the existence of a Maxwell kinetic term; namely, that v is Killing and not only conformal Killing. It then follows that all backgrounds admitting standard — i.e. quadratic — supersymmetric Yang-Mills theories, involve a v which is a genuine Killing vector. They are thus solutions of the $\mathcal{N}=1$ Poincaré supergravity — see e.g. [18, 19, 21, 23]. In particular, the case of $\mathbb{R}\times S^4$ is of special interest as the partition function on this space in the absence of additional background fields gives the superconformal index [5]. The relevant supersymmetry spinors appearing in the calculation define a vector v which is conformal Killing; and therefore the background is only a solution of conformal supergravity. As we will explicitly see below, it is easy to check that such a solution, which can be easily obtained by a simple change of coordinates in the spinors in [12], nicely fits in our general discussion above. If, on the other hand, one studies supersymmetric backgrounds on S^5 without additional background fields, one finds v to be Killing (see below as well). Thus such backgrounds can be regarded as a solution to conformal supergravity that are not obstructed by (4.2) and do thus admit a constant σ_B. In fact, it is easy to check this nicely reproduces the results of [2].

Eq. (4.2) shows that backgrounds admitting only a conformal Killing vector cannot support a standard gauge theory with a constant Maxwell kinetic term. As anticipated above, and explicitly described below, this is precisely the case of $\mathbb{R}\times S^4$, relevant for the computation of the index. Of course it is possible to solve (4.2) if one accepts that the Yang-Mills coupling is now position dependent. This way we can still think of the standard Yang-Mills action as a regulator to the index computation.[6] While this goes beyond the scope of this paper, one might imagine starting with the Yang-Mills theory on $\mathbb{R}^5$ where (4.2) can be satisfied for a constant σ_B. Upon conformally mapping $\mathbb{R}^5$ into $\mathbb{R}\times S^4$ the otherwise constant $\sigma_B = g_{\rm YM}^{-2}$ becomes $\sigma_B = g_{\rm YM}^{-2}\, e^{\tau}$, being τ the coordinate parametrizing $\mathbb{R}$. In the limit $g_{\rm YM}^{-2}\to 0$ we recover the conformal theory of [5]. One can imagine computing the supersymmetric partition function in this background. As the preserved spinors are just the same as in the $g_{\rm YM}^{-2}\to 0$ limit, the localization action, localization locus and one-loop fluctuations will be just the same as in the conformal case. While we leave the computation of the classical action for future work, it is clear that the limit $g_{\rm YM}^{-2}\to 0$ will reproduce the result in [5].

5 Existence of transversally holomorphic foliations

We will now discuss under which circumstances solutions to equations (2.1) and (2.2) define transversally holomorphic foliations (THF). Since we assumed $s\neq 0$ and v real, it follows that the CKV v is non-vanishing and thus that v defines a foliation on M. Using (A.6) one can show then that Θ^{ij} defines a triplet of almost complex structures on the four-

[6] One might wonder that the cubic lagrangian theory is enough. However, in some cases such as e.g. Sp gauge theories, such cubic lagrangian is identically zero.

dimensional horizontal space TM_H. Thus, given a non-vanishing section[7] of the su(2)$_R$ Lie algebra m_{ij} we can define an endomorphism on TM

$$(\Phi[m])^{\mu}{}_{\nu} \equiv (\det m)^{-1/2} m_{ij} (\Theta^{ij})^{\mu}{}_{\nu}. \tag{5.1}$$

This satisfies $\Phi[m]^2 = -\Pi$ and thus induces a decomposition of the complexified tangent bundle

$$T_{\mathbb{C}}M = T^{1,0} \oplus T^{0,1} \oplus \mathbb{C}v. \tag{5.2}$$

Any such decomposition is referred to as an almost Cauchy-Riemann (CR) structure. If an almost CR structure satisfies the integrability condition

$$[T^{1,0} \oplus \mathbb{C}v, T^{1,0} \oplus \mathbb{C}v] \subseteq T^{1,0} \oplus \mathbb{C}v, \tag{5.3}$$

one speaks of a THF.[8] Intuitively, m_{ij} determines how Φ is imbedded in Θ^{ij} and thus how $T^{1,0}$ is embedded in TM_H. If one forgets about the vertical direction v for a moment, the question of integrability of Φ is similar to the question under which circumstances a quaternion Kähler structure on a four-manifold admits an integrable complex structure.

To address the question of the existence of an m_{ij} satisfying (5.3) we follow the construction of [23] and define the projection operator

$$H^i{}_j = (\det m)^{-1/2} m^i{}_j - \imath \delta^i{}_j. \tag{5.4}$$

One can then show that

$$X \in T^{1,0} \oplus \mathbb{C}v \qquad \Leftrightarrow \qquad X^\mu H^i{}_j \Pi^\nu_\mu \gamma_\nu \epsilon^j = 0, \tag{5.5}$$

if the supersymmetry spinor ϵ^i satisfies a reality condition such as (A.3). Acting from the left with $\mathcal{D}_Y$ for $Y \in T^{1,0} \oplus \mathbb{C}v$ and antisymmetrizing over X, Y, one derives the spinorial integrability condition

$$[X, Y] \in T^{1,0} \oplus \mathbb{C}v \qquad \Leftrightarrow \qquad X^{[\mu} Y^{\nu]} \mathcal{D}_\mu (H^i{}_j \Pi^\rho_\nu \gamma_\rho \epsilon^j) = 0. \tag{5.6}$$

Note that $H^i{}_j$ satisfies $H^2 = -2\imath H$ and has eigenvalues 0 and $-2\imath$. Thus, $H^i{}_j \epsilon^j$ projects the doublet ϵ^i to a single spinor that is a linear combination of the two. It is this spinor that will define the THF.

To proceed, we first consider $X, Y \in T^{1,0}$. After substituting (3.4) and making repeatedly use of (5.5), one finds that the condition (5.6) reduces to the vanishing of

$$X^{[\mu} Y^{\nu]} (\partial_\mu H^i{}_j + [Q_\mu, H]^i{}_j) \gamma_n \epsilon^j. \tag{5.7}$$

[7] Since Φ is invariant under $m_{ij} \mapsto f m_{ij}$ for any non-vanishing function $f : M \to \mathbb{R}$ it might be more appropriate to think of m_{ij} as a ray in the three-dimensional su(2) vector space. From this point of view, m_{ij} is a map

$$m : M \to S^2 \subset \mathrm{su}(2).$$

[8] The similar integrability condition $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$ defines a CR manifold.

Similarly, the case $X \in T^{1,0}$, $Y = v$ leads to the condition that

$$X^m(\partial_v H^i{}_j + [\iota_v Q, H]^i{}_j)\gamma_m \epsilon^j \tag{5.8}$$

must be identically zero. Contracting both expressions with ϵ^j and symmetrizing over SU(2) indices, we conclude that the integrability condition (5.3) can only be satisfied if and only if

$$\mathcal{D}^Q_\mu H^i{}_j \equiv \partial_\mu H^i{}_j + [Q_\mu, H]^i{}_j = 0, \tag{5.9}$$

i.e. iff the projection $H^i{}_j$ is covariantly constant with respect to the connection defined by Q.

From the condition that $H^i{}_j$ be covariantly constant we derive the necessary condition that it is also annihilated by the action of the corresponding curvature tensor:

$$[R^Q_{\mu\nu}, H]^i{}_j = 0, \tag{5.10}$$

where $R^Q_{\mu\nu} = [\mathcal{D}^Q_\mu, \mathcal{D}^Q_\nu]$. Now, one can only solve (5.10) if the SU(2) curvature R^Q lies in a U(1) inside $\mathrm{SU}(2)_R$. Note that since the curvature R^Q arises from the intrinsic torsions, we can relate it to the Riemann tensor and $P_{\mu\nu}$ using (3.4). The resulting expression is not too illuminating however.

To conclude we will relate the integrability condition (5.9) to the findings of [23]. There it was found that solutions of the $\mathcal{N} = 1$ Poincaré supergravity of [31–33] define THFs if $m_{ij} = t_{ij}$ and $\forall X \in TM_H, \mathcal{D}_X t_{ij} = 0$. In other words, the unique choice for m_{ij} is the field t_{ij} appearing in the Weyl multiplet of that theory and the latter has to be covariantly constant (with respect to the usual $\mathrm{SU}(2)_R$ connection V^{ij}_μ) along the horizontal leaves of the foliation. To relate our results to this, consider the case where v is actually Killing. It follows that we can assume $\pounds_v H^i{}_j = 0$ and thus the vertical part of (5.9) takes the form of the first condition of [23], namely

$$[\iota_v Q, H]^i{}_j = 0. \tag{5.11}$$

Moreover, our general solutions (3.22) and (3.25) are solved by $W = 0$; while this solution is not unique, it makes the connection to [23] very eveident as it follows now that $\Pi(Q)^{ij} = \Pi(V)^{ij}$ and so the horizontal part of (5.9) reproduces the second condition from [23]:

$$\forall X \in T^{1,0} \qquad \mathcal{D}_X H^i{}_j = X^\mu(\partial_\mu H^i{}_j + [V_\mu, H]^i{}_j) = 0. \tag{5.12}$$

6 Examples

Let us now discuss some specific examples illustrating the general results from the previous sections.

6.1 Flat $\mathbb{R}^5$

Flat space admits constant spinors generating the Poincaré supersymmetries. In addition, we can consider the spinor generating superconformal supersymmetries $\epsilon^i = x_\mu \gamma^\mu \epsilon^i_0$, where

ϵ_0^i is constant. Let us see how these fit into our general set-up. For the Poincaré supersymmetries, it is clear that we just have $Q = P = V = T = 0$. For the superconformal spinors on the other hand, the gravitino and dilatino equations are solved by

$$\eta^i = -\imath\epsilon_0^i, \qquad T_{\mu\nu} = V_\mu^{ij} = D = 0. \tag{6.1}$$

The intrinsic torsions are

$$Q_\mu^{ij} = -\frac{2}{sx^2}x_\kappa\Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = \frac{1}{sx^2}(x\wedge v)_{\mu\nu}, \qquad P_{(\mu\nu)} = s^{-1}x_\kappa v^\kappa\delta_{\mu\nu}. \tag{6.2}$$

Note that $\Theta^{ij}_{\nu\rho}Q^\rho_{ij} = -\frac{3s}{x^2}\Pi^\sigma_\nu x_\sigma$ and thus

$$\frac{2}{3}2v_{[\mu}\Theta^{ij}_{\nu]\rho}Q^\rho_{ij} = \frac{s}{x^2}(\Pi^\rho_\mu v_\nu - \Pi^\rho_\nu v_\mu)x_\rho = \frac{s}{x^2}(x\wedge v)_{\mu\nu} = s^2 P_{[\mu\nu]}. \tag{6.3}$$

We don't only see that (3.13) is satisfied, yet also that the only contribution to the right hand side of that equation comes from $\frac{2}{3}2v_{[\mu}\Theta^{ij}_{\nu]\rho}Q^\rho_{ij}$ while it is exactly the term that vanishes, $\Theta^{ij}_{\mu\nu}v_\rho Q^\rho_{ij}$, that contributes in the in the Sasaki-Einstein case to be discussed below.

Note that the superconformal supersymmetries involve non-zero trace of P. Hence, these supersymmetries are broken by the background scalar VEV corresponding to g_{YM}^{-2}. This just reflects the general wisdom that the 5d YM coupling, being dimensionful, breaks conformal invariance.

6.2 $\mathbb{R}\times S^4$

Consider now $\mathbb{R}\times S^4$, with $\mathbb{R}$ parametrized by $x^5 = \tau$ and v not along $\frac{\partial}{\partial\tau}$. As described in [5] — where the explicit spinor solutions are written as well, the spinors satisfy

$$\nabla_\mu\epsilon^q = -\frac{1}{2}\gamma_\mu\,\gamma_5\epsilon^q\,, \qquad \nabla_\mu\epsilon^s = \frac{1}{2}\gamma_\mu\,\gamma_5\epsilon^s. \tag{6.4}$$

Here $\epsilon^{q,s}$ generate Poincaré and superconformal supersymmetries respectively. It is straightforward to see that these solutions fit in our general scheme with

$$Q_\mu^{ij} = \pm\frac{1}{2s}w_\kappa\Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = \mp\frac{1}{2s}\,(w\wedge v)_{\mu\nu}, \qquad P_{(\mu\nu)} = \mp\frac{1}{2s}\,w_\kappa v^\kappa\,g_{\mu\nu}, \tag{6.5}$$

where upper signs correspond to the ϵ_q while lower signs correspond to the ϵ_s. In addition we have defined $w = d\tau$. Note that the trace of P does not vanish, implying that v is conformal Killing. Thus this is a genuine solution of superconformal supergravity that cannot be embedded in $\mathcal{N} = 1$ Poincaré supergravity. Moreover, as discussed above, this implies that no (constant) Yang-Mills coupling can be turned on on this background (see [34] for a further discussion in the maximally supersymmetric case).

6.3 Topological twist on $\mathbb{R}\times M_4$

Manifolds of the form $\mathbb{R}\times M_4$ can be regarded as supersymmetric backgrounds at the expense of turning on a non-zero V such that the spinors are gauge-covariantly constant

$$\mathcal{D}_\mu\epsilon^i = 0. \tag{6.6}$$

To show that we consider $v = \partial_\tau$, being τ the coordinate parametrizing $\mathbb{R}$. Then, from (3.5), it follows that $P_{\mu\nu} = 0$. Furthermore, by choosing $V = Q$ — which translates into $W_\mu = 0$, $\Delta^{ij} = 0$ and implies $T_{\mu\nu} = 0$ — all the remaining constraints are automatically solved. This is nothing but the topological twist discussed in [12] (see also [35] for the maximally supersymmetric case; twisted theories on five manifolds were also considered in [29]). Note that since $P = 0$, in these backgrounds the Yang-Mills coupling can indeed be turned on.

6.4 $\mathrm{SU}(2)_R$ twist on M_5

If M_5 is not a direct product, one can still perform an $\mathrm{SU}(2)_R$ twist. For v Killing, the details of this can be found in [23]. One can perform an identical calculation for the conformal supergravity in question. In the case of a $\mathbb{R}$ or U(1) bundle over some M_4 for example, one finds T to be the curvature of fibration.

6.5 Sasaki-Einstein manifolds

For a generic Sasaki-Einstein manifold the spinor satisfies

$$\nabla_\mu \epsilon_i = -\frac{\imath}{2}\gamma_\mu(\sigma^3)_i{}^j\epsilon_j. \tag{6.7}$$

It follows that

$$P_{\mu\nu} = -\frac{\imath}{2}s^{-1}(\sigma^3_{ij}\Theta^{ij})_{\mu\nu}, \qquad Q^{ij}_\mu = -\frac{\imath}{2}s^{-1}v_\mu(\sigma^3)^{ij}. \tag{6.8}$$

Clearly

$$s^2 P_{[\mu\nu]} = -\frac{\imath}{2}s(\sigma^3_{ij}\Theta^{ij})_{\mu\nu} = \Theta^{ij}_{\mu\nu}Q^\rho_{ij}v_\rho. \tag{6.9}$$

Hence, upon taking $V^{ij}_\mu = 0 = T_{\mu\nu}$, we indeed have a solution of (3.13) and (3.14).

Note that the trace of P is vanishing, and hence in these backgrounds the Yang-Mills coupling can be turned on. This holds also for Sasakian manifolds. Super Yang-Mills theories on these were considered in e.g. [4].

6.6 S^5

The S^5 case is paticularly interesting as well, as it leads to the supersymmetric partition function [2, 3]. Not surprisingly, since S^5 can be conformally mapped into $\mathbb{R}^5$, the solution fits into our general discussion including two sets of spinors, one corresponding to the Poincaré supercharges and the other corresponding to the superconformal supercharges. Writing the S^5 metric as that of conformally S^5 as

$$ds^2 = \frac{4}{(1+\vec{x}^2)^2}\,d\vec{x}^2, \tag{6.10}$$

we find for the Poincare supersymmetries

$$Q^{ij}_\mu = \frac{1}{2s}x_\kappa\Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = -\frac{1}{2s}\,(x\wedge v)_{\mu\nu}, \qquad P_{(\mu\nu)} = -\frac{1}{2s}\,x_\kappa v^\kappa\, g_{\mu\nu}. \tag{6.11}$$

For the superconformal supercharges on the other hand, we find

$$Q^{ij}_\mu = -\frac{1}{2sx^2}x_\kappa\Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = \frac{1}{2sx^2}\,(x\wedge v)_{\mu\nu}, \qquad P_{(\mu\nu)} = \frac{1}{2sx^2}\,x_\kappa v^\kappa\, g_{\mu\nu}. \tag{6.12}$$

Note that in both these cases the trace of P is non-zero, so neither of these spinors are preserved if we deform the theory with a Yang-Mills coupling. Nevertheless it is possible to find a combination of supercharges which does allow for that. This can be easily understood by looking at the explicit form of the spinors, which in these coordinates is simply

$$\epsilon_q^i = \frac{1}{\sqrt{1+\vec{x}^2}}\epsilon_0^i, \qquad \epsilon_s^i = \frac{1}{\sqrt{1+\vec{x}^2}}\not{x}\eta_0^i, \tag{6.13}$$

being ϵ_0^i and η_0^i constant spinors. Considering for instance $\not{\nabla}\epsilon_q^i \supset P_{[\mu\nu]}\gamma^\mu\gamma^\nu\epsilon_q^i + P^\mu{}_\mu\epsilon_q^i$, we see that the term with $P_{[\mu\nu]}$ involves a contraction $\not{x}\epsilon_q^i$ which is basically ϵ_s^i. This suggests that one might consider a certain combination of ϵ_q and ϵ_s for which the effective P-trace is a combination of $P_{[\mu\nu]}$ and $P^\mu{}_\mu$ which might vanish. Indeed one can check that this is the case. Choosing for instance the Majorana doublet ξ^i constructed as

$$\xi^1 = \epsilon_q^1 + \epsilon_s^2, \qquad \xi^2 = \epsilon_q^2 - \epsilon_s^1, \tag{6.14}$$

it is easy to see that it satisfies

$$\nabla_\mu\epsilon_i = -\frac{\imath}{2}\gamma_\mu(\sigma^2)_i{}^j\epsilon_j; \tag{6.15}$$

that is, the same equation as that for the Sasaki-Einstein case. Therefore, borrowing our discussion above, it is clear that it admitts a Yang-Mills kinetic term. Indeed, this is corresponds, up to conventions, to the choice made in [2, 3] to compute the supersymmetric partition function.

7 Conclusions

In this paper we have studied general solutions to $\mathcal{N} = 2$ conformal supergravity. In the spirit of [13], these provide backgrounds admitting five-dimensional supersymmetric quantum field theories. The starting point of our analysis, being conformal supergravity, requires that such quantum field theories must exhibit conformal invariance. In particular, the action for vector multiplets must be the cubic completion of 5d Chern-Simons term instead of the standard quadratic Maxwell one. However, since the Yang-Mills coupling can be thought as a VEV for the scalar in a background vector multiplet, we can regard gauge theories as conformal theories conformally coupled to background vector multiplets whose VEVs spontaneously break conformal invariance. From this perspective it is very natural to consider superconformal supergravity as the starting point to construct the desired supersymmetric backgrounds.

We have described the most generic solution to $\mathcal{N} = 2$ five-dimensional conformal supergravity (see also [20]). By expanding spinor covariant derivatives in intrinsic torsions we have been able to find a set of algebraic equations (3.8), (3.15), (3.16), (3.24) together with a set of differential constraints (3.22), (3.25) characterizing the most general solution. Interestingly, the solutions admit transverse holomorphic foliations if the $\mathrm{SU}(2)_R$ connection R^Q "abelianizes" by lying along a U(1) inside $\mathrm{SU}(2)_R$, in agreement with the discussion in [22].

On general grounds, the only obstruction to the existence of supersymmetric backgrounds is the requirement of a conformal Killing vector. On the other hand we have showed that only when the vector becomes actually Killing a constant VEV for background vector multiplet scalars can be turned on. This shows that all cases where a Yang-Mills theory with standard Maxwell kinetic term can be supersymmetrically constructed are in fact captured by Poincaré supergravity. On the other hand, on backgrounds admitting only a conformal Killing vector we can still turn on a Yang-Mills coupling at the expense of being position-dependent. While this is certainly non-standard, in particular this allows to think of the quadratic part of the Yang-Mills action as the regulator in index computations.

Having constructed all supersymmetric backgrounds of $\mathcal{N} = 2$ superconformal supergravity, the natural next step would be the computation of supersymmetric partition functions. In particular, it is natural to study on what data they would depend along the lines of e.g. [36]. For initial progress in this direction see [19, 23]. We postpone such study for future work.

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A Conventions

We use the standard NE-SW conventions for $\mathrm{SU}(2)_R$ indices $\{i, j, k, l\}$ with $\epsilon^{12} = \epsilon_{12} = 1$. The charge conjugation matrix C is antisymmetric, hermitian and orthogonal, i.e. $C^* = C^T = -C = C^{-1}$. Its action on gamma matrices is given by $(\gamma^a)^* = (\gamma^a)^T = C\gamma^a C^{-1}$. In general we choose not to write the charge conjugation matrix explicitly; thus $\epsilon^i \eta^j = (\epsilon^i)^T C \eta^j$. Antisymmetrised products of gamma matrices are defined with weight one,

$$\gamma_{a_1 \ldots a_p} = \frac{1}{p!} \gamma_{[a_1} \cdots \gamma_{a_p]}, \tag{A.1}$$

yet contractions between tensors and gamma matrices are not weighted.

$$\gamma \cdot T = \gamma^{\mu\nu} T_{\mu\nu}. \tag{A.2}$$

In general, symmetrization $T_{(\mu_1 \ldots \mu_p)}$ and antisymmetrization $T_{[\mu_1 \ldots \mu_p]}$ are with weight one however.

One can impose a symplectic Majorana condition

$$\epsilon^{ij}(\epsilon^j)^* = C\epsilon^i, \tag{A.3}$$

yet as we mentioned in the main body of this paper it is generally sufficient for us to assume s to be non-vanishing and v to be real.

Using Fierz identities, one finds the following identities involving the spinor bilinear Θ^{ij}:

$$\Theta^{ij}_{\mu\nu}\Theta^{kl\mu\nu} = s^2(\epsilon^{ik}\epsilon^{jl} + \epsilon^{il}\epsilon^{jk}), \tag{A.4}$$

$$\Theta^{ij}_{\kappa\lambda}\Theta^{\mu\nu}_{ij} = \frac{s^2}{2}(\Pi^\mu_\kappa\Pi^\nu_\lambda - \Pi^\nu_\kappa\Pi^\mu_\lambda) - \frac{s}{2}\epsilon_{\kappa\lambda}{}^{\mu\nu\rho}v_\rho \tag{A.5}$$

$$\Theta^{ij}_{\mu\rho}\Theta^{kl\rho\nu} = -\frac{s^2}{4}(\epsilon^{ik}\epsilon^{jl} + \epsilon^{il}\epsilon^{jk})\Pi_\mu{}^\nu + \frac{s}{4}(\epsilon^{jk}\Theta^{il} + \epsilon^{ik}\Theta^{jl} + \epsilon^{jl}\Theta^{ik} + \epsilon^{il}\Theta^{jk})_\mu{}^\nu. \tag{A.6}$$

B Details of the computation

In this appendix we summarize the most relevant details of the computation that lead us to the two equations (3.13) and (3.14) and to the three differential equations (3.22), (3.24) and (3.25), that allow to determine Δ^{ij}, the scalar D and the vector W_κ.

B.1 Gravitino equation

In this subsection we furnish further details for the derivation of the equations (3.13) and (3.14). As explained in section 3 we rewrite the covariant derivative acting on the spinor ϵ^i as

$$\mathcal{D}_\mu\epsilon^i = \nabla_\mu\epsilon^i - V^{ij}_\mu\epsilon_j = P_{\mu\nu}\gamma^\nu\epsilon^i + (Q-V)^{ij}_\mu\epsilon_j. \tag{B.1}$$

Inserting this expression for the covariant derivative in the gravitino equation (2.5) we obtain

$$\begin{aligned}
0 =& \frac{3}{4}P_{[\mu\nu]}\gamma^\nu\epsilon^i + \frac{1}{8}\epsilon_{\mu\kappa\lambda\sigma\tau}P^{[\kappa\lambda]}\gamma^{\sigma\tau}\epsilon^i + \frac{5}{4}P_{(\mu\nu)}\gamma^\nu\epsilon^i - \frac{1}{4}\gamma_\mu P^\nu{}_\nu\epsilon^i \\
&+(Q^{ij}_\mu - V^{ij}_\mu)\epsilon_j - \frac{1}{4}\gamma_{\mu\nu}(Q^{\nu ij} - V^{\nu ij})\epsilon_j - \frac{\imath}{2}\epsilon_{\mu\kappa\lambda\sigma\tau}T^{\kappa\lambda}\gamma^{\sigma\tau}\epsilon^i - 3\imath T_{\mu\nu}\gamma^\nu\epsilon^i \\
=& \frac{5}{4}\left(P_{(\mu\nu)} - \frac{1}{5}g_{\mu\nu}P^\lambda{}_\lambda\right)\gamma^\nu\epsilon^i + (Q-V)^{ij}_\mu\epsilon_j - \frac{1}{4}\gamma_{\mu\nu}(Q-V)^{\nu ij}\epsilon_j \\
&+\frac{3}{4}(P_{[\mu\nu]} - 4\imath T_{\mu\nu})\gamma^\nu\epsilon^i + \frac{1}{8}\epsilon_{\mu\kappa\lambda\sigma\tau}(P^{[\kappa\lambda]} - 4\imath T^{\kappa\lambda})\gamma^{\sigma\tau}\epsilon^i.
\end{aligned} \tag{B.2}$$

We manipulate the previous expression, as discussed in section 3, multiplying it from the left by $\epsilon_i\gamma_\kappa$. In this way we obtain the equation (3.6). While we obtain the equation (3.7) multiplying the equation (B.2) by ϵ^j and symmetrizing in the indices i and j.

In order to recover the equation (3.13) we have to determine $(P-4\imath T)^+$ and $(P-4\imath T)^-$. Therefore we project the equation (3.6) on the horizontal space using the projector operator $\Pi^\mu_\nu = \delta^\mu_\nu - s^2v^\mu v_\nu$. We find

$$0 = \frac{5}{8}\Pi^\kappa_\mu\Pi^\lambda_\nu\left[(P-4\imath T)_{[\kappa\lambda]} + \frac{1}{2}\epsilon_{\kappa\lambda\sigma\tau\rho}(P-4\imath T)^{\sigma\tau}v^\rho\right] = \frac{5}{4}(P-4\imath T)^+. \tag{B.3}$$

This means that $\Pi(P-4\imath T)$ is anti-self dual. On the other hand contracting the equation (3.10) with $\Theta_{ij\kappa\lambda}$ and using the identity (A.5) we get

$$0 = s^3\left(\Pi_{\kappa\mu}\Pi_{\lambda\nu} - \frac{1}{2}s^{-1}\epsilon_{\kappa\lambda\mu\nu\rho}v^\rho\right)(P-4\imath T)^{[\kappa\lambda]} - 2s\Theta_{ij\kappa\lambda}(Q-V)^{ij}_\rho v^\rho. \tag{B.4}$$

Solving the previous expression we obtain $(P-4\imath T)^- = s^{-2}\Theta^{ij}\imath_v(Q-V)_{ij}$. Therefore we know all the components of $(P-4\imath T)$, since we have an equation for $(P-4\imath T)^+$, an equation for $(P-4\imath T)^-$ and finally an equation for $\imath_v(P-4\imath T)$. Putting these information together we recover the equation (3.13).

In order to determine the equation (3.14) we evaluate the projection of the equation (3.7), using the identities (3.9) and (3.10) we get

$$0 = \frac{1}{2}s\Pi^\nu_\mu(Q-V)^{ij}_\nu + \frac{1}{4}(Q-V)^{\nu(j}{}_k\Theta^{i)k}_{\mu\nu} + \frac{1}{6}s^{-1}\Theta^{ij}_{\mu\nu}\Theta^{\nu\rho}_{kl}(Q-V)^{kl}_\rho. \tag{B.5}$$

Using the identity (A.6) the previous expression becomes

$$0 = s\Pi_\mu{}^\nu(Q-V)_\nu{}^i{}_j + \frac{1}{2}[(Q-V)^\nu, \Theta_{\mu\nu}]^i{}_j. \tag{B.6}$$

Obtaining in this way the equation (3.14).

B.2 Dilatino equation

In this subsection we furnish further details regarding the derivation of the equations (3.22), (3.24) and (3.25). The most involved terms that appear in the equation (2.6) are

$$\begin{aligned}\mathcal{D}^\mu\mathcal{D}_\mu\epsilon^i =\ & \frac{1}{5}\nabla\!\!\!\!/\, P^\mu{}_\mu\epsilon^i - \gamma^\mu\nabla^\nu P_{[\mu\nu]}\epsilon^i + \frac{1}{5}(P^\mu{}_\mu)^2\epsilon^i + P_{[\mu\nu]}P^{[\mu\nu]}\epsilon^i \\ & - \nabla^\mu(Q-V)_\mu{}^i{}_j\epsilon^j - V_\mu{}^i{}_j(Q-V)^{\mu j}{}_k\epsilon^k + (Q-V)_\mu{}^i{}_jQ^{\mu j}{}_k\epsilon^k \\ & + \frac{2}{5}P^\kappa{}_\kappa\gamma^\mu(Q-V)^{ij}_\mu\epsilon_j - 2\gamma^\mu P_{[\mu\nu]}(Q-V)^{\nu ij}\epsilon_j\end{aligned} \tag{B.7}$$

and

$$\begin{aligned}\gamma_{\kappa\lambda\mu}T^{\kappa\lambda}\mathcal{D}^\mu\epsilon^i =\ & \gamma_{\kappa\lambda\mu\nu}T^{\kappa\lambda}P^{[\mu\nu]}\epsilon^i + \frac{3}{5}P^\kappa{}_\kappa T_{\mu\nu}\gamma^{\mu\nu}\epsilon^i - 2P_{[\mu\kappa]}T^\kappa{}_\nu\gamma^{\mu\nu}\epsilon^i \\ & + \gamma_{\kappa\lambda\mu}T^{\kappa\lambda}(Q-V)^{\mu ij}\epsilon_j,\end{aligned} \tag{B.8}$$

$$\gamma^\mu T_{\mu\nu}\mathcal{D}^\nu\epsilon^i = -P_{[\mu\nu]}T^{\mu\nu}\epsilon^i - P_{[\mu\kappa]}T^\kappa{}_\nu\gamma^{\mu\nu}\epsilon^i + \frac{1}{5}P^\kappa{}_\kappa T_{\mu\nu}\gamma^{\mu\nu}\epsilon^i + \gamma^\mu T_{\mu\nu}(Q-V)^{\nu ij}\epsilon_j. \tag{B.9}$$

The symmetric contraction. Multiplying the equation (2.6) by ϵ^j and symmetrizing in i and j we obtain

$$\begin{aligned}0 =\ & \frac{1}{8}\epsilon^{(i}\mathcal{D}^\mu\mathcal{D}_\mu\epsilon^{j)} + \frac{3\imath}{40}\epsilon^{(i}\gamma_{\kappa\lambda\mu}T^{\kappa\lambda}\mathcal{D}^\mu\epsilon^{j)} + \frac{11\imath}{40}\epsilon^{(i}\gamma^\mu T_{\mu\nu}\mathcal{D}^\nu\epsilon^{j)} \\ & + \frac{\imath}{4}\epsilon^{(i}\gamma_{\kappa\lambda\mu}\epsilon^{j)}\nabla^\mu T^{\kappa\lambda}.\end{aligned} \tag{B.10}$$

The individual components are

$$\epsilon^{(i}\mathcal{D}^\mu\mathcal{D}_\mu\epsilon^{j)} = \frac{s}{2}\left[\nabla^\mu(Q-V)^{ij}_\mu + (Q+V)_\mu{}^{(i}{}_k(Q-V)^{\mu j)k}\right] + \frac{1}{5}P^\kappa{}_\kappa v_\mu(Q-V)^{\mu ij} - v^\mu P_{[\mu\nu]}(Q-V)^{\nu ij}, \tag{B.11}$$

$$\epsilon^{(i}\gamma_{\kappa\lambda\mu}T^{\kappa\lambda}\mathcal{D}^\mu\epsilon^{j)} = \frac{3}{5}P^\kappa{}_\kappa T^{\mu\nu}\Theta^{ij}_{\mu\nu} - 2P_{[\mu\kappa]}T^\kappa{}_\nu\Theta^{ij\mu\nu} + \frac{1}{2}\epsilon_{\kappa\lambda\mu}{}^{\nu\rho}T^{\kappa\lambda}\Theta^{k(i}_{\nu\rho}(Q-V)^{\mu j)}{}_k, \tag{B.12}$$

$$\epsilon^{(i}\gamma^\mu T_{\mu\nu}\mathcal{D}^\nu\epsilon^{j)} = -P_{[\mu\kappa]}T^\kappa{}_\nu\Theta^{ij\mu\nu} + \frac{1}{5}P^\mu{}_\mu T^{\kappa\lambda}\Theta^{ij}_{\kappa\lambda} + \frac{1}{2}v_\mu T^{\mu\nu}(Q-V)^{ij}_\nu, \tag{B.13}$$

$$\epsilon^{(i}\gamma_{\kappa\lambda\mu}\epsilon^{j)}\nabla^\mu T^{\kappa\lambda} = -\frac{1}{2}\epsilon_{\kappa\lambda\mu}{}^{\nu\rho}\Theta^{ij}_{\nu\rho}\nabla^\mu T^{\kappa\lambda}. \tag{B.14}$$

Putting the various terms together we recover the expression (3.22).

The vector contraction. Multiplying the equation (2.6) with $\epsilon^i\gamma_\mu$ and contracting we obtain

$$0 = v_\mu\left(\frac{32D+R}{128} + \frac{1}{15}T_{\mu\nu}T^{\mu\nu}\right) + \frac{1}{8}\epsilon^i\gamma_\mu\mathcal{D}^\nu\mathcal{D}_\nu\epsilon_i + \frac{3\imath}{40}\epsilon^i\gamma_\mu\gamma_{\kappa\lambda\nu}T^{\kappa\lambda}\mathcal{D}^\nu\epsilon_i + \frac{11\imath}{40}\epsilon^i\gamma_\mu\gamma^\kappa T_{\kappa\lambda}\mathcal{D}^\lambda\epsilon_i + \frac{\imath}{4}\epsilon_\mu{}^{\nu\kappa\lambda\sigma}v_\sigma\nabla_\nu T_{\kappa\lambda} + \frac{\imath s}{2}\nabla^\nu T_{\mu\nu} - \frac{s}{5}\epsilon_\mu{}^{\kappa\lambda\sigma\tau}T_{\kappa\lambda}T_{\sigma\tau}. \tag{B.15}$$

The most involved terms are given by

$$\epsilon^i\gamma_\mu\mathcal{D}^\nu\mathcal{D}_\nu\epsilon_i = \frac{s}{5}\nabla_\mu P^\kappa{}_\kappa - s\nabla^\nu P_{[\mu\nu]} - \frac{2}{5}P^\kappa{}_\kappa\Theta^{ij}_{\mu\nu}(Q-V)^\nu_{ij} + 2\Theta^{ij}_{\mu\nu}P^{[\nu\rho]}(Q-V)_{\rho ij} + v_\mu\left[\frac{1}{5}(P^\kappa{}_\kappa)^2 + P_{[\mu\nu]}P^{[\mu\nu]} - \frac{1}{2}(Q-V)^{ij}_\nu(Q-V)^\nu_{ij}\right], \tag{B.16}$$

$$\epsilon^i\gamma_\mu\gamma_{\kappa\lambda\nu}T^{\kappa\lambda}\mathcal{D}^\nu\epsilon_i = s\epsilon_{\mu\kappa\lambda\sigma\tau}T^{\kappa\lambda}P^{[\sigma\tau]} + \frac{6}{5}P^\kappa{}_\kappa T_{\mu\nu}v^\nu - 2(P_{[\mu\rho]}T^\rho{}_\nu - P_{[\nu\rho]}T^\rho{}_\mu)v^\nu - (Q-V)_{\mu ij}\Theta^{ij}_{\kappa\lambda}T^{\kappa\lambda} - 2T_{\mu\nu}\Theta^{ij\nu\rho}(Q-V)_{\rho ij}, \tag{B.17}$$

$$\epsilon^i\gamma_\mu\gamma^\kappa T_{\kappa\lambda}\mathcal{D}^\lambda\epsilon_i = -v_\mu P_{[\kappa\lambda]}T^{\kappa\lambda} - (P_{[\mu\rho]}T^\rho{}_\nu - P_{[\nu\rho]}T^\rho{}_\mu)v^\nu + \frac{2}{5}P^\kappa{}_\kappa T_{\mu\nu}v^\nu - \Theta^{ij}_{\mu\kappa}T^{\kappa\lambda}(Q-V)_{\lambda ij}. \tag{B.18}$$

Finally putting the various terms together and projecting on the vertical component we recover the equation (3.24). While projecting on the horizontal component we recover the equation (3.25).

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6.4 Nekrasov-Shatashvili limit of 5D superconfromal index

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Nekrasov-Shatashvili limit of the 5D superconformal index

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We consider the Nekrasov-Shatashvili limit of the five-dimensional (5D) superconformal index and propose a novel prescription for selecting the finite contributions. Applying the latter to various examples of U(1) theories, we find that the 5D Nekrasov-Shatashvili index can be reproduced using recent techniques of Córdova and Shao, who related the 4D Schur index to the Bogomol'nyi-Prasad-Sommerfield (BPS) spectrum of the theory on the Coulomb branch. In this picture, the 5D instanton solitons are interpreted as additional flavor nodes in an associated 5D BPS quiver.

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I. INTRODUCTION AND SUMMARY

The superconformal index has proven to be an important tool in the study of superconformal field theories (SCFTs) in diverse dimensions [1,2]. In some cases, interesting limits of the index have been devised, which isolate contributions from particular subsets of operators and provide information about its different phases; see e.g Ref. [3]. Limits of the index also help in identifying algebraic structures hidden within special subsectors of the theory, a fact which has been put to remarkable effect in four and six dimensions [4].

In a closely related direction, recent work [5] established a connection between the so-called Schur limit of the four-dimensional (4D) index on the one hand[1] and a certain algebraic quantity associated with the BPS spectrum of particles on the Coulomb branch on the other—the trace of the Kontsevich–Soibelman (KS) operator—for a convincing number of 4D $\mathcal{N}=2$ SCFTs[2]; see also Ref. [8] for generalizations. In this fashion, one demonstrates that, for specific BPS subsectors, the operator spectrum of an SCFT is directly related to the particle spectrum of the same theory in a phase where the conformal symmetry has been broken.

In this paper, we would like to import some of these results to five-dimensional SCFTs [9–12]. Our first objective will be to define a limit of the five-dimensional (5D) superconformal index by turning off one of the two Ω-deformation parameters[3]; this is the limit first considered by Nekrasov and Shatashvili (NS) in a four-dimensional context [13]. Its naive implementation leads to a singular index, which calls for a prescription on how to extract the finite parts. This problem can in principle be addressed in a way similar to the original NS limit of [13]. However, the direct 5D extension of that recipe leads to a function of which the fugacity expansion does not necessarily involve integer coefficients. In turn, we propose a different 5D regularization which results in a fugacity expansion with integer coefficients for arbitrary gauge groups. In the Abelian case, our regularization clearly isolates contributions from states localized on a four-dimensional subspace of the Euclideanized spacetime. Moreover, it reproduces, at least for the perturbative sector, the large-orbifold limit of the gauge theory index of Ref. [14]. The latter effectively reduces the space down to a 4D geometry—of the form $M_3\times S^1$—where the contributions of vector and hypermultiplets become identical to the 4D Schur index and may hint toward an interesting connection with Ref. [4]. Although our limit does not lead to a counting of states preserving a larger fraction of supersymmetry,[4] it does lead to a factorization of the index into a "holomorphic" and "antiholomorphic" part for general 5D SCFTs. This factorization is reminiscent of the work of Iqbal and Vafa [15], where it also appeared as the starting point for connecting the 5D BPS-particle degeneracy[5] to the index, using the topological string.

With this last point in mind, our second objective will be to relate the NS limit of the 5D index to the work of Ref. [5]. For a number of Abelian examples, we will show that the NS index can be reproduced by the trace of the KS operator for a "5D BPS quiver." This quiver can be constructed straightforwardly by assigning a node for each "partonic BPS state."[6]; This involves a node corresponding to the instanton-soliton parton of the 5D theory, as well as a

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[1]Recent exact results on the 4D Schur index include Ref. [6].

[2]For an alternative calculation of the Schur index for Argyres-Douglas theories, see Ref. [7].

[3]Since the precise operator spectrum of the interacting 5D UV theories is unknown, one usually works with the realization of the index as a supersymmetric partition function on $S^4\times S^1$ with twisted boundary conditions for the various fields.

[4]Interesting limits of the 4D index with additional supersymmetry were originally considered in Ref. [3].

[5]Note that in 5D there also exist BPS strings.

node for each of the possible N_f hypermultiplets of the theory. The construction and study of the 5D BPS quiver for non-Abelian theories, and their possible connection to the NS index, is a question that we will leave open for future investigation. However, our Abelian results can already be thought of as a check of the proposal of Ref. [15], for a particular subsector of five-dimensional theories.

The rest of this article is organized as follows. In Sec. II, we will present the details of the NS limit for the 5D index, after briefly reviewing some background material necessary for our discussion. Then, in Sec. III, we will introduce the algebraic tools of Ref. [5] and use them to recover our index for U(1) theories with different matter content and values of the Chern-Simons coefficient. We will also discuss some directions for generalizing these results to non-Abelian gauge groups.

II. NEKRASOV-SHATASHVILI LIMIT OF THE 5D INDEX

A. Generalities

The superconformal index in five dimensions was first defined in Ref. [2] and computed using supersymmetric localization [16] for a variety of $\mathcal{N}=1$ theories in Ref. [17]. Recall that, using a Verma module construction, one can obtain all irreducible representations of the 5D superconformal algebra (SCA) $F(4)$ from irreducible representations of the maximal compact subalgebra $\mathfrak{so}(2)_E \oplus \mathfrak{so}(5) \oplus \mathfrak{su}(2)_R$. The latter are labelled by strings of quantum numbers denoting the highest weight state $\{\epsilon_0, R, h_1, h_2\}$, where h_1, h_2 are the Cartan generators of $\mathfrak{so}(5)$,[7] while ϵ_0 is the scaling dimension measured by the charge under $\mathfrak{so}(2)_E$. Finally, the $\mathfrak{su}(2)_R$ Cartan generator is denoted by R.[8]

In the radial quantization of the theory, where $S = Q^\dagger$, and for a particular choice of supercharge,[9] one can define

$$\delta \coloneqq \{Q, S\} = \epsilon_0 - h_1 - h_2 - 3R, \qquad (2.1)$$

which is a positive-definite quantity. The index is a partition function counting operators transforming in irreducible representations of the subalgebra of the SCA that (anti) commute with the above Q, S (these are $\frac{1}{8}$-BPS) and hence also δ—or equivalently, Irreducible representations of the commutant of (Q, S, δ) of the 5D SCA. It is straightforward to see that $h_1 + R$ and $h_2 + R$ commute with the above choice of δ, and as a result the most general, or "refined," index with respect to the supercharge Q is given by [2,17,18]

$$I = \mathrm{Tr}_{\mathcal{H}_{\delta=0}} (-1)^F p^{h_1+R} q^{h_2+R} \prod_a w_a^{\mathfrak{Q}_a} \mathfrak{q}^k. \qquad (2.2)$$

Here, the trace is taken over the Hilbert space of $\delta = 0$ operators, $F = 2h_1$ is the fermion number operator, p and q are fugacities keeping track of the elements of the commutant, and the w_a are additional fugacities for commuting charges $\mathfrak{Q}_a$, corresponding to possible global/gauge symmetries. One such commuting charge corresponds to a topological U(1) symmetry which is always present in the examples we are interested in: 5D gauge theories possess a conserved current, $*J = \frac{1}{8\pi^2}\mathrm{tr}(F \wedge F)$, and their spectrum contains instanton solitons, charged under the associated symmetry. This global symmetry plays an important role in five dimensions, where SCFTs with very interesting properties exist [9]; in many cases, it can combine with and enhance other symmetries (flavor, Lorentz); see e.g. Refs. [17,19,20]. Indeed, one can also include a fugacity $\mathfrak{q}$ in the index (2.2), which keeps track of the instanton charge k, where $|\mathfrak{q}| = 1$.

Via the state-operator map, the 5D index can alternatively be evaluated by a Euclidean path integral on $S^4 \times S^1$ with twisted boundary conditions for the various fields according to their charges [2,17,21]. The index then counts $\frac{1}{8}$-BPS states for the theory on the sphere. This functional integral can be evaluated in the IR theory[10] using localization [17], and the answer reduces to a gauge-group integral over the product of perturbative and nonperturbative contributions, schematically

$$I = \int [\mathrm{d}U] Z^{S^4}_{\mathrm{pert}} Z_{\mathrm{nonpert}}, \qquad (2.3)$$

with $[\mathrm{d}U]$ the unit-normalized Haar measure. The nonperturbative factor can be written as

$$Z_{\mathrm{nonpert}} = |Z_{\mathrm{Nek}}|^2, \qquad (2.4)$$

where Z_{Nek} is the Nekrasov instanton partition function [24]. The perturbative contribution is a modular quantity built out of the weak-coupling multiplets. The vector-multiplet and hypermultiplet contributions are given by

$$I_{V,H} = \mathrm{PE}[f_{V,H}], \qquad (2.5)$$

where PE refers to the plethystic exponential. The so-called single-letter indices appearing above in turn read[11]

[6]By this, we mean states with the lowest possible charges, i.e. ones that cannot be written as bound states of any other states.

[7]These are related to the Ω-deformation parameters ϵ_1, ϵ_2 in a simple way.

[8]As is common in the literature, we will use the same symbols for the Cartan generators and the corresponding charges, depending on the context.

[9]We follow the conventions and choices of Ref. [17].

[10]For a generic SCFT on $\mathbb{R} \times S^4$, it is possible to turn on supersymmetrically a position-dependent Yang-Mills coupling, interpolating between the SCFT and the IR gauge theory [22,23].

[11]For definiteness, we will assume that the hypermultiplet is in the fundamental of the gauge group.

$$f_V = -\frac{p+q}{(1-p)(1-q)}\chi_{\mathrm{Adj}},$$
$$f_H = \frac{\sqrt{pq}}{(1-p)(1-q)}(\chi_{\Box} + \chi_{\bar{\Box}}), \qquad (2.6)$$

with $\chi_{\mathcal{R}}$ denoting the character of a given representation $\mathcal{R}$.

B. NS index

Having set the stage, we would like to investigate whether there exist limits of the index (2.2) which only receive contributions from certain sectors of the theory, as e.g. is the case in 4D [3]. Note that, as opposed to other dimensions, the 5D index only depends on two fugacities. Moreover, these correspond to Cartans of SU(2) symmetries, a fact which underlies the $(p,q) \leftrightarrow (q^{-1}, p^{-1})$ and $(p,q) \leftrightarrow (q,p)$ invariance of the index; cf. Eq. (2.6). Thus, it is hard to imagine nontrivial regular limits as in Ref. [3]. Yet, this does not exclude interesting singular limits. In particular, following [13], we will focus on the NS limit of the index. Generically, the NS limit involves sending one of the two Ω-deformation parameters to zero, $\epsilon_1 \to 0$ while keeping the other one, ϵ_2, fixed. These parameters are chemical potentials for rotations in two real planes, $\mathrm{SO}(2)_{\epsilon_1} \times \mathrm{SO}(2)_{\epsilon_2} \subset \mathrm{SO}(5)$, and related to our choice of fugacities through $p = e^{-\epsilon_1}$ and $q = e^{-\epsilon_2}$. Hence, one can naively implement the NS limit directly at the level of the index, by considering

$$p \to 1 \quad \text{and} \quad q \to \text{fixed}. \qquad (2.7)$$

Although this definition is natural, it leads to divergences as can be immediately seen by applying it to the perturbative contributions (2.6). We therefore need to put forward a modified definition for taking the NS limit of the 5D index, which leads to finite contributions.

Toward that end, we follow Ref. [18] and rewrite the index of the full theory on $S^4 \times S^1$ in terms of two "hemisphere indices" on $D^4 \times S^1$ with Dirichlet boundary conditions, where $D^4 \subset S^4$ is half the sphere. The hemisphere index is in turn defined by

$$II = Z^{D^4}_{\mathrm{pert}} Z_{\mathrm{Nek}}. \qquad (2.8)$$

For the example of a single vector multiplet and a hypermultiplet in the fundamental representation, the perturbative piece reads

$$Z^{D^4}_{\mathrm{pert}} = \mathrm{PE}\left[-\frac{p+q}{(1-p)(1-q)}\chi_{\mathrm{Adj}} + \frac{\sqrt{pq}}{(1-p)(1-q)}\chi_{\Box}\right], \qquad (2.9)$$

where the gauge symmetry of the full index on $S^4 \times S^1$ is to be understood as a global boundary symmetry.

The full index is then computed by combining two such contributions and gauging the appropriate diagonal subgroup of said global symmetries to obtain

$$I = (I_V^{4D})^r \int [\mathrm{d}U] II \bar{II}, \qquad (2.10)$$

where the overline implies that one inverts all gauge/flavour fugacities. The term

$$I_V^{4\mathrm{D}} = \mathrm{PE}\left[-\frac{p}{1-p} - \frac{q}{1-q}\right] \qquad (2.11)$$

is a purely four-dimensional $\mathcal{N}=1$ vector-multiplet contribution coming from the boundary, and $r = \mathrm{rank}(G)$ is the gauge group rank.

We are now in the position to define the NS index as follows,

$$\text{NS index}: \; II^{\mathrm{NS}}(z_i, \mathfrak{q}; q) \coloneqq \mathrm{PE}\left[\lim_{p\to 1}(1-p)\mathrm{PE}^{-1}[II(z_i, \mathfrak{q}; p, q)]\right], \qquad (2.12)$$

such that

$$I^{\mathrm{NS}}(\mathfrak{q}; q) \coloneqq \int [\mathrm{d}U] II^{\mathrm{NS}}(z_i, \mathfrak{q}; q)\overline{II^{\mathrm{NS}}(z_i, \mathfrak{q}; q)}. \qquad (2.13)$$

Note that we have stripped off the (divergent in this limit) factors of $I_V^{4\mathrm{D}}$. We will come back to this below.

We stress that this definition of the NS limit is different from other versions where the PE in (2.12) is traded for a standard exponential and results in a function of which the fugacity expansion does not necessarily involve integer coefficients; see Refs. [13,25]. On the other hand, Eq. (2.13) does admit an expansion with integer coefficients, due to the use of the PE.

In the above, the z_i, $i = 1, \ldots, r$, are gauge/global symmetry fugacities, and the plethystic logarithm, PE^{-1}, is the inverse of the plethystic exponential, defined as

$$\mathrm{PE}^{-1}[g(t)] \coloneqq \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log[g(t^n)], \qquad (2.14)$$

with $\mu(n)$ the Möbius function. This factorization of the superconformal index in the NS limit is reminiscent of the discussion in Ref. [15], where the full index was calculated using the refined topological vertex formalism and related to the counting of BPS states on the Coulomb branch of the theory. We will see in the next section that the relationship to 5D BPS quivers can be quantified for $G = \mathrm{U}(1)$ through the formalism of Ref. [5].

1. Perturbative NS limit

Since our prescription for the NS limit (2.12) factorizes over the perturbative and nonperturbative contributions, let us first look at the former. From (2.9), it is straightforward to deduce that

$$Z_{\text{pert}}^{D^4,\text{NS}} = \text{PE}\left[-\frac{q}{(1-q)}\chi_{\text{Adj}} + \frac{\sqrt{q}}{(1-q)}\chi_{\Box}\right], \quad (2.15)$$

and consequently if we only focus on the perturbative sector,

$$\int [\text{d}U] Z_{\text{pert}}^{D^4,\text{NS}} \overline{Z_{\text{pert}}^{D^4,\text{NS}}} = \int [\text{d}U] \text{PE}\left[-\frac{2q}{(1-q)}\chi_{\text{Adj}} + \frac{\sqrt{q}}{(1-q)}(\chi_{\Box} + \chi_{\bar{\Box}})\right]. \quad (2.16)$$

This is tantamount to projecting out states with a nontrivial x_{++} dependence, as can be seen by taking the NS limit directly on the full 5D single-letter indices.

This requires an equivalent prescription for which it is convenient to introduce fugacities $x = \sqrt{pq}$, $y = \sqrt{q/p}$. Note that, after performing this substitution in Eq. (2.2), the exponents of the x and y fugacities are given respectively by $h_1 + h_2 + 2R = 2j_1 + 2R$ and $-h_1 + h_2 = -2j_2$. In terms of these, the NS index for the hypermultiplet can be implemented by taking $y \to x$. More precisely,

$$f_H^{\text{NS}} = \lim_{\epsilon_1 \to 0} \epsilon_1 f_H(x, x(1+\epsilon_1)). \quad (2.17)$$

In this fashion, the NS index picks out the coefficient of the $\frac{1}{\epsilon_1}$ pole in the naive $\epsilon_1 \to 0$ limit of f_H. Recall that for the free hypermultiplet the single particle index f_H can be understood in terms of letter counting using the state-operator map [17]. Using Table I, one immediately sees that f_H contains operators made out of letters of the form $\partial_{+\pm}^m \mathcal{O}$; here, $\mathcal{O}$ is a scalar or fermionic component of the hypermultiplet, and the derivatives are responsible for the factor $(1-p)(1-q) = (1-xy)(1-\frac{x}{y})$ appearing in the denominator of (2.6). In the limit $\epsilon_1 \to 0$, one such derivative becomes of zero weight. This results in a divergence in the limit $y \to x$, originating from an unrefinement in the index which now counts letters containing arbitrary powers of ∂_{++} with the same weight (zero). Defining the NS index through selecting the pole in (2.17) is tantamount to only accounting for the contribution with no derivatives.

TABLE I. The letters in the hypermultiplet and their respective charges.

	ϵ_0	(j_1, j_2)	R
q	$\frac{3}{2}$	(0, 0)	$\pm\frac{1}{2}$
ψ	2	$(\pm\frac{1}{2}, 0) \oplus (0, \pm\frac{1}{2})$	0
∂	1	$(\pm\frac{1}{2}, \pm\frac{1}{2}) \oplus (0, 0)$	0

Somewhat surprisingly, the vector-multiplet piece can also be given an IR-operator interpretation. In such a scenario, one can understand the single-letter vector-multiplet contribution as arising from components of the gaugino plus a tower of infinitely many derivatives. In the limit $\epsilon_1 \to 1$, not only the weight of a derivative but also one of the components of the gaugino become zero. These translate into singularities of the index, and our prescription amounts to regularizing them by discarding zero-weight letters.

Hence, at the level of implementation, the following single-letter functions can be used for the perturbative contributions in the NS limit:

$$f_V^{\text{NS}} = -\frac{2q}{(1-q)}\chi_{\text{Adj}}, \qquad f_H^{\text{NS}} = \frac{\sqrt{q}}{(1-q)}(\chi_{\Box} + \chi_{\bar{\Box}}). \quad (2.18)$$

We highlight that these single-letter terms are precisely the vector and hypermultiplet single-letter index contributions for the perturbative sector of $\mathcal{N} = 2$ four-dimensional theories in the Schur limit [3,6], which may hint at a connection with the results of Ref. [4]. It is also interesting to observe that the large-orbifold limit of Ref. [14] also led to perturbative contributions identical to those of the 4D Schur index.[12]

All in all, in the perturbative sector our NS limit discards states with a dependence on the x_{++} direction on D^4, along with the boundary $\mathcal{N} = 1$ vector-multiplet contributions I_V^{4D}. This is equivalent to using the single-letter expressions (2.18) directly in (2.5). We will next see that this interpretation extends to the nonperturbative sector for Abelian theories.

2. Nonperturbative NS limit

The result of the prescription (2.12) on the nonperturbative piece is somewhat more involved. This is due to the fact that, with the exception of the Abelian case, the Nekrasov partition function cannot be written as a PE of single-letter contributions but is evaluated as an expansion in powers of the instanton fugacity $\mathfrak{q}$,

$$Z_{\text{Nek}} = \sum_{k=0}^{\infty} \mathfrak{q}^k Z_{\text{Nek}}^{(k)} \quad \text{with} \quad Z_{\text{Nek}}^{(0)} = 1. \quad (2.19)$$

We will henceforth assume that the NS limit commutes with the instanton expansion and then use this along with (2.12) to get

[12] Recall that Ref. [14] considered the 5D theory on $S^4/\mathbb{Z}_n \times S^1$ in the large-n limit. This effectively dimensionally reduced the space down to a (singular) 4D geometry.

$$\begin{aligned}
Z^{\rm NS}_{\rm Nek}(z_i,\mathfrak{q};q) &= {\rm PE}\left[\lim_{p\to1}(1-p){\rm PE}^{-1}\left[\sum_{k=0}^{\infty}\mathfrak{q}^k Z^{(k)}_{\rm Nek}(z_i;p,q)\right]\right]\\
&= {\rm PE}\left[\lim_{p\to1}(1-p){\rm PE}^{-1}[1+\mathfrak{q}Z^{(1)}_{\rm Nek}(z_i;p,q)+\mathfrak{q}^2Z^{(2)}_{\rm Nek}(z_i;p,q)+O(\mathfrak{q}^3)]\right]\\
&= {\rm PE}\left[\lim_{p\to1}(1-p)\left(\mathfrak{q}Z^{(1)}_{\rm Nek}(z_i;p,q)\right.\right.\\
&\quad\left.\left.+\mathfrak{q}^2\left(Z^{(2)}_{\rm Nek}(z_i;p,q)-\frac{1}{2}Z^{(1)}_{\rm Nek}(z_i;p,q)^2-\frac{1}{2}Z^{(1)}_{\rm Nek}(z_i^2;p^2,q^2)+O(\mathfrak{q}^3)\right)\right]\\
&= 1+\mathfrak{q}\lim_{p\to1}(1-p)Z^{(1)}_{\rm Nek}(z_i;p,q)\\
&\quad+\mathfrak{q}^2\lim_{p\to1}\left((1-p)\left(Z^{(2)}_{\rm Nek}(z_i;p,q)-\frac{1}{2}Z^{(1)}_{\rm Nek}(z_i;p,q)^2-\frac{1}{2}Z^{(1)}_{\rm Nek}(z_i^2;p^2,q^2)\right)\right.\\
&\quad\left.+(1-p)^2\frac{Z^{(1)}_{\rm Nek}(z_i;p,q)^2}{2}+(1-p^2)\frac{Z^{(1)}_{\rm Nek}(z_i^2;p^2,q^2)}{2}\right)+O(\mathfrak{q}^3)\\
&=: \sum_{k=0}^{\infty}\mathfrak{q}^k Z^{{\rm NS},(k)}_{\rm Nek}(z_i;q). \qquad (2.20)
\end{aligned}$$

This proposal is obviously applicable to the case of $G = \mathrm{U}(1)$, where, as we will see shortly, the instanton expansion can be explicitly resummed into a PE. For example, for a pure U(1) theory, one has

$$Z_{\rm nonpert} = {\rm PE}\left[\frac{\sqrt{pq}}{(1-p)(1-q)}(\mathfrak{q}+\mathfrak{q}^{-1})\right]. \qquad (2.21)$$

In that context, the NS limit once again explicitly counts states which do not have any dependence on the x_{++} direction.[13] However, the definition (2.20) also makes sense for the case of non-Abelian gauge groups, where $Z^{(k)}_{\rm Nek}$ can be expanded in q to yield terms with integer coefficients, as expected for an index. We have explicitly checked this to sufficiently high order for $G = \mathrm{SU}(2)$.

As raised above, we should emphasize that a version of the NS limit for the Nekrasov partition function has already been considered in Ref. [25], along the lines of Ref. [13]. This is a different limit from the one discussed here, insofar as it involves replacing plethystic exponentials with exponentials and plethystic logarithms with logarithms. Our motivation for (2.12) stems from requiring finite coefficients in the fugacity expansion and mirroring the definition of the 4D limits of Ref. [3], which act directly on the single-letter indices.

[13]One can also ascribe an IR-operator interpretation to the Abelian instanton partition function, as the PE of single-letter contributions from instanton operators [19,23].

III. KONTSEVICH–SOIBELMAN OPERATORS AND BPS QUIVERS

Having provided our definition for the NS index, one can establish a connection with Ref. [5]. In that reference—see also Ref. [8]—it was conjectured that the 4D Schur index of a rank-r theory can be recovered in terms of quantities associated with the BPS quiver of the theory [26] through

$$I_{\rm KS} = (q)_\infty^{2r}{\rm Tr}[\mathcal{O}], \qquad (3.1)$$

where the Pochhammer symbol is defined as

$$(q)_0 = 1, \qquad (q)_n = \prod_{k=1}^{n}(1-q^k). \qquad (3.2)$$

Here, the quantity $\mathcal{O}$ is the KS operator associated with the BPS quiver of the four-dimensional gauge theory. Such a theory contains a set of BPS particles on the Coulomb branch labelled by a vector γ in the charge lattice Γ. Then, for each γ, one introduces a formal variable X_γ obeying a quantum torus algebra,

$$X_\gamma X_{\gamma'} = q^{\frac{\langle\gamma,\gamma'\rangle}{2}}X_{\gamma+\gamma'} = q^{\langle\gamma,\gamma'\rangle}X_{\gamma'}X_\gamma, \qquad (3.3)$$

where $\langle\cdot,\cdot\rangle$ is the (integer) Dirac pairing of charges in the lattice Γ, which can be read off from the BPS quiver. In terms of these X_γ, the KS operator can be explicitly written as

$$\mathcal{O} = \prod_\gamma E_q(X_\gamma), \qquad (3.4)$$

where E_q is the q-exponential function

$$E_q(z) = \prod_{i=0}^{\infty}(1+q^{i+\frac{1}{2}}z)^{-1} = \sum_{n=0}^{\infty}\frac{(-q^{\frac{1}{2}}z)^n}{(q)_n}. \tag{3.5}$$

For a theory without flavor, the trace of the quantum torus algebra is defined by its action on the formal variables X_γ,

$$\mathrm{Tr}[X_\gamma] = \begin{cases} 1 & \gamma = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

and extending linearly. For theories with flavor, there exist flavor charge vectors γ_f, which have zero Dirac pairing with all other $\gamma' \in \Gamma$, $\langle\gamma_f, \gamma'\rangle = 0$. Moreover, the definition of the trace needs to be modified to

$$\mathrm{Tr}[X_\gamma] = \begin{cases} \prod_i \mathrm{Tr}[X_{\gamma_{f_i}}]^{f_i(\gamma)} & \langle\gamma,\gamma'\rangle = 0 \quad \forall\ \gamma' \in \Gamma \\ 0 & \text{otherwise} \end{cases}, \tag{3.7}$$

where γ_{f_i} is an integral basis of flavor charges and $f_i(\gamma)$ is the flavor charges of γ. The $\mathrm{Tr}[X_{\gamma_{f_i}}]$ are free quantities that are to be identified with the flavor fugacities appearing in the index. Using the above machinery, the 4D Schur index can be read off from the BPS quiver [5].

In view of the similarities between the NS limit of the 5D index discussed above and the Schur index for an $\mathcal{N} = 2$ 4D theory with the same number of vector and hypermultiplets, it is natural to wonder whether a decomposition in terms of 5D BPS quiver data also exists. In fact, Iqbal and Vafa have used the topological string [15] to argue that the 5D BPS-particle spectrum reproduces the superconformal index.

We will next provide a simple but concrete realization of this idea, relating the NS index to the trace of the KS operator for a number of Abelian examples. At this point, we should make it clear that there exist no nontrivial Abelian fixed points in five dimensions, and one may be alarmed that the notion of the superconformal index is ill defined. However, the quantity Eq. (2.3), and its subsequent NS limit, is meaningful even for nonconformal theories, and it is this definition that we will use in the upcoming discussion.[14]

We have already seen that the existence of BPS instanton particles in 5D leads to index contributions with a new global fugacity, related to the topological charge. It is therefore natural to suspect that any 5D extension of the Schur-KS correspondence must involve a BPS quiver where at least one extra node, corresponding to the BPS instanton particle, is added.

Unlike four dimensions, the five-dimensional central charge is real, and the BPS states are divided into CPT-conjugate pairs. The states with the lowest possible charges (the "partonic" BPS states) comprise W-bosons and quarks, instanton solitons and magnetically charged BPS strings; see e.g. Ref. [12]. The existence of BPS strings makes the identification of the appropriate five-dimensional non-Abelian generalization of the BPS quiver subtle.[15] However, for Abelian theories with N_f flavors, BPS-string states are absent, and one can straightforwardly construct a 5D quiver comprising only of an instanton-particle node and a node for each of the N_f flavors, with no arrows extending between them.

In the following section, we will show that the Abelian NS index can be reexpressed to match the trace of the KS operator for the corresponding 5D BPS quiver. We will also comment on the possible extension to non-Abelian gauge groups.

A. Abelian theories

For Abelian theories, the nonperturbative contribution is particularly simple. This allows for a straightforward reinterpretation of their NS index in terms of quiver data. The instanton partition function for the U(1) theory with F flavors and Chern-Simons (CS) level κ can be borrowed from Ref. [17][16]:

$$Z^{(k)}_{\text{Nekrasov}} = \frac{(2i)^{k(F-3)}}{k!} \times \int \prod_{I=1}^{k}\frac{d\phi_I}{2\pi}\frac{e^{i\kappa\phi_I}(\sin\frac{\phi_I}{2})^F \prod_{I\neq J}\sin\frac{\phi_I-\phi_J}{2}\prod_{I,J}\sin\frac{\phi_I-\phi_J-2i\gamma_1}{2}}{\prod_{i=1}^{N}\sin\frac{\phi_I-\alpha_i-i\gamma_1}{2}\sin\frac{-\phi_I+\alpha_i-i\gamma_1}{2}\prod_{I,J}\sin\frac{\phi_I-\phi_J-i\gamma_1-i\gamma_2}{2}\sin\frac{\phi_I-\phi_J-i\gamma_1+i\gamma_2}{2}}. \tag{3.8}$$

[14] Having said that, "SU(1) theories" can exist at fixed points, since they correspond to pq brane webs which can be collapsed to an intersection of five branes at a point. For instance, a pure "SU(1)" theory can be engineered in the NS-D5 intersection and corresponds to a pure U(1) gauge theory where the perturbative vector multiplet is removed. The leftover instanton sector, behaving as a hypermultiplet, then still remains. Thus, our Abelian computations can be understood in terms of these SU(1) theories, which often appear in quiver tails (e.g. Ref. [27]).

[15] For example, the results in Ref. [15] suggest that only BPS particles are important in reproducing the index.

[16] Compared to that reference, we have taken the limits of chemical potentials $m_l \to 1$ for simplicity.

Recall that, as is well known, integrating out a massive flavor produces a shift to the CS level by a factor of $\Delta\kappa = \frac{\mathrm{sign}(m)}{2}$. As a consequence, odd F requires a half-integer κ. In order to take the NS limit of the index, we shall rewrite the above expression using the fugacities p and q, as well as a gauge fugacity u:[17]

$$p = e^{-(\gamma_1+\gamma_2)}, \qquad q = e^{-(\gamma_1-\gamma_2)}, \qquad u = e^{i\alpha}. \tag{3.9}$$

We will next consider specific cases by fixing the CS level and the number of flavors.

1. Pure $\mathbf{U(1)_{\pm 1}}$ theory

Let us consider the pure U(1) theory. The bound in Ref. [11] requires $|\kappa| = 0, 1$. Setting $\kappa = 1$, we find from Eq. (3.8)

$$Z^{(1)}_{\mathrm{Nekrasov}} = \frac{1}{u}\frac{pq}{(1-p)(1-q)}. \tag{3.10}$$

As discussed in Ref. [28], the instanton contributions should be invariant under a transformation that simultaneously sends $p \to 1/q$ and $q \to 1/p$; this is a transformation that is part of the superconformal group, under which the perturbative single-letter indices are invariant. However, as it stands, Eq. (3.10) is not invariant, and this presents a problem.

Recall that this issue typically arises whenever the corresponding brane configuration involves parallel external 5-brane legs. Indeed, in the case of $SU(N)_N$ theories, the brane web includes a pair of external parallel NS5 branes. In the process of computing the instanton contributions by decoupling the U(1) factor from the $U(N)_N$ theory, one finds that the naive result does not exhibit the expected $p \to 1/q$ and $q \to 1/p$ invariance. As first argued in Ref. [28], this noninvariance can be traced back to extra states left over from the naive truncation, which in the brane web description correspond to D-strings stretched between the parallel external NS5s. These can slide off to infinity and hence should not be taken into account.

The discarded contribution from Ref. [28] turns out to be precisely equal to the naive $U(1)_1$ instanton piece (3.10). As a result, going over the same brane web argument, we conclude that (3.10) corresponds to states which should not be counted in the 5D theory. Upon removing them, we are left with $Z^{(1)}_{\mathrm{Nekrasov}} = 0$, so that the full instanton contribution in this case is simply unity. Note that, had we chosen the other sign for the CS level, $\kappa = -1$, we would have found the same function upon taking $u \to u^{-1}$. This is tantamount to exchanging instantons with anti-instantons, and the previous discussion goes through unchanged.

All in all, this theory has a trivial instanton sector; the index is purely perturbative and coincides with the Schur index of a four-dimensional $\mathcal{N} = 2$ theory with the same gauge and flavor symmetries. Since there are no BPS particles in this rank-1 theory, the corresponding 5D BPS quiver is trivial. One can therefore simply express the answer in the general form of (3.1) by writing

$$I_{\mathrm{KS}} = (q)^2_\infty. \tag{3.11}$$

2. Pure $\mathbf{U(1)_0}$

In four dimensions, the Schur index of the pure U(1) theory at zero CS level, $\kappa = 0$, simply reads

$$I^{4\mathrm{D}} = \mathrm{PE}\left[-\frac{2q}{(1-q)}\right] = \prod_{n=1}^{\infty}(1-q^n)^2 = (q)^2_\infty. \tag{3.12}$$

In turn, the BPS quiver in 4D is trivial, and therefore

$$\mathrm{Tr}[\mathcal{O}] = 1. \tag{3.13}$$

This fits the pattern of Ref. [5], since from (3.1) one also recovers that $I_{KS} = (q)^2_\infty$.

Let us now go to five dimensions. The exact index of the pure U(1) theory in 5D was worked out in Ref. [29]. This is

$$I^{5\mathrm{D}}_{U(1)_0} = \mathrm{PE}\left[-\frac{p+q}{(1-p)(1-q)} + \frac{\sqrt{pq}(\mathfrak{q}+\mathfrak{q}^{-1})}{(1-p)(1-q)}\right]. \tag{3.14}$$

The first term is a free vector multiplet, while the second looks like a hypermultiplet with the gauge fugacities replaced by the instanton fugacities, $\mathfrak{q}$. We can therefore use (2.18) to infer the corresponding NS index

$$I^{5\mathrm{D,NS}}_{U(1)_0} = \mathrm{PE}\left[-\frac{2q}{(1-q)} + \frac{\sqrt{q}(\mathfrak{q}+\mathfrak{q}^{-1})}{(1-q)}\right]. \tag{3.15}$$

As the instanton contribution is similar to that of a hypermultiplet, and in view of the fact that a free hypermultiplet contributes a flavor node to the BPS quiver [5], it is natural to suspect that there is a 5D BPS quiver description containing one node and yielding the correct 5D NS index.

In order to confirm this prediction, let us first pause to consider the nonperturbative part of the index (3.15). Concentrating on instantons alone, one can rewrite their contribution as

$$\mathrm{PE}\left[\frac{\sqrt{q}\mathfrak{q}}{(1-q)}\right] = \sum_{m=0}^{\infty}\frac{(\sqrt{q}\mathfrak{q})^m}{\prod_{k=1}^{m}(1-q^k)} = E_q(-\mathfrak{q}), \tag{3.16}$$

where in the last step we used Eq. (3.5). As an aside, it is interesting to observe that the above expression can be

[17] The chemical potentials γ_1, γ_2 appearing here are not related to the vectors γ of the charge lattice Γ. We hope that this notation, which is compatible with the literature, will not cause confusion.

identified with the 5D ("K-theoretic") vortex partition function [30].[18] In fact, the NS limit of the full 5D index can be rewritten as

$$I^{\text{5D,NS}}_{\text{U(1)}_0} = \prod_{n=1}^{\infty}(1-q^n)^2 \prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}\mathfrak{q})^{-1} \prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}\mathfrak{q}^{-1})^{-1}, \tag{3.17}$$

which with the help of (3.5) can in turn be massaged into

$$\begin{aligned} I^{\text{5D,NS}}_{\text{U(1)}_0} &= (q)^2_\infty E_q(-\mathfrak{q}^{-1})E_q(-\mathfrak{q}) \\ &= (q)^2_\infty \text{Tr}[E_q(X_{-\gamma_f})E_q(X_{\gamma_f})]. \end{aligned} \tag{3.18}$$

The above expression is consistent with it originating from a 5D rank-1 theory with a BPS quiver consisting of a single flavor node. The corresponding quantum torus algebra is commuting, and the formal variable X_{γ_f} can be chosen such that $\text{Tr}[X_{\gamma_f}] = -\mathfrak{q}$.

3. $\mathbf{U(1)}_{-\frac{1}{2}}$ *with one flavor*

Our next example is a U(1) theory with one flavor at CS level $\kappa = -\frac{1}{2}$. The 5D index reads

$$I^{\text{5D}}_{\text{U(1)}_{-\frac{1}{2}}} = \int \frac{\mathrm{d}u}{u} Z_{\text{pert}} Z_{\text{nonpert}}, \tag{3.19}$$

where u is the U(1) gauge fugacity and the perturbative contribution, after massaging (2.18), is given by

$$Z_{\text{pert}} = \prod_{n=1}^{\infty}(1-q^n)^2 \prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}u)^{-1} \prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}u^{-1})^{-1}. \tag{3.20}$$

In order to find the full nonperturbative contribution, given by the plethystic exponential of the one-instanton term, let us begin by looking at the latter. This is given by

$$Z^{(1)}_{\text{Nek}} = \frac{\sqrt{pq}}{(1-p)(1-q)}(1-u\sqrt{pq}). \tag{3.21}$$

As in the $|\kappa| = 1$ case, the above expression is not invariant under a transformation which simultaneously sends $p \to 1/q$ and $q \to 1/p$. However, following Ref. [28] and introducing a correction factor,

$$\Delta = \frac{qu}{(1-p)(1-q)}, \tag{3.22}$$

we can write a new one-instanton partition function in terms of

$$Z'^{(1)}_{\text{Nek}} = Z^{(1)}_{\text{Nek}} + \Delta = \frac{\sqrt{pq}}{(1-p)(1-q)}. \tag{3.23}$$

This would suggest that the correct instanton sector contribution for $F = 1$ is the same as for the $F = 0$ case,

$$Z_{\text{nonpert}} = \text{PE}\left[\frac{\sqrt{pq}}{(1-p)(1-q)}(\mathfrak{q}+\mathfrak{q}^{-1})\right]. \tag{3.24}$$

By expanding to arbitrary order in the q fugacity, it is straightforward to check that the NS index is equivalent to

$$\begin{aligned} I^{\text{5D,NS}}_{\text{U(1)}_{-\frac{1}{2}}} &= (q)^2_\infty \sum_{k_1,k_2,r_1,r_2=0}^{\infty} \\ &\quad\times \frac{(-1)^{k_1+k_2+r_1+r_2} q^{\frac{k_1+k_2+r_1+r_2}{2}} (-\mathfrak{q})^{r_2-r_1} \delta_{k_1,k_2}}{(q)_{k_1}(q)_{k_2}(q)_{r_1}(q)_{r_2}} \\ &= (q)^2_\infty \text{Tr}[E_q(X_{-\gamma_f})E_q(X_{-\gamma})E_q(X_{\gamma_f})E_q(X_\gamma)]. \end{aligned} \tag{3.25}$$

In complete analogy with our previous discussion, the interpretation of this result in the language of Ref. [5] would be that the instanton provides a flavor charge γ_f, in addition to the charge lattice vector for the hypermultiplet, γ. This is consistent with having a 5D BPS quiver involving two nodes and no adjoining arrows.

4. *Maximally SUSY theory*

Consider adding to the U(1) vector multiplet a hypermultiplet in the adjoint representation. This is the content of the maximally supersymmetric (SUSY) theory.[19] One might naively think that the adjoint hypermultiplet decouples and as a result that the instanton contribution is simply that of the pure U(1) theory. This is, however, not the case, as the noncommutative deformation regulating the Nekrasov partition function couples zero modes of the U(1) adjoint hypermultiplet to the instantons. In fact, it turns out [31] that the instanton contribution is

$$\begin{aligned} Z_{\text{inst}} &= \text{PE}\left[\sum_{k=1}^{\infty}\mathfrak{q}^k z_{\text{sp}}\right] \quad \text{with} \\ z_{\text{sp}} &= -\frac{p+q}{(1-p)(1-q)} + 2\frac{\sqrt{pq}}{(1-p)(1-q)}. \end{aligned} \tag{3.26}$$

As stressed in Ref. [31], z_{sp} is equal to the contribution of a six-dimensional (6D) tensor multiplet. This constitutes a nontrivial check for the conjectured UV self-completion of the maximally SUSY 5D theory into the (2, 0) theory [32]. Note that the expression for z_{sp} above is exactly that of an

[18] The second part of Eq. (3.16) is to be compared with Eq. (3.16) of Ref. [30] or its generalization Eq. (2.40).

[19] Although this theory has $\mathcal{N} = 2$ supersymmetry, we can still study it using 5D $\mathcal{N} = 1$ tools.

Abelian vector plus an adjoint hypermultiplet. The latter is the full perturbative contribution of the 5D maximally SUSY theory, i.e.

$$Z_{\mathrm{pert}} = \mathrm{PE}[z_{\mathrm{sp}}]. \tag{3.27}$$

Moreover, in the NS limit, one can reexpress

$$\mathrm{PE}[\mathfrak{q}^k f_H] = \left(\prod_{m=0}^{\infty}(1-\mathfrak{q}^k q^{m+\frac{1}{2}})^{-1}\right)^2 = (E_q(-\mathfrak{q}^k))^2, \tag{3.28}$$

while

$$\mathrm{PE}[\mathfrak{q}^k f_V] = \prod_{m=0}^{\infty}(1-\mathfrak{q}^k q q^m)^2 = (\mathfrak{q}^k q; q)^2, \tag{3.29}$$

where $(a;b)$ stands for the q-Pochhammer symbol.[20] The full index is given by

$$I_{\mathrm{U}(1)}^{\mathrm{MaxSUSY}} = Z_{\mathrm{pert}} Z_{\mathrm{inst}} \bar{Z}_{\mathrm{inst}}, \tag{3.30}$$

where the overline implies an inversion of the instanton fugacity. This prescription—which we stress is just the direct implementation of the results of Ref. [17] and strongly supported by nontrivial checks, including the emergence of the enhanced flavor symmetries in the case of E_{N_f+1} theories—amounts to writing

$$\bar{Z}_{\mathrm{inst}} = \mathrm{PE}\left[\sum_{k=1}^{\infty}\mathfrak{q}^{-k} z_{\mathrm{sp}}\right], \tag{3.31}$$

and Eq. (3.30) can be nicely repackaged into[21]

$$I_{\mathrm{U}(1)}^{\mathrm{MaxSUSY}} = \prod_{k=-\infty}^{\infty}(\mathfrak{q}^k q; q)^2 E_q(-\mathfrak{q}^{-k}) E_q(-\mathfrak{q}^k). \tag{3.32}$$

This expression does not have a strict 5D BPS quiver interpretation. However, its form is rather suggestive: the collection of instantons corresponds to BPS states at threshold associated with the Kaluza-Klein modes that uplift the theory to 6D [31]. As such, one may expect that these would provide an infinite tower of flavor nodes, each parametrized by integer multiples of a fundamental charge, $\mathfrak{q}^n$, which is what we seem to find. However, the q-Pochhammer symbol, expected to arise from the vector-multiplet contribution, also depends on $\mathfrak{q}^n$. It is tempting to speculate that this is due to the flavor fugacity combinations $\mathfrak{q}^n$ being remnants of a 6D Lorentz fugacity.

B. Toward non-Abelian theories

It is natural to ask whether there exists a non-Abelian extension of the correspondence between the NS index of a 5D SCFT and the trace of the KS operator for an associated BPS quiver, but we have thus far not been successful in constructing any such examples. Having a closed-form expression for the nonperturbative part of the non-Abelian NS index—the perturbative part reduces trivially to the 4D Schur index—would be helpful in pursuing this direction. Although the NS limit of the Abelian K-theoretic Nekrasov partition function coincides with the K-theoretic vortex partition function—cf. under Eq. (3.16)—explicitly applying our prescription (2.12) to non-Abelian gauge groups quickly produces an answer which disagrees with the $\mathfrak{q}$-expansion of any K-theoretic vortex partition function.

However, there may be another way forward using dualities. The instanton partition function—the 4D limit of the non-Abelian 5D Nekrasov partition function[22]—has a well-defined NS limit, originally discussed in Refs. [13,25], of which our prescription (2.12) is a natural generalization. As can be seen by expanding in the instanton fugacity, and simultaneously for small but nonzero ϵ_1, ϵ_2, Eq. (2.20) becomes

$$\begin{aligned} Z_{\mathrm{Nek}}^{\mathrm{NS}} &\to 1 + \mathfrak{q}\lim_{\epsilon_2\to 0}\epsilon_2 Z_{\mathrm{inst}}^{(1)} \\ &\quad + \lim_{\epsilon_2\to 0}\mathfrak{q}^2\left(\frac{\epsilon_2(\epsilon_2-1)}{2}(Z_{\mathrm{inst}}^{(1)})^2 + \epsilon_2 Z_{\mathrm{inst}}^{(2)}\right) + O(\mathfrak{q}^3) \\ &= \exp\left[\lim_{\epsilon_2\to 0}\epsilon_2 \log Z_{\mathrm{inst}}(\mathfrak{q};\epsilon_1,\epsilon_2)\right], \end{aligned} \tag{3.33}$$

precisely the expression appearing in Ref. [13]. In that reference, the resultant partition function was identified as the nonperturbative contribution to the twisted superpotential for some associated two-dimensional theory. Subsequently, the authors of Ref. [33] also linked the full 2D twisted superpotential—the NS limits of the full perturbative plus nonperturbative partition functions of the 4D theory—with the twisted superpotential for a different, dual 2D theory. Interestingly, the latter theory can in certain cases—e.g. the Abelian example—be interpreted as the world volume description for a 2D defect in the Higgs branch of the original 4D theory. The partition

[20] The $(a;b)$ q-Pochhammer symbol is defined as $(a;b) \coloneqq \prod_{j=0}^{\infty}(1-ab^j)$.

[21] Note that, by taking into account Eqs. (3.26), (3.27), and (3.31) and naively resumming the instanton expansions, it looks like the total partition function is $\mathrm{PE}[0]=1$. However, this conclusion is incorrect, since for this to happen each series is implicitly resummed in a different regime, while here $|\mathfrak{q}|=1$.

[22] This is known as the "homological limit" (see e.g. Ref. [30] and references therein), and in notation where one has made explicit the dependence of the fugacities on the Euclideanized time radius, $p=e^{-\beta\epsilon_1}$, $q=e^{-\beta\epsilon_2}$, corresponds to taking $\beta\to 0$. In this limit, the full "K-theoretic" version of the Nekrasov partition function we have been using thus far reduces to the 4D instanton partition function.

function for these defects is the well-known vortex partition function, which has a natural K-theoretic lift up to 5D.[23] It would be interesting to closely study similar 5D → 4D → 2D → 2D → 4D → 5D chains for more complicated theories. This in turn could lead to identifying closed-form expressions for the NS limits of non-Abelian instanton contributions and shed light on how to proceed with the non-Abelian extension of the NS-KS correspondence presented in this section.

Another closely related task would be to investigate whether the 5D NS index we have defined admits an alternative (and possibly simpler) description associated with some lower-dimensional structure, along the lines of Ref. [4]. In this respect, the similarity of our prescription to the large orbifold limit of Ref. [14] may hint toward such a connection. We will leave the answers to these questions as open problems for future research.

[23] The details of this K-theoretic vortex partition function depend on how the embedding of the defect breaks the gauge group, i.e. on the choice of Levi subgroup; see e.g. Ref. [30] and references therein for details. See also Ref. [34] for new results on the calculation of 5D instanton partition functions in the presence of defects.

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7. Conclusions and outlook

In this thesis work we studied different aspects of five and lower dimensional quantum field theories. As we reviewed in section 3 $5d$ QFT can be at fixed point and admit a gravity dual. This allowed us to begin the study of holographic RG flows in the context of $5d\ \mathcal{N}=1$ using the corresponding gravity dual geometry. As illustrated in the article 6.1 we did it solving the equations of motion near the boundary and we classified the operators that are taking a VEV in the dual QFT according to their conformal dimension and other quantum numbers. As in the $4d$ case we found a correlation between the particular kind of RG flow, that can be triggered by a mesonic or baryonic operator and the brane configuration. However, differently from the $4d$ case, the singularity arising at $\alpha = 0$ in the metric is necessary in order to fix the correct boundary conditions and determine the conformal dimensions of the operators. Moreover in the case of a RG flow triggered by a baryonic operator we were able to identify the modules of the VEV of the condensate and to identify the corresponding Goldstone boson. However for simplicity we considered only a $\mathbb{Z}_2$ orbifold of the initial $5d$ theory.

Moreover $5d$ supersymmetric QFTs on a curved background play an important role in the computation of the partition function and other observables of the theory. Therefore in the article 6.3 we studied the conditions that must be satisfied to define the above theories on a Riemann manifold. We performed such analysis following the method illustrated in section 4.3. This means that we coupled the $5d$ conformal supergravity with $5d$ conformal matter (vector multiplets and hypermultiplets) and then we took the rigid limit of this configuration. We discovered that the necessary and sufficient condition is the existence of a conformal Killing vector. Moreover we found that most of the backgrounds admit a more interesting geometric structure called *transversally holomorphic foliation*.

Then we focused on a particular kind of background, namely $\mathbb{S}^1 \times \mathbb{S}^4$, that is relevant for the computation of the $5d$ superconformal index and we consider a particular limit of such quantity the so called *Nekrasov-Shatashvili limit* in the article 6.4. We found that in general the above limit is ill-defined.

Therefore we introduced a consistent prescription which ensures that all the coefficients of the index, once the limit has been taken, are integers. Moreover we showed that such limit reproduces, at least in the case of abelian gauge theories, the Schur limit of $4d$ $\mathcal{N} = 2$ SCI. In terms of the BPS quiver data the instantons play the role of a further global symmetry node in the quiver diagram.

Moreover, as we reviewed in section 4.1, the SCI receives non perturbative contributions due to instanton corrections. For this reason the characterization of the moduli space of instantons is of great importance. In the article reported in section 6.2 we extended such analysis in the context of self-dual instantons on $\mathbb{C}P^2$. We used the corresponding ADHM-like construction that is described by a $3d$ $\mathcal{N}$=2 gauge theory. In particular we began the study, from a physical point of view, of the further directions that characterize the *resolved moduli space* of instantons. Furthermore we clarified the relation between the Hilbert Series of the moduli space of instantons on $\mathbb{C}P^2/\mathbb{Z}_n$ and the Hilbert Series of the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_n$ for unitary, orthogonal and symplectic gauge groups.

In the future it would be interesting to extend the analysis performed for a $5d$ QFT on a curved background also in the context of $6d$ theories coupled with the corresponding conformal supergravity and clarify if it holds a similar condition. Moreover also the characterization of the moduli space of instantons on different kind of curved manifolds could be very useful due to the important role played by such spaces in the application of the localization technique.

8. Conclusiones y pronósticos

En este tesis hemos estudiado diferentes aspectos de teorías cuánticas de campos en cinco y menos dimensiones. Como hemos revisado en la sección 3 una teoría cuántica de campos en $5d$ se puede encontrar en punto fijo y admite un dual gravitatorio. Esto nos has permitido empezar el estudio del *"RG flows"* para teorías de campos en $5d$ con $\mathcal{N} = 1$ usando la correspondiente geometría del dual gravitatorio. Como hemos ilustrado en el artículo 6.1 lo hemos hecho encontrado una solución de las ecuaciones del movimiento cerca de la frontera y hemos clasificado los operadores que toman un valor esperado en la teoría de campos dual determinado la correspondiente dimensión conforme y otros números cuánticos. En analogía con el caso en $4d$ hemos encontrado una correlación entre la tipología de flujo del GR, que puede ser empezado por un operador mesonico o bariónico y la configuración particular en términos de branas que describe la teoría. Sin embargo, a diferencia del caso en $4d$, la singularidad de la métrica en $\alpha = 0$ es necesaria para fijar las condiciones correctas a la frontera y encontrar la dimensión conforme de los operadores. Además en el caso de un flujo del GR empezado por un operador bariónico hemos identificado el modulo del valor esperado del condensado y el correspondiente bosón de Goldstone. Sin embargo, por simplicidad hemos considerado solo un orbifold $\mathbb{Z}_2$ de la teoría inicial en $5d$.

Por otra parte para poder calcular observables como la función de partición es importante construir la teoría de campos en una variedad curva arbitraria. Por lo tanto en el artículo 6.3 hemos estudiado las condiciones que tienen que estar satisfechas para definir dichas teorías sobre una variedad de Riemann. Hemos desarrollado dicha análisis siguiendo el procedimiento introducido en la sección 4.3. Esto ha sido implementado acoplando la supergravedad conforme en $5d$ con materia conforme en $5d$ (multipletes vectoriales y hypermultipletes) y luego hemos tomado el *rigid limit* de dicha configuración. Hemos descubierto que la condición necesaria y suficiente, que tiene que ser satisfecha para preservar supersimetría, es la existencia de un vector de Killing conforme. Además hemos descubierto que la mayoría de los backgrounds admite una estructura geométrica mas interesante llamada *transversally holomorphic foliation.*

Luego nos hemos concentrados en un tipo particular de *background*, dado por $\mathbb{S}^1 \times \mathbb{S}^4$, que es relevante para el calculo del índice superconforme en $5d$. Hemos estudiado y hemos considerado un límite particular de dicha cantidad llamado límite de *Nekrasov-Shatashvili.* en el artículo 6.4. Henos descubierto que dicho limite en general está mal definido. Por lo tanto hemos introducido una prescripción consistente que asegura que todos los coeficientes del índice, una vez que ha sido tomado el límite, son números enteros. Por otra parte, hemos mostrado que dicho límite reproduce, por los menos en el caso de teorías de gauge abelianas, el límite de Schur del índice superconforme de una teoría en $4d$ con $\mathcal{N} = 2$. Haciendo uso del correspondiente *quiver BPS* los instantones pueden ser interpretados como un nodo extra de simetría global del diagrama.

Además, como hemos revisado en la sección 4.1, el indice superconforme recibe correcciónes no perturbativas debidas a instantones. Por esto la caracterización del *moduli space* de los instantones es de gran importancia. En el artículo reportado en la sección 6.2 hemos analizado diferentes aspectos del *moduli space* de los instantones sobre $\mathbb{C}P^2$ usando la correspondiente construcción ADHM descrita por una teoría en $3d$ con $\mathcal{N} = 2$. En particular hemos empezado el estudio, usando un punto de vista físico, de las direcciones que caracterizan el *resolved moduli space* de los instantones. Por otra parte hemos clarificado la relación entre la Serie de Hilbert del *moduli space* de los instantones sobre $\mathbb{C}P^2/\mathbb{Z}_n$ y la Serie de Hilbert del *moduli space* de los instantones sobre $\mathbb{C}^2/\mathbb{Z}_n$ para grupos de simetría local unitarios, ortogonales y simplécticos.

En futuro podría ser interesante extender el análisis desarrollado para una teoría de campos en $5d$ sobre una variedad curva también en el contexto de teorías de campos en $6d$ acopladas con la correspondiente supergravedad y clarificar si una condición parecida tiene que estar satisfecha. Además la caracterización del *moduli space* de los instantones sobre diferentes tipos de variedades curvas puede ser muy útil debido al importancia que estos espacios tienen cuando se aplica la técnica de calculo denominada *localization.*

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