

Multivariable codes in principal ideal polynomial quotient rings with applications to additive modular bivariate codes over \mathbb{F}_4

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Abstract

In this work, we study the structure of multivariable modular codes over finite chain rings when the ambient space is a principal ideal ring. We also provide some applications to additive modular codes over the finite field \mathbb{F}_4 .

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1 Introduction

Multivariable codes over a finite field \mathbb{F}_q are a natural generalization of several classes of codes, including cyclic, negacyclic, constacyclic, polycyclic and abelian codes. Since these particular families have also been considered over finite chain rings (e.g., over Galois rings), we proposed in [16, 17] constructions of multivariable codes over them. As with classical cyclic codes over finite fields, the modular case (i.e., codes with repeated roots) is much more difficult to handle than the semisimple case (i.e., codes with non-repeated roots). In

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this sense, different authors have studied the properties of cyclic, negacyclic, constacyclic and polycyclic modular codes over finite chain rings. Among these codes, those contained in an ambient space which is a principal ideal ring admit a relatively simple description, quite close to that of semisimple codes. This feature has been recently used in the description of abelian codes over a finite field [9], and in the description of modular additive cyclic codes over \mathbb{F}_4 [8]. As a natural continuation of these works, in this paper we consider the structure of multivariable modular codes over finite chain rings when the ambient space is a principal ideal ring.

2 Finite chain rings and codes over them

An associative, commutative, unital, finite ring R is called *chain ring* if it has a unique maximal ideal M and it is principal (i.e, generated by an element a). This condition is equivalent [5, Proposition 2.1] to the fact that the set of ideals of R is the chain (hence its name) $\langle 0 \rangle = \langle a^t \rangle \subsetneq \langle a^{t-1} \rangle \subsetneq \dots \subsetneq \langle a^1 \rangle = M \subsetneq \langle a^0 \rangle = R$, where t is the nilpotency index of the generator a . The quotient ring $\overline{R} = R/M$ is a finite field \mathbb{F}_q where $q = p^l$ is a prime number power. Examples of finite chain rings include Galois rings $GR(p^n, l)$ of characteristic p^n and p^{nl} elements (here $a = p$, and $t = n$) and, in particular, finite fields ($\mathbb{F}_q = GR(p, l)$) [18].

Multivariable codes over finite chain rings, i.e., ideals of the ring $\mathcal{R} = R[X_1, \dots, X_r] / \langle t_1(X_1), \dots, t_r(X_r) \rangle$, where $t_i(X_i) \in R[X_i]$ are monic polynomials, were introduced in [16, 17]. These codes generalize the notion of multivariable codes over a finite field \mathbb{F}_q , as presented in [21], and include well-known families of codes over a finite chain ring alphabet. For instance *cyclic* ($r = 1, t_1(X_1) = X_1^{e_1} - 1$), *negacyclic* ($r = 1, t_1(X_1) = X_1^{e_1} + 1$), *constacyclic* ($r = 1, t_1(X_1) = X_1^{e_1} + \lambda$), *polycyclic* ($r = 1$) and *abelian codes* ($t_i(X_i) = X_i^{e_i} - 1, \forall i = 1, \dots, r$) [5, 12]. Properties of multivariable codes over a finite chain ring depend on the structure of the ambient ring \mathcal{R} . So, in [16] a complete account of codes was given when the polynomials $\overline{t}_i(X_i) \in \mathbb{F}_q[X_i]$ have no repeated roots (the so-called *semisimple* or *serial* case). On the other hand, as a first approach to the *repeated-root* (or *modular*) case, Canonical Generating Systems [19] were considered in [17]. Unfortunately, the description is not as satisfactory as in the semisimple case. This situation agrees with that of cyclic, negacyclic, constacyclic and polycyclic repeated-root codes. Different authors have dedicated their efforts to provide a better understanding of these codes over finite chain rings (see, for instance [6, 1, 22, 12]).

One important feature of semisimple codes is that all of them can be generated by a single codeword, i.e., they can be regarded as principal ideals in \mathcal{R} . This property is not generally true in the modular case, and it partly explains the reason why these codes are more difficult to describe. However, that of all the ideals in \mathcal{R} are principal is not equivalent to the semisimple condition. Instead, it is equivalent to the fact that its nilradical is principal [4, Lemma 3]. As it was shown in [4, Theorem 2], we have the following characterization (see also [12, Theorem 5.2], [22, Theorem 3.2], [17, Theorem 1]).

Theorem 1. *The ring $\mathcal{R} = R[X_1, \dots, X_r] / \langle t_1(X_1), \dots, t_r(X_r) \rangle$ is a principal ideal ring (PIR) if and only if one of the following conditions is satisfied:*

1. *If R is a Galois ring $GR(p^n, l)$, then the number of polynomials for which $\bar{t}_i(X_i)$ is not square-free is at most one. Moreover, if R is not a finite field (i.e., $n > 1$), and $\bar{t}_i(X_i)$ is not square-free with*

$$t_i(X_i) = g(X_i)h(X_i) + au(X_i)$$

where $\bar{g}(X_i)$ is the square-free part of $\bar{t}_i(X_i)$, then $\bar{u}(X_i)$ and $\bar{h}(X_i)$ are coprime polynomials.

2. *If R is not a Galois ring, then $r = 1$, and $\bar{t}_1(X_1)$ is square-free.*

Example 1. *Let us consider the ring $R = \mathbb{Z}/4\mathbb{Z}$, which is the Galois ring $GR(4, 1)$, and the polynomials $t_1(X_1) = X_1^2 + 1$ and $t_2(X_2) = X_2^7 - 1$. Following Theorem 1, $t_1(X_1)$ can be written as $t_1(X) = (X_1 - 1)^2 + 2X_1$. Since $\bar{h}(X_1) = X_1 + 1$ and $\bar{u}(X_1) = X_1$ are coprime polynomials, then $\mathcal{R} = R[X_1, X_2] / \langle X_1^2 + 1, X_2^7 - 1 \rangle$ is a principal ideal ring. Notice that the ring $R[X_1] / \langle X_1^2 + 1 \rangle$ is also a principal ideal ring and its ideals are negacyclic codes.*

The principal ideal property has been recently used in the description of modular abelian codes over a finite field [9], and in the description of modular *additive* cyclic codes over \mathbb{F}_4 (i.e., additive subgroups of the ambient ring $\mathbb{F}_4[X_1] / \langle X_1^{e_1} - 1 \rangle$, e_1 even) [8]. As a natural continuation of these works, in this paper we consider the structure of multivariable modular codes over finite chain rings when the ambient space is a principal ideal ring. Since this ring is a polynomial quotient ring we will call it a *principal ideal polynomial quotient ring (PIPQR)*. Our aim is to achieve a complete description of them and their properties.

3 Multivariable modular codes in PIPQRs

From now on we will restrict our attention to multivariable codes over a finite chain ring in an ambient space \mathcal{R} which is a PIR, i.e., to multivariable codes in PIPQRs. We will impose the modular (or repeated-root) condition, as the semisimple case was fully treated in [16]. Hence, from Theorem 1, R must be a Galois ring $GR(p^n, l)$, and there exists exactly one index $i = 1, \dots, r$ such that $\bar{t}_i(X_i)$ has repeated-roots. We shall assume w.l.o.g. that $i = 1$. Let $\bar{t}_1(X_1) = \prod_{j=1}^s \bar{g}_j(X_1)^{k_j}$ be the unique decomposition in powers of coprime irreducible polynomials $\bar{g}_j(X_1) \in \mathbb{F}_q[X_1]$ of degree r_j . Then, because of Hensel's lemma [18, Theorem XIII.4], there exist monic $G_j(X_1) \in R[X_1]$ pair-wise coprime polynomials such that $t_1(X_1) = \prod_{j=1}^s G_j(X_1)$, and $G_j(X_1) = g_j(X_1)^{k_j} + pu_j(X_1)$ (i.e. $t_1(X_1)$ is decomposed as a product of primary coprime polynomials). As in [12, Section 5] we may assume w.l.o.g. that $g_j(X_1)$ is monic and $r_j k_j > \deg u_j(X_1)$.

Hence, $g(X_1) = \prod_{j=1}^s g_j(X_1)$ is such that $\bar{g}(X_1)$ is the square-free part of $t_1(X_1) = g(X_1)h(X_1) + pu(X_1)$, where $h(X_1) = \prod_{j=1}^s g_j(X_1)^{k_j-1}$, and

$u(X_1) = \sum_{j=1}^s u_j(X_1) \prod_{l \neq j} g_l(X_1)^{k_l} + p\Delta(X_1)$, for some $\Delta(X_1) \in R[X_1]$. If R is not a finite field, then the principal condition is equivalent to $\bar{u}(X_1) = \sum_{j=1}^s \bar{u}_j(X_1) \prod_{l \neq j} \bar{g}_l(X_1)^{k_l}$ nonzero and coprime with $\bar{h}(X_1) = \prod_{j=1}^s \bar{g}_j(X_1)^{k_j-1}$. This means that $\bar{g}_j \nmid \bar{u}_j$, whenever $k_j \geq 2$.

A first question that can be asked is whether it is possible to obtain analogues of *abelian codes in principal ideal group algebras* [9] in this setting. The answer is not, as the following corollary of Theorem 1 shows.

Corollary 1. *If $\mathcal{R} = R[X_1, \dots, X_r] / \langle X_1^{e_1} - 1, \dots, X_r^{e_r} - 1 \rangle$ is a principal ideal polynomial quotient ring (PIPQR), then either R is a finite field \mathbb{F}_q (i.e., \mathcal{R} is a principal ideal group algebra, PIGA) or we are in the semisimple case.*

Proof. If we are not in the semisimple case, then R is a Galois ring, $\bar{t}_1(X_1) = X_1^{e_1} - 1$ has repeated-roots, and so $e_1 = p^{l_1} m_1$, with $l_1 \geq 1$ and $p_1 \nmid m_1$. Besides, if R is not a finite field, then $t_1(X_1) = g(X_1)h(X_1) + pu(X_1)$, with $g(X_1) = X_1^{m_1} - 1$, $h(X_1) = 1 + X_1^{m_1} + \dots + X_1^{(p^{l_1}-1)m_1}$, $u(X_1) = 0$. But this is a contradiction with Theorem 1, because $\bar{g}(X_1)$ is the square-free part of $\bar{t}_1(X_1)$, and $\bar{h}(X_1), \bar{u}(X_1)$ are not coprime polynomials. \square

As the abelian case is fully treated in [9] we will impose the condition that \mathcal{R} is not a group ring henceforth. Even though we cannot have abelian codes in our setting it is interesting to mention how abelian codes in PIGAs are viewed in [9]. In such a paper the ring \mathcal{R} is a group algebra $\mathbb{F}_q[G]$ over an abelian finite group G which is a direct product of a cyclic p -Sylow B and a complementary subgroup A . A one-to-one correspondence between the ring $\mathbb{F}_q[G]$ and the group ring $\mathcal{S}[B]$ (where $S = \mathbb{F}_q[A]$ is a semisimple group ring) is used to describe all the codes (i.e., ideals) in the former ring (see sections II and III in [9]). We want to use the same type of approach in our case: adjoin the semisimple part to the base ring and use its decomposition as sum of finite chain rings [16] to describe the original PIPQR. Let us begin this technique with the univariable case, i.e., with the description of polycyclic codes.

Proposition 1. *Let $\mathcal{R} = R[X_1] / \langle t_1(X_1) \rangle$ be a PIPQR such that $\bar{t}_1(X_1) \in \mathbb{F}_q[X_1]$ has repeated-roots (in particular, R is a Galois ring $GR(p^n, l)$). Then, \mathcal{R} is a direct sum of finite chain rings \mathcal{R}_j . For each of these rings, the maximal ideal has nilpotency index nk_j and the residual field $\overline{\mathcal{R}}_j$ is the finite field $\mathbb{F}_{q^{r_j}}$ ($q = p^l$).*

Proof. We will follow [18] through [12] (ambient structure of polycyclic codes).

With the previous notation $\mathcal{R} = \bigoplus_{j=1}^s \mathcal{R}_j$, where $\mathcal{R}_j = \langle \widetilde{G}_j(X_1) \rangle \cong R[X_1] / \langle G_j(X_1) \rangle$

and $\widetilde{G}_j(X_1) = \prod_{l \neq j} G_l(X_1)$ [12, Theorem 5.1]. Now, with the same argument of [22, Theorem 3.2] we can say that \mathcal{R}_j is local PIR with maximal ideal $\langle p\widetilde{G}_j(X_1), g_j(X_1)\widetilde{G}_j(X_1) \rangle = \langle g_j(X_1)\widetilde{G}_j(X_1) \rangle$. Therefore \mathcal{R}_j is a finite chain ring and $\overline{\mathcal{R}}_j \cong \mathbb{F}_q[X_1] / \langle \bar{g}_j(X_1) \rangle$ is a field extension of \mathbb{F}_q of degree r_j . Because of the proof of [18, Lemma XVII.4], $|\mathcal{R}_j| = (q^{r_j})^{w_j}$, where w_j is the nilpotency

index of $g_j(X_1)\widetilde{G}_j(X_1)$. But since $G_j(X_1)$ is a monic polynomial of degree $k_j r_j$, we have that $|\mathcal{R}_j| = (q^n)^{k_j r_j}$, and so $w_j = nk_j$. \square

Remark 1. In view of [18, Theorem XVII.5] we have the following description of the ambient space ring \mathcal{R} as a direct sum of finite chain rings (cf. [9, Equation (II.5)]):

$$\mathcal{R} \cong \bigoplus_{j=1}^s R_j[X_1] / \langle \gamma_j(X_1), p^{n-1} X_1^{k_j} \rangle$$

where $R_j = GR(p^n, l r_j)$, and $\gamma_j(X_1) \in R_j[X_1]$ is an Eisenstein polynomial of degree k_j of the form $X_1^{k_j} + p f_j(X_1)$. Moreover, for each factor the set of nonzero ideals is $\{ \langle p^i X_1^l \rangle \mid 0 \leq i \leq n-1, 0 \leq l \leq k_j-1 \}$

Now, let us describe all possible univariable codes (cf. [16, Corollaries 3.11, 3.12]).

Corollary 2. If $\mathcal{R} = R[X_1] / \langle t_1(X_1) \rangle$ is a PIPQR such that $\bar{t}_1(X_1) \in \mathbb{F}_q[X_1]$ has repeated-roots (in particular, R is a Galois ring $GR(p^n, l)$), then any code $\mathcal{K} \triangleleft \mathcal{R}$ is a sum of ideals of the form

$$\langle p^{i_j} g_j(X_1)^{c_j} \widetilde{G}_j(X_1) \rangle,$$

where $(i_j, c_j) = (n, 0)$ or $0 \leq i_j \leq n-1, 0 \leq c_j \leq k_j-1, 1 \leq j \leq s$. Hence, there exists a family of polynomials $H_1, \dots, H_n \in R[X_1]$ such that

$$\mathcal{K} = \langle H_1, p H_2, \dots, p^{n-1} H_n \rangle = \left\langle \sum_{i=0}^{n-1} p^i H_{i+1} \right\rangle \quad (1)$$

Moreover,

$$|\mathcal{K}| = |\bar{R}|^{\sum_{j=1}^s r_j (nk_j - c_j - i_j k_j)}$$

and there exist $\prod_{j=1}^s (nk_j + 1)$ repeated-root codes in \mathcal{R} .

Example 2. (Example 1 cont'd). In the special case of $R[X_1] / \langle X_1^2 + 1 \rangle$, we have $g_1(X_1) = X_1 - 1, G_1(X_1) = t_1(X_1)$ and $\widetilde{G}_1 = 1$. Moreover, since $(X_1 - 1)^2 \equiv 2X_1 \pmod{t_1(X_1)}$ and X_1 is a unit in \mathcal{R} , the ideal $\langle (X_1 - 1)^2 \rangle$ is equal to $\langle 2 \rangle$. So, $R[X_1] / \langle X_1^2 + 1 \rangle$ is a finite chain ring with ideals (i.e. negacyclic codes)

$$R[X_1] / \langle X_1^2 + 1 \rangle \supseteq \langle X_1 - 1 \rangle \supseteq \langle 2 \rangle \supseteq \langle 2(X_1 - 1) \rangle \supseteq 0.$$

In the following result we adjoin the semisimple part to the ring R in order to describe the original PIPQR with the help of Proposition 1.

Theorem 2. Let $\mathcal{R} = R[X_1, \dots, X_r] / \langle t_1(X_1), \dots, t_r(X_r) \rangle$ be a PIPQR such that $\bar{t}_1(X_1) \in \mathbb{F}_q[X_1]$ has repeated-roots (in particular, R is a Galois ring $GR(p^n, l)$). Then \mathcal{R} is a direct sum of finite chain rings $R_{C,j}[X_1] / \langle \gamma_j(X_1), p^{n-1} X_1^{k_j} \rangle$, where $R_{C,j}$ is a Galois extension of the coefficient ring R and $\gamma_j(X_1)$ is an Eisenstein polynomial of degree k_j .

Proof. The ring \mathcal{R} is isomorphic to the tensor product

$$R[X_1]/\langle t_1(X_1) \rangle \otimes (R[X_2, \dots, X_r]/\langle t_2(X_2), \dots, t_r(X_r) \rangle).$$

Since $t_2(X_2), \dots, t_r(X_r)$ have simple roots only, from [16, Theorem 3.9], there exists an isomorphism

$$\varphi : \bigoplus_{C \in \mathcal{C}} Q_C \longrightarrow R[X_2, \dots, X_r]/\langle t_2(X_2), \dots, t_r(X_r) \rangle$$

where

$$\mathcal{C} = \{ \{ (\mu_2^s, \dots, \mu_r^s) \mid s \in \mathbb{N} \} \mid \bar{t}_i(\mu_i) = 0, i = 2, \dots, r \} \quad (2)$$

is the set of all cyclotomic classes of the roots of $\bar{t}_2(X_2), \dots, \bar{t}_r(X_r)$. Each $Q_C = GR(p^n, l \cdot |C|)$ is a Galois extension of R contained in $R[X_2, \dots, X_r]/\langle t_2(X_2), \dots, t_r(X_r) \rangle$. Then, since $t_1(X_1) \in R[X_1]$, φ induces an isomorphism

$$\hat{\varphi} : \bigoplus_{C \in \mathcal{C}} Q_C[X_1]/\langle t_1(X_1) \rangle \longrightarrow \mathcal{R}.$$

As a consequence of Proposition 1, \mathcal{R} can be written as a direct sum of finite chain rings $R_{C,j}[X_1]/\langle \gamma_j(X_1), p^{n-1}X_1^{k_j} \rangle$, where $R_{C,j}$ is a Galois extension of Q_C and so, of R . \square

Now, we can generalize Corollary 2 to the multivariable case.

Corollary 3. *If $\mathcal{R} = R[X_1, \dots, X_r]/\langle t_1(X_1), \dots, t_r(X_r) \rangle$ is a PIPQR such that $\bar{t}_1(X_1) \in \mathbb{F}_q[X_1]$ has repeated-roots (in particular, R is a Galois ring $GR(p^n, l)$). Let us suppose that for each $C \in \mathcal{C}$ (see equation (2)),*

$$t_1(X_1) = \prod_{j=1}^s \prod_{m=1}^{s_{j,C}} G_{j,m}^C(X_1)$$

is the decomposition of $t_1(X_1)$ as the product of primary coprime polynomials in $Q_C[X_1]$ (i.e. $\overline{G_{j,m}^C}(X_1) = \overline{g_{j,m}^C}(X_1)^{k_j}$, where $\overline{g_{j,m}^C}(X_1) \in \mathbb{F}_{q^{|C|}}[X_1]$ is irreducible). Then any code $\mathcal{K} \triangleleft \mathcal{R}$ is a sum of ideals of the form

$$\hat{\varphi} \left(\left\langle p^{i_{j,m}} g_{j,m}^C(X_1)^{c_{j,m}} \widetilde{G_{j,m}^C}(X_1) \right\rangle \right),$$

where $(i_{j,m}, c_{j,m}) = (n, 0)$ or $0 \leq i_{j,m} \leq n-1$, $0 \leq c_{j,m} \leq k_j - 1$, $1 \leq m \leq s_{j,C}$, $1 \leq j \leq s$, $C \in \mathcal{C}$. Hence, there exists a family of polynomials $H_1, \dots, H_n \in R[X_1, \dots, X_r]$ such that

$$\mathcal{K} = \langle H_1, pH_2, \dots, p^{n-1}H_n \rangle = \left\langle \sum_{i=0}^{n-1} p^i H_{i+1} \right\rangle \quad (3)$$

Moreover,

$$|\mathcal{K}| = |\bar{R}|^{\sum_{C \in \mathcal{C}} \sum_{j=1}^s \sum_{m=1}^{s_{j,C}} \deg g_{j,m}^C(nk_j - c_{j,m} - i_{j,m}k_j)}$$

and there exist $\prod_{C \in \mathcal{C}} \prod_{j=1}^s (nk_j + 1)^{s_{j,C}}$ repeated-root codes in \mathcal{R} .

The results on the Hamming distance of linear codes (and in particular of serial multivariable codes) over finite chain rings contained in [10, Section 2.1], [20, Section 4] and [17, Theorem 2] can be applied in this context to multivariable codes in PIPQRs. However, because of the special simple description of these codes, we can use the same ideas of [16, Section 3.3] to compute their Hamming distance.

Proposition 2. *In the conditions of Corollary 3, $d(\mathcal{K}) = d(\mathcal{L})$, where \mathcal{L} is the code $\langle \overline{H}_1, \dots, \overline{H}_t \rangle$ in $\mathbb{F}_q[X_1, \dots, X_r] / \langle \overline{t}_1(X_1), \dots, \overline{t}_r(X_r) \rangle$.*

Hence, the results on the Hamming distance of the codes \mathcal{L} (i.e, on multivariable codes with repeated-roots over a finite field) which can be found in [21] can be lifted to our codes.

Example 3. *(Example 1 cont'd). The factorization of $X_2^7 - 1 = (X_2 - 1)(X_2^3 + 2X_2^2 + X_2 + 3)(X_2^3 + 3X_2^2 + 2X_2 + 3)$ into basic irreducible polynomials over $\mathbb{Z}/4\mathbb{Z}$ provides the following decomposition of \mathcal{R} (see Theorem 2)*

$$\mathcal{R} \cong R[X_1] / \langle X_1^2 + 1 \rangle \oplus GR(4, 3)[X_1] / \langle X_1^2 + 1 \rangle \oplus GR(4, 3)[X_1] / \langle X_1^2 + 1 \rangle.$$

Each summand is a finite chain ring (cf. Example 2), and so any negacyclic code can be written as the direct sum of three ideals.

Let us consider the code $\mathcal{K} = \langle (X_1 - 1)(X_2^4 + X_2^3 - 3X_2^2 + 2X_2 + 3) \rangle$. The polynomial $X_2^4 + X_2^3 - 3X_2^2 + 2X_2 + 3$ is, up to units, an orthogonal idempotent of \mathcal{R} . Namely, it generates the third summand of the previous decomposition. Since $(X_1 - 1)^2 \equiv 2X_1 \pmod{t_1(X_1)}$ and X_1 is a unit of \mathcal{R} , we deduce that $\mathcal{K} = \langle H_1, 2H_2 \rangle$ with $H_1 = (X_1 - 1)(X_2^4 + X_2^3 - 3X_2^2 + 2X_2 + 3)$ and $H_2 = X_2^4 + X_2^3 - 3X_2^2 + 2X_2 + 3$ (see Corollary 3). Thus, the code $\langle \overline{H}_1, \overline{H}_2 \rangle$ contains a codeword with Hamming weight 4 and, by Proposition 2, the Hamming distance of \mathcal{K} is at most 4. Direct computations with Sage [23] show that this is the actual minimum distance of the code. On the other hand, observe that the Hamming distance of the code $\overline{\mathcal{K}} = \langle (X_1 + 1)(X_2^4 + X_2^3 + X_2^2 + 1) \rangle \triangleleft \mathbb{Z}/2\mathbb{Z}[X_1, X_2] / \langle (X_1 + 1)^2, X_2^7 + 1 \rangle$ is 8 (to see this, apply the isometry of [21, Proposition 45] and check [3, Table 1]).

Example 4. *As a variation of the previous example, let us take the same ambient space \mathcal{R} and construct the code \mathcal{L} generated by the codeword $(X_1 - 1)(X_2^3 + 2X_2^2 + X_2 - 1)$. This code can be seen as a product code of the negacyclic code generated by $X_1 - 1$ and the (punctured) \mathbb{Z}_4 -base linear code of the Kerdock code of length 16.*

Example 5. *(Generalized Kerdock code) Let $R = \mathbb{Z}/4\mathbb{Z}$ and $S = GR(2^2, m)$, with m prime, a Galois extension of R . Let $U = 1 + 2R = \langle \eta_1 \rangle$ be the group of units of R . Let $\lambda \in S$ be a generator of $\Gamma^*(S)$, with $\Gamma(S)$ the Teichmüller coordinate set of S , and let $\text{Tr} : S \rightarrow R$ be the trace function of S onto R . According to [11],*

$$\mathcal{K} = \left\{ \sum_{i_1=0}^{2^m-1} \sum_{i_2=0}^1 ((\text{Tr}(\xi\lambda^{i_1}) + \beta)\eta^{i_2})X_1^{i_1}X_2^{i_2} \mid \xi \in S, \beta \in R \right\}$$

is an ideal of the ambient space $R[X_1, X_2]/\langle X_1^{2^m-1} - 1, X_2^2 - 1 \rangle$ known as generalized Kerdock code. We can regard the ambient space as a direct sum of rings of the form $T[X_2]/\langle X_2^2 - 1 \rangle$, with T a suitable Galois extension of R . Such rings are not principal ideal rings, since their nilradical, $\langle 2, X_2 + 1 \rangle$, is not principal (see [13, Proposition 4.4]). Thus, the generalized Kerdock code is not a PIPQR. Also, notice that $X_2^2 - 1 = (X_2 - 1)^2 - 2(X_2 - 1)$ and so, statement 1 of Theorem 1 is not satisfied.

4 Additive modular codes over \mathbb{F}_4 from PIPQRs

Additive modular codes over \mathbb{F}_4 can be used to construct quantum error correcting codes, as shown in [15]. However, except in very special cases (for instance, the univariable cyclic modular codes analyzed in [8]), the description of such codes seems quite difficult. As an application of the study of the multivariable codes of the previous section, we obtain a complete description of additive modular codes in PIPQRs.

In the framework presented in [15], additive modular codes can be seen as \mathcal{A}_2 -additive submodules of the algebra \mathcal{A}_q , where $\mathcal{A}_q = \mathbb{F}_q[X_1, \dots, X_r]/\langle t_1(X_1), \dots, t_r(X_r) \rangle$ and $t_1(X_1), \dots, t_r(X_r)$ have coefficients in \mathbb{F}_2 . Since the finite field \mathbb{F}_q , with $q = 2, 4$, is the Galois ring $GR(2, r)$ with $r = 1, 2$ respectively, the algebra \mathcal{A}_q is a PIPQR if and only if the polynomials $t_2(X_2), \dots, t_r(X_r)$ are square-free. In such a case, from Corollary 3, the algebra \mathcal{A}_q can be decomposed into a direct sum of ideals (see also [15, Theorem 1]). Let us recall some definitions and results from [14] in order to describe such a decomposition.

The set \mathcal{C}_2 of 2-classes of the roots of $t_2(X_2), \dots, t_r(X_r)$ (take $q = 2$ in equation (2)) is a disjoint union of two subsets according to their relation to the set \mathcal{C}_4 of 4-classes (take $q = 4$ in the same equation). The first subset, \mathcal{C}_2^o , contains the classes $C_2(\mu)$ such that $C_2(\mu) \in \mathcal{C}_4$, i.e., classes with odd cardinality. The second subset, \mathcal{C}_2^e , contains the classes that split in \mathcal{C}_4 , i.e., those $C_2(\mu)$ with even cardinality such that $C_2(\mu) = C_4(\mu) \cup C_4(\mu^2)$.

Theorem 3. *Let $\mathcal{A}_q = \mathbb{F}_q[X_1, \dots, X_r]/\langle t_1(X_1), \dots, t_r(X_r) \rangle$, $q = 2, 4$, be a PIPQR such that $t_i(X_i) \in \mathbb{F}_2[X_i]$ for all $i = 1, \dots, r$, and such that only $t_1(X_1)$ has repeated-roots. Let us suppose that for each $C \in \mathcal{C}_2$ the polynomial $t_1(X_1)$ factorizes as the product of primary coprime polynomials in $\mathbb{F}_{2^{|C|}}[X_1]$ as (cf. Corollary 3)*

$$t_1(X_1) = \prod_{j=1}^s \left(\prod_{m=1}^{s_{j,C}} g_{j,m}^C(X_1) \right)^{k_j}$$

Then:

1. \mathcal{A}_2 is a direct sum of finite chain rings $\mathcal{K}_{j,m}^C \cong \mathbb{F}_{2^{|C| \deg g_{j,m}^C}}[Z]/\langle Z^{k_j} \rangle$.
2. \mathcal{A}_4 is a direct sum of ideals $\mathcal{I}_{j,m}^C$ which are free $\mathcal{K}_{j,m}^C$ -modules of rank 2.
3. Any additive modular code \mathcal{D} is direct sum of subcodes $\mathcal{D}_{j,m}^C$ which are $\mathcal{K}_{j,m}^C$ -submodules of $\mathcal{I}_{j,m}^C$.

Proof. 1. This is a direct consequence of Proposition 1 and Theorem 2.

2. The proof depends on the cardinality of each class $C \in \mathcal{C}_2$, and the degree of the polynomial $g_{j,m}^C(X_1)$.

- (a) If $C \in \mathcal{C}_2^o$ and $g_{j,m}^C(X_1)$ has odd degree, then $C \in \mathcal{C}_4$ and $g_{j,m}^C(X_1)$ is also irreducible in $\mathbb{F}_{4^{|C|}}[X_1]$. Therefore, because of Proposition 1, there exists an ideal $\mathcal{I}_{j,m}^C \cong \mathbb{F}_{4^{|C| \deg g_{j,m}^C}}[Z]/\langle Z^{k_j} \rangle$ in \mathcal{A}_4 , which is clearly a free $\mathcal{K}_{j,m}^C$ -module of rank 2.
- (b) If $C \in \mathcal{C}_2^e$ and $g_{j,m}^C(X_1)$ has even degree, then $C \in \mathcal{C}_4$ and $g_{j,m}^C(X_1)$ splits as the product of two irreducible polynomials $g_{j,m,1}^C(X_1), g_{j,m,2}^C(X_1)$ of the same degree $\frac{1}{2} \deg g_{j,m}^C$ in $\mathbb{F}_{4^{|C|}}[X_1]$. Therefore, because of Proposition 1, there exists an ideal in \mathcal{A}_4

$$\mathcal{I}_{j,m}^C = \mathcal{I}_{j,m,1}^C \oplus \mathcal{I}_{j,m,2}^C \cong \left(\mathbb{F}_{4^{|C| \frac{1}{2} \deg g_{j,m}^C}}[Z]/\langle Z^{k_j} \rangle \right)^2$$

which can be seen as a free $\mathcal{K}_{j,m}^C$ -module of rank 2 (cf. [15, Proposition 3]).

- (c) If $C \in \mathcal{C}_2^e$, then $C = D \cup E$, with $D, E \in \mathcal{C}_4$, and $|D| = |E| = \frac{|C|}{2}$. Hence, $g_{j,m}^C(X_1)$ is also irreducible in $\mathbb{F}_{4^{|D|}}[X_1]$, and so there exists an ideal in \mathcal{A}_4

$$\mathcal{I}_{j,m}^C = \mathcal{I}_{j,m}^D \oplus \mathcal{I}_{j,m}^E \cong \left(\mathbb{F}_{4^{\frac{|C|}{2} \deg g_{j,m}^C}}[Z]/\langle Z^{k_j} \rangle \right)^2$$

which again is a free $\mathcal{K}_{j,m}^C$ -module of rank 2.

3. The proof is similar to [14, Theorem 2]. □

Let us illustrate this theorem with a concrete example. It provides, via [2, Theorem 2], a way to construct a quantum-error-correcting code with parameters $[[8, 4, 2]]$. This code has an optimal distance for its length and dimension according to [7], and it can be fully described as an additive modular code in a PIPQR as we shall see now.

Example 6. Consider the binary polynomials $t_1(X_1) = (X_1 + 1)^2, t_2(X_2) = X_2^2 + X_2 + 1$. Theorem 3 give us an isomorphism $\mathcal{A}_2 \cong \mathbb{F}_4[Z]/\langle Z^4 \rangle$ (here $\mathbb{F}_4 = \mathbb{F}_2[X_2]$, and $Z = X_1 + 1$), and a direct sum decomposition $\mathcal{A}_4 \cong \mathbb{F}_4[Z]/\langle Z^4 \rangle \oplus w\mathbb{F}_4[Z]/\langle Z^4 \rangle$. The additive modular code \mathcal{D} generated by the codeword $c = w + w^2X_1 + wX_1^2 + w^2X_1^3$ can be seen as the submodule $\langle Z^2 + wZ^3 \rangle \leq \mathcal{A}_4$. Magma computations [23] show that this code is self-orthogonal w.r.t. bilinear form considered in [14, Section 4], and that the smallest weight of the codewords in $\mathcal{D}^\perp \setminus \mathcal{D}$ is 2. This provides the conditions to construct the $[[8, 4, 2]]$ quantum-error-correcting code mentioned above.

As an application of this theorem we will finally count the number of additive modular codes in PIPQRs.

Corollary 4. *Under the hypothesis of the previous theorem:*

1. *The total number of additive modular codes in \mathcal{A}_4 is*

$$\prod_{C \in \mathcal{C}_2} \prod_{j=1}^s \prod_{m=1}^{s_{C,j}} \left(1 + k_j + \left(\frac{2^{\delta_{j,m}^C} + 1}{2^{\delta_{j,m}^C} - 1} \right) \left(\frac{2^{k_j \delta_{j,m}^C} - 1}{2^{\delta_{j,m}^C} - 1} - k_j + 2^{k_j \delta_{j,m}^C} - 1 \right) \right)$$

where $\delta_{j,m}^C = |C| \deg g_{j,m}^C(X_1)$.

2. *Of these, only*

$$\prod_{C \in \mathcal{C}_2} \prod_{j=1}^s \prod_{m=1}^{s_{C,j}} \left(1 + \left(\frac{2^{\delta_{j,m}^C} + 1}{2^{\delta_{j,m}^C} - 1} \right) (2^{k_j \delta_{j,m}^C} - 1) \right)$$

codes can be generated by a single codeword.

3. *Any of those codes can be generated at most by two codewords.*

Proof. Combine Theorem 3 and [15, Theorem 3]. □

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