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Abstract

In this paper, the possibility of introducing random field theory into the cross-entropy algorithm is studied. Cross-entropy algorithm is an optimization process that Walsh and González (2009) use to estimate the stiffness distribution of a structure given a set of displacements. Although this method has been successfully tested, many lines of improvement are still opened. Random field theory is incorporated into the algorithm in an attempt to account for spatial variability of stiffness throughout the structure. For this purpose, a correlation function, variable in space, is defined and, as a result, a modification of the algorithm is proposed. The modified algorithm is then tested using numerical simulations in a scenario consisting of a simply supported beam.

Keywords: structural health, damage assessment, static testing, cross-entropy method, optimization algorithm, random field theory, spatial variability.

1 Introduction

The need for assessment of bridge structures is an urgent demand [1] that has provided the impetus for the development of many damage detection methods [2, 3, 4, 5, 6]. In this context, Walsh and González [7] propose a method to predict the distribution of stiffness throughout the structure based on the finite element method (FEM), cross-entropy (CE) and static measurements. This technique has proven to be valid, under certain conditions, both in numerical simulations [7, 8] and lab experiments [9]. One of the assumptions of this technique is that the stiffness of a specific finite element is independent from other elements. However, it appears to be reasonable to think that a level of correlation will exist between the stiffness of different structural elements.

In this regard, random field theory is a statistical approach that allows treating a certain property that varies in space (one, two or three dimensional) as a set of

correlated random variables. Correlation represents a way of measuring the degree of linear dependency between variables. In the case of flexural stiffness, it seems logical to think that random field theory could be used to describe its spatial variation. The use of random fields to describe spatial variability has already been explored by Stewart and others to study pitting corrosion in reinforced concrete beams [10, 11, 12].
In this paper, random field theory is incorporated into the original CE-based damage detection algorithm [7]. The modified algorithm is then tested through numerical simulations. Finally, results of the simulations are discussed.

2 Cross-entropy algorithm

The CE algorithm is attributed to Reuven Rubinstein [13] and it can be classified as an optimization algorithm, within the family of genetic algorithms. It was first applied to structural damage detection by Brian Walsh and Arturo González in 2009 [7]. The first applications of the algorithm use input data from static loading tests although it can easily be extended to the use of dynamic data too [14]. This paper focuses in the use of static data, measured deflections in particular, which are obtained at a number of points in the structure. In this case, the response of the structure relies solely on the static equilibrium Equation (1):

$$f = K \cdot u \tag{1}$$

where f represents the force vector, K is the global stiffness matrix and u is a vector that contains the displacements of the structure. Knowing both f and K, the stated problem has a unique solution, i.e., there is only one set of displacements that satisfies the equation. However, knowing f and u only for a limited number of degrees of freedom, the calculation of K is a problem with multiple solutions, meaning that there are many variations of the stiffness matrix that can solve the equilibrium equation. The CE-based damage detection algorithm poses an iterative procedure that attempts to find the stiffness profile that best fits the static deflections used as inputs. Since damage can be characterized as a loss of stiffness over a certain area, the solution provided by the algorithm can determine whether or not structural damage is present, identifying its location and magnitude.
Prior to the application of the CE optimisation procedure, the algorithm requires the formulation of a discretised FEM of the structure that accounts for features such as geometry and loading state. The location and magnitude of the applied loads is considered to be known. Boundary conditions are also assumed to be known but they may be modelled using springs whose constant could be modified. Regarding the level of discretisation, the number of elements depends on each particular case but it should always be chosen so that points where deflections have been measured match nodes of the FEM.
Stiffness values for each element are treated as normal random variables by the algorithm. Accordingly, the first step consists in setting the initial parameters of the corresponding probability distributions: mean and standard deviation. Standard deviation is chosen to be the same for all elements whereas mean is chosen

randomly in the range of typical expected stiffness. The next step of the algorithm involves the creation of different versions of the structure which are called ‘trial beams’. Trial beams are created by assigning a stiffness value to each element, sampled randomly from its probability distribution. As a result, stiffness profiles of the trial beams are different, covering a wide range of possible distributions of stiffness.
Then, for each trial beam, deflections are calculated using Equation (1) and compared to the measured ones by determining the sum of square errors. This comparison evaluates which trial beams lead to a deflection profile more similar to the target one. Trial beams are arranged according to this criterion and the ones in the top percentage are retained. The selected trial beams contain, for each particular element, a diversity of stiffness values that represent a sample of their distributions. However, since not all trial beams are being considered, this sample corresponds to a new distribution, which will be characterised by parameters different from those in the previous iteration. As a consequence, probability distributions are updated by calculating mean and standard deviation, for each element, of the sample of top trial beams, to be used in the next iteration of the algorithm
The algorithm continues iterating until the convergence criterion is met. It is assumed that the algorithm converges when the rate of change of error over the last ten iterations falls below a predefined value. At the point of convergence, a tool known as ‘injection’ is applied to the solution of stiffness distributions. This term refers to the act of artificially widen distributions by increasing the value of the standard deviation, whose value decreased naturally throughout iterations. The purpose of the injection is to help the algorithm to avoid local minima and its number is a parameter of the algorithm set by the user. The first two injections reset standard deviations at their initial values and the magnitude of the following ones decreases inversely proportional to the number of injections yet to be applied. Once all injections have been applied, next convergence is considered to be the final one. When using injection, only the standard deviation is typically modified, except when convergence is achieved for the second time. At that point, mean is recalculated for each element based on the means of adjacent elements to profit from one of the biggest strengths of the algorithm, the accuracy on the determination of the global average stiffness value.

3 Random field theory

Random fields can be used to describe various phenomena or properties when a probabilistic approach is more suitable than a deterministic one. They can be applied to multiple scales in space and time and they can also serve to different kind of purposes [15]. A random field independent of time can be referred to as a random medium, where the value of the property described by the field at a certain point is deterministic. However, that value is actually unknown unless it has been measured, a task that is normally unfeasible for all points of the field [15].
In order to model a structure, i.e., a beam, as a random field, it is necessary to discretise the continuous reality into a finite number of elements. Then, the value of the analysed property for each element is treated as a random variable, which can be

discrete or continuous. As a consequence, it is considered that the value of the field does not change inside the element and that the random variable is assigned to its centroid (midpoint method [12]). There will be as many random variables as the number of elements used to discretise the structure.
Given two random variables (*X, Y*), their inter-dependence affects the shape of their joint probability distribution. In particular, the degree of linear dependence is measured by means of the correlation coefficient ($\rho_{X,Y}$) in Equation (2).

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} \tag{2}$$

where *Cov*[*X*,*Y*] represents the covariance of *X* and *Y* while σ_X and σ_Y correspond to the standard deviation of *X* and *Y*, respectively. The value of the correlation coefficient ranges from -1 to +1, being ρ = |1| perfect correlation (positive or negative) and ρ = 0, no correlation. Figure 1 shows the joint probability distribution of two variables; both variables are assumed to have same mean ($\mu_X = \mu_Y = 100$) and standard deviation ($\sigma_X = \sigma_Y = 1$).

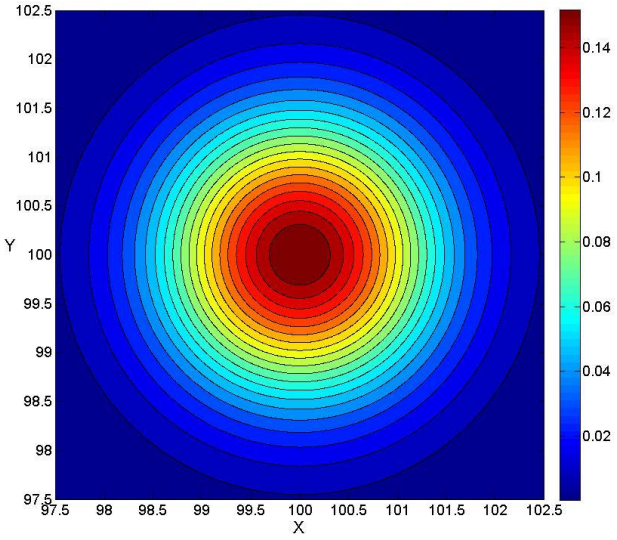

Figure 1: Two normally distributed variables with no correlation

It can be noticed that contours of equal probability density are centred at the point (μ_X, μ_Y) with respect to which there is radial symmetry. In this case, no correlation between random variables has been considered. However, if variables are assumed to be correlated, being 0.75 the value of the coefficient of Equation (2), the distribution of probability is modified as illustrated by Figure 2.

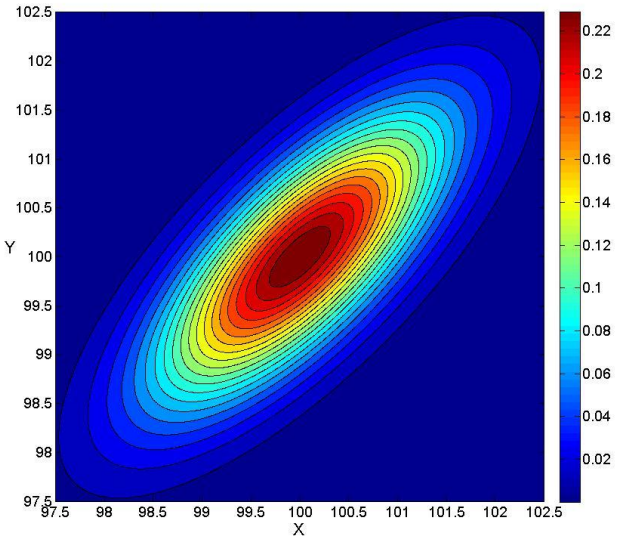

Figure 2: Two normally distributed variables with correlation

Due to the effect of correlation, the contours are not symmetrical anymore and combinations of values along the diagonal are more probable than off-diagonal ones [16]. It can also be noticed that more probability is concentrated around the centre point (μ_X, μ_Y). The value of the coefficient described by Equation (2) has been selected so the effect of correlation is exaggerated to serve as an example.

As a result of the previous analysis it is clear that, in order to fully characterize a random field, it is necessary to establish a correlation function that accounts for the degree of correlation between all the variables of the random field. In a homogeneous random field, the value of this function will only depend on the distance between elements [15]. In this paper, a correlation function, adopted for a 1-D field, of the following form [16] is considered:

$$\rho = \rho_0 + (1-\rho_0)\cdot e^{-\left(\frac{\tau_x}{d_x}\right)^2} \tag{3}$$

where ρ_0 represents a constant component of the correlation function, τ_x is the distance between centroids of elements and d_x is a parameter that is related to the distance at which the second term of the function begins to be irrelevant. The latter can be defined in terms of the scale of fluctuation (θ_x) [15] as $d_x = \theta_x / \pi^{1/2}$. This function assigns higher correlation coefficients to elements close to each other while its value decreases as τ_x increases. The impact of different combinations of the parameters (ρ_0, d_x) in the shape of the correlation function is shown in Figure 3.

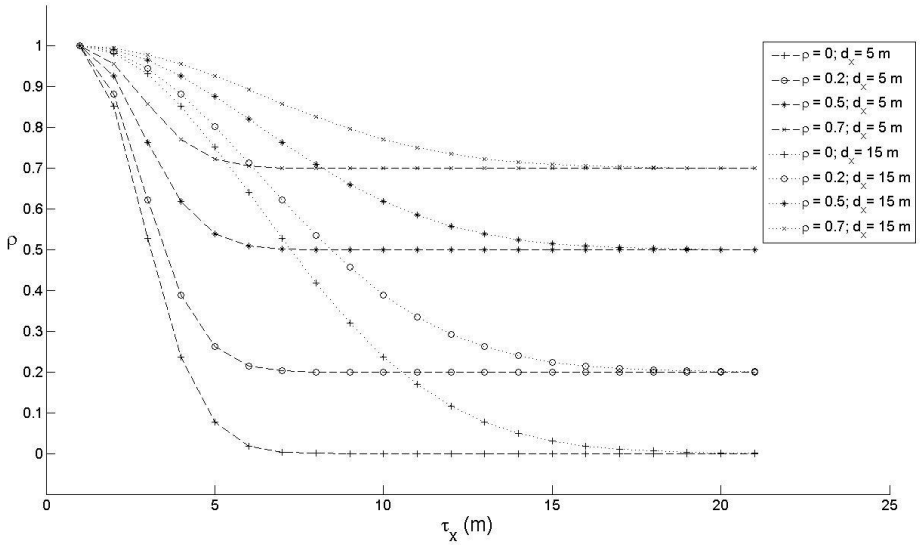

Figure 3: Correlation coefficient of Equation (3) for a range of parameters

4 Cross-entropy algorithm with random field theory

From the description of CE algorithm in section 2, it can be deduced that, to some extent, flexural stiffness (EI) was already been treated as a random field. The approach made in the algorithm is probabilistic rather than deterministic since stiffness at element centroids is considered to be a normal random variable, characterised by its mean and standard deviation. This fits the definition of random field: a property (EI) is described as a random field (one-dimensional in space for a beam) where observations of that property are made at discrete points (element centroids). Values of the random field at each of these points are random variables and observations are particular values of those random variables. A certain set of observations for all variables constitutes a realization of the random field [15]. Hence, the outcome of the algorithm could be seen as the stiffness random field that best fits the inputs, which were measured displacements of the structure.

On the other hand, CE algorithm did not consider any correlation between stiffness of different elements in the iterative process. However, in a real structure, given that flexural stiffness depends solely on material properties and cross section geometry, the variation of this property from one point of the structure to another is bounded by physical and structural limitations. As a result, it could be assumed that variables of the random field are correlated and that the correlation coefficient would account for a realistic modelling of stiffness spatial variability. This implies modifying the process by which trial beams are generated in the algorithm. When correlation was not considered, trial beams were created by randomly extracting from each probability distribution as many values as the number of trial beams. This process was repeated independently for each element of the structure; that is, in order to generate one trial beam, a number of normal distributions were accessed

consecutively. However, if correlation is assumed to exist, stiffness for each element should be considered to be a normal variable being part of a multivariate normal distribution. As a consequence, CE algorithm is modified so that one trial beam is created just by accessing once this joint distribution, obtaining a vector of stiffness values (one per element of the beam). This process is repeated as many times as the number of trial beams set to be created.
The multivariate normal distribution is characterised by a vector of means and a covariance matrix. The process for obtaining means and standard deviations remains unaltered but, since correlation is being considered, it is necessary to determine the covariance matrix. In this matrix, which is always symmetric and semi-positive definite, diagonal elements represent variances while non-diagonal elements correspond to covariances. Knowing the standard deviations and assuming a correlation function as defined by Equation (3), elements of the matrix can be computed from Equation (2) by clearing the covariance:

$$\sigma_{ij} = \rho_{ij} \cdot \sigma_i \cdot \sigma_j \quad \forall i, j \in [1, n] \tag{4}$$

where σ_{ij} represents the element of the covariance matrix in row i and column j, ρ_{ij} is the correlation coefficient between elements i and j according to Equation (3), σ_i and σ_j correspond to the standard deviation of elements i and j respectively, and, finally, n is the total number of elements in which the beam has been discretised.
It should be noticed that for diagonal elements ($i = j$) the correlation coefficient will always be equal to one, since the correlation of one variable with respect to itself is absolute, and thus, variance is calculated as the square of the standard deviation, which is how it is statistically defined:

$$\sigma_{ii} = 1 \cdot \sigma_i \cdot \sigma_i = \sigma_i^2 \quad \forall i \in [1, n] \tag{5}$$

where $\sigma_{ii} = \sigma_i^2$ represents the variance of element i, calculated as the square of its standard deviation, σ_i.

5 Numerical simulations

The algorithm is initially tested using the theoretical exact response of a 16 m long beam with constant section and homogenous material, i.e., constant flexural stiffness throughout the beam. Stiffness value is chosen so that deflection at mid-span would resemble that of a real bridge structure recently tested by the authors in León (Spain). The stiffness of that structure was found to be approximately 9750 MN·m^2.
The finite element beam model is considered to be simply supported at both ends. Deflections are simulated every 0.5 m using Equation (1), which results into a total of 31 control points evenly spaced along the structure. Finally, one concentrated load is considered to be acting at mid-span with a value of 400 MN.
The selected parameters for the CE algorithm are as follows. The number of trial beams is 10000, which is a number that balances computational time and efficiency of the algorithm according to experience. The top percentage of these trial beams

used to update distributions is 10%. The discretised model of the beam is created using 2 elements per meter, so that every node of the model would correspond to a point were deflection has been simulated. Also, the number of convergences until the stop of the algorithm is fixed at four, which means that injection is applied three times. The rate of change of error for which convergence is considered to be met is set at 0.0001. Initial standard deviation for all probability distributions is set at 975 $MN \cdot m^2$, 10% of the stiffness value mentioned above, and initial mean is randomly calculated in the range of that same stiffness value.

Regarding the incorporation of random field theory into the CE algorithm, a number of approaches have been adopted: (I) correlation is introduced in the algorithm right from the start, so trial beams are created using the multivariate normal distribution in all iterations; (II) the correlation is introduced in a more advanced iteration and it was chosen to do this after the second convergence, since at that point means are re-calculated for all elements as explained in section 2. As a result, during the first two convergences, values are drawn from independent normal distributions whereas for the last two, values are drawn from the multivariate normal distribution; (III) a final scenario is considered where correlation is only applied after the last injection.

Several values are considered for the parameters of the correlation function defined by Equation (3) with the scale of fluctuation (d_x) ranging from 0.25 m to 16 m. The constant component of the function (ρ_0) varies between 0 and 1, in increments of 0.1.

6 Results

In a first group of tests, stiffness distributions are calculated five times (which differ due to the random nature of the algorithm) using the input model described in section 5 and the original CE algorithm, that is, without considering any kind of correlation. Figure 4 shows the stiffness profile for each of five applications of the algorithm as well as their average.

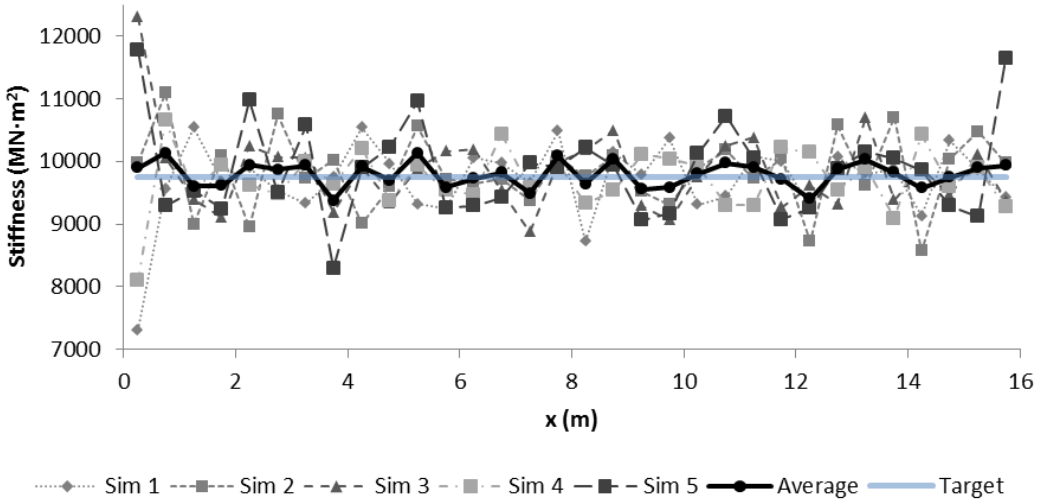

Figure 4: Stiffness profiles, no correlation

The stiffness profiles represented in Figure 4 are built using the mean values of the element stiffness probability distributions corresponding to the best iteration of the algorithm (the one with the least sum of square errors), since mean is considered their most representative value. It can be seen that the algorithm is successful in predicting the stiffness profile of the structure. Predictions for stiffness values at the supports are less accurate, particularly for one single application of the algorithm; this is reflected by the values of standard deviations associated to the final probability distributions, which are greater near the boundaries of the structure [8, 9]. The average of five simulations is computed in order to smooth the final solution, especially near the supports. When the outputs of several applications of the algorithm are averaged, the uncertainties near the supports are reduced considerably. The root mean square error for the average of five simulations is 206.81 MN·m^2 (2.12% error) while the average error for individual simulations is 616.65 MN·m^2 (6.32% error). Maximum errors correspond in both cases to elements 1 or 2 (supports): 3.89% for the average and 26.14% for a single simulation.
In a second group of tests, correlation is incorporated to the algorithm following the approach (I) described in section 5. In order to decide the ideal values of the parameters (ρ_0, d_x) of the correlation function, simulations are conducted with several combinations of those values in the ranges mentioned before. Again, the average of five applications of the algorithm is computed for each combination of parameters. Figure 5 represents a contour plot of all the tested cases where axis *z* corresponds to the root mean square error of the stiffness prediction.

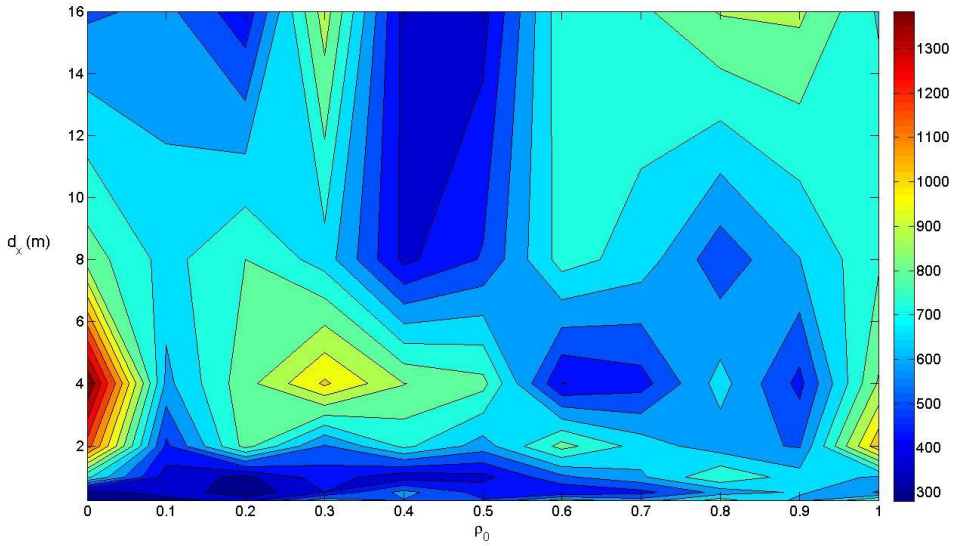

Figure 5: Stiffness prediction error depending on correlation function parameters

Intuitively, one would think that a combination providing a minimum error would appear in the figure, which then could be selected as the optimum combination of values for the correlation parameters. However, the figure reveals that this is not the case, which in part could be due to the random nature of the algorithm; although, in fact, stiffness predictions made by the algorithm are slightly less accurate when

allowing for correlation. For example, the smallest error in the contour plot of Figure 5 is 280.71 MN·m^2 (2.88% error for case ρ_0 = 0 and d_x = 0.25 m) compared to 206.81 MN·m^2 (2.12% error) for the average profile in Figure 4. On the other hand, the largest error in Figure 5 corresponds to case ρ_0 = 0 and d_x = 4 m and its value is 1457.55 MN·m^2 (14.95% error). This is clearly shown in Figure 6, which represents the stiffness profiles for one of the cases (ρ_0 = 0.2 and d_x = 1 m).

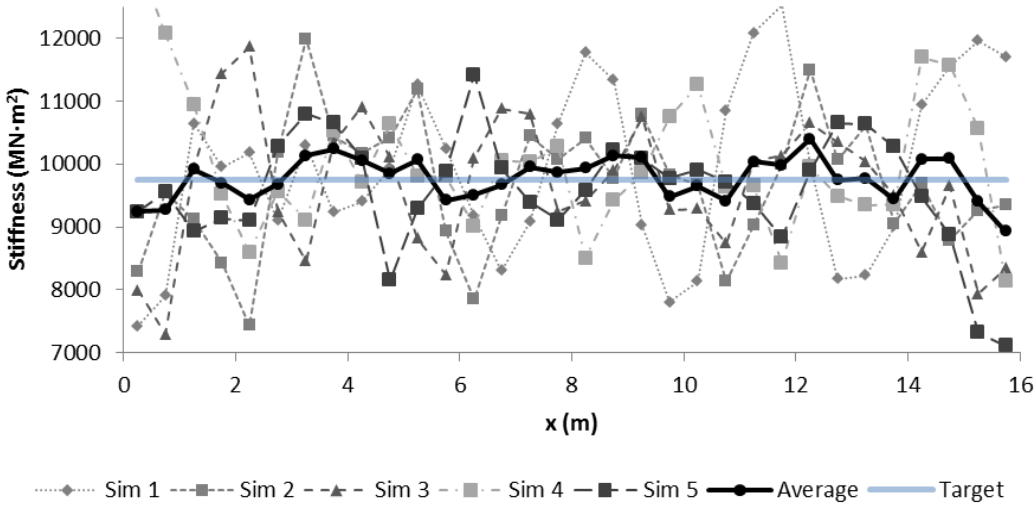

Figure 6: Stiffness profiles, initial model of correlation

In Figure 6, individual stiffness profiles for all simulations show greater variability than those in Figure 4. Here the average error for individual simulations is 1112.39 MN·m^2 (11.41% error). The average profile is also less accurate than the one obtained without introducing correlation.

Finally, considering these results it is decided to delay the application of correlation within the algorithm. This third group of tests is based on the assumption that if the correlation function is applied once the stiffness distributions are closer to the target values, the behaviour of the algorithm will improve. In a first attempt, correlation is not applied until after the second injection (approach (II) defined in section 5) and, in a second attempt, it is only applied between the third injection and final convergence (approach (III) defined in section 5). In the last case, the predicted stiffness profiles are similar to the ones in Figure 4 (without correlation) and the root mean square error of the stiffness prediction is in the same order of magnitude. Figure 7 shows the stiffness profiles of the average of five simulations for multiple combinations of the correlation function parameters.

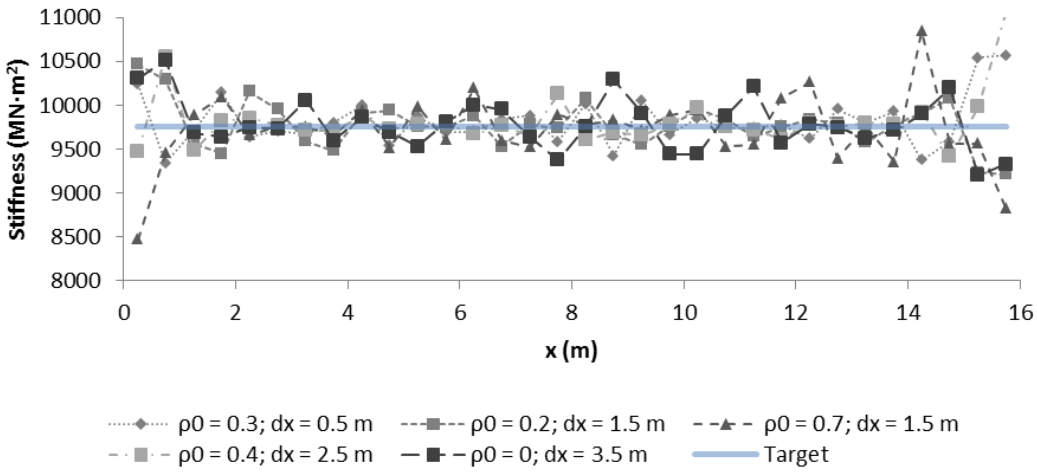

Figure 7: Stiffness profiles, final model of correlation

As a summary, Table 1 shows the average of the predicted stiffness values throughout the structure and root mean square errors (RMS), both in absolute value and percentage, for the cases in Figure 4, 6 and 7. Errors in Table 1 correspond in all cases to the average of five simulations.

Cases		Average stiffness value ($MN \cdot m^2$)	RMS ($MN \cdot m^2$)	RMS (%)
Original algorithm		9790.33	206.81	2.12
Initial model of correlation (I)	$\rho_0 = 0.2$; $d_x = 1\ m$	9768.51	335.39	3.44
Final model of correlation (III)	$\rho_0 = 0.3$; $d_x = 0.5\ m$	9812.01	284.53	2.92
	$\rho_0 = 0.2$; $d_x = 1.5\ m$	9778.52	268.54	2.75
	$\rho_0 = 0.7$; $d_x = 1.5\ m$	9715.38	406.25	4.17
	$\rho_0 = 0.4$; $d_x = 2.5\ m$	9814.58	305.37	3.13
	$\rho_0 = 0$; $d_x = 3.5\ m$	9785.02	297.30	3.05

Table 1: Comparison between models

It can be noticed from Table 1 how close the average stiffness predicted by the algorithm is to the target one (9750 MN·m2). The delay in incorporating correlation into the CE algorithm (approach (III)) has led to more accurate results than using correlation from the starting iteration (approach (I)), although the third case that illustrates approach (III) is the exception and it is an example of the effect of the random nature of the algorithm. However, the level of accuracy is within the same range as those results obtained without considering any correlation at all, as it is shown in Table 1. The latter is likely due to convergence being rapidly achieved after the last injection with correlation hardly having any impact on the results. The number of iterations where correlation is applied is small so results are almost the same as the ones obtained with the original version of the algorithm.

7 Conclusions

In the past, cross-entropy has been used to predict the distribution of stiffness within a structure subjected to a static loading test, and this prediction is dependant on the number, location and accuracy of available measurement points. The overall stiffness of the structure (average value) is predicted accurately, but predictions of stiffness for individual elements exhibit larger errors due to their sensitivity to the random nature of the algorithm. In order to address the latter, the average of the predictions obtained from several applications of the algorithms is used as the solution (which is typically closer to the true value and leads to a smoother profile than individual predictions).

This paper has incorporated random field theory into the algorithm in an attempt to account for the spatial variability of stiffness in a structure. The initial expectations were that the stiffness profiles calculated by the new algorithm would reduce the error in of the prediction of elementary stiffness; however, this has not been the case. It is clear that the real stiffness of a structure can be seen as a random field, whose spatial variability can be modelled using a correlation function similar to Equation (3). Nevertheless, an accurate representation of reality by the random field requires knowing the true value of stiffness at a number of points, not only the shape of its correlation function. In the cross-entropy algorithm, the real stiffness is precisely the unknown and its value is updated at each iteration. Therefore, if the correlation function is applied to the intermediate random values of stiffness that are generated in the cross-entropy iterative procedure, the assumed random field happens to be unable to represent the actual stiffness distribution. As a result, further research aims to adapt the algorithm to be able to predict not only the stiffness profile producing the closer deflection to the measured one but also the random field giving the better resemblance with reality.

Acknowledgments

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