Tesis doctoral

Universidad de Oviedo

- Programa de Doctorado en Matemáticas y Estadística -

Expanding Baker Maps: Una Herramienta Para El Estudio De Bifurcaciones Homoclínicas Asociadas A Difeomorfismos 3-D

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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

1.- Título de la Tesis	
Español/Otro Idioma: ***Expanding Baker Maps: una herramienta para el estudio de bifurcaciones homoclínicas en difeomorfismos 3-D***	Inglés: ***Expanding Baker Maps: A First Tool To Study Homoclinic Bifurcations Of 3-D Diffeomorphisms***

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RESUMEN (en español)

Bajo el título "***Expanding Baker Maps: una herramienta para el estudio de bifurcaciones homoclínicas en difeomorfismos 3-D***" esta tesis abre las puertas de un interesante campo de los Sistemas Dinámicos: el estudio de la dinámica que surge cuando se despliega una tangencia homoclínica genérica en una familia de difeomorfismos definidos en variedades de dimensión tres.

Si bien este problema ha sido profundamente abordado en el caso de tangencias bidimensionales, hasta hace escaso tiempo poco o nada se sabía en el marco tridimensional.

En un trabajo del profesor Joan Carles Tatjer publicado en 2002 se da por primera vez la expresión de las familias límite retorno asociadas a dicho despliegue de tangencias homoclinicas 3-D. Posteriormente, en dos trabajos en colaboración entre el profesor Antonio Pumariño y el propio Tatjer, se vislumbra numéricamente la amplia fauna de atractores extraños que aparecen en el despliegue de estas tangencias. Sin embargo, resultados analíticos que expliquen esta naturaleza caótica han tenido que esperar hasta la publicación de los siguientes artículos:

[1] A. Pumariño, J. A. Rodríguez, J. C. Tatjer and E. Vigil, ***Piecewise linear bidimensional maps as models of return maps for 3D-diffeomorphisms***, Progress and Challenges in Dynamical Systems, Springer, 54, 351-366 (2013).
[2] A. Pumariño, J. A. Rodríguez, J. C. Tatjer and E. Vigil, ***Expanding Baker Maps as models for the dynamics emerging from 3D-homoclinic bifurcations***. Discrete and continuous dynamical systems series B. 19 - 2, 523-541 (2014).
[3] A. Pumariño, J. A. Rodríguez, J. C. Tatjer and E. Vigil, ***Chaotic dynamics for 2-D tent maps***, Nonlinearity, 28, 407-434 (2015).
[4] José F. Alves, A. Pumariño and E. Vigil, ***Statistical stability for multidimensional piecewise expanding maps,*** *preprint* (2014).

Sin duda, el avance más profundo en esta línea de investigación ha sido la definición de lo que hemos bautizado como *Expanding Baker Maps* (*EBMs*), aplicaciones que, como el propio nombre intenta explicar, reproducen el método que utiliza un panadero al amasar el pan: un dominio bidimensional es constantemente plegado y estirado hasta obtener el producto final. En nuestros términos, el producto final no es más que el atractor que surge en la correspondiente dinámica. Atractor entendido como producto final al iterar una dinámica disipativa. Si bien este tipo de dinámica (pliegue y expansión) había sido descubierto y profundamente estudiado en la década de los sesenta en dimensión uno, el caso que nos ocupa supone un hito en el ámbito de los sistemas dinámicos bidimensionales. Al igual que estas dinámicas de plegamiento y estiramiento unidimensional han sido de extrema

importancia en el estudio de tangencias homoclínicas en dimensión dos, nuestras *EBMs* bidimensionales se posicionan sin lugar a dudas como una herramienta fundamental en el estudio de bifurcaciones homoclínicas tridimensionales. Con este espíritu nos hemos atrevido a bautizar a los diferentes atractores que hemos encontrado como *atractores tipo magdalenas*, *atractores tipo rosca* y *atractores tipo hogaza*. Si bien parecen términos un poco bucólicos, éstos han tenido buena aceptación en la Comunidad Matemática, lo cual parece de esperar al ver las correspondientes configuraciones geométricas que cada uno de ellos presenta.

Asimismo, los contenidos de esta tesis muestran por primera vez un ejemplo de familia de transformaciones bidimensionales no triangulares que presentan atractores extraños bidimensionales persistentes (sobreviven en un conjunto abierto de parámetros).

RESUMEN (en Inglés)

Under the title "***Expanding Baker Maps: A First Tool To Study Homoclinic Bifurcations Of 3-D Diffeomorphisms***" this thesis explores an interesting part of the Dynamical Systems: the study of the dynamics emerging when a family of diffeomorphisms unfolds a homoclinic tangency in a three-dimensional manifold.

Even though this problem has been deeply considered in the case of two-dimensional tangencies, short time until little or nothing was known in three-dimensional framework.

In a paper of Professor Joan Carles Tatjer published in 2002, the expression for the family of limit return maps associated to 3-D homoclinic tangencies is given for the first time.

Later, in two papers in collaboration between Professor Antonio Pumariño and Tatjer himself, the wide fauna of strange attractors appearing in the unfolding of these tangencies is numerically discovered. However, analytical results explaining this chaotic nature have had to wait until the publication of the following papers:

[1] A. Pumariño, J. A. Rodríguez, J. C. Tatjer and E. Vigil, ***Piecewise linear bidimensional maps as models of return maps for 3D-diffeomorphisms***, Progress and Challenges in Dynamical Systems, Springer, 54, 351-366 (2013).
[2] A. Pumariño, J. A. Rodríguez, J. C. Tatjer and E. Vigil, ***Expanding Baker Maps as models for the dynamics emerging from 3D-homoclinic bifurcations***. Discrete and continuous dynamical systems series B. 19 - 2, 523-541 (2014).
[3] A. Pumariño, J. A. Rodríguez, J. C. Tatjer and E. Vigil, ***Chaotic dynamics for 2-D tent maps***, Nonlinearity, 28, 407-434 (2015).
[4] José F. Alves, A. Pumariño and E. Vigil, ***Statistical stability for multidimensional piecewise expanding maps,*** *preprint* (2014).

Without doubt, the most important advancement in this line of research has been the definition of what we have called *Expanding Baker Maps* (*EBMs*). These maps, as the name tries to explain, reproduce the method used by a baker to knead the bread: a two-dimensional domain is constantly bending and stretching until the final product is obtained. In our terms, the final product is not more than the attractor that arises in the corresponding dynamics. Attractor understood as the final product obtained by iterating a dissipative dynamics. While this type of dynamics (fold and expansion) had been discovered and thoroughly studied in the sixties in dimension one, the case is a milestone in the field of two-dimensional dynamical systems. In the same way as these dynamics have been extremely important in the study of homoclinic tangencies in two dimensions, our two-dimensional *EBMs* are undoubtedly placed as a fundamental tool in the study of three-dimensional homoclinic bifurcations. In this spirit we have dared to baptize the different attractors that we have found as *fairy cakes attractors*, *bread roll attractors* and *country bread attractors*. While these terms seem somewhat bucolic, they have been well accepted in Mathematics Community, overall due to the shape of the numerically

obtained strange attractors.

Also, the contents of this thesis show a first example of two-dimensional family of non-skew-product maps displaying two dimensional persistent strange attractors (surviving in an open set of parameters).

SR. DIRECTOR DE DEPARTAMENTO DE MATEMÁTICAS/
SR. PRESIDENTE DE LA COMISIÓN ACADÉMICA DEL PROGRAMA DE DOCTORADO EN MATEMÁTICAS Y ESTADÍSTICA

A mi familia, por ser los pilares de todo esto a pesar de que mi trabajo os suene a sirio. *Y en especial a Totó, que pusiste la semilla con tus juegos de ingenio y me enseñaste que, hasta el último momento, la vida es aprender.*

Doctoral Dissertation
University of Oviedo

- PhD program in Mathematics and Statistic -

Expanding Baker Maps: A First Tool To Study Homoclinic Bifurcations of 3-D Diffeomorphisms

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Contents

Introduction

For dissipative dynamics, *chaos* is frequently defined as the existence of strange attractors in the phase space. This work deals with the existence of two-dimensional strange attractors for a one-parameter family of two-dimensional piecewise linear maps defined on certain triangle. Furthermore, these strange attractors are persistent in an open set of parameters. Let us introduce some definitions.

Definition **I.1** *Given a transformation f defined on a manifold $\mathcal{M} \subset \mathbb{R}^N$, an $f-$invariant set $\mathcal{K}$ is said to be* ***transitive*** *if there is some $f-$orbit dense in $\mathcal{K}$. Equivalently, for any pair of open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{K}$ there exists a natural number n such that $f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$.*

Definition **I.2** *An* ***attractor*** *for a transformation f defined on a manifold $\mathcal{M}$ is a compact, $f-$invariant and transitive set $\mathcal{K}$ whose stable set*

$$W^s(\mathcal{K}) = \{Q \in \mathcal{M} : d(f^n(Q), \mathcal{K}) \to 0 \ as \ n \to \infty\}$$

has non-empty interior. An attractor is said to be ***strange*** *if it contains a dense orbit $\{f^n(\widetilde{Q}) : \ n \geq 0\}$ displaying exponential growth of the derivative, i.e., there exists some constant $c > 0$ such that, for every $n \geq 0$,*

$$\|Df^n(\widetilde{Q})\| \geq \exp(cn).$$

By *two-dimensional strange attractors* we mean those strange attractors for which the sum of the Lyapounov exponents along a dense orbit is positive. In this work, in fact, our strange attractors will be even "stranger", due to the fact that the Lyapounov exponent in any direction will be positive for a full Lebesgue measure set of initial conditions in the phase space.

The simplest example of strange attractor is provided by the one-parameter family of one-dimensional tent maps $\{\lambda_\mu\}_{\mu \in (1,2]}$ being $\lambda_\mu : [0,1] \to [0,1]$ the map given by

$$\lambda_\mu(x) = \begin{cases} \mu x & , \quad if \ x \in [0, \frac{1}{2}] \\ \mu(1-x) & , \quad if \ x \in [\frac{1}{2}, 1]. \end{cases}$$

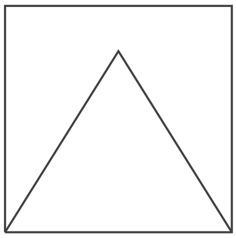


Figure I.1: Tent map

The interval $[\mu(1-\frac{\mu}{2}), \frac{\mu}{2})]$ is an strange attractor of λ_μ for $\sqrt{2} < \mu \leq 2$.

As in our case, these strange attractors are fully persistent (in an open set of parameters) and the number of positive Lyapounov exponents coincide with the dimension of the ambient manifold.

It is not hard to see that each one of the above defined tent maps can be seen as the composition of two maps. The first one folds (by the middle) the interval $[0,1]$ onto the interval $[0,\frac{1}{2}]$. The second one is a linear expansion from the interval $[0,\frac{1}{2}]$ onto the interval $[0,\frac{\mu}{2}]$ by a factor μ.

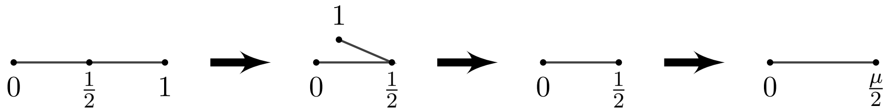


Figure I.2: Tent map dynamics

One of the main objectives of this memoir is to carry out these geometrical ideas to dimension two.

Let us now introduce the family $\{\Lambda_t\}_{t\in[0,1]}$ being $\Lambda_t : \mathcal{T} \to \mathcal{T}$ the two-dimensional map given by

$$\Lambda_t(x,y) = \begin{cases} (t(x+y), t(x-y)) & \text{, if } (x,y) \in \mathcal{T}_0 \\ (t(2-x+y), t(2-x-y)) & \text{, if } (x,y) \in \mathcal{T}_1 \end{cases} \tag{I.1}$$

where $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ and

$$\begin{aligned} \mathcal{T}_0 &= \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, 0 \leq y \leq x\}, \\ \mathcal{T}_1 &= \{(x,y) \in \mathbb{R}^2 : \ 1 \leq x \leq 2, 0 \leq y \leq 2-x\}. \end{aligned} \tag{I.2}$$

Let us firstly announce that this family of two-dimensional piecewise linear maps is closely related to the return maps arising when certain homoclinic bifurcations are unfolded for families of three-dimensional diffeormorphisms, as we will see along this work. In any case, as we will recall along this Introduction, the same holds for the above defined one-dimensional tent maps when homoclinic bifurcations are unfolded in a two-dimensional scenario.

Each one of these maps Λ_t can be seen as the composition of two maps. The first one, given by $\mathcal{F}_{\mathcal{C},O} : \mathcal{T} \to \mathcal{T}_0$ where

$$\mathcal{F}_{\mathcal{C},O}(x,y) = \begin{cases} (x,y) & , \; if \;\; (x,y) \in \mathcal{T}_0 \\ (2-x,y) & , \; if \;\; (x,y) \in \mathcal{T}_1 \end{cases} ,$$

folds the triangle $\mathcal{T}$ onto $\mathcal{T}_0$. The second one, given by $A_t : \mathcal{T}_0 \to \mathcal{T}$ where

$$A_t(x,y) = \begin{pmatrix} t & t \\ t & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{T},$$

is a linear transformation from $\mathcal{T}_0$ into $\mathcal{T}$.

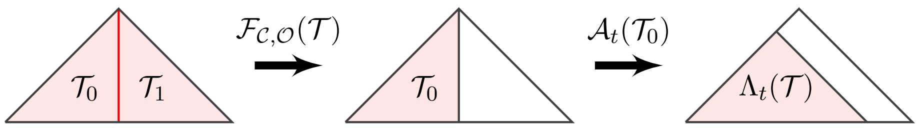


Figure I.3: Dynamic of Λ_t

Before entering into details we will state the main result of this memoir, see Theorem I.6. To this end, we begin with some definitions.

Definition **I.3** *Let $f : \mathcal{M} \to \mathcal{M}$ be a transformation, $\mathcal{M} \subset \mathbb{R}^N$. Then, f is said to be* ***strongly topologically mixing*** *in an invariant set $\mathcal{A}$ if for any open subset $\mathcal{U} \subset \mathcal{A}$ there is some natural number k such that $f^k(\mathcal{U}) = \mathcal{A}$.*

It is easy to see that if f is strongly topologically mixing in an invariant set $\mathcal{A}$, then $\mathcal{A}$ is transitive.

Definition **I.4** *Let $f : \mathcal{M} \to \mathcal{M}$ be a measurable transformation, $\mathcal{M} \subset \mathbb{R}^N$.*

a) A probability measure μ on the Borel sets of $\mathcal{M}$ is said to be ***f-invariant*** *if*

$$\mu(f^{-1}(\mathcal{B})) = \mu(\mathcal{B})$$

for any Borel set $\mathcal{B} \subset \mathcal{M}$.

b) If $\mu(\mathcal{B}) = 0$ whenever $m_N(\mathcal{B}) = 0$, where m_N denotes the Lebesgue measure on the Borel sets of $\mathbb{R}^N$, then f is called ***absolutely continuous****.*

c) A f-invariant probability measure μ is said to be ***ergodic*** *if*

$$\mu(\mathcal{B})\mu(\mathcal{M} \setminus \mathcal{B}) = 0$$

whenever $f^{-1}(\mathcal{B}) = \mathcal{B}$.

Let us remark that if μ is an absolutely continuous probability measure then there exists an m_N-integrable function $h \geq 0$ called the ***density*** of μ with respect to m_N.

DEFINITION **I.5** *Let I be a metric space and $\{f_t\}_{t\in I}$ a family of transformations $f_t : \mathcal{M} \to \mathcal{M}$ such that each f_t has a unique absolutely continuous f_t-invariant probability measure with density h_t. The family $\{f_t\}_{t\in I}$ is said to be* ***statistically stable*** *if h_t converges to $h_{t'}$ in the L^1-norm whenever $t \to t'$.*

THEOREM **I.6 (Main Theorem)** *For every $t \in (t_0, 1]$ being $t_0 = \frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}} \approx 0.882$, the map Λ_t exhibits a strange attractor $\mathcal{R}_t \subset \mathcal{T}$. Moreover the map Λ_t is strongly topologically mixing in $\mathcal{R}_t$, the periodic orbits are dense in $\mathcal{R}_t$ and $\mathcal{R}_t$ is a two dimensional strange attractor: there exists a dense orbit of Λ_t in $\mathcal{R}_t$ with two positive Lyapounov exponents. Furthermore, $\mathcal{R}_t$ supports a unique absolutely continuous invariant and ergodic probability measure and the family $\{\Lambda_t\}_{t\in(t_0,1]}$ is statistically stable.*

Now, let us explain the relationship between the family Λ_t and the dynamics emerging when certain homoclinic tangencies are unfolded in a three-dimensional framework.

I.1 Homoclinic dynamics

From the time of Poincaré, the homoclinic scenario has been usually identified with the presence of a rich amount of complicated dynamics. Specially from the sixties to the present day a great effort was done in order to clarify all the possible chaotic behaviours emerging when a homoclinic tangency is unfolded in the two dimensional setting. By this unfolding, we mean the creation of homoclinic orbits associated to a periodic saddle point (a more detailed explanation will be done in Chapter 1). More concretely, let $\mathcal{M}$ be a surface, $f : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism and let us assume the existence of a periodic saddle point p whose invariant manifolds intersect in some homoclinic point q. A natural question in this scenario was: How the existence of this homoclinic point affects to the dynamics? Or, more precisely: How the dynamics changes after a homoclinic orbit is created?

Most of the results concerning this problem start at the same point: By using the existence of *limit return maps* associated to the unfolding of a two-dimensional homoclinic tangency. In a few words, limit return maps explain the asymptotic behaviour of high iterates of the diffeomorphism when it is

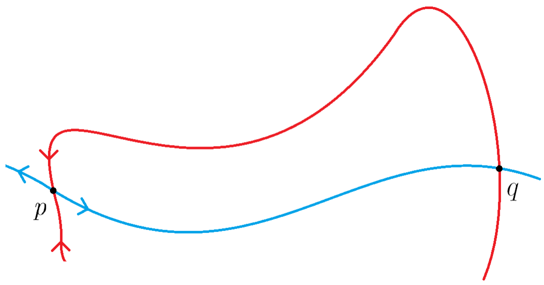


Figure I.4: Two-dimensional homoclinic tangency

restricted to certain neighbourhoods of the homoclinic orbit. Under generic assumptions, in the two-dimensional dissipative case these limit return maps are given by

$$F_a(x_1, x_2) = (1 - ax_1^2, 0)$$

and therefore the core of the respective dynamics corresponds to the well-known quadratic family $f_a(x) = 1 - ax^2$.

Since we are interested in attractors, let us recall the result obtained by L. Mora and M. Viana in [33] (see also Theorem I.7). This result guarantees the existence of physically persistent strange attractors when a generic homoclinic tangency associated to a dissipative periodic point p is unfolded in dimension two. In this sense, we point out that in [25] and [7] a deep study of the "chaotic" behaviour of the quadratic family f_a for a positive Lebesgue measure set of parameters was made and, moreover, the techniques and results in those papers were extensively used in [33]. By ***physically persistent strange attractors*** we mean persistent in the sense of probability measure. See again Theorem I.7 for details.

We also want to stress that the results in [33] have been stated in the two-dimensional scenario, and it is natural to quote ourselves if similar results prevail in higher dimensions. As one can see in [47] and [48], the numerical evidence show us that many different strange attractors arise when certain classes of three-dimensional homoclinic tangencies are unfolded. As in the two-dimensional case, the starting point to study these homoclinic three-dimensional phenomena is again the existence of limit return maps associated to the unfolding of these three-dimensional tangencies. We refer the reader to [51], where these families of limit return maps were constructed. Nevertheless, a brief review will be done in Chapter 1.

As it was expected, as in the unfolding of two-dimensional tangencies, these limit return maps are not linear. In fact, these limit return maps are given by

$$T_{a,b}(\widetilde{x},\widetilde{y}) = (a + \widetilde{y}^2, \widetilde{x} + b\widetilde{y}). \tag{I.3}$$

(see Section I.3 for a more detailed explanation).

However, for the special values of the parameters $a = -4$ and $b = -2$ the respective limit return map is conjugate to a piecewise affine map defined on certain triangle (see Chapter 2, Section 2.1, or [47]). The same holds in the lower-dimensional case; i.e., when $\dim \mathcal{M} = 2$, the limit return map $f_2(x) = 1 - 2x^2$ is also conjugate to the piecewise affine map (*tent map*) $\lambda_2(x) = 1 - 2|x|$ and this fact is also useful to obtain the Main Theorem in [33]. Both maps, f_2 and λ_2 present sensitivity with respect to initial conditions. A map $f : I \to I$ defined on an interval I is said to have ***sensitivity with respect to initial conditions*** if there exists $\epsilon > 0$ such that for every $x \in I$ and for any neighbourhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ with $|f^n(x) - f^n(y)| > \epsilon$.

As it was pointed out in [12] (page 145), the desire to compare maps with sensitivity to simple maps (piecewise affine maps) has a long history for the diffeomorphisms of the circle, see [24]. In the case of transformations defined on intervals let us firstly recall a result of Milnor-Thurston, [31], which asserts that every continuous, piecewise monotone map with positive topological entropy is semiconjugate to a continuous, piecewise linear map with constant slope and with the same entropy. We refer the reader also to a previous result of Parry ([41]) and to Section 8, Chapter II, in [30] where this kind of results are extensively treated. Furthermore, from Theorem II.7.12 in [12] if $f_a(x) = 1 - ax^2$ has no stable periodic orbits and no restrictive central points then there exists $a' \in (\sqrt{2}, 2]$ such that f_a and $\lambda_{a'}$ are conjugate (recall that $\lambda_{a'}(x) = 1 - a'|x|$). We refer the reader to Section II.7 in [12] for details.

Hence, one of the main purposes of this memoir is to perform a family of two-dimensional tent maps playing the same role as λ_a does for the lower-dimensional case. This family $\{\Lambda_t\}_t$ of two-dimensional tent maps is constructed in Chapter 3, Section 3.2. Every one of the members of this family are *Expanding Baker Maps*.

The definition of *Expanding Baker Map* is given in Chapter 3, Section 3.1. In a few words, an *Expanding Baker Map* folds some domain of $\mathbb{R}^2$ and after that expands the folded region, just a baker does when he kneads the dough. As we have seen at the beginning of this Introduction, this is just what the map Λ_t introduced in (I.1) does on the triangle $\mathcal{T}$.

Let us finish this section by pointing out that the family of limit return maps defined in (I.3) was earlier introduced in a series of papers, [18] and [19], by Gonchenko, Shilnikov and Turaev, see also [20] and [21]. In these papers the authors studied parameter families $\{X_\mu\}_\mu$ of dynamical systems unfolding a homoclinic tangency in any dimension bigger than two. In those cases, they obtained two-parameter families of limit return maps, but one of the parameters depends on the multipliers of the involved saddle periodic orbit. However, we are going to deal with two-parameter families of diffeomorphisms defined in a three-dimensional manifold unfolding a generalized homoclinic tangency (see the definition in Chapter 1) and, as a consequence, in our case both parameters a and b in the definition of $T_{a,b}$ depend on the geometry of the tangency.

Furthermore, under the change in coordinates

$$X = \widetilde{y}\ ,\ Y = \widetilde{x} + b\widetilde{y}$$

the map $T_{a,b}$ given in (I.3) takes the form

$$T_{a,b}(X, Y) = (Y, a + X^2 + bY).$$

These maps were also introduced in a book of Mira et al. where the authors develop an extensive study on noninvertible quadratic maps (see [32]).

In fact, as one can check in Theorem 1.18, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a quadratic map such that

i) $f \circ f$ is also quadratic

ii) f does not have any invariant linear foliation

then, f is linearly conjugate to the map $(x, y) \mapsto (a + bx - y^2, x)$ for some real numbers a and b (see Theorem 2 in [51] for details).

I.2 The two-dimensional case revisited

Let $\mathcal{M}$ be a two-dimensional manifold and let us consider one-parameter families of diffeomorphisms $\{f_\mu\}_\mu$, $f_\mu : \mathcal{M} \to \mathcal{M}$, such that f_{μ_0} has a homoclinic point q_{μ_0} associated to a hyperbolic saddle point p_{μ_0}. Moreover, for $\mu = \mu_0$, we assume that the stable and the unstable manifolds of p_{μ_0} display a quadratic tangency at q_{μ_0}. It is said that the family $\{f_\mu\}_\mu$ ***generically unfolds the homoclinic tangency*** if, for $\mu < \mu_0$ the invariant manifolds of the saddle point p_μ (the analytic continuation of p_{μ_0}) has no intersections and for $\mu > \mu_0$ there exist transverse homoclinic points associated to p_μ. See Figure I.5.

One of the main purposes during the last four decades was to determine the prevalence or not of hyperbolic dynamics for values of the parameter $\mu > \mu_0$. One of the main results in this sense was proved by L. Mora and M. Viana in [33] where they proved the following theorem.

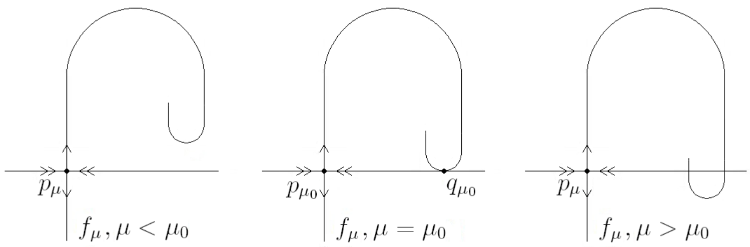


Figure I.5: Unfolding a two-dimensional homoclinic tangency

THEOREM **I.7** *Let $\{f_\mu\}_\mu$ be a generic family of diffeomorphisms defined on a surface, unfolding a homoclinic tangency of f_{μ_0}. Suppose that f_{μ_0} is area dissipative at the saddle point involved in the tangency. Then there exists a positive measure set of values of μ for which f_μ has strange attractors of Hénon type.*

In order to give a definition of *Hénon type attractor* we must previously remark that the above important result, as well as many other interesting ones, see e.g. the results by Colli ([13]), Newhouse ([36]), Yorke-Alligood ([58]) or Palis-Yoccoz ([40]) among many others, is strongly based on the existence of families of *limit return maps* associated to the unfolding of the homoclinic tangency.

Let us briefly describe the ***renormalization scheme*** in the two-dimensional scenario (see [52] and also [38]): for a nice and complete overview on this topic we refer the reader to [56]. Given a generic one-parameter family of dissipative diffeomorphisms $\{f_\mu\}_\mu,\ f_\mu : \mathcal{M} \to \mathcal{M}$, such that f_{μ_0} has a generic homoclinic tangency (the tangency is quadratic and is generically unfolded) then, for each sufficiently large $n \in \mathbb{N}$, there exists a neighbourhood U_n near the homoclinic point q satisfying the following properties:

i) $f_\mu^n(U_n) \cap U_n \neq \varnothing$.

ii) f_μ^n restricted to U_n is conjugate to certain map $F_{a,n} = \Phi_n \circ f_\mu^n \circ \Phi_n^{-1}$, $a = a(\mu)$, defined on certain domain of $\mathbb{R}^2$.

iii) $F_{a,n} \to F_a$ as $n \to \infty$, where F_a is the so called ***limit return map***.

Moreover, for the 2D-case the family of limit return maps can be written as (see [52]):

$$(x_1, x_2) \mapsto (a - x_1^2, 0).$$

For $a > 1$ this family of endomorphisms is completely equivalent to the following one

$$F_a(x_1, x_2) = (1 - ax_1^2, 0). \qquad \text{(I.4)}$$

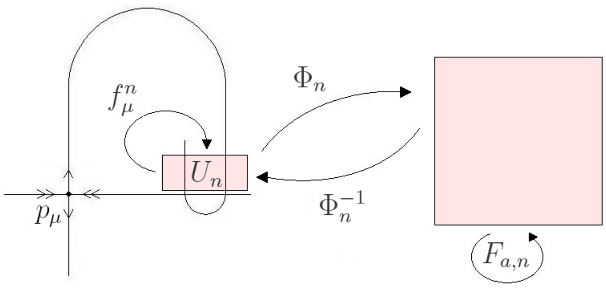


Figure I.6: Two-dimensional renormalization scheme

This means that a convenient (high) power of f_μ restricted to certain neighbourhood U_n near the homoclinic point is conjugate to a (small) perturbation of a one-dimensional map.

The Hénon-family $h_{a,b}(x,y) = (1 - ax^2 + y, bx)$ for $b > 0$ can be easily written as

$$h_{a,b}(x,y) = (1 - ax^2 + \sqrt{b}y, \sqrt{b}x)$$

in such a way that, for small values of b the proper Hénon-family is also a small perturbation of (I.4). This is the main reason why the strange attractors arising in [33] are called Hénon-like attractors: They appear in the dynamics of small perturbations of (I.4).

Let us recall that (I.4) also works as a family of limit return maps in higher-dimensional settings. For instance, under the assumption of *sectionally dissipativeness*, that is, if $\dim \mathcal{M} = n$ and $f_\mu : \mathcal{M} \to \mathcal{M}$ unfolds a homoclinic tangency associated to a periodic saddle point p whose eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfy

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_{n-1}| < 1 < |\lambda_n| \ , \ |\lambda_{n-1}\lambda_n| < 1,$$

then in [55] it is proved that the respective family of limit return maps is given by

$$F_a(x_1, x_2, \ldots, x_n) = (1 - ax_1^2, 0, \ldots, 0).$$

This fact is also used in [39] and [49] in order to extend to any dimension those classical two-dimensional results due to Newhouse, see [34] and [35]. Moreover, it was proved in [19] that the Newhouse wild hyperbolic sets (see [34]) exist for systems arbitrary C^2-close to any system with any homoclinic tangency.

In view of the above results, it seems clear that a good knowledge of the dynamics of the quadratic family $f_a(x) = 1 - ax^2$ has been essential to obtain the results in [8] and [33].

In fact, many arguments for proving the existence of persistent strange attractors in the two-dimensional homoclinic setting (see [33]) are based (in any case inspired) in those ones leading to prove the existence of a positive Lebesgue measure set of parameters for which the orbit of the critical point of $f_a(x) = 1 - ax^2$ presents exponential grow of the derivative, see [25] and [7].

Finally, we point out that a first good approximation to understand the dynamics of this famous one-dimensional quadratic family should be firstly study the dynamics of the family of tent maps given by $\lambda_a(x) = 1-a|x|$. It is known, as we have already pointed out, that for many values of the parameter (for instance, for $a = 2$) both families display the same kind of dynamical behaviour. Namely, the fact that for $a = 2$ both dynamical systems are conjugate is strongly used for obtaining the results in [7].

I.3 Return maps for certain class of $3-$D diffeomorphisms

In [51] families of limit return maps are obtained for generic two-parameter families of three-dimensional diffeomorphisms unfolding a generalized homoclinic tangency. Namely, let us consider a two-parameter family $\{f_{a,b}\}_{a,b}$ of three-dimensional diffeomorphisms having a hyperbolic saddle fixed point p_0 for $(a, b) = (0, 0)$ satisfying the following properties:

1.- The family $\{f_{a,b}\}_{a,b}$ satisfies the linearization assumption (generic condition for families of diffeomorphisms having saddle fixed points).

2.- The eigenvalues λ_1, λ_2 and λ_3 are real numbers satisfying $0 < |\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. We remark that from this assumption the corresponding fixed point is never sectionally dissipative.

3.- The invariant manifolds of p_0 have a *generalized homoclinic tangency* which unfolds generically (we only remark here that in the set of two-parameter families of three-dimensional diffeomorphisms unfolding homoclinic tangencies those ones unfolding a **generalized** homoclinic tangency are generic).

We refer the reader to Chapter 1 in this memoir or [51] for the definition of *generalized homoclinic tangency* and the statement of the *linearization assumption.*

In [51] (see Theorem 1) it is proved that, under these conditions, a two-parameter family $\{\widetilde{f}_{\widetilde{a},\widetilde{b}}\}$ of limit return maps can be constructed associated to the homoclinic orbit and is given by

$$\lim_{n\to\infty} F_{\widetilde{a},\widetilde{b},n}(\widetilde{x},\widetilde{y},\widetilde{z}) = \widetilde{f}_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{y},\widetilde{z}) = (\widetilde{z},\widetilde{a}+\widetilde{b}\widetilde{y}+\widetilde{z}^2,\widetilde{y}).$$

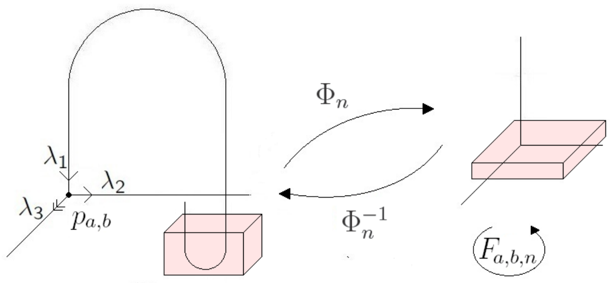


Figure I.7: Three-dimensional renormalization scheme

Let us point out that, for each $\left(\widetilde{a},\widetilde{b}\right) \in \mathbb{R}^2$, every point in $\mathbb{R}^3$ “falls” by one iteration of the map $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ into the surface

$$C_{\widetilde{a},\widetilde{b}} = \left\{(\widetilde{x},\widetilde{y},\widetilde{z}) : \widetilde{y} = \widetilde{a}+\widetilde{b}\widetilde{z}+\widetilde{x}^2\right\}.$$

Hence, $C_{\widetilde{a},\widetilde{b}}$ is invariant by $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ and it is enough to study the dynamics of $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ on $C_{\widetilde{a},\widetilde{b}}$. It is not difficult to see that the map restricted to $C_{\widetilde{a},\widetilde{b}}$ is conjugate to the family of two-dimensional maps defined on $\mathbb{R}^2$ by

$$T_{a,b}(\widetilde{x},\widetilde{y}) = (a+\widetilde{y}^2,\widetilde{x}+b\widetilde{y})$$

where we have written, in order to avoid excessive notation, $a=\widetilde{a}$ and $b=\widetilde{b}$.

Remark **I.8** *There is another case of generalized homoclinic tangency when the unstable invariant manifold of the fixed point is one-dimensional but the fixed point is not sectionally dissipative (see [51] and [17]).*

The dynamical behaviour of this family of limit return maps is rather complicated as was numerically pointed out in [48] and, in particular, the attractors exhibited by $T_{a,b}$ for a large set of parameters seem to be two-dimensional strange attractors (the sum of the Lyapounov exponents being positive). Moreover, in [47] a curve of parameters $(a(t),b(t))$ has been constructed in such a way that the respective transformation $T_{a(t),b(t)}$ has an invariant region in $\mathbb{R}^2$ homeomorphic to the triangle $\mathcal{T}$ (we also refer the reader to [48] where the authors numerically construct a large open set of parameters (a, b) for which $T_{a,b}$ has an invariant region). This curve of parameters is given by

$$\mathcal{G} = \{(a(t), b(t)) = (-\frac{1}{4}t^3(t^3 - 2t^2 + 2t - 2), -t^2 + t) : t \in \mathbb{R}\}. \tag{I.5}$$

In fact, the above curve of parameters corresponds to those ones for which there exists a straight line in $\mathbb{R}^2$ which is invariant under $T^2_{a,b}$. This fact was useful in [47] for proving the existence of an invariant domain $\mathcal{D}_t$ for $T_{a,b}$. The shape of $\mathcal{D}_t$ is shown in Figure I.8. Moreover, the curve of parameters (I.5) contains the point $(-4, -2) = (a(2), b(2))$ and, for this special value, $T_{-4,-2}$ is conjugate to the non-invertible piecewise affine map

$$\widetilde{\Lambda}(\Phi, \Psi) = (\pi - |\pi - \Phi - \Psi|, \Phi - \Psi)$$

defined on the triangle

$$\widetilde{\mathcal{T}} = \{(\Phi, \Psi) : 0 \leq \Phi \leq \pi, 0 \leq \Psi \leq \Phi\}$$

(see [47]). As one can see in Chapter 2, Section 2.1 (see also [47]), the conjugation $\Gamma_2 : \mathcal{D}_2 \to \widetilde{\mathcal{T}}$, between $T_{-4,-2}$ and $\tilde{\Lambda}$, is given by

$$\Gamma_2(\tilde{x}, \tilde{y}) = \left(\arccos\left(\frac{\tilde{y} - \sqrt{\tilde{y}^2 + 8\tilde{y} - 4\tilde{x}}}{4} \right), \arccos\left(\frac{\tilde{y} + \sqrt{\tilde{y}^2 + 8\tilde{y} - 4\tilde{x}}}{4} \right) \right). \tag{I.6}$$

After a new change in coordinates

$$(x, y) = \Xi(\Phi, \Psi) = (\frac{1}{\pi}(\Phi + \Psi), \frac{1}{\pi}(\Phi - \Psi)) \tag{I.7}$$

it is easy to see that one may write the above map $\tilde{\Lambda}$ in the following way:

$$\Lambda(x, y) = \begin{cases} (x + y, x - y) & , \text{if } (x, y) \in \mathcal{T}_0 \\ (2 - x + y, 2 - x - y) & , \text{if } (x, y) \in \mathcal{T}_1 \end{cases} \tag{I.8}$$

where $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ is the triangle defined in (I.2).

Let us observe that the map Λ given in (I.8) coincides with the map Λ_t defined in (I.1) when $t = 1$.

As was pointed out in [47], the map Λ displays the same nice properties of the one-dimensional tent map $\lambda_2(x) = 1 - 2|x|$. In particular, the consecutive pre-images of the critical line $\mathcal{C} = \{(x, y) \in \mathcal{T} : x = 1\}$, denoted by

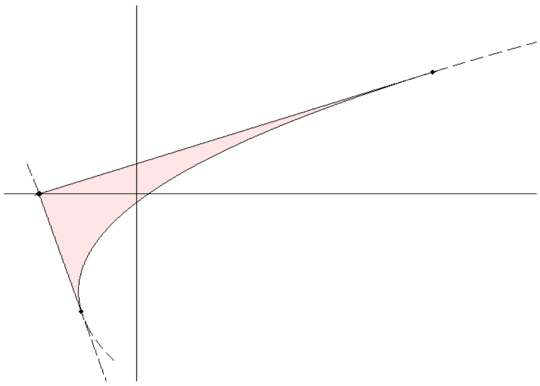


Figure I.8: The shape of $\mathcal{D}_t$ for $t = 1.8$.

$\{\Lambda^{-n}(\mathcal{C})\}_{n\in\mathbb{N}}$, define a sequence of partitions (whose diameter tends to zero as n goes to infinity) of $\mathcal{T}$ leading us to conjugate Λ (and therefore $T_{-4,-2}$) to a one sided *shift* with two symbols (see [47] for a complete explanation). Furthermore, for every initial point $Q_0 \in \mathcal{T}$ whose orbit never visits the critical line the Lyapounov exponent of Λ along the orbit of Q_0 is positive (in fact, it coincides with $\frac{1}{2}\log 2$) in all nonzero direction and the same holds for the limit return map $T_{-4,-2}$. Finally, it is not hard to construct an absolutely continuous ergodic invariant measure for Λ and therefore for $T_{-4,-2}$ (see [47]).

These basically were the main reasons why the authors called Λ the ***two-dimensional tent map***.

A first approach in order to simplify the dynamics of the limit return maps $T_{a,b}$ could be to apply the change in coordinates Γ_2 given in (I.6) to any map $T_{a(t),b(t)}$, with $(a(t), b(t)) \in \mathcal{G}$ (see (I.5)). Recall that Γ_2 conjugates $T_{-4,-2}$ with Λ (up to the universal change Ξ defined in (I.7)).

To this end, let us previously define

$$\Gamma_{1,t} : (\widetilde{x}, \widetilde{y}) \to \Gamma_{1,t}(\widetilde{x}, \widetilde{y}) = \left(\frac{16}{t^4}x + \frac{16(2-t)}{t^3}y + 4 - \frac{8}{t}, \frac{8}{t^3}y\right)$$

mapping the invariant domain $\mathcal{D}_t$ of $T_{a(t),b(t)}$ into the invariant domain $\mathcal{D}_2$ of $T_{-4,-2}$, see [48] for details. Doing so, and denoting by

$$\widetilde{F}_t = \Xi \circ \Gamma_2 \circ \Gamma_{1,t} \circ T_{a(t),b(t)} \circ \Gamma_{1,t}^{-1} \circ \Gamma_2^{-1} \circ \Xi^{-1}$$

(see Proposition 3.4 in Chapter 3, Section 3.2), certain dynamical properties of $\widetilde{F}_t$ are proved and, moreover, in Proposition 3.6 (Chapter 3, Section 3.2) it is demonstrated that if $G : \mathcal{T} \to \mathcal{T}$ is a piecewise linear map satisfying all these dynamical properties then $G = \Lambda_t$ for some t, being $\{\Lambda_t\}_t$ the family of piecewise linear maps given in (I.1).

These are the main reasons (see Remark 3.7) why we work with Λ_t along this memoir.

I.4 Different kinds of attractors (breads)

It is easy to see that for $t \in [0, 1/\sqrt{2})$ the origin is a global attractor for Λ_t (see Lemma 3.8 in Chapter 3, Section 3.2). On the other hand, for $t \in (1/\sqrt{2}, 1]$ one can easily obtain a fixed point

$$P_t = \left(\frac{2t(2t+1)}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1} \right),$$

for Λ_t in $\mathcal{T}_1$. Let us distinguish between three cases.

a) FAIRY CAKES ATTRACTORS.-

Let us assume that $\frac{1}{\sqrt{2}} < t < (\frac{1}{4})^{\frac{1}{5}}$. In this set of parameters, we obtain attractors formed by several pieces, as it is shown in Figure I.9(a). Repelling periodic orbits Q_t of period eight arise when $t = \frac{1}{\sqrt{2}}$. We can not speak about Hopf bifurcation due to the absence of smoothness, but we point out that, for every $\frac{1}{\sqrt{2}} < t \leq 1$, the eigenvalues λ_1 and λ_2 of the fixed point P_t satisfy

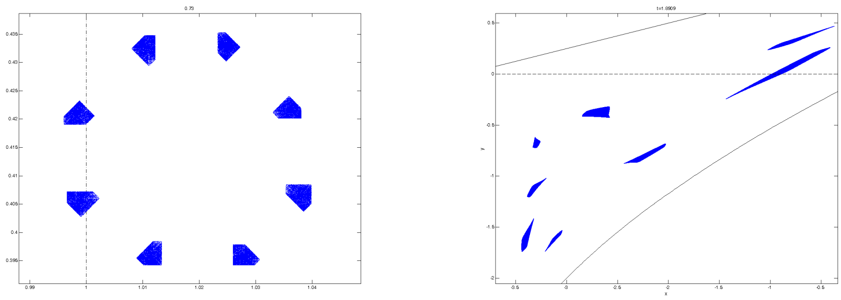
$$\lambda_1 = -t + ti \qquad \lambda_2 = -t - ti$$

and therefore we have $\lambda_1^8 = \lambda_2^8$ is a real positive number.

As was already pointed out in [48], eight-pieces attractors were also observed for $T_{a,b}$. For instance, if we restrict ourselves to the curve of parameters $(a(t), b(t))$ given in (I.5), we obtain for $t = 1.8909$ the attractor exhibited in Figure I.9(b). Let us remark that this eight-pieces attractor also contains a eight periodic orbit which emerges due to a previous Hopf bifurcation (see [48] for details).

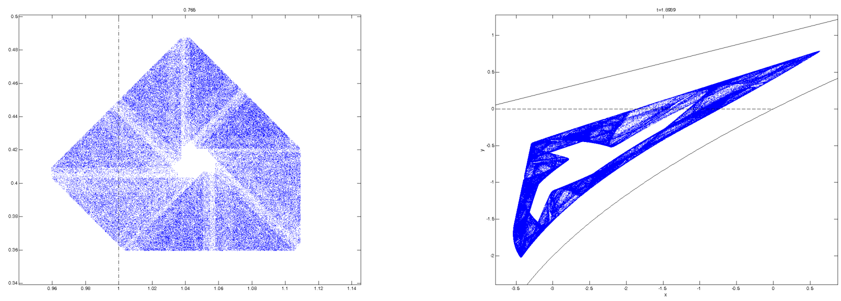
b) BREAD ROLL ATTRACTORS.-

If $(\frac{1}{4})^{\frac{1}{5}} < t < (\frac{1}{2})^{\frac{1}{3}}$, then Λ_t displays a one-piece strange attractor with a hole, as we can see in Figure I.10(a). This kind of attractors were also observed for the family $T_{a(t),b(t)}$ in [48]. For instance, for $t = 1.88817$ we numerically obtain the attractor showed in Figure I.10(b).



(a) Attractor for $\Lambda_t,\ t = 0.73$ (b) Attractor for $T_{a(t),b(t)},\ t = 1.8909$

Figure I.9: 8-pieces attractors



(a) Attractor for $\Lambda_t,\ t = 0.765$ (b) Attractor for $T_{a(t),b(t)},\ t = 1.88817$

Figure I.10: 1-piece attractors with a hole

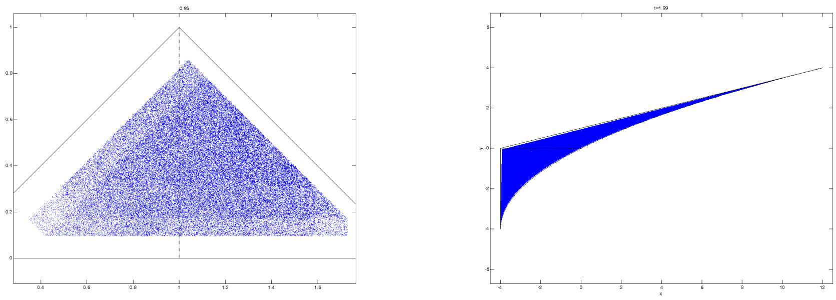
c) COUNTRY BREAD ATTRACTORS.-

Finally, if $(\frac{1}{2})^{\frac{1}{3}} < t \leq 1$, then Λ_t displays a one-piece strange attractors without holes (see Figure I.11(a)) and this kind of attractors are also observable for $T_{a(t),b(t)}$. For instance, for $t = 1.88904$ we obtain the attractor showed in Figure I.11(b).

Hence, it seems clear that $T_{a(t),b(t)}$ and Λ_t display the same "kind" of attractors for certain values of the parameters.

This memoir is organized as follows. In Chapter 1 we recover from [51] those results leading to the construction of the family of return maps given in (I.3).

Chapter 2 is devoted to describe basic properties of $T_{a,b}$ putting special emphasis in the dynamics when $a = -4$ and $b = -2$. These results can be found in [47] and [48]. The chapter ends by stating two "difficult?" conjec-



(a) Attractor for $\Lambda_t,\ t = 0.80$ (b) Attractor for $T_{a(t),b(t)},\ t = 1.88904$

Figure I.11: 1-piece attractors without holes

tures which probably keep us busy during the next decades.

In Chapter 3 we prove the first part of the Main Theorem (see Theorem I.6). We construct the maximal invariant set $\mathcal{R}_t$ for $\Lambda_t,\ t \in (t_0, 1]$, and we prove that Λ_t is strongly topologically mixing on $\mathcal{R}_t$. Moreover, we demonstrate the existence of a unique absolutely continuous, invariant and ergodic measure μ_t for $\Lambda_t,\ t \in (t_0, 1]$. Those results can be found in [44], [45] and [46]. The existence of μ_t is strongly based on the results given in [10], [11], [50], [53] and [54]. In this way, the family $\Lambda_t,\ t \in (t_0, 1]$, provides an example for the results obtained in those papers.

Finally, in Chapter 4 we prove the statistically stability of $\{\Lambda_t\}_{t\in(t_0,1]}$. All the arguments come from [5].

We want to finally remark those contents from the Appendix of Chapter 3 (see Chapter 3, Section 3.5). Although the results of the Main Theorem are restricted to the *country bread attractor case*, we have some partial results ("feelings") on the *fairy cakes attractor case*. We pursue results related to the renormalization techniques used in dimension one and, in this line, we hope to reach them in a near future.

We particularly remark Proposition 3.41 which tell us that Λ_t^2 can also be seen as a new Expanding Baker Map but with two folds before just one expansion. This theoretically means that a baker can obtain the same bread in two different ways: he can fold the dough, expand it, fold it in another direction and expand the dough once again or he can fold the dough repeatedly several times and applies just one expansion at the end.

For the *bread roll attractor case* we have no interesting result yet but, in the future, time will tell…

Chapter 1

Return maps for the $3-D$ case

Given a two-parameter family of three-dimensional diffeomorphisms with a saddle fixed point, suppose that there exists a homoclinic intersection. The aim of this section is to obtain certain "manageable" maps allowing us to describe the dynamics in a neighbourhood of the homoclinic tangency. These are called *limit return maps* and we will introduce some definitions and theorems leading us to obtain an expression for them (see [51] for the sake of completeness).

Let us begin by recalling several concepts which will be crucial in our study.

Definition **1.1** *Let $\mathcal{K}$ be a $N-$dimensional smooth manifold and $f : \mathcal{K} \to \mathcal{K}$ a diffeomorphism. Suppose that $p \in \mathcal{K}$ is a fixed point of f and denote by $\lambda_1, \ldots, \lambda_N$ the eigenvalues of $Df(p)$.*

1. *We say that p is* ***dissipative*** *if the product of the eigenvalues is less than 1 in absolute value.*

2. *We say that p is* ***sectionally dissipative*** *if p is dissipative and $|\lambda_i\lambda_j| < 1$ for all $i, j \in \{1, \ldots, N\}$ such that $i \neq j$.*

3. *The* ***index of stability*** *of p is the number of eigenvalues whose absolute value is less than 1.*

4. *We say that p is a* ***saddle*** *if $|\lambda_i| \neq 1$ for all $i \in \{1, \ldots, N\}$ and its index of stability is different from 0 and N.*

Let us now introduce the definitions of stable and unstable manifolds. These are invariant manifolds and their existence is proved by the Stable Manifold Theorem (see [37]).

Definition **1.2** *Let $\mathcal{K}$ be a $N-$dimensional smooth manifold and $f : \mathcal{K} \to \mathcal{K}$ a diffeomorphism. Suppose that $p \in \mathcal{K}$ is a saddle fixed point of f. We define the* ***stable invariant manifold*** *and the* ***unstable invariant manifold****, that we denote by $W^s(f,p)$ and $W^u(f,p)$ respectively, by*

$$W^s(f,p) = \{q \in \mathcal{K} : \lim_{n\to\infty} f^n(q) = p\}, \quad W^u(f,p) = W^s(f^{-1},p)$$

If it is not necessary we will not write explicitly the dependence on f.

Furthermore, we are interested in other invariant objects. From now on, given a smooth manifold $\mathcal{N} \subset \mathcal{K}$ we will write

$$L_i(\mathcal{N}) = \{(q,L) : q \in \mathcal{N},\ dim L = i,\ L \leq T_qM,\ T_q\mathcal{N} \cap L \neq \{0\}\}.$$

Recall that we will consider three-dimensional diffeomorphisms, so let f be a diffeomorphism defined on a smooth manifold $\mathcal{K}$ such that $dim\mathcal{K} = 3$. Suppose that p is a saddle fixed point and $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$, being $\lambda_i,\ i = 1,2,3$, the eigenvalues of $Df(p)$. We denote by E^{cu} (respectively E^{ss}) the invariant linear subspace of T_pM associated to the eigenvalues λ_2 and λ_3 (respectively λ_1). Let $F_1 : L_1(W^s(p)) \to L_1(W^s(p))$ and $F_2 : L_2(W^u(p)) \to L_2(W^u(p))$ be the maps induced by f on the respective spaces.

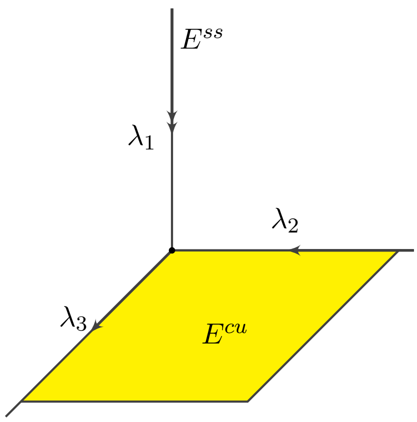


Figure 1.1:

It is easy to check that (p, E^{ss}) is a fixed point of F_1 with eigenvalues $\lambda_1,\ \lambda_2$ and $\lambda_1^{-1}\lambda_2$ (see [38]), so the corresponding stable manifold $W^s(p, E^{ss})$ has dimension two. On the other hand, (p, E^{cu}) is a fixed point of F_2 with eigenvalues λ_3 and $\lambda_1\lambda_2^{-1}$, so $W^u(p, E^{cu})$ has dimension one.

Using these invariant manifolds we can give the following definitions.

Definition **1.3** *Let f be as before.*

1. *If* $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ *the foliation* $\mathcal{F}^{ss}(p)$ *induced by* $W^s(p, E^{ss})$ *is called the* ***strong stable foliation*** *of* p*. Moreover, the* ***unstable invariant tangent bundle*** *of* p *is the vector bundle* $W^u(p, E^{cu})$*.*

2. *If* $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$ *the* ***strong unstable foliation*** *of* p*, denoted by* $\mathcal{F}^{uu}(p)$*, is the strong stable foliation of* p *with respect to* f^{-1}*. In the same way, the* ***stable invariant tangent bundle*** *of* p *is the unstable invariant tangent bundle of* p *associated to* f^{-1}*.*

Before giving the main results of this section, we need to introduce three definitions. First of all, we will define the notion of *limit return map* associated to a homoclinic tangency. Secondly, we will introduce a *linearization assumption* simplifying the study of families of diffeomorphisms, but this is not an indispensable assumption. Finally, we will see conditions for a family of diffeomorphisms to have a *generalized quadratic homoclinic tangency unfolding generically* related to the parameters.

DEFINITION **1.4** *Let* $\{f_a\}_{a\in\mathcal{V}}$ *be a smooth family of diffeomorphisms in some dimensional manifold* $\mathcal{K}$*. Suppose that for* $a = a_0$ *there exists a homoclinic orbit* O_0 *for some dissipative fixed point. We say that the family* $\{f_a\}_{a\in\mathcal{V}}$ *has a family of* ***limit return maps*** *associated to the homoclinic orbit* O_0*, in the* $\mathcal{C}^l$ *topology, if there exist a point* $q \in O_0$ *and a natural number* n_0 *such that for any positive integer* $n \geq n_0$ *there exist reparametrizations* $a = M_n(\widetilde{a})$ *of the* a *variable and* $\widetilde{a}-$ *dependent coordinate transformations* $x = \Psi_{n,\widetilde{a}}(\widetilde{x})$ *satisfying the following properties:*

1. *For each compact set* K *in the* $(\widetilde{a}, \widetilde{x})$ *space the images of* K *under the maps*
$$(\widetilde{a}, \widetilde{x}) \mapsto (M_n(\widetilde{a}), \Psi_{n,\widetilde{a}}(\widetilde{x}))$$
converge, for $n \to \infty$*, to* (a_0, q) *(in the* (a, x) *space).*

2. *The domains of the maps*
$$(\widetilde{a}, \widetilde{x}) \mapsto (\widetilde{a}, \Psi^{-1}_{n,\widetilde{a}} \circ f^n_{M_n(\widetilde{a})} \circ \Psi_{n,\widetilde{a}})$$
converge, for $n \to \infty$*, to all of* $\mathbb{R}^{m+k}$*, and the maps converge, in the* $\mathcal{C}^l$ *topology, to some map of the form*
$$(\widetilde{a}, \widetilde{x}) \mapsto (\widetilde{a}, \widetilde{f}_{\widetilde{a}}(\widetilde{x})).$$

In such case, the map $\widetilde{f}_{\widetilde{a}}$ *will be called limit return map and* $\{\widetilde{f}_{\widetilde{a}}\}_{\widetilde{a}\in\mathbb{R}^k}$ *a family of limit return maps associated to the homoclinic orbit* O_0*.*

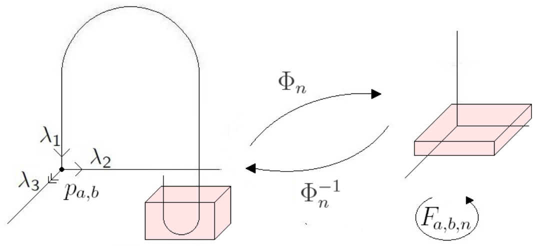


Figure 1.2: Family of limit return maps in a three-dimensional scenario

From now on, we suppose that the family $\{f_a\}_{a\in\mathcal{V}}$ ($\mathcal{V}$ being an open subset of $\mathbb{R}^k$) is defined on a smooth manifold $\mathcal{K}$ of dimension three, p_a is a saddle fixed point for all $a \in \mathcal{V}$ and the eigenvalues of p_a satisfy $|\lambda_{a,1}| < |\lambda_{a,2}| < |\lambda_{a,3}|$. Moreover $|\lambda_{a,1}| < 1 < |\lambda_{a,3}|$.

Definition **1.5** *We say that $\{f_a\}_{a\in\mathcal{V}}$ satisfies the **linearization assumption** with respect to the map of fixed points p_a if there exists an open subset $\mathcal{U} \subset \mathbb{R}^{3+k}$ and a $\mathcal{C}^s$ ($s \geq 3$) map $\mathbb{x} : \mathcal{U} \mapsto \mathcal{K}$ such that:*

1. $\Psi(\mathcal{U}) = \mathcal{U}$ *being*
$$\Psi(x_1, x_2, x_3, a) = (\lambda_{a,1}x_1, \lambda_{a,2}x_2, \lambda_{a,3}x_3, a)$$
for $(x_1, x_2, x_3, a) \in \mathbb{R}^{3+k}$.

2. $\mathcal{U}$ *is either a neighbourhood of*
$$\{(x_1, x_2, x_3) : \ (x_1^2 + x_2^2)x_3 = 0\} \times \mathcal{V}$$
if $|\lambda_{a,1}| < |\lambda_{a,2}| < 1 < |\lambda_{a,3}|$, *or a neighbourhood of*
$$\{(x_1, x_2, x_3) : \ (x_2^2 + x_3^2)x_1 = 0\} \times \mathcal{V}$$
if $|\lambda_{a,1}| < 1 < |\lambda_{a,2}| < |\lambda_{a,3}|$.

3. $\mathbb{x}(0, 0, 0, a) = p_a$ *for all* $a \in \mathcal{V}$.

4. $\mathbb{x}(\lambda_{a,1}x_1, \lambda_{a,2}x_2, \lambda_{a,3}x_3, a) = f_a(\mathbb{x}(x_1, x_2, x_3, a))$ *for all* $(x_1, x_2, x_3, a) \in \mathcal{U}$.

The linearization assumption is a generic condition (open and dense) for families of diffeomorphisms having saddle fixed points. We note that, if the family is linearizable, the invariant manifolds have natural parametrizations: for example, if $|\lambda_{a,1}| < |\lambda_{a,2}| < 1 < |\lambda_{a,3}|$ then

$$
\begin{aligned}
W^s(p_a) &= \{q \in \mathcal{K} : \; q = \mathbb{x}(t_1, t_2, 0, a) \text{ for some } (t_1, t_2) \in \mathbb{R}^2\} \\
W^u(p_a) &= \{q \in \mathcal{K} : \; q = \mathbb{x}(0, 0, t_3, a) \text{ for some } t_3 \in \mathbb{R}\}
\end{aligned}
$$

Without loss of generality, when we take a family $\{f_a\}_{a \in \mathcal{V}}$ of diffeomorphisms satisfying the linearization assumption we can suppose that $\mathcal{K} = \mathbb{R}^3$ and $p_a = 0$ for all $a \in \mathcal{V}$. Moreover, we can consider the sets

$$
\begin{aligned}
\mathcal{A}_1 &= \{(t_1, t_2, t_3) \in \mathbb{R}^3 : \; (t_1^2 + t_2^2)t_3 = 0 = 0\} \times \mathcal{V} \\
\mathcal{A}_2 &= \{(t_1, t_2, t_3) \in \mathbb{R}^3 : \; (t_2^2 + t_3^2)t_1 = 0 = 0\} \times \mathcal{V} \\
\mathcal{B}_1 &= \{(t_1, t_2, 0) : \; t_1^2 + t_2^2 \leq 3\} \\
\mathcal{B}_2 &= \{(t_1, 0, 0) : \; |t_1| \leq 3\}.
\end{aligned}
$$

The linearization assumption allow us to suppose (via a change in coordinates and a reparametrization if necessary) that if $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ (resp. $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$) there exists a C^k-map ($k \geq 3$) $\mathbf{x}_f$ defined on a neighbourhood $\mathcal{U}_f$ of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) such that:

1. If $(t_1, t_2, t_3, a) \in \mathcal{U}_f$ then $(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3) \in \mathcal{U}_f$ and $f_a(\mathbf{x}_f(t_1, t_2, t_3)) = \mathbf{x}_f(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a)$.
2. For some neighbourhood $\mathcal{U}_{1,f}$ of $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) the map $\mathbf{x}_f(\cdot, \cdot, \cdot, a)$ restricted to $\mathcal{U}_{1,f}$ is the identity, for all $a \in \mathcal{V}$.

Now we are really focused on the unfolding of homoclinic tangencies and we will define several concepts which are necessary to understand this phenomenon. Let us start by defining what is a *quadratic tangency* between the invariant manifolds, which requires the definition of *quadratic tangency* between curves and surfaces and a generic condition that concerns the position of an orbit inside a two dimensional invariant manifold.

DEFINITION **1.6** *Let $\{C_a\}_{a \in \mathcal{V}}$ be a smooth family of smooth regular curves, $C_a \subset \mathbb{R}^3$, and $\{S_a\}_{a \in \mathcal{V}}$ be a smooth family of smooth regular surfaces, $S_a \subset \mathbb{R}^3$, depending on a parameter $a \in \mathcal{V} \subset \mathbb{R}$, being $\mathcal{V}$ an open subset such that, for $a = a_0$, C_a and S_a intersect at a point p_0. We say that C_a and S_a have a* ***quadratic tangency*** *at p_0 (or a contact of order 1) which unfolds generically with a at $a = a_0$ iff there exists some smooth change of variables such that, in the new variables, $p_0 = (0, 0, 0)$, $S_a = \{(x, y, z) \in \mathcal{U} : \; z = 0\}$ (where $\mathcal{U}$ is a neighbourhood of $(0, 0, 0)$), C_a is represented by the parametrized curve $\gamma(t, a) = (x(t, a), y(t, a), z(t, a))$ and the following properties hold:*

1. $z(0, a_0) = 0$
2. $D_1 z(0, a_0) = 0$
3. $D_{1,1} z(0, a_0) \neq 0$
4. $D_2 z(0, a_0) \neq 0$

DEFINITION **1.7** *Let f be a three-dimensional diffeomorphism defined on a neighbourhood of a hyperbolic saddle fixed point p with eigenvalues λ_i, $i = 1, 2, 3$ which are supposed to be real and different in absolute value. We denote by Ψ the linear map $\Psi(x_1, x_2, x_3, a) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, a)$. Let W and $\widetilde{W} \subset W$ be, respectively, the two dimensional (stable or unstable) invariant manifold of p and the strong (stable or unstable) manifold contained in W. We say that q is in* ***general position*** *if $q \notin \widetilde{W}$ and there exists some smooth conjugacy F between f and Ψ such that the orbit $O(\Lambda, F(q))$ has at least one point out of the coordinate axes.*

In many cases the condition for an orbit to be in general position is equivalent to be in the two-dimensional stable or unstable invariant manifold but not in the corresponding strong one-dimensional manifold.

Recall that we assume that the family $\{f_a\}_{a \in \mathcal{V}}$ is defined on a smooth manifold $\mathcal{K}$ of dimension three, p_a is a saddle for all $a \in \mathcal{V}$ and the eigenvalues of p_a satisfy $|\lambda_{a,1}| < |\lambda_{a,2}| < |\lambda_{a,3}|$, $|\lambda_{a,1}| < 1 < |\lambda_{a,3}|$.

DEFINITION **1.8** *We say that $\{f_a\}_{a \in \mathcal{V}}$ has a* ***quadratic homoclinic tangency*** *at $q \in W^s(p_{a_0}) \cap W^u(p_{a_0})$ unfolding generically with $a = a_0$ if the manifolds $W^s(p_a)$ and $W^u(p_a)$ have a quadratic tangency at q which unfolds generically at $a = a_0$ and q is in general position.*

A typical quadratic homoclinic tangency is shown in Figure 1.3.

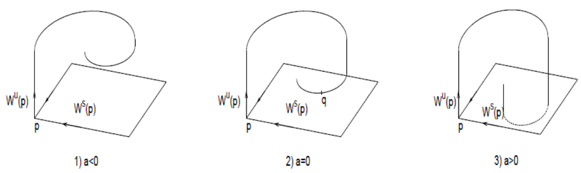


Figure 1.3: Example of quadratic homoclinic tangency

The next step is to define what is a generalized homoclinic transversality, so we begin with the local definition of *foliation*.

Definition **1.9** *Let $S \subset \mathbb{R}^3$ a surface and $\phi = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ a partition of S into disjoints sets called* ***leaves***. *The partition ϕ is called a* ***foliation*** *if there is a parametrization $\mathbb{x} : I \times J \subset \mathbb{R}^2 \to \mathbb{R}^3$ of S such that $\mathcal{L}_\alpha = \mathbb{x}(I \times \{c_\alpha\})$ where I and J are open intervals, $S = \mathbb{x}(I \times J)$ and c_α is a constant for each $\alpha \in A$. We call the pair (S, ϕ) a* ***foliated surface***.

In order to define the concept of *generalized transversality*, we take a family of foliated surfaces $\{(S_a, \phi_a)\}_{a \in \mathcal{V}}$, being $\mathcal{V} \subset \mathbb{R}$ an open set, and a family of vector bundles $\{\mathcal{V}_a\}_{a \in \mathcal{V}}$ whose zero section is a curve $\gamma = \gamma(t, a)$ and with fibers that are tangent planes to the curve. We can associate to each fiber of the bundle a vector $v_a(t) = v(t, a) \in T_{\gamma_a(t)}\mathbb{R}^3 \setminus T_{\gamma_a(t)}\gamma_a$, such that v is a smooth map, in such a way $v_a(t)$ and $\gamma_a(t)$ generate the fiber. In this situation we say that the family of vector fields $\{v_a\}_{a \in \mathcal{V}}$ along the family of curves $\{\gamma_a\}_{a \in \mathcal{V}}$ is associated to the family of tangent vector bundles $\{\mathcal{V}_a\}_{a \in \mathcal{V}}$.

Definition **1.10** *We say that $\{(S_a, \phi_a)\}_{a \in \mathcal{V}}$ and $\{\mathcal{V}_a\}_{a \in \mathcal{V}}$ have a* ***generalized transversal intersection*** *at $p \in S_{a_0}$ which unfolds generically with a at $a = a_0 \in \mathcal{V}$ iff in some coordinate system, for which the leaves of the foliation are the straight lines $z = 0$, $y = c$, the following properties are satisfied:*

1. $p = \gamma(0, a_0)$
2. $z(0, a_0) = 0$
3. $D_1 z(0, a_0) \neq 0$
4. *If* $T(t, a) = v_2(t, a)D_1 z(t, a) - v_3(t, a)D_1 y(t, a) = 0$ *then* $T(0, a_0) = 0$
5. $D_1 z(0, a_0)D_2 T(0, a_0) - D_2 z(0, a_0)D_1 T(0, a_0) \neq 0$

Here $v = (v_1, v_2, v_3)$ is a family of vector fields associated to the family of vector bundles and we write $\gamma(t, a) = (x(t, a), y(t, a), z(t, a))$.

This definition means that, at the point p of generalized transversality, the corresponding surface and curve have a transversal intersection and the fiber of the vector bundle and the leaf of the foliation at p are tangent. This latter property is lost when we change the value of the parameter.

Definition **1.11** *We say that the family $\{f_a\}_{a \in \mathcal{V}}$ has a* ***generalized homoclinic transversality*** *at $q \in W^s(p_{a_0}) \cap W^u(p_{a_0})$ unfolding generically with $a = a_0$ if*

1. *When* $|\lambda_{a,2}| < 1$, *the family of foliated surfaces*

$$\{(W^s(p_a), \mathcal{F}^{ss}(p_a))\}_{a\in\mathcal{V}}$$

and the family of invariant tangent vector bundles

$$\{(W^u(p_a), E^{cu}(p_a))\}_{a\in\mathcal{V}}$$

have a generalized transversal intersection at q *which unfolds generically with* a *at* $a = a_0$. *Moreover,* $q \in W^s(p_{a_0})$ *is in general position.*

2. *When* $|\lambda_{a,2}| > 1$, *the invariant manifolds of* p_a *with respect to the family* $\{f_a^{-1}\}_{a\in\mathcal{V}}$ *have a generalized transversal intersection at* q *which unfolds generically with* a *at* $a = a_0$.

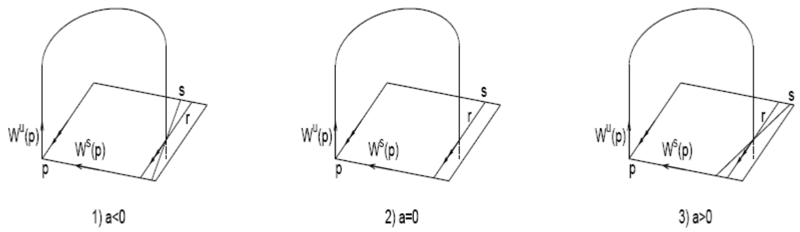


Figure 1.4: Generalized transversality. The line from p with two arrows represents the strong stable manifold $W^{ss}(p)$, the line r is a leaf of the strong stable foliation of $W^s(p)$ and the line s represents the intersection of $W^s(p)$ with the fiber of the unstable invariant vector bundle at the point of intersection of the invariant manifolds. In 1) and 3) r and s are different and in 2) (corresponding to the generalized homoclinic transversality) they coincide. Observe that in all the cases the invariant manifolds have transversal intersection.

As before, we first study a new type of tangency between a two-parameter family of foliated surfaces $\{(S_{a,b}, \phi_{a,b}\}_{(a,b)\in\mathcal{V}}$ and a two-parameter family of vector bundles $\{\mathcal{V}_{a,b}\}_{(a,b)\in\mathcal{V}}$ having a regular parametrized curve $\gamma_{a,b} = \gamma_{a,b}(t) = (x(t,a,b), y(t,a,b), z(t,a,b))$ as a zero section and tangent planes to $\gamma_{a,b}(t)$ as fibers. Here, $\mathcal{V} \subset \mathbb{R}^2$ is an open neighbourhood of $(a,b) = (a_0,b_0)$. Recall that a smooth family of vector fields $\{v_{a,b}\}_{(a,b)\in\mathcal{V}}$ associated to $\{\mathcal{V}_{a,b}\}_{(a,b)\in\mathcal{V}}$ satisfies that, for all t, $v_{a,b}(t) = (v_1(t,a,b), v_2(t,a,b), v_3(t,a,b))$ and $\gamma_{a,b}(t)$ generate the corresponding fiber of $\mathcal{V}_{a,b}$.

Definition **1.12** *We say that* $\{(S_{a,b},\phi_{a,b}\}_{(a,b)\in\mathcal{V}}$ *and* $\{\mathcal{V}_{a,b}\}_{(a,b)\in\mathcal{V}}$ *have a* ***generalized tangency of type I*** *at* $p_0=\gamma_{a_0,b_0}(t_0)\in S_{a_0,b_0}$ *which unfolds generically with* (a,b) *at* (a_0,b_0) *if there exists*

a) a coordinate system (x,y,z) *for which* $S_{a,b}$ *is the plane* $z=0$ *and the leaves of the foliation are the straight lines* $z=0,\ y=c$*;*

b) a two-parameter family of vector fields $\{v_{a,b}\}_{(a,b)\in\mathcal{V}}$ *associated to the fibers* $\{\mathcal{V}_{a,b}\}_{(a,b)\in\mathcal{V}}$ *such that*

1. $z(t_0,a_0,b_0)=0,\ D_1z(t_0,a_0,b_0)=0,\ D_1y(t_0,a_0,b_0)=0$
2. $D_{11}z(t_0,a_0,b_0)\neq 0,\ v_3(t_0,a_0,b_0)\neq 0$
3. $D_{11}y(t_0,a_0,b_0)v_3(t_0,a_0,b_0)-D_{11}z(t_0,a_0,b_0)v_2(t_0,a_0,b_0)\neq 0$
4. $$\begin{vmatrix} D_1z(t_0,a_0,b_0) & D_2z(t_0,a_0,b_0) & D_3z(t_0,a_0,b_0)\\ D_{11}z(t_0,a_0,b_0) & D_{12}z(t_0,a_0,b_0) & D_{13}z(t_0,a_0,b_0)\\ D_{11}y(t_0,a_0,b_0) & D_{12}y(t_0,a_0,b_0) & D_{13}y(t_0,a_0,b_0)\end{vmatrix}\neq 0$$

Definition **1.13** *We say that* $\{(S_{a,b},\phi_{a,b}\}_{(a,b)\in\mathcal{V}}$ *and* $\{\mathcal{V}_{a,b}\}_{(a,b)\in\mathcal{V}}$ *have a* ***generalized tangency of type II*** *at* $p_0=\gamma_{a_0,b_0}(t_0)\in S_{a_0,b_0}$ *which unfolds generically with* (a,b) *at* (a_0,b_0) *if there exists*

a) a coordinate system (x,y,z) *for which* $S_{a,b}$ *is the plane* $z=0$ *and the leaves of the foliation are the straight lines* $z=0,\ y=c$*;*

b) a two-parameter family of vector fields $\{v_{a,b}\}_{(a,b)\in\mathcal{V}}$ *associated to the fibers* $\{\mathcal{V}_{a,b}\}_{(a,b)\in\mathcal{V}}$ *such that*

1. $z(t_0,a_0,b_0)=0,\ D_1z(t_0,a_0,b_0)=0,\ v_3(t_0,a_0,b_0)=0$
2. $D_{11}z(t_0,a_0,b_0)\neq 0,\ D_1y(t_0,a_0,b_0)\neq 0$
3. $D_1y(t_0,a_0,b_0)D_1v_3(t_0,a_0,b_0)-D_{11}z(t_0,a_0,b_0)v_2(t_0,a_0,b_0)\neq 0$
4. $$\begin{vmatrix} D_1z(t_0,a_0,b_0) & D_2z(t_0,a_0,b_0) & D_3z(t_0,a_0,b_0)\\ D_{11}z(t_0,a_0,b_0) & D_{12}z(t_0,a_0,b_0) & D_{13}z(t_0,a_0,b_0)\\ D_1v_3(t_0,a_0,b_0) & D_2v_3(t_0,a_0,b_0) & D_3v_3(t_0,a_0,b_0)\end{vmatrix}\neq 0$$

Both definitions do not depend on changes of coordinates and parameters, and on the associated vector field $v_{a,b}$.

We are ready to define *generalized homoclinic tangencies* for two-parameter families of diffeomorphisms. Let $\{f_{a,b}\}_{(a,b)\in\mathcal{V}}$, $\mathcal{V}\subset\mathbb{R}^2$ being an open set, be a smooth two-parameter family of three-dimensional diffeomorphisms having a saddle fixed point $p = p(a,b)$ for $(a,b) \in \mathcal{V}$, with real eigenvalues $\lambda_i,\ i = 1,2,3$, such that $|\lambda_1| < |\lambda_2| < |\lambda_3|$.

DEFINITION **1.14** *We say that* $\{f_{a,b}\}_{(a,b)\in\mathcal{V}}$ *has a* ***generalized quadratic homoclinic tangency of type I (respectively type II)*** *at*

$$q \in W^s(p(a_0,b_0)) \cap W^u(p(a_0,b_0))$$

unfolding generically with (a,b) *at* $(a,b) = (a_0,b_0)$ *if:*

1. *When* $|\lambda_2| < 1$, *the family of foliated surfaces*

$$\{(W^s(p(a,b)), \mathcal{F}^{ss}(p(a,b)))\}_{(a,b)\in\mathcal{V}}$$

and the family of invariant tangent vector bundles

$$\{(W^u(p(a,b)), E^{cu}(p(a,b)))\}_{(a,b)\in\mathcal{V}}$$

have a generalized tangency of type I (resp. type II) at q *which unfolds generically with* (a,b) *at* $(a,b) = (a_0,b_0)$. *Moreover,* q *is in general position.*

2. *When* $|\lambda_2| > 1$, *the invariant manifolds of* $p(a,b)$ *with respect to the family* $\{f^{-1}_{a,b}\}_{(a,b)\in\mathcal{V}}$ *have a generalized quadratic homoclinic tangency of type I (resp. type II) at* q *unfolding generically with* (a,b) *at* $(a,b) = (a_0,b_0)$.

We refer the reader to [51] for a complete description of this phenomenon.

Before formulating the main results of this section, we need to introduce the definition of *Bogdanov-Takens bifurcation.*

DEFINITION **1.15** *Let* $\{f_{a,b}\}_{(a,b)\in\mathcal{V}}$, $\mathcal{V} \subset \mathbb{R}^2$ *being an open neighbourhood of* (a_0,b_0), *be a smooth two-parameter two-dimensional family of diffeomorphisms, having a fixed point* p_0 *for* $(a,b) = (a_0,b_0)$ *with two eigenvalues equal to* 1 *and* $Df_{a_0,b_0}(p_0)$ *different from the identity. We say that* $\{f_{a,b}\}_{(a,b} \in \mathcal{V}$ *has a Bogdanov-Takens bifurcation at* p_0 *which unfolds generically with* (a,b) *at* (a_0,b_0) *if via an smooth change of variables and a reparametrization, we can obtain the family of maps*

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u+v \\ v+\mu+u^2+v(\nu+\widetilde{\gamma}u)+\widetilde{c}v^2+\widetilde{r}_3(u,v) \end{pmatrix}$$

where p_0 *is transformed in* $(0,0)$, $\mu = \mu(a,b)$ *and* $\nu = \nu(a,b)$ *are the new parameters such that* $(\mu(a_0,b_0), \nu(a_0,b_0)) = (0,0)$ *and* $\left|\frac{\partial(\mu,\nu))}{\partial(a,b)}\right| \neq 0$ *when* $(a,b) = (a_0,b_0)$, $\widetilde{\gamma}$ *and* $\widetilde{c}$ *are functions of* (a,b) *such that* $\widetilde{\gamma}(a_0,b_0) \neq 2$. *Moreover,* $\widetilde{r}_3 = \widetilde{r}_3(u,v,a,b)$ *is a smooth function whose Taylor expansion around* $(u,v)=(0,0)$, *for each fixed* (a,b), *begins at degree 3.*

We are now in conditions to formulate the theorems supporting this section. The first of these theorems describes the dynamic behaviour of a two-parameter family of three-dimensional diffeomorphisms having a generalized quadratic homoclinic tangency. The second one gives us a classification of certain quadratic maps in $\mathbb{R}^2$ and $\mathbb{R}^3$. Both proofs can be found in [51].

THEOREM **1.16** *Let* $\{f_{a,b}\}_{(a,b)\in\mathcal{V}}$ *(*$\mathcal{V} \subset \mathbb{R}^2$*) be a family of three dimensional diffeomorphisms having a hyperbolic dissipative fixed point* p_0 *for* $(a,b) = (0,0)$ *with eigenvalues* λ_1, λ_2 *and* λ_3. *Assume that either*

- $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ *and* $|\lambda_2\lambda_3| > 1$ *(Case A)*
- $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$ *(Case B)*

Moreover, we suppose that the invariant manifolds of p_0 *have a generalized homoclinic tangency which unfolds generically and the family* $f_{a,b}$ *satisfies the linearization assumption. Then:*

1. *For* n *large enough there are values of the parameter* (a_n,b_n) *for which the map* $f_{a,b}$ *undergoes a generic* $n-$*periodic Bogdanov-Takens bifurcation. Moreover* $(a_n,b_n) \to (0,0)$ *when* $n \to \infty$.
2. *There exists a family of limit return maps* $\{\widetilde{f}_{\widetilde{a},\widetilde{b}}\}_{(\widetilde{a},\widetilde{b})\in\mathbb{R}^2}$ *associated to the generalized homoclinic tangency, such that*

   (a) *If* $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$, $|\lambda_1\lambda_3| < 1$ *and* $|\lambda_2\lambda_3| > 1$ *then*

   $$\widetilde{f}_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{y},\widetilde{z}) = (\widetilde{z}, \widetilde{b}\widetilde{z}, \widetilde{a} + \widetilde{y} + \widetilde{z}^2)$$

   or

   $$\widetilde{f}_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{y},\widetilde{z}) = (0, \widetilde{z}, \widetilde{a} + \widetilde{b}\widetilde{y} + \widetilde{z}^2)$$

   depending on the type of generalized homoclinic tangency (I or II resp).

   (b) *If* $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$ *then*

   $$\widetilde{f}_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{y},\widetilde{z}) = (\widetilde{z}, \widetilde{a} + \widetilde{b}\widetilde{y} + \widetilde{z}^2, \widetilde{y})$$

3. $(a, b) = (0, 0)$ *is in the closure of the set of parameter values for which there exist differentiable invariant circles. In case A, the circles can be of attracting or saddle type (attracting or repelling inside the centre manifold), depending on the geometry of the tangency. In case B the circles are always attracting.*

4. $(a, b) = (0, 0)$ *is in the closure of the set of parameter values for which there are attracting periodic orbits.*

5. *There is a set E of parameter values such that its intersection with any neighbourhood of* $(0, 0)$ *has positive Lebesgue measure, and for* $(a, b) \in E$ *the diffeomorphism* $f_{a,b}$ *exhibits a strange attractor near the orbit of tangency.*

6. *There are open sets* $\mathcal{U} \subset \mathbb{R}^2$*, arbitrarily near* $(0, 0)$ *such that for a generic* $(a, b) \in \mathcal{U}$ *the map* $f_{a,b}$ *has infinitely many sinks.*

Remark **1.17** *The two quadratic maps appearing in the previous theorem (case A) are conjugate if* $\widetilde{b} \neq 0$. *When* $\widetilde{b} = 0$ *they are not conjugate but if we restrict the first map to the plane* $\widetilde{y} = 0$ *and the second to* $\widetilde{x} = 0$ *we obtain the same two-dimensional map. These restrictions are logistic maps in both cases.*

At first glance, one can see that the limit return maps in Theorem 1.16 (Case A) are essentially Hénon maps. On the other hand, the maps in Theorem 1.16 (Case B) (when the unstable invariant manifold has dimension two) are the natural setting in which topologically two-dimensional strange attractors must show up.

Theorem **1.18** 1. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be a quadratic map such that* $f \circ f$ *is also quadratic. Suppose that f does not have any invariant linear foliation. Then it is linearly conjugate to the map*

$$(x, y) \mapsto (a + bx - y^2, x)$$

for some real numbers a and b.

2. *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *be a quadratic map such that* $f \circ f$ *is also quadratic and having constant Jacobian. Suppose that f does not have any invariant linear foliation. Then it is linearly conjugate to the map*

$$(x, y, z) \mapsto (a + bx + z - y^2, x, cy)$$

for some real numbers a, b and c.

Chapter 2

The family $T_{a,b}$

Let us assume $\{f_{a,b}\}_{(a,b)\in\mathcal{V}}$ ($\mathcal{V}$ being an open subset of $\mathbb{R}^2$) a family of diffeomorphisms defined on $\mathbb{R}^3$. Suppose $p_{a,b}$ is a fixed saddle point for $f_{a,b}$ for all $a \in \mathcal{V}$ and the eigenvalues of $Df_{a,b}$ at $p_{a,b}$ satisfy $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. Moreover:

1. The point $p_{a,b}$ is dissipative
2. The invariant manifolds of p_0 have a generalized homoclinic tangency which unfolds generically
3. The family $\{f_{a,b}\}_{(a,b)\in\mathcal{V}}$ satisfies the linearization assumption

Thus, Theorem 1.16 allows us to ensure the existence of a family of limit return maps defined by

$$\widetilde{f}_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{y},\widetilde{z}) = (\widetilde{z}, \widetilde{a} + \widetilde{b}\widetilde{y} + \widetilde{z}^2, \widetilde{y}) \tag{2.1}$$

associated to $\{f_{a,b}\}_{(a,b)}$. It is easy to see that every point in $\mathbb{R}^3$ “falls” by one iteration of the map $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ into the surface

$$\widetilde{\mathcal{C}}_{\widetilde{a},\widetilde{b}} = \{(\widetilde{x},\widetilde{y},\widetilde{z}) : \ \widetilde{y} = \widetilde{a} + \widetilde{b}\widetilde{z} + \widetilde{x}^2\}.$$

So $\widetilde{\mathcal{C}}_{\widetilde{a},\widetilde{b}}$ is an invariant surface by $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ and it is enough to study the dynamics of these maps on $\widetilde{\mathcal{C}}_{\widetilde{a},\widetilde{b}}$. That is, it is enough to consider the family of two-dimensional endomorphisms

$$(g^{-1}_{\widetilde{a},\widetilde{b}} \circ \widetilde{f}_{\widetilde{a},\widetilde{b}} \circ g_{\widetilde{a},\widetilde{b}})(\widetilde{x},\widetilde{z}) = (\widetilde{z}, \widetilde{a} + \widetilde{b}\widetilde{z} + \widetilde{x}^2)$$

being $g_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{z}) = (\widetilde{x}, \widetilde{a} + \widetilde{b}\widetilde{z} + \widetilde{x}^2, \widetilde{z})$ a parametrization of $\widetilde{\mathcal{C}}_{\widetilde{a},\widetilde{b}}$.

Lastly, we take the change in coordinates

$$x = \widetilde{z} - \widetilde{b}\widetilde{x}, \; y = \widetilde{x}$$

in order to write the above family of transformations as $\{T_{a,b}\}_{(a,b)}$ being

$$T_{a,b}(x, y) = (a + y^2, x + by) \tag{2.2}$$

where we write (a, b) instead of $(\widetilde{a}, \widetilde{b})$ for simplifying the notation.

It is easy to check that $T_{a,b}$ has two fixed points $P_{a,b}$ and $\widetilde{P}_{a,b}$ given by

$$\begin{aligned} P_{a,b} = & \left(\tfrac{1-b}{2}(1 - b + \sqrt{(1-b)^2 - 4a}), \tfrac{1}{2}(1 - b + \sqrt{(1-b)^2 - 4a})\right) \\ \widetilde{P}_{a,b} = & \left(\tfrac{1-b}{2}(1 - b - \sqrt{(1-b)^2 - 4a}), \tfrac{1}{2}(1 - b - \sqrt{(1-b)^2 - 4a})\right) \end{aligned} \tag{2.3}$$

whenever $(1 - b)^2 \geq 4a$. Recall that we are looking for strange attractors, so it will be convenient to find some invariant domains and these points will be a great help in our search. The simplest case in which we can prove the existence of these domains is the case $b = 0$. Note that $T_{a,0}(x, y) = (a+y^2, x)$ and $T^2_{a,0} = (x, y) = (a+x^2, a+y^2)$. In particular, it follows that $T_{a,0}(\mathcal{J}_a) = \mathcal{J}_a$ for all $a \in [-2, -1]$, being $\mathcal{J}_a = [a, a + a^2] \times [a, a + a^2]$. In addition, it is possible to prove the existence of two-dimensional strange attractors (in a one-dimensional persistent way) as stated in the next proposition.

Proposition **2.1** *For any $0 < c < \log 2$ there exists $a_0 > -2$ and a positive Lebesgue measure set E of values of the parameter a contained in $[-2, a_0]$ such that for any $a \in E$ there exists a dense orbit $\bigcup_{n\in\mathbb{N}}\{T^n_{a,0}(Q_0)\}$ in $\mathcal{J}_a$ and a natural number n_0 such that*

$$\|DT^n_{a,0}(Q_0)v\| \geq \exp(\frac{cn}{2})$$

for any unit vector v and $n \geq n_0$.

Proof. Let us denote by L_a the map $x \to L_a(x) = x^2 + a$. First, we note that if f is a density of an invariant measure μ of the map L_a then $g(x, y) = f(x)f(y)$ is a density of an invariant measure of $T_{a,0}$ and the corresponding measure is the product measure. Moreover, from [59] for all $a_0 > -2$ there is a positive Lebesgue measure set E of values of the parameter a contained in $[-2, a_0]$ such that for any $a \in E$ the map L_a has an absolutely continuous and strongly mixing invariant measure μ, such that

$$supp(\mu) = [a, a + a^2].$$

This implies that for $a \in E$ the map $T_{a,0}$ has an absolutely continuous and ergodic invariant measure $\widetilde{\mu} = \mu \times \mu$ (see [57]), with support equal to $\mathcal{J}_a$. In particular, $T_{a,0|\mathcal{J}_a}$ is transitive (see [29]).

Suppose now that there exists a set $\mathcal{S} \subset \mathcal{J}_a$ such that $\widetilde{\mu}(\mathcal{S}) = 1$ and if $x \in \mathcal{S}$ then its $T_{a,0}-$orbit is dense in $\mathcal{J}_a$ (we will prove this result in Lemma 2.2). Let us define $W \subset [a, a + a^2]$ as the set of points $x \in [a, a + a^2]$ for which $\lambda(x) > 0$, where

$$\lambda(x) = \limsup_{n\to\infty} \frac{1}{n} \log |DL_a^n(x)|.$$

As μ is absolutely continuous and ergodic there exists $W_1 \subset W$ such that $\mu(W_1) = 1$ and

$$\lambda(x) = l = \int \log |DL_a| d\mu > 0,$$

for all $x \in W_1$ (see Theorem 3.2, page 366 of [30]). Hence, if $v \in \mathbb{R}^2$ is a non-zero vector, the Lyapunov exponent of $T_{a,0}$ at $(x, y) \in W_1 \times W_1$ in the direction of v is always equal to $l/2 > 0$. As $\widetilde{\mu}(W_1 \times W_1) = 1$, we deduce that $(W_1 \times W_1) \cap \mathcal{S} \neq \emptyset$, meaning that there exists a dense orbit for which all the Lyapunov exponents are positive. We only have to take into account the behaviour of the Lyapunov exponent for the logistic map when a is close to -2, wherewith the proposition is proved. ■

In order to finish the proof we will demonstrate that the set $\mathcal{S}$ really exists.

Lemma **2.2** *There exists a set $\mathcal{S} \subset \mathcal{J}_a$ such that $\widetilde{\mu}(\mathcal{S}) = 1$ and if $x \in \mathcal{S}$ then its $T_{a,0}-$orbit is dense in $\mathcal{J}_a$.*

Proof. First, we observe that if $\mathcal{U} \subset \mathcal{J}_a$ is open and non-empty, then $\widetilde{\mu}(\cup_{n\geq 0} T_{a,0}^{-n}(\mathcal{U})) > 0$. Let $\{\mathcal{U}_n\}_{n\geq 0}$ be a countable basis of open sets. If we define

$$\mathcal{S}_{i,n} = \bigcup_{j\geq i} T_{a,0}^{-j}(\mathcal{U}_n), \;\; \mathcal{S}_n = \bigcup_{i\geq 0} \mathcal{S}_{i,n}$$

it follows that

i) $\mathcal{S}_{i,n} = T_{a,0}^{-1}(\mathcal{S}_{i-1,n}) \subset \mathcal{S}_{i-1,n}$ for all $i \in \mathbb{N}$.

ii) $\widetilde{\mu}(\mathcal{S}_n) = \lim_{i\to\infty} \widetilde{\mu}(\mathcal{S}_{i,n}) = \widetilde{\mu}(\mathcal{S}_{i,n}) > 0$, for all $i \geq 0$ and $n \geq 0$, because the measure is invariant.

iii) $\mathcal{S}_n$ is a $T_{a,0}-$invariant set and, as $\widetilde{\mu}$ is ergodic, $\widetilde{\mu}(\mathcal{S}_n) = 1$ for all $n \geq 0$.

iv) $\mathcal{S}_n$ is residual, because as $T_{a,0}$ is transitive, then $\mathcal{S}_{i,n}$ is open and dense, for all $n \geq 0$ and $i \geq 0$ (see Proposition 2.8 in [29])

Now, if we define

$$\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$$

then, obviously, $\widetilde{\mu}(\mathcal{S}) = 1$ and if $x \in \mathcal{S}$ then its $\omega-$limit coincides with $\mathcal{J}_a$. Indeed, it is enough to prove that for all $n \geq 0$ there are infinitely many values $m > 0$ for which $T^m_{a,0}(x) \in \mathcal{U}_n$. But $x \in \mathcal{S}$ implies that $x \in \mathcal{S}_n$, that is

$$x \in \bigcap_{i \geq 0} \bigcup_{j \geq 0} T^{-j}_{a,0}(\mathcal{U}_n)$$

which means that $x \in T^{-m}_{a,0}(\mathcal{U}_n)$ for infinitely many values m, and for these values $T^m_{a,0} \in \mathcal{U}_n$. ■

Note that

$$T^2_{a,0}(x, y) = (a + x^2, a + y^2)$$

has invariant lines. Therefore, a possible method for finding more values of the parameters (a, b) for which $T_{a,b}$ has invariant domains would be to detect those parameters for which $T^2_{a,b}$ has invariant curves.

We consider the parameter curve $\mathcal{G}$ given by

$$\mathcal{G} = \{(a(t), b(t)) : \ t \in \mathbb{R}, \ a(t) = -\frac{t^3}{4}(t^3-2t^2+2t-2), \ b(t) = -t^2+t\}. \quad (2.4)$$

From now on, we take $(a, b) = (a(t), b(t)) \in \mathcal{G}$. Denoting by

$$\alpha = \alpha(t) = \frac{t}{2}(1 - t) \ , \ \beta = \beta(t) = t^2,$$

it is easy to check that the line

$$\mathcal{L}^1 = \mathcal{L}^1(t) = \{(x, y) \in \mathbb{R}^2 : x = \alpha + \beta y\}$$

is invariant for $T^2_{a,b}$ and $P_{a,b} \in \mathcal{L}^1$ (see (2.3)). We denote by $\mathcal{C}^1 = \mathcal{C}^1(t)$ the segment in $\mathcal{L}^1$ joining $P_{a,b}$ with the point $L = (\alpha, 0)$.

Under these conditions, the curve $\mathcal{L}^2$ given by

$$\mathcal{L}^2 = \mathcal{L}^2(t) = \{(x, y) \in \mathbb{R}^2 : x = \left(\frac{y - \alpha}{b + \beta}\right)^2 + a\}$$

verifies that $P_{a,b} \in \mathcal{L}^2$ and $T_{a,b}(\mathcal{C}^1) \subset \mathcal{L}^2$. Let $Q_{a,b}$ be the unique point (different from $P_{a,b}$) such that $T_{a,b}(Q_{a,b}) = P_{a,b}$. Then, $Q_{a,b} \in \mathcal{L}^2$. We can define $\mathcal{C}^2 = \mathcal{C}^2(t)$ as the segment in $\mathcal{L}^2$ joining $P_{a,b}$ with $Q_{a,b}$. Then, $T_{a,b}(\mathcal{C}^2) \subset \mathcal{L}^1$.

Finally, we denote by $\mathcal{L}^3$ the line

$$\mathcal{L}^3 = \mathcal{L}^3(t) = \{(x,y) \in \mathbb{R}^2 : x = \alpha - (2b+\beta)y\}.$$

Due to the fact that $L = (\alpha, 0)$ and $Q_{a,b}$ belong to $\mathcal{L}^3$, we can define the segment $\mathcal{C}^3 = \mathcal{C}^3(t) \subset \mathcal{L}^3$ joining these points. It is easy to check that $T_{a,b}(\mathcal{C}^3) \subset \mathcal{L}^2$.

So let $\mathcal{D}_t$ be the bounded set limit by $\mathcal{C}^1$, $\mathcal{C}^2$ and $\mathcal{C}^3$ (see Figure 2.1). We denote by $\mathcal{C}_T$ the critical line, i.e., the straight line $y = 0$. We are interested in those parameters for which $\mathcal{D}_t$ is a $T_{a,b}-$invariant set (recall that $a = a(t)$ and $b = b(t)$).

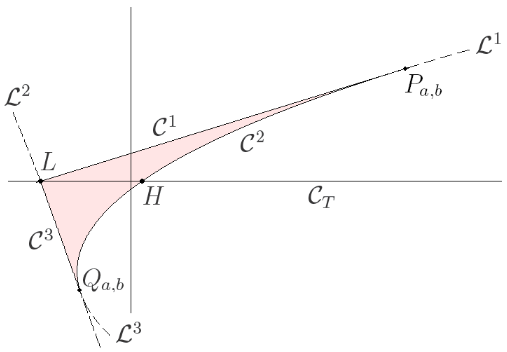


Figure 2.1: Invariant domain for $T_{a,b},\ (a,b) \in \mathcal{G}$

PROPOSITION **2.3** *The above defined domain $\mathcal{D}_t$ satisfies*

$$T_{a(t),b(t)}(\mathcal{D}_t) \subset \mathcal{D}_t$$

for all $t \in [0,2]$.

Proof. It is easy to see that $T_{a,b}(\mathcal{D}_t) \subset \mathcal{D}_t$ whenever $T_{a,b}(\mathcal{C}_T \cap \mathcal{D}_t) \subset \mathcal{D}_t$, and that the set $\mathcal{C}_T \cap \mathcal{D}_t$ coincides with the segment joining L with the point $H = H(t) = ((\alpha/(b+\beta))^2 + a, 0)$. Therefore, it is straightforward to see that the condition $T_{a,b}(\mathcal{C}_T \cap \mathcal{D}_t) \subset \mathcal{D}_t$ is fulfilled whenever

$$\left(\frac{\alpha}{b+\beta}\right)^2 + a \geq 0 \tag{2.5}$$

(recall that the first image by $T_{a,b}$ of any segment in the critical line is a vertical straight segment and that the first image of the point H belongs to $\mathcal{L}^1$ while the first image of the point L belongs to $\mathcal{L}^2$). Now, the bigger value of t for which the condition (2.5) holds corresponds to $t = 2$. ■

We want to point out that if $t = 2$ then $T_{a(2),b(2)}(\mathcal{D}_2) = \mathcal{D}_2$. This is a very special case and we will describe it later.

We introduce now a numerical analysis for certain region $\widetilde{\mathcal{G}} \subset \mathbb{R}^2$ for which $T_{a,b}$ has an invariant domain. This region $\widetilde{\mathcal{G}} \subset \mathbb{R}^2$ is shown in Figure 2.2, where we distinguish between four subdomains according to the following criterion:

1. The region in blue (pale grey in the B/W version) represents those parameters for which the attractor reduces to a periodic point.

2. The region in green (intermediate grey in the B/W version) represents those parameters for which one attractor presents a zero Lyapunov exponent and therefore we expect the attractor becomes a finite union of closed curves.

3. The region in red (dark grey in the B/W version) represents those parameter for which the sum and the product of the two Lyapunov exponents of an attractor is negative (one-dimensional strange attractors).

4. The region in black represents those parameters for which the sum of the Lyapunov exponents of an attractor is positive (two-dimensional strange attractors).

2.1 Properties of $T_{-4,-2}$

As we have already mentioned, the map $T_{a(2),b(2)} = T_{-4,-2}$ has some special properties. The most important one is stated in the next proposition.

Proposition **2.4** *The map $T_{-4,-2}$ is conjugated to a shift σ defined on a subset of the space of sequences of two symbols*

$$\Sigma = \{ \ \{x_n\}_{n\in\mathbb{N}} : \ x_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\}$$

Before proving Proposition 2.4, let us recall that $f_2(x) = 1 - 2x^2$ is a *Misiurewicz map* such that all the positive images of the critical point belong

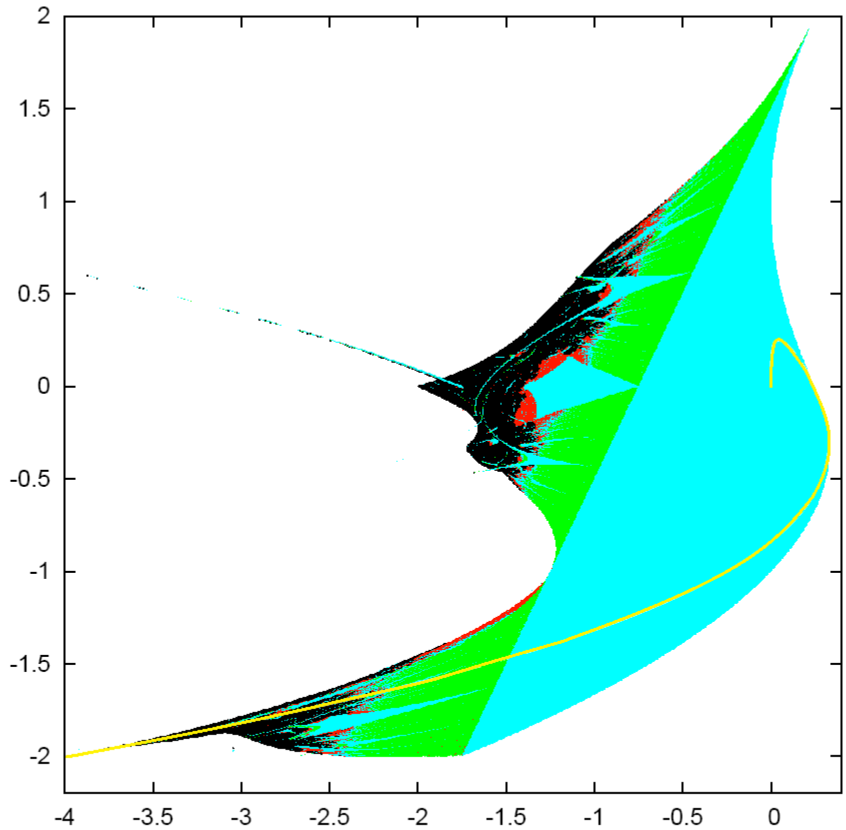


Figure 2.2: Region of parameters (a, b) for which $T_{a,b}$ has attractors. In yellow, the curve $\mathcal{G}$.

to the boundary of the interval $[-1, 1]$ (the maximal invariant domain of f_2). This is one of the reasons why f_2 can be conjugate to the *tent map*

$$\lambda_2 : z \in [-1, 1] \mapsto \lambda_2(z) = 1 - 2|z|$$

and this fact gives a lot of information about the orbits of f_2, see for instance [12]. As we have seen, it holds that $T_{-4,-2}(\mathcal{C}_T \cap \mathcal{D}_2) = \mathcal{C}^3$ and (see Figure 2.1)

$$T_{-4,-2}(\mathcal{C}^3) = \mathcal{C}^2, \; T_{-4,-2}(\mathcal{C}^2) = \mathcal{C}^1, \; T_{-4,-2}(\mathcal{C}^1) = \mathcal{C}^2,$$

i.e., all the positive iterates of any critical point belong to the boundary of the maximal invariant domain $\mathcal{D}_2$. This would be enough in order to check that $T_{-4,-2}$ is conjugate to some two-dimensional map displaying all the nice properties exhibited by the one-dimensional tent map (for instance, constant Jacobian outside critical points). In fact, it is easy to see that the restriction of $T^2_{-4,-2}$ to its invariant segment $\mathcal{C}^1$ is conjugate to the transformation

$$\theta \in [0,4] \mapsto \lambda(\theta) = (2-\theta)^2,$$

which is also a Misiurewicz unimodal map conjugate to a one-dimensional standard tent map.

In order to extend these ideas to the whole domain of definition of $T_{-4,-2}$ let us define on $\mathcal{D}_2$ the change in coordinates

$$(x,y) \to (\widetilde{x},\widetilde{y}) = \left(\frac{y - \sqrt{y^2+8y-4x}}{2}, \frac{y + \sqrt{y^2+8y-4x}}{2} \right)$$

Under this change in variables, the invariant domain $\mathcal{D}_2$ becomes the triangle

$$\widetilde{\mathcal{D}}_2 = \{(\widetilde{x},\widetilde{y}) : -2 \leq \widetilde{x} \leq 2,\ \widetilde{x} \leq \widetilde{y} \leq 2\},$$

and the respective transformation $\widetilde{T}$ is defined by

$$\widetilde{T}(\widetilde{x},\widetilde{y}) = \left(\frac{\widetilde{x}\widetilde{y} - \sqrt{(4-\widetilde{x}^2)(4-\widetilde{y}^2)}}{2}, \frac{\widetilde{x}\widetilde{y} + \sqrt{(4-\widetilde{x}^2)(4-\widetilde{y}^2)}}{2} \right).$$

We can consider the new change in coordinates defined by

$$(\widetilde{x},\widetilde{y}) \to (\phi,\psi) = \left(arccos(\frac{\widetilde{x}}{2}), arccos(\frac{\widetilde{y}}{2}) \right)$$

transforming $\widetilde{\mathcal{D}}_2$ into the triangle

$$\widetilde{\mathcal{T}} = \{(\phi,\psi) :\ 0 \leq \phi \leq \pi,\ 0 \leq \psi \leq \phi\}. \tag{2.6}$$

In these new coordinates, $T_{-4,-2}$ is written as

$$\widetilde{\Lambda}(\phi,\psi) = (\pi - |\pi - \phi - \psi|, \phi - \psi). \tag{2.7}$$

We will denote by Γ_2 the composition of the above changes in coordinates. Note that Γ_2 is given by

$$\Gamma_2(x,y) = \left(arccos\left(\frac{y - \sqrt{y^2+8y-4x}}{4} \right), arccos\left(\frac{y + \sqrt{y^2+8y-4x}}{4} \right) \right) \tag{2.8}$$

and one can see that Γ_2^{-1} is defined by

$$(x,y) = \Gamma_2^{-1}(\phi,\psi) = (4(\cos\phi + \cos\psi + \cos\phi\cos\psi), 2(\cos\phi + \cos\psi)), \tag{2.9}$$

so Γ_2^{-1} is a C^1- map defined on $\widetilde{\mathcal{T}}$. On the other hand, $\Gamma_2(\mathcal{C}_T) = \mathcal{C}_{\widetilde{\Lambda}}$ being

$$\mathcal{C}_{\widetilde{\Lambda}} = \{(\phi,\psi) \in \widetilde{\mathcal{T}} : \ \phi + \psi = \pi\}$$

the critical line of $\widetilde{\Lambda}$. Further, the set $\bigcup_{n\in\mathbb{N}} \widetilde{\Lambda}^n(\mathcal{C}_{\widetilde{\Lambda}})$ is contained in the boundary of $\widetilde{\mathcal{T}}$ and outside this critical set $\mathcal{C}_{\widetilde{\Lambda}}$, the entries of the matrix $D\widetilde{\Lambda}$ are constants. Therefore, the map $\widetilde{\Lambda}$ would display the same properties as the one-dimensional tent map.

We are ourselves in conditions to prove Proposition 2.4.

Proof. (Proposition 2.4) Let $\Sigma = \{ \ \{x_n\}_{n\in\mathbb{N}} : \ x_i \in \{0,1\}$ for all $i \in \mathbb{N}\}$ and let σ be the shift transformation

$$\sigma : \{x_n\}_{n\in\mathbb{N}} \in \Sigma \mapsto \{y_n\}_{n\in\mathbb{N}} \in \Sigma$$

defined by $y_n = x_{n+1}$. We will prove that there exists a set $\Sigma^* \subset \Sigma$ and a map

$$h : \Sigma^* \to \widetilde{\mathcal{T}}$$

such that $h \circ \sigma_{|\Sigma^*} = \widetilde{\Lambda} \circ h$. As a consequence of this fact, the map $T_{-4,-2}$ will be conjugated to the map σ (via the composition of Γ_2 and h).

First of all, note that $\mathcal{C}_{\widetilde{\Lambda}}$ divides $\widetilde{\mathcal{T}}$ into two disjoint triangles $\mathcal{A}_0$ and $\mathcal{A}_1$ defined by

$$\mathcal{A}_0 = \{(\phi,\psi) \in \widetilde{\mathcal{T}} : \phi + \psi \leq \pi\}, \ \mathcal{A}_1 = \{(\phi,\psi) \in \widetilde{\mathcal{T}} : \phi + \psi > \pi\}$$

and, furthermore, it follows that $\widetilde{\Lambda}(\mathcal{A}_0) = \widetilde{\mathcal{T}}$ and $\widetilde{\Lambda}(\mathcal{A}_1) = \widetilde{\mathcal{T}} \setminus \mathcal{C}_{\widetilde{\Lambda}}$.

Now, let us consider the first two preimages of $\mathcal{C}_{\widetilde{\Lambda}}$, which are the straight segments in $\widetilde{\mathcal{T}}$ given by the equations

$$\phi = \frac{\pi}{2}, \ \psi = \frac{\pi}{2}.$$

These preimages of $\mathcal{C}_{\widetilde{\Lambda}}$ divide each one of the sets $\mathcal{A}_0$ and $\mathcal{A}_1$ into two new triangles that we will denote by $\mathcal{A}_{00}$, $\mathcal{A}_{01}$, $\mathcal{A}_{10}$ and $\mathcal{A}_{11}$ according to the following rule:

$$\mathcal{A}_{ij} \subset \mathcal{A}_i \text{ and } \widetilde{\Lambda}(\mathcal{A}_{ij}) = \mathcal{A}_j$$

for $i,j \in \{0,1\}$.

In this way, for each $n \in \mathbb{N}$ we get a partition

$$\{\mathcal{A}_{x_1x_2\ldots x_n} : \ x_i \in \{0,1\}, \ i = 1,\ldots,n\}$$

of $\widetilde{\mathcal{T}}$ by 2^n triangles recurrently defined by the following two conditions:

1. $\mathcal{A}_{x_1x_2\dots x_n} \subset \mathcal{A}_{x_1x_2\dots x_{n-1}}$

2. $\widetilde{\Lambda}^{n-1}(\mathcal{A}_{x_1x_2\dots x_n}) = \widetilde{\Lambda}^{n-2}(\mathcal{A}_{x_2x_3\dots x_n}) = \dots = \widetilde{\Lambda}(\mathcal{A}_{x_{n-1}x_n}) = \mathcal{A}_{x_n}$

(see Figure 2.3 as an example).

For each $\{x_n\}_{n\in\mathbb{N}} \in \Sigma$, we define

$$A_{\{x_n\}} = \bigcap_{i\in\mathbb{N}} \mathcal{A}_{x_1x_2\dots x_i}.$$

Note that the triangles $\mathcal{A}_{x_1x_2\dots x_j}$ are convex sets, so it follows that $A_{\{x_n\}} \in \widetilde{\mathcal{T}}$ for each sequence $\{x_n\}_{n\in\mathbb{N}} \in \Sigma$.

On the other hand, we introduce the subset Σ^* of Σ defined by the following three conditions.

1. If $x_2 = 1$ and $x_j = 0$ for every $j > 2$, then $\{x_n\}_{n\in\mathbb{N}} \in \Sigma^*$ if and only if $x_1 = 0$. We are describing by this condition the orbit of the point $(\pi, 0)$.

2. If there exists $n_0 > 2$ for which $x_{n_0} = 1$ and $x_j = 0$ for every $j > n_0$, then $\{x_n\}_{n\in\mathbb{N}} \in \Sigma^*$ if and only if $x_{n_0-1} = x_{n_0-2} = 0$. We are describing the orbit of the preimages of $(\pi, 0)$.

3. If there exists $n_0 > 1$ for which $x_{n_0} = 1$ and $x_{n_0+2j} = 0$ for every $j > 1$, then $\{x_n\}_{n\in\mathbb{N}} \in \Sigma^*$ if and only if $x_{n_0-1} = 0$. We are describing the orbit of every preimage of each critical point different from $(\pi/2, \pi/2)$ and $(\pi, 0)$.

Then, one can show that the map

$$h : \{x_n\} \in \Sigma^* \mapsto h(\{x_n\}) = A_{\{x_n\}} \in \widetilde{\mathcal{T}}$$

is one to one and onto. Moreover, h is continuous when considering the metric in Σ^* induced by the distance

$$d(\{x_n\}, \{y_n\}) = \sum_{i\in\mathbb{N}} \frac{1}{2^i}|x_i - y_i|$$

and $h \circ \sigma_{|\Sigma^*} = \widetilde{\Lambda} \circ h$. ■

As a consequence of the Proposition 2.4, we may derive several properties for $T_{-4,-2}$ and $\widetilde{\Lambda}$. These properties are the main reason why the map $\widetilde{\Lambda}$ is called the *two-dimensional tent map.*

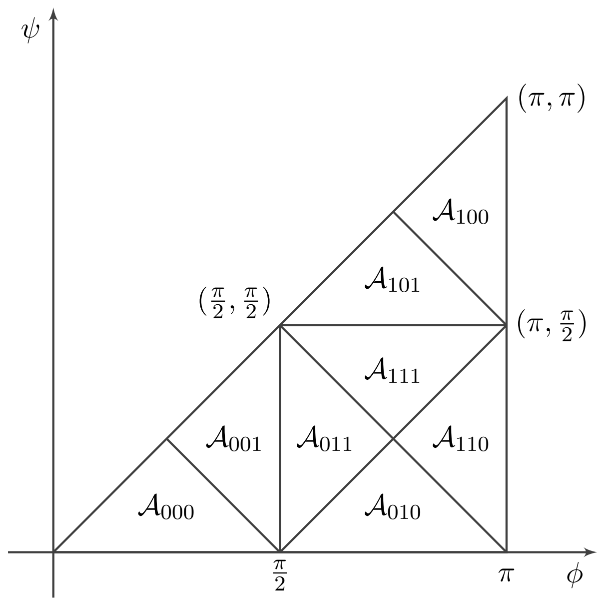


Figure 2.3: The partition of $\widetilde{\mathcal{T}}$ for $n = 3$

1. The map $T_{-4,-2}$ has periodic orbits of period p for all $p \in \mathbb{N}$ and the set of periodic orbits is dense in $\mathcal{D}_2$. The same holds for $\widetilde{\Lambda}$ on $\widetilde{\mathcal{T}}$. In addition, for each $n \in \mathbb{N}$ the maps $T^n_{-4,-2}$ and $\widetilde{\Lambda}^n$ are not C^1-conjugate to any map of the type $H(x,y) = (H_1(x), H_2(x,y))$.

2. The two fixed points of $T_{-4,-2}$, $P_{-4,-2} = (12,4)$ and $\widetilde{P}_{-4,-2} = (-3,-1)$, satisfy
$$\overline{W^u(P_{-4,-2})} = \overline{W^u(\widetilde{P}_{-4,-2})} = \mathcal{D}_2$$
and the two fixed points of $\widetilde{\Lambda}$, $(0,0)$ and $(4\pi/5, 2\pi/5)$, satisfy
$$\overline{W^u((0,0))} = \overline{W^u((4\pi/5, 2\pi/5))} = \widetilde{\mathcal{T}}.$$

3. The map $T_{-4,-2}$ has dense orbits in $\mathcal{D}_2$ and the map $\widetilde{\Lambda}$ has dense orbits in $\widetilde{\mathcal{T}}$. On the other hand, no dense orbit of $\widetilde{\Lambda}$ intersects the critical line $\mathcal{C}_{\widetilde{\Lambda}}$ and, if the orbit of (ϕ_0, ψ_0) is dense in $\widetilde{\mathcal{T}}$, the two Lyapunov exponents of $\widetilde{\Lambda}$ along $\{\widetilde{\Lambda}^n(\phi_0,\psi_0) : \ n \in \mathbb{N}\}$ are positive. More concretely, it is straightforward to check that, for every unit vector v, one has
$$\|D\widetilde{\Lambda}^n(\phi_0,\psi_0)v\| = (\sqrt{2})^n.$$
Therefore, according to Definition I.2 the whole $\widetilde{\mathcal{T}}$ is a strange attractor for the transformation $\widetilde{\Lambda}$.

Therefore, we easily obtain the following result.

PROPOSITION **2.5** *For almost all $Q \in \mathcal{D}_2$ and all $v \in \mathbb{R}_2$ different from zero, the Lyapunov exponent of Q with respect to $T_{-4,-2}$, in the direction of v, is* $\frac{1}{2}\log 2$.

In addition, some ergodic properties for the maps $T_{-4,-2}$ and $\widetilde{\Lambda}$ can be proved, see Corollary 8 in [47].

PROPOSITION **2.6** *The Lebesgue measure is ergodic with respect to $\widetilde{\Lambda}$ and $T_{-4,-2}$ has an invariant absolutely continuous and ergodic measure defined by the density*

$$\overline{\rho}(x,y) = \frac{1}{\pi^2\sqrt{(4+x-4y)(\frac{1}{4}y^2+2y-x)(x+4)}}.$$

To finish this chapter, we state two conjectures. The first one establishes the existence, with positive probability in the curve of parameters $\mathcal{G}$ given in (2.4), of two-dimensional strange attractors with two positive Lyapunov exponents for the respective transformation $T_{a(t),b(t)}$ defined in $\mathcal{D}_t$. The second conjecture claims that the existence of two-dimensional strange attractors would be a phenomenon of positive probability in the space of parameters when a two-parameter family of three-dimensional diffeomorphisms unfolds a homoclinic tangency in case B (in (2.1) the associated limit return maps are written). This would be a natural extension of the results given in [33] for two-dimensional diffeomorphisms.

CONJECTURE **2.7** *For any $0 < c < \frac{1}{2}\log 2$ there exists $t_0 < 2$ and a positive Lebesgue measure set E of values of the parameter t contained in $[t_0, 2]$ such that for any $t \in E$ there exists a dense orbit $\bigcup_{n\in\mathbb{N}} \{T^n_{a,b}(x_0,y_0)\}$ in $\mathcal{D}_t$, $(a,b) = (a(t), b(t)) \in \mathcal{G}$, such that*

$$\|DT^n_{a,b}(x_0,y_0)\boldsymbol{v}\| \geq \exp(cn)$$

for any unit vector $\mathbf{v}$.

CONJECTURE **2.8** *Let $\{f_{a,b}\}_{a,b}$ be a two-parameter family of three-dimensional diffeomorphisms and suppose that f_{a_0,b_0} has a dissipative saddle fixed point p_0. Assume that the family $\{f_{a,b}\}_{a,b}$ satisfies the linearization assumption and the eigenvalues λ_1, λ_2 and λ_3 of $Df_{a_0,b_0}(p_0)$ satisfy $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. If the invariant manifolds of p_0 have a generalized homoclinic tangency for $(a,b) = (a_0,b_0)$ which unfolds generically, then there exists a positive measure set E of parameter values near (a_0,b_0) such that for $(a,b) \in E$ the diffeomorphism $f_{a,b}$ exhibits a strange attractor with two positive Lyapunov exponents.*

At least, this was the natural route in many other lower-dimensional contexts: For proving the persistence of strange attractors for families of two-dimensional diffeomorphisms (see [8], [33], [42], [43] ...) take all the possible information of expansiveness in some related family of one-dimensional transformations (see [7], [25], Chapter 2 in [42] ...).

For an extensive study on a numerical analysis directed to obtain the attractors arising in the invariant domain $\mathcal{D}_t$ for parameters $(a(t), b(t)) \in \mathcal{G},\ t \in (0, 2)$, we refer the reader to [48].

Chapter 3

Chaotic $2D-$tent maps

3.1 Expanding Baker Maps: Definition and the simplest case

In this section we will introduce the definition of *Expanding Baker Map* and we will describe how our piecewise affine map $\widetilde{\Lambda}$ given in (2.7) can be seen as one of these special maps.

Expanding Baker Maps are characterized by a particular dynamic: they fold and expand some domain. We will restrict ourselves to Expanding Baker Maps defined on compact connected convex domains in $\mathbb{R}^2$, but the definition could be extended to more general domains in $\mathbb{R}^2$ and even to domains in $\mathbb{R}^n$ ($n \neq 2$).

First of all, we need to state the definition of *fold*.

Definition **3.1** *Let $\mathcal{K} \subset \mathbb{R}^2$ be a compact connected convex domain with nonempty interior, P a point in $\mathcal{K}$ and $\mathcal{L}$ a line in $\mathbb{R}^2$ with $\mathcal{L} \cap int(\mathcal{K}) \neq \emptyset$ and $P \notin \mathcal{L}$. Then $\mathcal{L}$ divides $\mathcal{K}$ into two subsets denoted by $\mathcal{K}_0$ and $\mathcal{K}_1$, being $\mathcal{K}_0$ the one with $P \in \mathcal{K}_0$. We define the **fold** $\mathcal{F}_{\mathcal{L},P}$ as the map*

$$\mathcal{F}_{\mathcal{L},P}(Q) = \begin{cases} Q & , \ if \ Q \in \mathcal{K}_0 \\ \overline{Q} & , \ if \ Q \in \mathcal{K}_1 \end{cases}$$

where $\overline{Q}$ denotes the symmetric point of Q with respect to $\mathcal{L}$.

We have to point out that not every fold will be considered and this is the reason why we introduce the concept of *admissible fold*.

Definition **3.2** *In the above conditions, we will say that $\mathcal{F}_{\mathcal{L},P}$ is an **admissible fold** if $\mathcal{F}_{\mathcal{L},P}(\mathcal{K}) = \mathcal{K}_0$.*

Now, let us put $\mathcal{L} = \mathcal{L}_1$. We may ask whether we can fold again the set $\mathcal{K}_0$. We may consider a line $\mathcal{L}_2$ with $\mathcal{L}_2 \cap int(\mathcal{K}_0) \neq \emptyset$ and $P \notin \mathcal{L}_2$ dividing $\mathcal{K}_0$ into two new subsets $\mathcal{K}_{00}$ and $\mathcal{K}_{01}$, with $P \in \mathcal{K}_{00}$. Assuming that $\mathcal{F}_{\mathcal{L},P}$ is an admissible fold, we may successively define a sequence of admissible folds $\mathcal{F}_{\mathcal{L}_3,P} \ldots \mathcal{F}_{\mathcal{L}_n,P}$ being

$$\mathcal{F}_{\mathcal{L}_1,P} : \mathcal{K} \to \mathcal{K}_0 \ , \ \mathcal{F}_{\mathcal{L}_i,P} : \mathcal{K}_{0\overset{i-1}{\cdots}0} \to \mathcal{K}_{0\overset{i}{\cdots}0},$$

$\mathcal{K}_{0\overset{i}{\cdots}0} \subset \mathcal{K}_{0\overset{i-1}{\cdots}0}$ and $P \in \mathcal{K}_{0\overset{i}{\cdots}0}$ for all $i = 1 \ldots n$. After this finite number of admissible folds $\mathcal{F}_{\mathcal{L}_1,P} \ldots \mathcal{F}_{\mathcal{L}_n,P}$ are consecutively constructed we obtain a final folded region $\mathcal{K}_{0\overset{n}{\cdots}0}$.

We are now in conditions to define the concept of *Expanding Baker Map.*

Definition **3.3** *Let $\mathcal{K} \subset \mathbb{R}^2$ be a compact connected convex domain with nonempty interior. Let P be a point in $\mathcal{K}$ and $\{\mathcal{F}_{\mathcal{L}_1,P} \ldots \mathcal{F}_{\mathcal{L}_n,P}\}$ a sequence of admissible folds of $\mathcal{K}$. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an expanding linear map, i.e., $|det(A)| > 1$. Let us consider*

$$\widetilde{A} : Q \in \mathbb{R}^2 \mapsto \widetilde{A}(Q) = P + A(Q - P).$$

Then, if $\widetilde{A}(\mathcal{K}_{0\overset{n}{\cdots}0}) \subset \mathcal{K}$ we define the ***Expanding Baker Map*** *(EBM for short) $\Gamma : \mathcal{K} \to \mathcal{K}$ associated to $P,\ A,\ \mathcal{L}_1, \ldots, \mathcal{L}_n$ as*

$$\Gamma = \widetilde{A} \circ \mathcal{F}_{\mathcal{L}_n} \circ \ldots \circ \mathcal{F}_{\mathcal{L}_1}.$$

For short, we will denote

$$\Gamma = EBM(\mathcal{K}, \mathcal{L}_1, \ldots, \mathcal{L}_n, P, A).$$

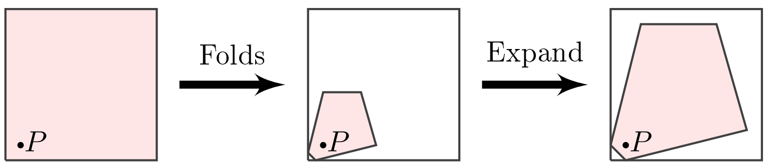


Figure 3.1: Example of the dynamic of any *EBM*

Let us now recover from (2.7) the two-dimensional tent map $\widetilde{\Lambda}$ conjugated to the limit return map $T_{-4,-2}$. Note that after a new change in coordinates

$$\Xi : (\Phi, \Psi) \to (x, y) = (\frac{1}{\pi}(\Phi + \Psi), \frac{1}{\pi}(\Phi - \Psi)) \qquad (3.1)$$

one may write the above map $\widetilde{\Lambda}$ as

$$\Lambda(x,y) = \begin{cases} (x+y, x-y), & \text{if } (x,y) \in \mathcal{T}_0 \\ (2-x+y, 2-x-y) & \text{if } (x,y) \in \mathcal{T}_1 \end{cases} \tag{3.2}$$

defined on the triangle $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$, where

$$\begin{aligned} \mathcal{T}_0 &= \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, 0 \le y \le x\}, \\ \mathcal{T}_1 &= \{(x,y) \in \mathbb{R}^2 : \ 1 \le x \le 2, 0 \le y \le 2-x\}. \end{aligned} \tag{3.3}$$

Let us denote by $\mathcal{O}$ the point $(0,0)$ and by $\mathcal{C}$ the critical line $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 : x = 1\}$. It is clear that $\mathcal{T}$ is a compact and convex polygonal domain and Λ may be written as the composition of two maps. The first one will be denoted by $\mathcal{F}_{\mathcal{C},\mathcal{O}} : \mathcal{T} \to \mathcal{T}_0$ and it is the admissible fold defined as follows,

$$\mathcal{F}_{\mathcal{C},\mathcal{O}}(x,y) = \begin{cases} (x,y) & , \ if \ (x,y) \in \mathcal{T}_0 \\ (2-x,y) & , \ if \ (x,y) \in \mathcal{T}_1 \end{cases} \tag{3.4}$$

The second one is given by the linear map $A_1 : (x,y) \in \mathcal{T}_0 \mapsto (x+y, x-y) \in \mathcal{T}$ represented by the matrix

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then $\Lambda = EBM(\mathcal{T}, \mathcal{C}, \mathcal{O}, A_1)$ (see Figure 3.2).

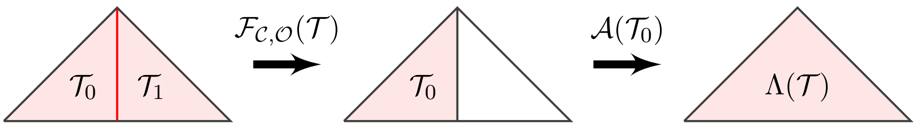


Figure 3.2: Dynamic of Λ

3.2 The one-parameter family $\{\Lambda_t\}_t$

From Chapter 2, Section 2.1, we know that we may write

$$\Lambda = \Xi \circ \Gamma_2 \circ T_{-4,-2} \circ \Gamma_2^{-1} \circ \Xi^{-1}$$

where

$$T_{a,b}(x,y) = (a + y^2, x + by)$$

is the family of limit return maps given in (2.2),

$$\Gamma_2(x,y) = \left(\arccos\left(\frac{y-\sqrt{y^2+8y-4x}}{4}\right), \arccos\left(\frac{y+\sqrt{y^2+8y-4x}}{4}\right)\right)$$

is the change in coordinates given at (2.8) conjugating $T_{-4,-2}$ with $\widetilde{\Lambda}$ (see (2.7)) and

$$\Xi(\Phi,\Psi) = (\frac{1}{\pi}(\Phi+\Psi), \frac{1}{\pi}(\Phi-\Psi))$$

is the change in coordinates given at (3.1) conjugating $\widetilde{\Lambda}$ with Λ (see (3.2)).

A first approach in order to simplify the dynamics of $T_{a(t),b(t)}$, with

$$\mathcal{G} = \{(a(t),b(t)) = (-\frac{1}{4}t^3(t^3-2t^2+2t-2), -t^2+t) : t \in \mathbb{R}\}$$

(see (2.4)), could be to apply to $T_{a(t),b(t)}$ the same sequence of changes in coordinates.

Of course, first of all we must apply the following (new) change in coordinates

$$\Gamma_{1,t} : (x,y) \to (\overline{x},\overline{y}) = \left(\frac{16}{t^4}x + \frac{16(2-t)}{t^3}y + 4 - \frac{8}{t}, \frac{8}{t^3}y\right)$$

mapping the invariant domain $\mathcal{D}_t$ of $T_{a(t),b(t)}$ into the invariant domain $\mathcal{D}_2$ of $T_{-4,-2}$, see [48] for details.

Doing so, and defining for every $t \in [0,2]$

$$\widetilde{F}_t = \Xi \circ \Gamma_2 \circ \Gamma_{1,t} \circ T_{a(t),b(t)} \circ \Gamma_{1,t}^{-1} \circ \Gamma_2^{-1} \circ \Xi^{-1},$$

we obtain a family of maps $\widetilde{F}_t : \mathcal{T} \to \mathbb{R}^2$ being, as usual, $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ (see (3.3)).

Proposition **3.4** *For every $t \in [0,2]$, $\widetilde{F}_t$ satisfies:*

1. $\widetilde{F}_2 = \Lambda$,
2. $\widetilde{F}_t(\mathcal{T}) \subset \mathcal{T}$,
3. *The critical set of $\widetilde{F}_t$ is $\mathcal{C} = \{(x,y) \in \mathcal{T} : x = 1\}$.*
4. *$\widetilde{F}_t(\mathcal{C})$ is a straight segment with slope -1 joining a point U_d in $\mathcal{L}_d = \{(x,y) \in \mathcal{T} : x = y\}$ with a point U_x in $\mathcal{L}_x = \{(x,y) \in \mathcal{T} : y = 0\}$.*

5. *$\widetilde{F}_t$ restricted to $\mathcal{T}_0$ (or restricted to $\mathcal{T}_1$) is a homeomorphism. Moreover, $\widetilde{F}_t(0,0) = \widetilde{F}_t(2,0) = (0,0)$, $\widetilde{F}_t(1,0) = U_d$ and $\widetilde{F}_t(1,1) = U_x$.*

Proof. It is clear that $\widetilde{F}_2 = \Lambda$. Now, since $T_{a(t),b(t)}(\mathcal{D}_t) \subset \mathcal{D}_t$, $\Gamma_{1,t}(\mathcal{D}_t) = \mathcal{D}_2$ and $(\Xi \circ \Gamma_2)(\mathcal{D}_2) = \mathcal{T}$ we have $\widetilde{F}_t(\mathcal{T}) \subset \mathcal{T}$.

Third statement easily follows bearing in mind that the critical set of $T_{a(t),b(t)}$ is $\widetilde{C} \cap \mathcal{D}_t$, $\widetilde{C} = \{(\overline{x}, \overline{y}) : \overline{y} = 0\}$ and that

$$\Gamma_{1,t}(\widetilde{C} \cap \mathcal{D}_t) = \widetilde{C} \cap \mathcal{D}_2, \qquad (\Xi \circ \Gamma_2)(\widetilde{C} \cap \mathcal{D}_2) = \{(x,y) \in \mathcal{T} : x = 1\} = C.$$

The fourth statement follows if we prove that

$$(\Gamma_t \circ T_{a(t),b(t)} \circ \Gamma_t^{-1})(\Xi^{-1}(\mathcal{C}))$$

is a vertical segment in $\widetilde{\mathcal{T}}$ (see (2.6)) joining a point in $\{(\Phi,\Psi) \in \widetilde{\mathcal{T}} : \Psi = 0\}$ with a point in $\{(\Phi,\Psi) \in \widetilde{\mathcal{T}} : \Phi = \Psi\}$. Let us first note that $\Xi^{-1}(\mathcal{C}) = \{(\Phi,\Psi) \in \widetilde{\mathcal{T}} : \Phi + \Psi = \pi\}$. Then, an easy calculation gives

$$(\Gamma_t \circ T_{a(t),b(t)} \circ \Gamma_t^{-1})(u, \pi - u) = (\arccos(1-t), \arccos(1 - t\cos^2 u)).$$

Therefore, one obtains $U_d = (\frac{1}{\pi}\arccos(1-t), \frac{1}{\pi}\arccos(1-t))$ and $U_x = (\frac{2}{\pi}\arccos(1-t), 0)$.

Finally, the last statement is an easy consequence of the previous ones. ■

Now, we define the family $\{\Lambda_t\}_{t\in[0,1]}$ by

$$\Lambda_t(x,y) = \begin{cases} (t(x+y), t(x-y)), & \text{if } (x,y) \in \mathcal{T}_0 \\ (t(2-x+y), t(2-x-y)) & \text{if } (x,y) \in \mathcal{T}_1 \end{cases} \tag{3.5}$$

REMARK **3.5** *We have that $\Lambda_t = EBM(\mathcal{T}, \mathcal{C}, \mathcal{O}, A_t)$ being $\mathcal{T}$, $\mathcal{C}$, $\mathcal{O}$ as usual and*

$$A_t = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}.$$

It is easy to see that every Λ_t satisfies all the properties stated in Proposition 3.4 for $\widetilde{F}_t$. Moreover, the following result can be easily proved:

PROPOSITION **3.6** *If $G : \mathcal{T} \to \mathcal{T}$ is a piecewise linear map and satisfies all the statements in Proposition 3.4, then $G = \Lambda_t$ for some t.*

REMARK **3.7** *In view of Proposition 3.4 and Proposition 3.6, Λ_t becomes the best choice in the piecewise linear setting for describing the dynamics of the limit return maps $T_{a(t),b(t)}$.*

Let us continue by giving a brief description of the dynamical behaviour of our family of two-dimensional maps $\{\Lambda_t\}_t$. First of all, note that the triangle $\mathcal{T}$ is invariant under Λ_t for every $0 \leq t \leq 1$. Moreover, if $t < \frac{1}{\sqrt{2}}$ the origin is the unique fixed point for Λ_t and it is a global attractor.

LEMMA **3.8** *For every $t \in [0, \frac{1}{\sqrt{2}})$ the origin is a global attractor for Λ_t.*

Proof. It is enough to prove that $\Lambda_t^2(\mathcal{T}) \subset \mathcal{T}_0$ and take into account that Λ_t restricted to $\mathcal{T}_0$ is a linear contraction for every $t \in [0, \frac{1}{\sqrt{2}})$. With this aim in mind let us point out that $\Lambda_t(\mathcal{T})$ is the triangle with vertices (t, t), $(2t, 0)$ and $(0, 0)$. Hence, if $2t < 1$ then the Lemma easily follows because $\Lambda_t(\mathcal{T}) \subset \mathcal{T}_0$.

Therefore, let us assume $2t \geq 1$ and let us observe that in this case, $\Lambda_t(\mathcal{T}) \cap \mathcal{T}_1$ is the triangle with vertices $(1, 0)$, $(1, 2t-1)$ and $(2t, 0)$. We will also denote by

$$\widetilde{\mathcal{T}}_0 = \{Q \in \mathcal{T}_0 : \Lambda_t(Q) \in \mathcal{T}_1\}$$
$$\widetilde{\mathcal{T}}_1 = \{Q \in \mathcal{T}_1 : \Lambda_t(Q) \in \mathcal{T}_1\}.$$

It is easy to check that these sets $\widetilde{\mathcal{T}}_0$ and $\widetilde{\mathcal{T}}_1$ are triangles. Furthermore, they are symmetric with respect to the critical line $\mathcal{C}$, being the vertices of $\widetilde{\mathcal{T}}_0$ the points $(\frac{1}{2t}, \frac{1}{2t})$, $(1, \frac{1-t}{t})$ and $(1, 1)$ (see Figure 3.3). Then, for proving that $\Lambda_t^2(\mathcal{T}) \subset \mathcal{T}_0$ it will be enough to check that, for $i = 0, 1$, it holds

$$\widetilde{\mathcal{T}}_i \cap \Lambda_t(\mathcal{T}) = \emptyset. \tag{3.6}$$

In order to prove (3.6) for $i = 1$ we have to ensure that

$$2t - 1 < \frac{1-t}{t}$$

but this is completely equivalent to $t < \frac{1}{\sqrt{2}}$. On the other hand, to prove (3.6) for $i = 0$ it is enough to check that $t < \frac{1}{2t}$ or $2t^2 < 1$ which is again the condition for a parameter t to belong to $[0, \frac{1}{\sqrt{2}})$. ■

Now, suppose that $t \in [\frac{1}{\sqrt{2}}, 1)$. The origin becomes a repelling node and a new fixed point P_t, given by

$$P_t = \left(\frac{2t(2t+1)}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1}\right), \tag{3.7}$$

arises in $\mathcal{T}_1$. The eigenvalues of this point are

$$\lambda_1 = -t(1-i) \qquad \lambda_2 = -t(1+i)$$

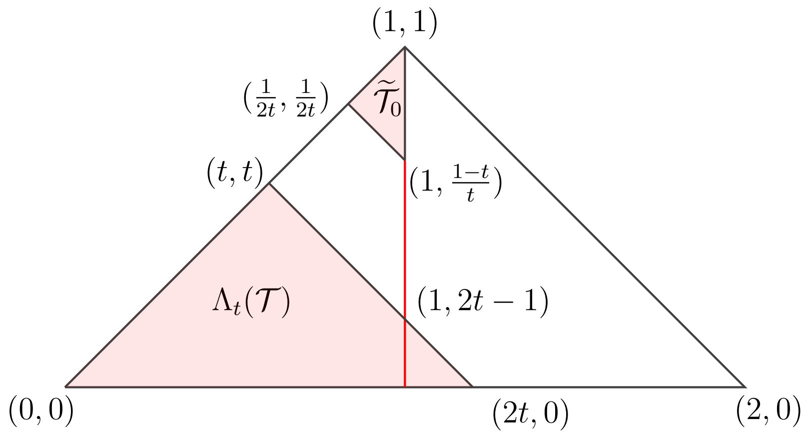


Figure 3.3: Dynamics for $t \in [0, \frac{1}{\sqrt{2}})$

so P_t is always a repelling focus.

The case $t = \frac{1}{\sqrt{2}}$ is special because $\Lambda_{\frac{1}{\sqrt{2}}}$ preserves area; in fact, there exists a line of fixed points given by $\{(x,y) \in \mathcal{T}_0 : \ y = (\sqrt{2}-1)x\}$ and the rest of points in $\mathcal{T}_0$ are periodic points of period two. On the other hand, for every $t > \frac{1}{\sqrt{2}}$ the Lyapounov exponents of any point which does not belong to any preimage of the critical line $\mathcal{C}$, in any non-zero direction, is $\log(\sqrt{2}t)$ and all the periodic orbits with no critical points are repelling.

For a brief explanation on the attractors exhibited by our family of two-dimensional tent maps Λ_t when $t \in (\frac{1}{\sqrt{2}}, 1]$ we refer the reader to the Introduction.

3.3 An attracting set of Λ_t

The aim of this section is to construct the maximal attracting set for Λ_t, see (3.5), for any $t \in (\frac{1}{\sqrt{2}}, 1)$. As an example, the attractor for $t = 0.9$ is showed in Figure 3.4. From now on, we will denote by $\overline{AB}$ the straight segment joining two different points $A, B \in \mathbb{R}^2$. Unless otherwise specified, we will also denote by A_j the j-th iterate of a point A by the map Λ_t.

Let us denote by $V = (1,0)$ and $H = (1,1)$. Then, the critical set of Λ_t coincides with $\overline{VH}$. We consider $\overline{V_1H_1}$ the first image of this critical set, being $V_1 = (t,t)$ and $H_1 = (2t,0)$, which is a straight segment with slope -1 and it intersects with the critical set at the point $J = (1, 2t-1)$, see Figure 3.5. Let us consider $J_1 = (2t^2, 2t(1-t))$ and $J_2 = (2t(1+t-2t^2), 2t(1-t))$. Since

$\overline{JJ_1}$ is contained in $\mathcal{T}_1$, its image is again a straight segment $\overline{J_1J_2}$ with null slope and the point J_2 belongs to $\mathcal{T}_0$, so we can consider $M = (1, 2t(1-t))$ the intersection between $\overline{J_1J_2}$ and the critical set $\mathcal{C}$. It is easy to see that the symmetric segment of $\overline{J_2M}$ by $\mathcal{C}$ is contained in $\overline{MJ_1}$. Hence, since Λ_t is symmetric with respect to $\mathcal{C}$, $\Lambda_t(\overline{J_1J_2}) = \Lambda_t(\overline{MJ_1}) = \overline{M_1J_2}$, with

$$M_1 = (t + 2t^2(1-t), t - 2t^2(1-t)).$$

This last straight segment has slope $+1$ and one can check that $M_1 \in \mathcal{T}_1$. Hence, $\overline{M_1J_2}$ intersects with the critical set at the point $K = (1, 1-4t^2+4t^3)$. Since $K \in \overline{MJ}$, $K_1 \in \overline{M_1J_1}$ being

$$K_1 = (2t - 4t^3(1-t), 4t^3(1-t))$$

and $\overline{K_1M_2} = \Lambda_t(\overline{KM_1})$ is a vertical straight segment with $M_2 \in \overline{MJ_1}$ and

$$M_2 = (2t - 4t^3(1-t), 2t(1-t)).$$

Finally, let us take $\overline{K_2M_3} = \Lambda_t(\overline{K_1M_2})$. This is a straight segment with slope -1 contained in $\mathcal{T}_0$, being

$$\begin{aligned} M_3 &= (2t - 2t^3 + 4t^4(1-t), 2t - 4t^2 + 2t^3 + 4t^4(1-t)) \\ K_2 &= (2t - 2t^2 + 8t^4(1-t), 2t(1-t)) \end{aligned}$$

We now denote by $\mathcal{C}_1 = \overline{M_1K_1}$, $\mathcal{C}_2 = \overline{M_2K_2}$, $\mathcal{C}_3 = \overline{M_3M_1}$, $\mathcal{C}_4 = \overline{K_1M_2}$ and $\mathcal{C}_5 = \overline{K_2M_3}$. One has that $\mathcal{C}_i \subset \Lambda_t^i(\mathcal{C})$ and the union of these five segments bound a compact connected pentagonal domain of $\mathcal{T}$ which will be called $\mathcal{R}_t$. We also define $\mathcal{C}_0 = \overline{MK} = \mathcal{C} \cap \mathcal{R}_t$ and we will divide $\mathcal{R}_t$ into the sets $\mathcal{R}_{t,0} = \mathcal{R}_t \cap \mathcal{T}_0$ and $\mathcal{R}_{t,1} = \mathcal{R}_t \cap \mathcal{T}_1$ (see Figure 3.4).

Remark **3.9** *For $t = 1$ one has that $M_1 = J = K = H$, $M_2 = J_1 = K_1 = H_1$, and so on. Therefore, in this case we have that $\mathcal{R}_t$ is the whole triangle $\mathcal{T}$.*

Lemma **3.10** *For every $\frac{1}{\sqrt{2}} < t \leq 1$, $\mathcal{R}_t$ is invariant by Λ_t. Moreover, if $\frac{1}{\sqrt{2}} < t < 1$, then the stable set of $\mathcal{R}_t$ coincides with the interior of $\mathcal{T}$.*

Proof. First of all, we will prove that $\Lambda_t(\mathcal{R}_{t,0}) \subset \mathcal{R}_t$. In fact, we will demonstrate something stronger (which will be used below) by checking that the image of the triangle with vertices M, J_2 and K is contained in $\mathcal{R}_t$, see Figure 3.5. To this end, since Λ_t is linear on this triangle and $\mathcal{R}_t$ is a convex set, it is enough to prove that the image of its vertices belong to $\mathcal{R}_t$. This follows by taking into account that $\Lambda_t(\overline{KJ_2})$ is a horizontal segment joining

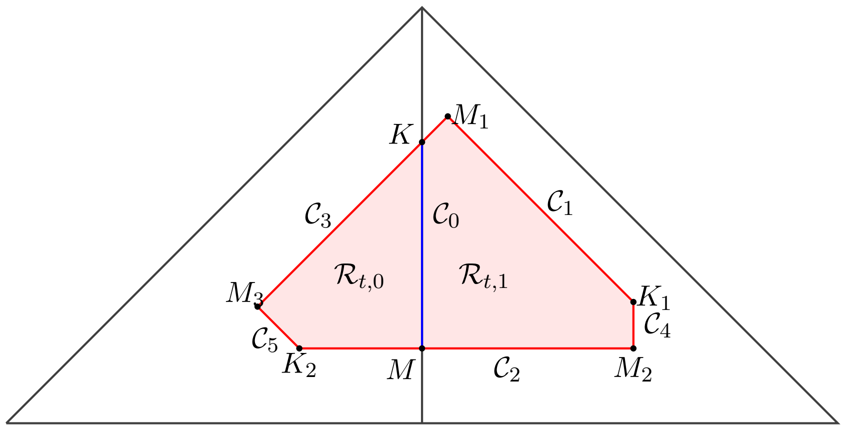


Figure 3.4: The attracting set $\mathcal{R}_t$

K_1 with certain point J_3 of $\mathcal{C}_3$. Then, it is enough to obtain that $J_3 \in \overline{M_3K}$ or, in other words, that the ordinate of K_1 is greater or equal than the ordinate of M_3. Consequently, we must check that $\eta(t) = (2t^2-1)(t-1)^2 \geq 0$ for $t \in (\frac{1}{\sqrt{2}}, 1)$. Therefore, the result is proved.

On the other hand, the equality $\Lambda_t(\mathcal{R}_{t,1}) = \mathcal{R}_t$ directly follows from the fact that Λ_t is linear also in $\mathcal{R}_{t,1}$. Therefore, $\Lambda_t(\mathcal{R}_t) = \mathcal{R}_t$.

Now, let us take $\frac{1}{\sqrt{2}} < t < 1$, Q any point in $int(\mathcal{T})$ and prove that for some natural number n, one has $\Lambda_t^n(Q) \in \mathcal{R}_t$. To this end let us first observe that, for every t, the boundary of $\mathcal{T}$ is invariant and, moreover, since $t \neq 1$ then the interior of $\mathcal{T}$ is also invariant. Therefore, if $Q \in int(\mathcal{T})$, then for an infinite number of iterates n_k one must have $\Lambda_t^{n_k}(Q) \in \mathcal{T}_1$ because the orbit of Q can not visit the boundary of $\mathcal{T}$ and it can not remain for an infinite number of successive iterates in $\mathcal{T}_0$. Let us divide the set $(\mathcal{T}_1 \cap \Lambda_t(\mathcal{T})) \setminus \mathcal{R}_t$ into three different zones: The triangle $\mathcal{U}$ with vertices K, J and M_1, the triangle $\Lambda_t(\mathcal{U})$ and the polygonal open region $\mathcal{V}$ with vertices J_1, H_1, V and M. See again Figure 3.5. In the first part of this proof we have demonstrated that $\Lambda_t^3(\mathcal{U}) \subset \mathcal{R}_t$ hence, now, it is enough to prove that every orbit passing through $\mathcal{V}$ converges to $\mathcal{R}_t$. We may assume that if $\Lambda_t^n(Q) \in \mathcal{V}$, then $\Lambda_t^{n-1}(Q) \in \mathcal{T}_0$ because the pre-image of $\mathcal{V}$ in $\mathcal{T}_1$ (the symmetric of the pre-image of $\mathcal{V}$ in $\mathcal{T}_0$ with respect to the critical line) is outside $\Lambda_t(\mathcal{T})$. Moreover, $\Lambda_t(\mathcal{V})$ is the polygonal region with vertices J_2, H_2, V_1 and M_1. Hence, $\Lambda_t(\mathcal{V}) \subset \mathcal{T}_0 \cup \mathcal{U}$. Now the result is consequence of the following three claims:

Claim 1.- If some point Q satisfies that Q, Q_1 and Q_2 belong to $int(\mathcal{T}_0)$ then, since $Q = (x, y)$ and $Q_2 = (x_2, y_2)$ belong to the same line $y = mx$, $0 < m < 1$, and $x_2 > x$, one has $dist(Q_2, \overline{OH}) > dist(Q, \overline{OH})$.

Claim 2.- If some point Q satisfies $Q \in \mathcal{T}_0$, $Q_1 \in \mathcal{V}$, $Q_2 \in \mathcal{T}_0$ then, for every $t \in (\frac{1}{\sqrt{2}}, 1)$, $dist(Q_2, \overline{OH}) > dist(Q, \overline{OH})$. This fact is due to the expansiveness of Λ_t for every $t \in (\frac{1}{\sqrt{2}}, 1)$. In fact, Λ_t expands all the segments not intersecting the critical line by a factor $\sqrt{2}t$.

Claim 3.- If some point $Q \in \mathcal{V}$ satisfies $Q_i \in \mathcal{T}_0$, $i = 1, \ldots, k$ and $Q_{k+1} \in \mathcal{V}$, then k is odd, and $dist(Q_{k+2}, \overline{OH}) > dist(Q_1, \overline{OH})$. This last claim easily follows if we take into account that, since $J \in \overline{V_1 M_1}$ then $J_{-1} \in \overline{VM}$ and therefore $\Lambda_t^{-2j}(\mathcal{V}) \cap \Lambda_t(\mathcal{V}) = \emptyset$, for every $j \in \mathbb{N}$.

Therefore, for every point Q in the interior of $\mathcal{T}$ one has $Q_n = \Lambda_t^n(Q) \in (\mathcal{T}_1 \cap \Lambda_t(\mathcal{T})) \setminus \mathcal{V}$, for some $n \in \mathbb{N}$ and thus the result is proved. ■

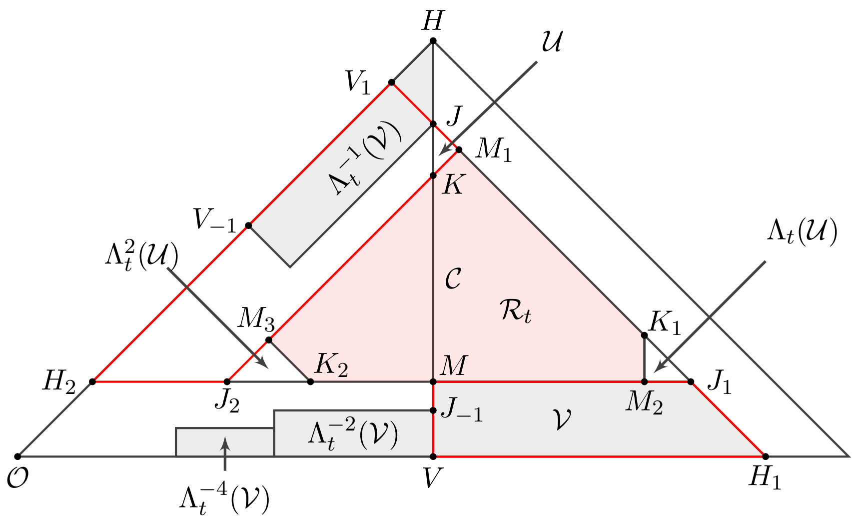


Figure 3.5: The maximal attracting set for Λ_t

At this point, we are going to construct a partition for the maximal attracting set $\mathcal{R}_t$ that will be very useful. Recall that we have considered $\mathcal{R}_t = \mathcal{R}_{t,0} \cup \mathcal{R}_{t,1}$, so we can define the sets

$$\Theta_0 = \mathcal{R}_{t,0} \ ,$$
$$\Theta_{-k} = (\Lambda_t^{-1}(\Theta_{-(k-1)})) \cap \mathcal{R}_{t,1} \ \ , k \in \mathbb{N}.$$

As we have seen in (3.7), there exists a fixed point of Λ_t in $\mathcal{R}_{t,1}$ for every $t \in (\frac{1}{\sqrt{2}}, 1)$. From the fact that $\Lambda_{t|\mathcal{R}_{t,1}}$ is a linear expansion, the Λ_t-orbit of any point in $\mathcal{R}_t$ must visit $\mathcal{R}_{t,0} = \Theta_0$ (except for the fixed point P_t, see (3.7)). Hence,

$$\mathcal{R}_t \setminus \{P_t\} = \bigcup_{k=0}^{\infty} \Theta_{-k}.$$

Moreover, for each $k \in \mathbb{N}$, we can define the set

$$\mathcal{A}_k = \mathcal{R}_t \setminus \bigcup_{j=0}^{k-1} \Theta_{-j} = \{P_t\} \cup \bigcup_{j=k}^{\infty} \Theta_j \tag{3.8}$$

which is a pentagonal domain containing P_t such that $\Lambda_t(\mathcal{A}_k) = \mathcal{A}_{k-1}$ and $\Lambda_t^k(\mathcal{A}_k) = \mathcal{R}_t$. The family $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ is a basis of neighbourhoods of the fixed point P_t.

Remark **3.11** *The map* Λ_t^k *linearly sends* $\mathcal{A}_k$ *into* $\mathcal{R}_t$ *and moreover,*

$$\Lambda_t^k(\Theta_{-j}) = \Theta_{-j+k}$$

for every $j \geq k$*. Furthermore, since* Λ_t *rotates (under an angle* $\rho = -\frac{3\pi}{4}$*) and expands (uniformly in any direction) any set contained in* $\mathcal{R}_{t,1}$*, it easily follows that* $\{\Theta_{-j}\}_{j=k}^{\infty}$ *is a partition of* $\mathcal{A}_k$ *which coincides (of course up to a linear change in coordinates) with the original partition* $\{\Theta_{-j}\}_{j=0}^{\infty}$ *of* $\mathcal{R}_t$*. See Figure 3.6 where the set* $\mathcal{R}_t$ *is rotated by an angle* $\rho = -\frac{3\pi}{4}$ *and observe the shape of* $\{\Theta_{-j}\}_{j=1}^{\infty}$ *as a partition of* $\mathcal{A}_1$ *compared with the shape of* $\{\Theta_{-j}\}_{j=0}^{\infty}$ *as a partition of* $\mathcal{R}_t$*.*

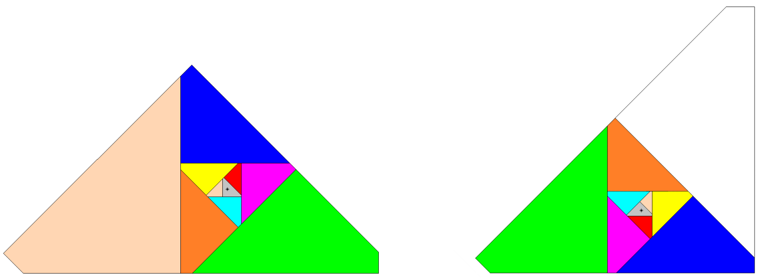


Figure 3.6: Self-similarities of the partition of $\mathcal{R}_t$

Now, let us consider the straight segments given by

$$\mathcal{C}_0 = \mathcal{C} \cap \mathcal{R}_t \ ,$$
$$\mathcal{C}_{-k} = (\Lambda_t^{-1}(\mathcal{C}_{-(k-1)})) \cap \mathcal{R}_{t,1} \ \ , \ k \in \mathbb{N}.$$

(see Figure 3.7). Recall that the sets $\mathcal{C}_j$, for $j = 1, \ldots, 5$, were already defined when we constructed the set $\mathcal{R}_t$. Now, if two sets $\mathcal{C}_i$ and $\mathcal{C}_j$ intersect for some $i, j \in \mathbb{Z}$ with $i < j \leq 5$, we will denote by

$$P_{i,j} = \mathcal{C}_i \cap \mathcal{C}_j.$$

In this way, we have that $\Lambda_t(P_{i,j}) = P_{i+1,j+1}$. Moreover, for every $j \leq 2$, $\mathcal{C}_j$ intersects $\mathcal{C}_{j-3}$, $\mathcal{C}_{j-2}$, $\mathcal{C}_{j+2}$ and $\mathcal{C}_{j+3}$ and it is easy to see that

$$\mathcal{C}_j = \overline{P_{j,j+2}P_{j,j+3}}.$$

Definition **3.12** *For each $k \in \{0\} \cup \mathbb{N}$ we define the map*

$$\mathbb{S}_k : Q \in \Theta_{-k} \mapsto \mathbb{S}_k(Q) \in \mathbb{S}_k(\Theta_{-k})$$

being $\mathbb{S}_k(Q)$ the symmetric of Q with respect to $\mathcal{C}_{-k}$.

Lemma **3.13** *For every $k = 0, 1, 2 \ldots$ the following statements hold:*

i) $\mathbb{S}_k(\Theta_{-k}) \subset \mathcal{A}_{k+1} \subset \mathcal{R}_{t,1}$.

ii) For every $Q \in \Theta_{-k}$, $\Lambda_t^{k+1}(Q) = \Lambda_t^{k+1}(\mathbb{S}_k(Q))$.

iii) For every $t \in (\frac{1}{\sqrt[3]{2}}, 1)$, $P_t \in \mathbb{S}_k(\Theta_{-k})$. Consequently, there exists a point $P_k^\star \in \Theta_{-k}$ such that $\mathbb{S}_k(P_k^\star) = P_t$. Moreover, this point $P_k^\star$ satisfies $\Lambda_t(P_{k+1}^\star) = P_k^\star$, $\Lambda_t(P_0^\star) = P_t$ and $\Lambda_t^k(P_k^\star) = P_t$.

Proof. From Remark 3.11, it is enough to prove the first statement for $k = 0$. But, for proving that $\mathbb{S}_0(\Theta_0) \subset \mathcal{A}_1 = \mathcal{R}_{t,1}$ it suffices to observe that the point K belongs to the segment $\overline{M_3M_1}$ and therefore $\mathbb{S}_0\left(\overline{M_3K}\right) \subset \mathcal{A}_1$ (see Figure 3.5).

The second statement easily follows because for each $Q \in \Theta_{-k}$, the points Q and $\mathbb{S}_k(Q)$ are symmetric with respect to $\mathcal{C}_{-k}$ and, since $\Lambda_t^j\left(\overline{Q\mathbb{S}_k(Q)}\right) \cap \mathcal{C}_0 = \emptyset$, for $j = 0, 1, \ldots, k-1$, we have that $\Lambda^k(Q)$ and $\Lambda^k(\mathbb{S}_k(Q))$ are symmetric with respect to $\mathcal{C}_0$. Hence, $\Lambda^{k+1}(Q) = \Lambda^{k+1}(\mathbb{S}_k(Q))$.

Once again due to Remark 3.11, it is enough to prove the third statement only for $k = 0$; i. e., that $P_t \in \mathbb{S}_0(\Theta_0)$, for every $t \in (\frac{1}{\sqrt[3]{2}}, 1)$. Let us first compute the pre-image of P_t in $\mathcal{T}_0$ (observe that this pre-image has to be the symmetric of P_t with respect to the critical line):

$$P_0^\star = (\frac{2t+2}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1}). \tag{3.9}$$

Then, we must obtain the value of the parameter t for which this point $P_0^\star$ belongs to the straight segment $\mathcal{C}_3 = \overline{M_3M_1}$. As this segment takes part of the line given by $\mathcal{L}_{\mathcal{C}_3} = \{(x, y) \in \mathbb{R}^2 : \; y = x + 4t^2(t-1)\}$ we must solve the equation $\sigma(t) = 0$, with $\sigma(t) = 4t^5 - 2t^3 - 2t^2 + 1$. It holds that $\sigma(\frac{1}{\sqrt[3]{2}}) = 0$ and $\sigma(t) \neq 0$ for every $\frac{1}{\sqrt[3]{2}} < t \leq 1$. The fact that each $P_k^\star$ is a pre-image of P_t is now consequence of the second statement. ∎

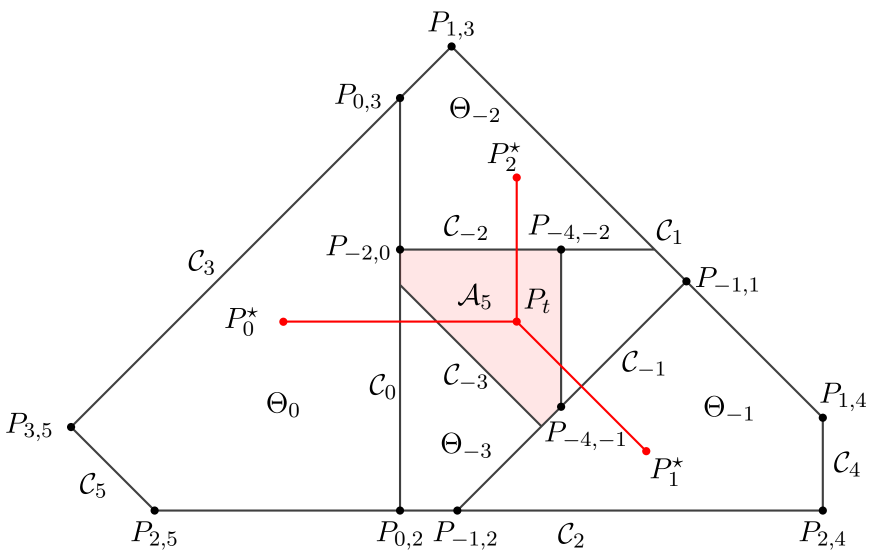


Figure 3.7: The partition of $\mathcal{R}_t$

3.4 The Country Bread Attractor case

As usual, we will denote by $B(q,r)$ the ball in $\mathbb{R}^2$ centered at the point q with radius r. Let us begin by stating the following result, whose proof is trivial by definition of Λ_t (see (3.5)).

Lemma **3.14** *For every $t \in [0,1]$ if $\mathcal{B} = B(q,r)$ is a ball in $\mathcal{R}_t$ with $\mathcal{B} \cap \mathcal{C}_0 = \emptyset$, then $\Lambda_t(\mathcal{B})$ is also a ball with $\Lambda_t(\mathcal{B}) = B(\Lambda_t(q), \sqrt{2}tr)$.*

From now on, we will consider

$$t_0 = \frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}} \approx 0.882.$$

The aim of this section is to prove the next theorem.

Theorem **3.15** *For every $t \in (t_0, 1)$ the map Λ_t exhibits a strange attractor $\mathcal{R}_t \subset \mathcal{T}$. Moreover the map Λ_t is strongly topologically mixing in $\mathcal{R}_t$, the periodic orbits are dense in $\mathcal{R}_t$, and $\mathcal{R}_t$ supports a unique absolutely continuous invariant and ergodic probability measure. Furthermore, $\mathcal{R}_t$ is a two dimensional strange attractor: There exists a dense orbit of Λ_t in $\mathcal{R}_t$ with two positive Lyapounov exponents.*

Recall that an attractor is a compact, invariant and transitive set whose stable set has non-empty interior. Now, $\mathcal{R}_t$ is compact and, as we have seen in Lemma 3.10, it is invariant and its stable set coincides with the interior of $\mathcal{T}$. So if we want to prove that $\mathcal{R}_t$ is an attractor it only remains to be demonstrated that $\mathcal{R}_t$ is transitive, i.e., Λ_t is transitive in $\mathcal{R}_t$. However, we will prove that the map Λ_t is strongly topologically mixing in $\mathcal{R}_t$ for $t \in (t_0, 1)$ (something stronger). Hence we need to demonstrate Proposition 3.16, whose proof will be given in Subsection 3.4.1. Let us recall that $t_0 \approx 0.882$.

Proposition **3.16** *Let $t \in (t_0, 1)$ and $\mathcal{B} = B(q, r)$ be a ball with $\mathcal{B} \subset \mathcal{R}_t$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. Then, at least one of the two situations holds:*

A) There exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$, or

B) There exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ with $\widetilde{\mathcal{B}} \subset \mathcal{B}$ such that:

i) $\widetilde{\mathcal{B}} \cap \mathcal{C}_0 = \emptyset$.

ii) If n is the first natural number with $\Lambda_t^n(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 \neq \emptyset$, then $\Lambda_t^n(\widetilde{\mathcal{B}})$ is again a ball, $\Lambda_t^n(\widetilde{\mathcal{B}}) = B(\Lambda_t^n(\widetilde{q}), (\sqrt{2}t)^n \widetilde{r})$, with $(\sqrt{2}t)^n \widetilde{r} > r$.

Let us assume by the moment that Proposition 3.16 is proved. Note that item B) can not occur up to the infinity, so the previous proposition easily implies the following corollary.

Corollary **3.17** *Let $t \in (t_0, 1)$. If $\mathcal{U}$ is any open set in $\mathcal{R}_t$, then there exists $n \in \mathbb{N}$ such that $\Lambda_t^n(\mathcal{U}) = \mathcal{R}_t$; i. e., Λ_t is strongly topologically mixing in $\mathcal{R}_t$.*

Proof. Given any open set $\mathcal{U}$ in $\mathcal{R}_t$, let us take any ball $\mathcal{B}$ contained in $\mathcal{U}$. Let n_1 be the first time for which $\Lambda_t^{n_1}(\mathcal{B})$ intersects the critical line $\mathcal{C}_0$. Observe that, from Lemma 3.14, $\mathcal{B}_1 = \Lambda_t^{n_1}(\mathcal{B})$ is again a ball and we can apply Proposition 3.16 to $\mathcal{B}_1$ in order to obtain either a natural number $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}_1) = \mathcal{R}_t$ (in this case we have finished) or a ball $\widetilde{\mathcal{B}_1} = B(\widetilde{q_1}, \widetilde{r_1}) \subset \mathcal{B}_1$ satisfying that $\widetilde{\mathcal{B}_1} \cap \mathcal{C}_0 = \emptyset$ and, if n_2 is the first natural number for which $\Lambda_t^{n_2}(\widetilde{\mathcal{B}_1}) \cap \mathcal{C}_0 \neq \emptyset$, then $\mathcal{B}_2 = \Lambda_t^{n_2}(\widetilde{\mathcal{B}_1})$ is again a ball and $area(\mathcal{B}_2) > area(\mathcal{B}_1)$. The finiteness of the diameter of $\mathcal{R}_t$ allows us to conclude (after a finite number of steps) the existence of a natural number $n \in \mathbb{N}$ such that $\Lambda_t^n(\mathcal{B}) = \mathcal{R}_t$. ■

Now, the denseness of the periodic points of Λ_t in $\mathcal{R}_t$ is easily obtained from Corollary 3.17. Namely, if $\mathcal{U}$ is any open set in $\mathcal{R}_t$, then we may construct a compact set $\mathcal{K}$ contained in $\mathcal{U}$ and a natural number n such that Λ_t^n is linear in $\mathcal{K}$ and $\Lambda_t^n(\mathcal{K}) = \mathcal{R}_t$. Therefore, there must exist a periodic point in $\mathcal{K}$ for Λ_t.

Moreover, some results about the existence of absolutely continuous invariant measures (*ACIMs* for short) can be applied to our family (3.5). In the one-dimensional setting a result of Lasota and Yorke (see [27]) ensures the existence of *ACIMs* for a class of piecewise C^2 expanding maps on the interval. A version of this result was obtained by Keller in [26] for certain piecewise expanding maps in the two-dimensional case. In [11], Buzzi proved the existence of *ACIMs* for expanding piecewise real-analytic maps of the plane.

At this point, we will recover the following definitions from [11].

DEFINITION **3.18** *An* ***arc*** *is a map* $\gamma : [0,1] \to \mathbb{R}^2$ *which is one-to-one, at least of class* C^1 *and such that* $\|\gamma'(t)\| \neq 0$ *for each* $t \in [0,1]$*. An* ***analytic piece*** *of* $\mathbb{R}^2$ *is a non-empty and bounded open subset of* $\mathbb{R}^2$*, the boundary of which is a finite union of analytic arcs.*

DEFINITION **3.19** *A* ***piecewise analytic map*** *of the plane is a map* $f : \mathcal{Y} \to \overline{\mathcal{Y}}$ *such that:*

(1) $\mathcal{Y} = \bigcup_{\mathcal{K}\in\mathcal{P}} \mathcal{K}$ *where* $\mathcal{P}$ *is a finite collection of pairwise disjoint analytic pieces of* $\mathbb{R}^2$*.*

(2) for each $\mathcal{K} \in \mathcal{P}$*, the restriction* $f : \mathcal{K} \to f(\mathcal{K})$ *can be extended to* $f_{\mathcal{K}} : \mathcal{U} \to \mathcal{V}$*, with* $f_{\mathcal{K}}$ *an analytic diffeomorphism between neighbourhoods* $\mathcal{U}$*,* $\mathcal{V}$ *of* $\overline{\mathcal{K}}$ *and* $\overline{f(\mathcal{K})}$ *.*

Such a map is called ***expanding*** *if*

$$\inf_{x\in\mathcal{Y}} \min_{v\in S^1} \|f'(x)\cdot v\| > 1.$$

Taking $\mathcal{P} = \{int(\mathcal{T}_0), int(\mathcal{T}_1)\}$, the facts that $\mathcal{T}_i$ are triangles and $\Lambda_{t|\mathcal{T}_i}$ are linear implies that, for every $t \in (\frac{1}{\sqrt{2}}, 1)$, the map Λ_t is a expanding piecewise analytic map of the plane.

Now, in order to prove that, for every $t \in (t_0, 1)$, the map Λ_t has a unique ergodic *ACIM* it is enough to recall that Λ_t is topologically transitive and apply the first statement of the Main Theorem in [11].

A result of Saussol (see [50]) gives us even more information. In [50] the author works with certain class of multi-dimensional piecewise expanding maps (not necessarily piecewise linear). This set of maps is defined at Section 2 in [50]. This definition contains five conditions, $(PE1) - (PE5)$. Our maps Λ_t do not satisfy condition $(PE5)$. Nevertheless, we will take advantage of Lemma 2.2 in [50]. This lemma ensures that, if some map f satisfies $(PE1) - (PE3)$ and certain extra condition (involving the so-called *weighted* *multiplicity* and the *dilatation coefficient*) holds, then some iterate of f satisfies $(PE1)-(PE5)$ and therefore we may apply all the results obtained in [50] to this iterate of f. This will be enough for our purposes. Hence, let us begin by describing conditions $(PE1)-(PE3)$ from Section 2 in [50].

From now on, for $\mathcal{K}\subset\mathbb{R}^N$ and $\varepsilon>0$ we will denote by B_ε the set

$$B_\varepsilon(\mathcal{K})=\{x\in\mathbb{R}^N : d_N(x,\mathcal{K})\leq\varepsilon\},$$

where d_N is the Euclidean distance in $\mathbb{R}^N$.

DEFINITION **3.20** *Let $\mathcal{M}\subset\mathbb{R}^N$ be a compact set with $\overline{int(\mathcal{M})}=\mathcal{M}$ and $f:\mathcal{M}\to\mathcal{M}$. Assume that there exist at most countably many disjoint open sets $\mathcal{U}_i$ and $\mathcal{V}_i$ such that $\mathcal{V}_i\supset\overline{\mathcal{U}}_i$ and maps $f_i:\mathcal{V}_i\to\mathbb{R}^N$ such that $f_{i|\mathcal{U}_i}=f_{|\mathcal{U}_i}$ for each i. We say that f satisfies conditions (PE1), (PE2) and (PE3) if there exist constants $c,\varepsilon_0>0$ and $0<\alpha\leq 1$ such that:*

(PE1) $f_i(\mathcal{V}_i)\supset B_{\varepsilon_0}(f(\mathcal{U}_i))$ *for each i;*

(PE2) for each i, $f_i\in C^1(\mathcal{V}_i)$, f_i is one-to-one and $f_i^{-1}\in C^1(f_i(\mathcal{V}_i))$. Moreover, for all i and every $\varepsilon\leq\varepsilon_0$ we assume that

$$|detDf_i^{-1}(x)-detDf_i^{-1}(y)|\leq c|detDf_i^{-1}(z)|\varepsilon^\alpha,$$

whenever $z\in f_i(\mathcal{V}_i)$ and $x,y\in B_{\varepsilon_0}(z)\cap f_i(\mathcal{V}_i)$;

(PE3) $m_N(\mathcal{M}\setminus\bigcup_i\mathcal{U}_i)=0$, *where m_N stands for the Lebesgue measure in $\mathbb{R}^N$.*

To check that Λ_t satisfies $(PE1)-(PE3)$ it suffices to consider $\mathcal{M}=\mathcal{T}$, $\mathcal{U}_1=int(\mathcal{T}_0)$, $\mathcal{U}_2=int(\mathcal{T}_1)$, $\mathcal{V}_i$ small neighbourhoods of $\overline{\mathcal{U}}_i$ and

$$\begin{aligned}(\Lambda_t)_1:(x,y)\in\mathcal{V}_1 &\mapsto (t(x+y),t(x-y))\in\mathbb{R}^2\\ (\Lambda_t)_2:(x,y)\in\mathcal{V}_2 &\mapsto (t(2-x+y),t(2-x-y))\in\mathbb{R}^2\end{aligned}$$

(see (3.5)). The existence of ε_0 satisfying $(PE1)$ is evident and, since $(\Lambda_t)_i$ are linear, $(PE2)$ holds. Condition $(PE3)$ is obviously fulfilled.

Now, let us deal with the extra condition announced before. We begin by defining the weighted multiplicity of a map f. Let $\mathcal{P}$ be a finite (finite will be enough for our purposes) collection of pairwise disjoint bounded open subsets of $\mathbb{R}^N$. Set $\mathcal{Y}=\bigcup_{\mathcal{K}\in\mathcal{P}}\mathcal{K}$ and let $f:\mathcal{Y}\to\overline{\mathcal{Y}}$ be a piecewise linear (piecewise linear will be enough for our purposes) map with respect to $\mathcal{P}$ (this means that f is linear in each subset $\mathcal{K}$). Let us take the sequence of partitions $\mathcal{P}^n$ for the nth iterate of the partition $\mathcal{P}$, i. e. the elements in $\mathcal{P}^n$ are given by

$$\mathcal{K}^0\cap f^{-1}\mathcal{K}^1\cap\cdots\cap f^{-n+1}\mathcal{K}^{n-1},\qquad \mathcal{K}^0,\ldots,\mathcal{K}^{n-1}\in\mathcal{P}.$$

Then one may define for each $n \in \mathbb{N}$, the number

$$mult(\mathcal{P}^n) = \max_{x \in \mathcal{Y}} card\{\mathcal{K} \in \mathcal{P}^n : x \in \overline{\mathcal{K}}\}.$$

Let us remark that in the one-dimensional case, we have $mult(\mathcal{P}^n) \leq 2$ for every piecewise linear map defined on an interval, but, of course, this is no longer true in higher dimensions.

DEFINITION **3.21** *In the above conditions, the* ***weighted multiplicity*** *of a map f is defined by*

$$h_{mult}(\mathcal{P}, f) = \lim_{n \to \infty} \frac{1}{n} \log mult(\mathcal{P}^n).$$

On the other hand, the ***dilatation coefficient*** *of f is defined according to [50] by*

$$\delta(f) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in f^n(\mathcal{Y})} \|Df^{-n}(x)\|,$$

where the norm of the derivative is taken along each smooth branch of f^{-n}.

We point out that for Λ_t, see (3.5), the collection $\mathcal{P}$ of pairwise disjoint and bounded open subsets of $\mathbb{R}^2$ is obtained by considering, once again, $\mathcal{P} = \{int(\mathcal{T}_0), int(\mathcal{T}_1)\}$. Therefore, $\overline{\mathcal{Y}} = \mathcal{T}$ and, for every t, the elements in $\mathcal{P}_n$ are polygonal domains (for instance, for $t = 1$, they are triangles) whose boundaries are made of straight segments which only can be horizontal, vertical or with slope 1 or -1. Thus, for each t, the numbers $mult(\mathcal{P}^n)$ remain bounded (for instance by eight). Then, in our case, $h_{mult}(\mathcal{P}, \Lambda_t) = 0$. It is also easy to see that, for our case, $\delta(\Lambda_t) = -\log(\sqrt{2}t)$. Therefore, for every $t \in (\frac{1}{\sqrt{2}}, 1)$, we have $h_{mult}(\mathcal{P}, \Lambda_t) + \delta(\Lambda_t) < 0$. Therefore, we may apply the following result (see Lemma 2.2 in [50]):

LEMMA **3.22** *Let f be a piecewise invertible C^1 map with a partition into smooth components $\mathcal{P}$ such that $(PE1) - (PE3)$ hold for some $0 < \alpha \leq 1$. Suppose that the boundary of the partition is included in a finite number of C^1 compact embedded submanifolds.*

If $h_{mult}(\mathcal{P}, f) + \delta(f) < 0$ then some iterate of the map satisfies $(PE1) - (PE5)$.

Therefore we have that, for every $t \in (\frac{1}{\sqrt{2}}, 1)$, there exists some $n \in \mathbb{N}$, such that the map Λ_t^n satisfies $(PE1) - (PE5)$. From Theorems 5.1 and 5.2 in [50] the map Λ_t^n has *ACIMs*, each one of them being a convex combination of a fixed, finite collection of ergodic ones. According to Lemma 3.1 in [50] the associated density of any $ACIM$ μ for Λ_t^n is bounded away from zero in some ball.

Hence the support of such measure μ always contains a ball. Since for every $t \in (t_0, 1)$, Λ_t^n is strongly topologically mixing the support of any $ACIM$ μ for Λ_t^n coincides with $\mathcal{R}_t$. Thus, Proposition 5.1 in [50] implies the existence of a unique ergodic $ACIM$ for Λ_t^n. Therefore, there exists only one $ACIM$ for Λ_t^n. Of course, this measure must be ergodic and it must be supported on $\mathcal{R}_t$. Finally, from Proposition 3.4 in [50] we deduce that the associated density of such $ACIM$ is bounded and therefore the $ACIM$ is finite.

Now, from Main Theorem in [11] we know that Λ_t admits $ACIMs$; but, since every $ACIM$ for Λ_t is an $ACIM$ for Λ_t^n we conclude that Λ_t only admits a unique $ACIM$ which we denote by μ_t. Of course, μ_t is finite and $supp(\mu_t) = \mathcal{R}_t$. According to Main Theorem in [11], μ_t must be ergodic.

Now, let us show the existence of at least one dense Λ_t-orbit on $\mathcal{R}_t$ not visiting the critical set $\mathcal{C}_0$. This is enough to conclude that the attractor is strange because one easily obtains that the Lyapounov exponent of Λ_t along any orbit not visiting the critical line coincides with $\log(\sqrt{2}t)$ in all nonzero direction, so both exponents are positive. To this end, we denote by μ_t the (unique) ergodic $ACIM$ for Λ_t described in the last paragraph and by m_N we now denote the usual Lebesgue measure in $\mathbb{R}^N$. Then the existence of such orbit directly follows from the next result.

Lemma **3.23** *For every $t \in (t_0, 1)$ the following statements hold:*

i) $\mu_t(\widetilde{\mathcal{C}}) = 0$, *where* $\widetilde{\mathcal{C}} = \{Q \in \mathcal{R}_t : \Lambda_t^j(Q) \in \mathcal{C}_0$, *for some* $j \in \mathbb{N}\}$.

ii) There exists a set $\mathcal{S} \in \mathcal{R}_t$ such that $\mu_t(\mathcal{S}) = 1$ and if Q_0 belongs to $\mathcal{S}$ then its Λ_t-orbit is dense in $\mathcal{R}_t$.

Proof. The first statement follows taking into account that μ_t is absolutely continuous with respect to the Lebesgue measure and that $m_N(\widetilde{\mathcal{C}}) = 0$. The second statement can be proved in the same way as Lemma 2.2. ■

Now, in order to finish the proof of Theorem 3.15, it still remains to prove Proposition 3.16.

3.4.1 Proof of Proposition 3.16

Let q be a point in $\mathbb{R}^2$, r a positive real number and α_1, α_2 real numbers with $0 \le \alpha_1 < \alpha_2 \le 2\pi$. We denote by $CS(q, r, \alpha_1, \alpha_2)$ the circular sector defined by

$$CS(q, r, \alpha_1, \alpha_2) = \{Q \in \mathbb{R}^2 : Q = q + r' \exp(i\alpha), (r', \alpha) \in [0, r] \times [\alpha_1, \alpha_2]\}$$

We are interested in obtain the ball of maximum radius inside certain circular sectors. The proof of next lemma is easy.

LEMMA **3.24** *Let $\mathcal{B} = B(q, r)$ be a ball and $\Gamma = CS(q, r, \alpha_1, \alpha_2)$ be a circular sector with $0 \le \alpha_1 < \alpha_2 \le 2\pi$. Then,*

i) If $\alpha_2 - \alpha_1 = \pi$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \frac{r}{2})$ contained in Γ.

ii) If $\alpha_2 - \alpha_1 = \frac{\pi}{2}$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, (\sqrt{2} - 1)r)$ contained in Γ.

iii) If $\alpha_2 - \alpha_1 = \frac{\pi}{4}$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \delta_0 r)$ contained in Γ, being $\delta_0 = \frac{\sin\frac{\pi}{8}}{1+\sin\frac{\pi}{8}} \approx 0.2767$.

Now, let us take a ball $\mathcal{B} = B(q, r)$ with $\mathcal{B} \subset \mathcal{R}_t$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. We may assume that the center q belongs to $\mathcal{R}_{t,1}$, because in the case in which $q \in \mathcal{R}_{t,0} = \Theta_0$ we may work with $\mathcal{B}^*$, being $\mathcal{B}^*$ the symmetric of $\mathcal{B}$ with respect to the critical line $\mathcal{C}_0$. Of course, $\mathcal{B}^*$ is a ball centered at $\mathcal{R}_{t,1}$. Moreover, the successive iterates of $\mathcal{B}$ and $\mathcal{B}^*$ coincide due to the fact that Λ_t is symmetric with respect to the critical line $\mathcal{C}_0$. Therefore, if anyone of the two items of Proposition 3.16 is proved for $\mathcal{B}^*$, then the same item is proved for $\mathcal{B}$.

To prove Theorem 3.15 it will be useful to control the maximal distance between each point $P_k^\star$, see Lemma 3.13, and the boundary of the respective Θ_{-k}.

Let us denote by $\partial(\Theta_{-k})$ the boundary of Θ_{-k}, see Figure 3.7, and define

$$\theta_k = \max_{Q \in \partial(\Theta_{-k})} dist(P_k^\star, Q). \tag{3.10}$$

To bound this distance it is enough to control θ_0, the maximal distance between $P_0^\star$ to the boundary of Θ_0. This is due to the fact that, given two points A and B in Θ_{-k}, then $\Lambda_t^k(A)$ and $\Lambda_t^k(B)$ belong to Θ_0 and $dist(A, B) = (\sqrt{2}t)^{-k} dist(\Lambda_t^k(A), \Lambda_t^k(B))$. Therefore since $\Lambda_t^k(P_k^\star) = P_0^\star$,

$$\theta_0 = (\sqrt{2}t)^k \theta_k.$$

Moreover, since Θ_0 is a convex polygonal domain one has that θ_0 coincides with the maximum distance between $P_0^\star$ and the four vertices of the boundary of Θ_0, see Figure 3.7. That is, $\theta_0 = \max\{\theta_{0,3}, \theta_{0,2}, \theta_{2,5}, \theta_{3,5}\}$ where

$$\theta_{0,3} = dist(P_0^\star, P_{0,3}) \quad , \quad \theta_{2,5} = dist(P_0^\star, P_{2,5})$$
$$\theta_{0,2} = dist(P_0^\star, P_{0,2}) \quad , \quad \theta_{3,5} = dist(P_0^\star, P_{3,5})$$

(see Figure 3.7).

Then, in the proof of Lemma 3.26 we will use the following result.

Lemma **3.25** *There exist* $t_{0,2} \approx 0.8478$, $t_{0,3} \approx 0.9112$ *such that*

i) $\theta_0 = \theta_{0,2}$ *if* $t \in [1/\sqrt[3]{2}, t_{0,2}]$.

ii) $\theta_0 = \theta_{0,3}$ *if* $t \in [t_{0,2}, t_{0,3}]$.

iii) $\theta_0 = \theta_{3,5}$ *if* $t \in [t_{0,3}, 1]$.

Proof. The coordinates of $P_{0,3}(=K)$, $P_{0,2}(=M)$, $P_{2,5}(=K_2)$ and $P_{3,5}(=M_3)$, see Figures 3.5 and 3.7, were given in Section 3.3. The coordinates of $P_0^\star$ were given at (3.9). Then, one may check that

$$\begin{aligned}
\theta_{0,3}^2 &= 2\alpha(1-2t+2t^2-4t^3+4t^4) \\
\theta_{0,2}^2 &= \alpha(1-2t+2t^2) \\
\theta_{2,5}^2 &= 4\alpha(1-2t+4t^2-4t^3+2t^4-4t^5+4t^6-8t^7+8t^8) \\
\theta_{3,5}^2 &= 4\alpha(1-2t+2t^2-2t^3+2t^6-4t^7+4t^8)
\end{aligned} \tag{3.11}$$

where $\alpha = \alpha(t) = \frac{(1-2t^2)^2}{1+2t+2t^2}$. Then, a numerical analysis in the interval of parameters $[1/\sqrt[3]{2}, 1]$, allows us to conclude the statement of the lemma. ■

We split the proof of Proposition 3.16 in several cases according in which Θ_{-k} the center of the ball q is situated. The proof of the following result can be found in [46].

Lemma **3.26** *Let* $t \in (t_0, 1]$ *and* $\mathcal{B} = B(q, r)$ *be a ball in* $\mathcal{R}_t$, *then*

i) If $q \in \Theta_{-1}$, *then* $\mathcal{B} \cap \mathcal{C}_0 = \emptyset$.

ii) If $q \in \Theta_{-k}$ *with* $k = 4$ *or* $k \geq 6$ *and* $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$, *then there exists* $j \in \mathbb{N}$ *such that* $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

Proof. If $\mathcal{B} = B(q, r) \subset \mathcal{R}_t$ and $q \in \Theta_{-1}$, then $\mathcal{B} \subset \Theta_{-1} \cup \mathbb{S}_1(\Theta_{-1})$, being as usual, $\mathbb{S}_1(Q)$ the symmetric of a point $Q \in \Theta_{-1}$ with respect to $\mathcal{C}_{-1}$. Now, see Figure 3.7, since $P_{-1,2} \in \overline{P_{0,2}P_{2,4}}$ one has that $\mathbb{S}_1(\Theta_{-1}) \cap \mathcal{C}_0 = \emptyset$ and the first statement follows.

In order to prove the second statement we will firstly check that, if $q \in \Theta_{-k}$, then $P_k^\star \in \mathcal{B}$. To this end, let us recall that we have denoted by θ_k the maximum distance between $P_k^\star$ to the boundary of Θ_{-k}. Hence, let us now consider $d_k = dist(\Theta_{-k}, \mathcal{C}_0)$ and observe that it will be enough to check that $\theta_k < d_k$ for $k = 4$ and $k \geq 6$. Of course, if $\mathcal{B} = B(q, r)$ satisfies $q \in \Theta_{-k}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$ then it necessarily holds that $r \geq d_k > \theta_k$ and therefore $P_k^\star \in \mathcal{B}$. Recall that $\theta_k = (\sqrt{2}t)^{-k}\theta_0$ satisfies $\theta_{k+1} < \theta_k$ for every k. Let us suppose that we have proved

$$\theta_k < d_k \text{ for } k \in \{4, 6, 8\}. \tag{3.12}$$

Then, since $d_6 = dist(P_{-6,-3}, \mathcal{C}_0)$ and $d_7 = dist(P_{-7,-5}, \mathcal{C}_0)$ with $P_{-6,-3} \in \Theta_{-8}$ and $P_{-7,-5} \in \Theta_{-7}$, one has $d_7 > d_6 > \theta_6 > \theta_7$. Moreover, since $d_8 = dist(P_{-5,-3}, \mathcal{C}_0) < d_k$ for every $k > 8$ we also have $d_k > d_8 > \theta_8 > \theta_k$. Hence, it is enough to prove (3.12).

A direct calculation gives

$$\begin{aligned} d_4 &= \frac{-1+2t-4t^3+4t^4}{4t^4} \\ d_6 &= \frac{-1+2t-8t^5+8t^6}{8t^6} \\ d_8 &= \frac{1-2t+2t^2-8t^4+8t^5}{8t^5} \end{aligned}$$

while $\theta_k = (\sqrt{2}t)^{-k}\theta_0$, being $\theta_0 = \theta_{0,3} = dist(P_0^\star, P_{0,3})$ for $t \in (t_0, t_{0,3}]$ and $\theta_0 = \theta_{3,5} = dist(P_0^\star, P_{3,5})$ for $t \in [t_{0,3}, 1]$, with $t_{0,3} \approx 0.9112$ (see Lemma 3.25). The expression of $\theta_{0,3}$ and $\theta_{3,5}$ were given in (3.11). Now, a numerical analysis shows that, for every $t \in (t_0, 1]$, $\theta_k < d_k$ for $k = 4, 6$ and $k = 8$. Therefore we have proved that $P_k^\star \in \mathcal{B}$.

Hence, there exists a sufficiently small neighbourhood $U \subset \mathcal{B}$ of $P_k^\star$ such that $\Lambda_t^k(U)$ is a neighbourhood of the fixed point P. This easily follows taking into account that the orbit of $P_k^\star$ never visits the critical set. Now, let us recall the sets $\mathcal{A}_m = \mathcal{R}_t \setminus \bigcup_{j=0}^{m-1} \Theta_{-j}$, see again (3.8) and Figure 3.7. These sets are a basis of neighbourhoods of P and moreover $\Lambda_t^m(\mathcal{A}_m) = \mathcal{R}_t$. Take m large enough so that $\mathcal{A}_m \subset \Lambda_t^k(U)$. Then, one has $\Lambda_t^{k+m}(U) = \mathcal{R}_t$. ∎

Now, we will study the cases in which $q \in \Theta_{-k}$ being $k = 2, 3, 5$. In that cases it will be necessary to use the following result, whose proof is given in Subsection 3.4.3.

Lemma **3.27** *Let $t \in (t_0, 1)$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0$.*

i) If $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$ then there exists $j \in \mathbb{N}$ such that

$$\Lambda_t^j(\mathcal{B} \cap \Theta_0) = \mathcal{R}_t.$$

ii) If $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$ then there exists $j \in \mathbb{N}$ such that

$$\Lambda_t^j(\mathcal{B} \cap \Theta_0) = \mathcal{R}_t.$$

We start by studying the case $q \in \Theta_{-5}$.

Proposition **3.28** *Let $t \in (t_0, 1)$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathcal{R}_t$ with $q \in \Theta_{-5}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. Then the following statements hold:*

i) If $\mathcal{B} \cap \mathcal{C}_{-5} \neq \emptyset$, then there exists $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

ii) If $\mathcal{B} \cap \mathcal{C}_{-5} = \emptyset$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ contained in $\mathcal{B} \cap \Theta_{-5}$ such that

$$\Lambda_t^6(\widetilde{\mathcal{B}}) = B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6 \widetilde{r})$$

with $(\sqrt{2}t)^6 \widetilde{r} > r$.

Proof. Let us take a ball $\mathcal{B} = B(q, r)$ with $q \in \Theta_{-5}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. Firstly, let us assume that $\mathcal{B} \cap \mathcal{C}_{-5} \neq \emptyset$. Let us consider the set $\mathcal{K} = \mathcal{B} \cap \Theta_{-5}$. Then, $\Lambda_t^5(\mathcal{K})$ is a subset of Θ_0 such that there exists a ball $\mathcal{B}^*$ in $\mathbb{R}^2$ centered at $\Lambda_t^5(q) \in \Theta_0$ with $\Lambda_t^5(\mathcal{K}) = \mathcal{B}^* \cap \Theta_0$. Moreover, $\mathcal{B}^* \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$. Therefore, applying the second statement of Lemma 3.27, there exists $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^* \cap \Theta_0) = \Lambda_t^{j+5}(\mathcal{K}) = \mathcal{R}_t$.

Let us assume that $\mathcal{B} \cap \mathcal{C}_{-5} = \emptyset$. Then, it is always possible to define a circular sector $CS(q, r, \alpha_1, \alpha_2)$ contained in $\mathcal{B} \cap \Theta_{-5}$, with $\alpha_2 - \alpha_1 = \frac{\pi}{4}$, see Figure 3.8. Applying Lemma 3.24 item iv) we may choose a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \delta_0 r)$, with $\delta_0 \approx 0.2767$, contained in the above circular sector, and therefore in $\mathcal{B} \cap \Theta_{-5}$. Hence, $\Lambda_t^j(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, 1, \ldots, 5$ and henceforth, from Lemma 3.14, $\Lambda_t^6(\widetilde{\mathcal{B}})$ is the ball $B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6 \delta_0 r)$. Finally, it is easy to check that

$$(\sqrt{2}t)^6 \delta_0 > 1$$

for every $t \in (t_0, 1)$. ∎

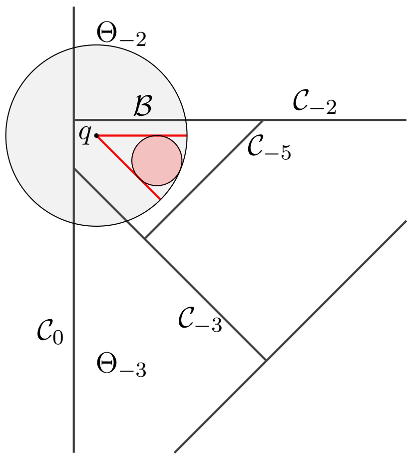


Figure 3.8: A circular sector for the case $q \in \Theta_{-5}$

Now, let us consider the case in which $q \in \Theta_{-3}$. Hence, let us assume $\mathcal{B} = B(q,r)$ to be a ball in $\mathcal{R}_t$ satisfying $q \in \Theta_{-3}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. We will make use of the point $P_{-3,0} = \mathcal{C}_{-3} \cap \mathcal{C}_0$. In fact, we will distinguish between the case in which $P_{-3,0} \in \mathcal{B}$ (Proposition 3.30) and the case in which $P_{-3,0} \notin \mathcal{B}$ (Proposition 3.31). In order to avoid tedious notation, from now on we will denote by $K_{-3} = P_{-3,0}$, $Z_{-3} = P_{-5,-3}$ and by W_{-3} the intersection between $\mathcal{C}_0$ and $\mathbb{S}_3^{-1}(\mathcal{C}_{-5})$. Note that W_{-3} always exists because $P \in \mathbb{S}_3(\Theta_{-3}) \subset \mathcal{A}_4$ (see Lemma 3.13 i)). Then, we may construct an isosceles right triangle $\Delta_{-3} \subset \Theta_{-3}$ with vertices W_{-3}, K_{-3} and Z_{-3} (see Figure 3.9). Let us also denote by Δ the triangle contained in Θ_0 with vertices W, K and Z, being $W = \Lambda_t^3(W_{-3})$, $K = \Lambda_t^3(K_{-3})$ and $Z = \Lambda_t^3(Z_{-3})$. Observe that $\Lambda_t^3(\Delta_{-3}) = \Delta$. In fact, the point K was earlier defined in Subsection 3.3.

We will use the following result whose proof is given in Subsection 3.4.2.

Lemma **3.29** *Let $t \in (t_0, 1)$. If $\mathcal{B} = B(q,r)$ is a ball in $\mathbb{R}^2$ with $q \in \Delta$ such that $K \in \mathcal{B}$ and $\mathcal{B} \cap \overline{ZW} \neq \emptyset$, then there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B} \cap \Delta) = \mathcal{R}_t$.*

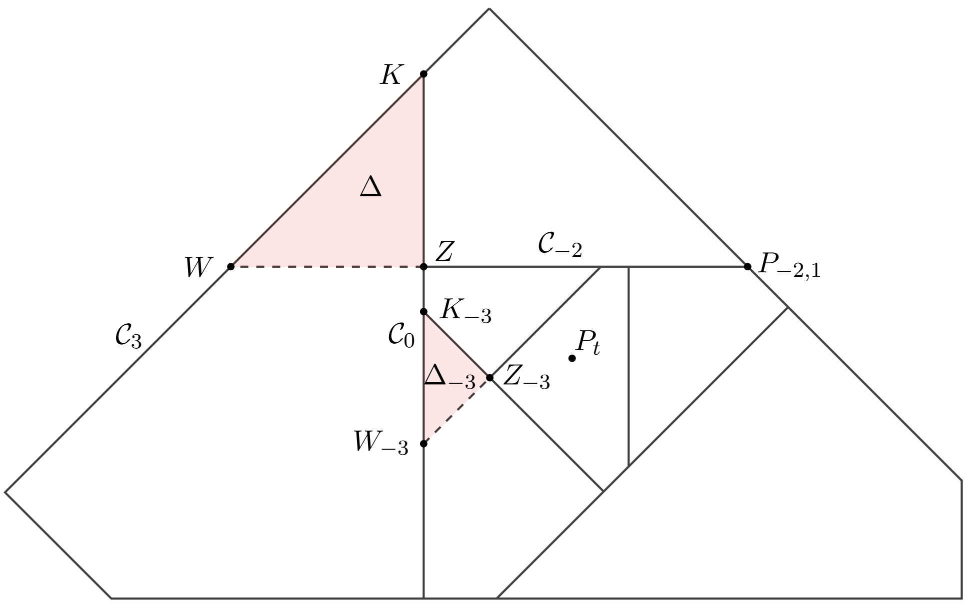


Figure 3.9: The triangles Δ and Δ_{-3}

Proposition **3.30** *Let $t \in (t_0, 1)$ and $\mathcal{B} = B(q,r)$ be a ball in $\mathcal{R}_t$ with $q \in \Theta_{-3}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. If $K_{-3} \in \mathcal{B}$, then the following statements hold:*

i) If $\overline{Z_{-3}W_{-3}} \cap \mathcal{B} = \emptyset$*, then there exists a ball* $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ *contained in* $\mathcal{B} \cap \Delta_{-3}$ *such that* $\Lambda_t^i(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$*, for* $i = 0, \ldots, 5$*,* $\Lambda_t^6(\widetilde{\mathcal{B}}) = B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6\widetilde{r})$ *and* $(\sqrt{2}t)^6\widetilde{r} > r$*.*

ii) If $\overline{Z_{-3}W_{-3}} \cap \mathcal{B} \neq \emptyset$*, then there exists* $j \in \mathbb{N}$ *such that* $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$*.*

Proof. If $\overline{Z_{-3}W_{-3}} \cap \mathcal{B} = \emptyset$, then we may always consider the circular sector $\Gamma = CS(q, r, \frac{3\pi}{2}, \frac{7\pi}{4})$ which is contained in $\mathcal{B} \cap \Delta_{-3}$.

Hence, $\mathbb{S}_3(\Gamma) = CS(\mathbb{S}_3(q), r, \frac{7\pi}{4}, 2\pi)$ is a circular sector contained in Θ_{-5}. So, we may pick a ball $\mathcal{B}^* = B(q^*, \delta_0 r) \subset \mathbb{S}_3(\Gamma)$ such that $\mathcal{B}^* \subset \Theta_{-5}$. Then, $\Lambda_t^j(\mathcal{B}^*) \cap \mathcal{C}_0 = \emptyset$, for $i = 0, \ldots, 5$, $\Lambda_t^6(\mathcal{B}^*) = B(\Lambda_t^6(q^*), (\sqrt{2}t)^6\delta_0 r)$, with $(\sqrt{2}t)^6\delta_0 > 1$ for every $t \in (t_0, 1)$. To conclude the proof of the first statement it is enough to take $\widetilde{\mathcal{B}} = \mathbb{S}_3^{-1}(\mathcal{B}^*) = B(\widetilde{q}, \delta_0 r)$, with $\widetilde{q} = \mathbb{S}_3^{-1}(q^*)$. From the second statement of Lemma 3.13, we have that $\Lambda_t^6(\widetilde{\mathcal{B}}) = \Lambda_t^6(\mathcal{B}^*) = B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6\delta_0 r)$.

Now, let us assume that $\overline{Z_{-3}W_{-3}} \cap \mathcal{B} \neq \emptyset$, with $\mathcal{B} = B(q, r)$. Let $\mathcal{K} = \mathcal{B} \cap \Delta_{-3}$. Then $\Lambda_t^3(\mathcal{K})$ is a subset of Δ such that there exists a ball $\mathcal{B}^*$ in $\mathbb{R}^2$ centered at $\Lambda_t^3(q) \in \Delta$ with $\Lambda_t^3(\mathcal{K}) = \mathcal{B}^* \cap \Delta$. Moreover, $K \in \mathcal{B}^*$ and $\mathcal{B}^* \cap \overline{ZW} \neq \emptyset$, then applying Lemma 3.29, there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}^* \cap \Delta) = \Lambda_t^{j+3}(\mathcal{K}) = \mathcal{R}_t$. Hence $\Lambda_t^{j+3}(\mathcal{B}) = \mathcal{R}_t$. ■

It still remains to analyse the case $K_{-3} \notin \mathcal{B}$.

PROPOSITION **3.31** *Let* $t \in (t_0, 1)$ *and* $\mathcal{B} = B(q, r)$ *be a ball in* $\mathcal{R}_t$ *with* $q \in \Theta_{-3}$ *and* $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$*. If* $K_{-3} \notin \mathcal{B}$ *then the following statements hold:*

i) If $\mathcal{B}$ *does not intersect* $\mathcal{C}_{-1}$ *and* $\mathcal{C}_{-3}$ *at the same time, there is a ball* $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ *contained in* $\mathcal{B} \cap \Theta_{-3}$ *such that*

$$\Lambda_t^4(\widetilde{\mathcal{B}}) = B(\Lambda_t^4(\widetilde{q}), (\sqrt{2}t)^4\widetilde{r}),$$

and $(\sqrt{2}t)^4\widetilde{r} > r$*.*

ii) If $\mathcal{B}$ *intersects* $\mathcal{C}_{-1}$ *and* $\mathcal{C}_{-3}$*, then there exists* $j \in \mathbb{N}$ *such that* $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$*.*

Proof. Let us first assume that $\mathcal{B}$ does not intersect $\mathcal{C}_{-1}$ and $\mathcal{C}_{-3}$. Then there is a circular sector $CS(q, r, \alpha_1, \alpha_2)$ contained in $\mathcal{B} \cap \Theta_{-3}$ with $\alpha_2 - \alpha_1 = \pi$. Then, from the first statement of Lemma 3.24 there is a ball

$$\widetilde{\mathcal{B}} = B(\widetilde{q}, \frac{r}{2}) \subset \mathcal{B} \cap \Theta_{-3}.$$

Hence $\Lambda_t^i(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$, for $i = 0, \ldots, 3$, $\Lambda_t^4(\widetilde{\mathcal{B}}) = B(\Lambda_t^4(\widetilde{q}), (\sqrt{2}t)^4\frac{r}{2})$, with $2t^4 > 1$ for every $t \in (t_0, 1)$.

Now, let us assume that $\mathcal{B}\cap\mathcal{C}_{-3}\neq\emptyset$ and $\mathcal{B}\cap\mathcal{C}_{-1}=\emptyset$ (the case $\mathcal{B}\cap\mathcal{C}_{-3}=\emptyset$ and $\mathcal{B}\cap\mathcal{C}_{-1}\neq\emptyset$ can be demonstrated in the same way). Let us denote by $q_1=q+r\exp(i\varphi_1)$ and $\widetilde{q}_1=q+r\exp(i\widetilde{\varphi}_1)$ the intersections between the boundary of $\mathcal{B}$ with $\mathcal{C}_0$ and by $q_2=q+r\exp(i\varphi_2)$ and $\widetilde{q}_2=q+r\exp(i\widetilde{\varphi}_2)$ the intersections between the boundary of $\mathcal{B}$ with $\mathcal{C}_{-3}$. See Figure 3.10. Since $\varphi_2-\varphi_1>0$, $\widetilde{\varphi}_1-\widetilde{\varphi}_2\geq 0$ and $\varphi_2-\varphi_1-(\widetilde{\varphi}_1-\widetilde{\varphi}_2)=\pi/2$ it holds that $\varphi_2-\varphi_1\geq\pi/2$. Then we may construct a circular sector $CS(q,r,\alpha_1,\alpha_2)\subset\mathcal{B}\cap\Theta_{-3}$ with $\alpha_2-\alpha_1=\pi/2$.

From the third statement of Lemma 3.24 there exists a ball

$$\widetilde{\mathcal{B}}=B(\widetilde{q},(\sqrt{2}-1)r)\subset\mathcal{B}\cap\Theta_{-3}$$

such that $\Lambda_t^4(\widetilde{\mathcal{B}})=B(\Lambda_t^4(q),(\sqrt{2}t)^4(\sqrt{2}-1)r)$, with $4t^4(\sqrt{2}-1)>1$, for every $t\in(t_0,1)$. So the first statement is proved. Let us remark that this last inequality holds for every $t>t_0=\frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}}$ and this fact gives us the value of t_0 in Theorem 3.15.

Now, for proving the second statement let us assume that $\mathcal{B}=B(q,r)$ satisfies $q\in\Theta_{-3}$, $\mathcal{B}\cap\mathcal{C}_{-j}\neq\emptyset$ for $j=0,1,3$. We take $\mathcal{K}=\mathcal{B}\cap\Theta_{-3}$. Then, the set $\Lambda_t^3(\mathcal{K})$ is a subset of Θ_0 such that there exists a ball $\mathcal{B}^*$ in $\mathbb{R}^2$ centered at $\Lambda_t^3(q)\in\Theta_0$ with $\Lambda_t^3(\mathcal{K})=\mathcal{B}^*\cap\Theta_0$. Moreover, $\mathcal{B}^*\cap\mathcal{C}_i\neq\emptyset$, for $i=0,2,3$. Therefore, applying the first statement of Lemma 3.27, we conclude that there is $j\in\mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}^*\cap\Theta_0)=\Lambda_t^{j+3}(\mathcal{K})=\mathcal{R}_t$. Hence $\Lambda_t^{j+3}(\mathcal{B})=\mathcal{R}_t$. ■

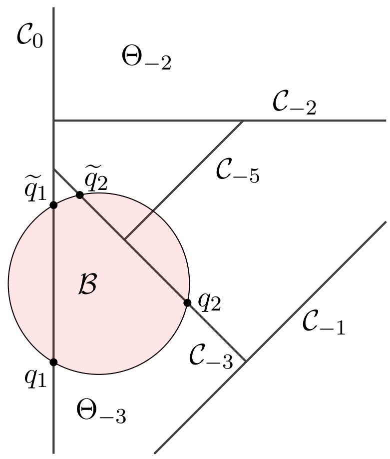


Figure 3.10: The case $\mathcal{B}\cap\mathcal{C}_{-3}\neq\emptyset$, $\mathcal{B}\cap\mathcal{C}_{-1}=\emptyset$

In order to finish the proof of Proposition 3.16 it remains to see the case in which we have a ball $\mathcal{B}=B(q,r)$ under the assumptions of Proposition 3.16 with $q\in\Theta_{-2}$. Let us consider the set $\mathcal{C}_{-2}^*=\mathcal{C}_{-2}\cup\mathbb{S}_0^{-1}(\mathcal{C}_{-2})=\overline{P_{-2,1}W}$, see Figure 3.9. Let us take $\mathcal{B}^* = B(q^*, r)$ the symmetric of $\mathcal{B}$ with respect to $\mathcal{C}^*_{-2}$. Then:

1) $q^* = \mathbb{S}_2(q) \in \mathcal{A}_3 = \mathcal{R}_t \setminus \bigcup_{j=0}^{2} \Theta_{-j}$.

2) $\mathcal{B}^* \cap \mathcal{C}_0 \neq \emptyset$.

3) For every $j > 2$, $\Lambda_t^j(\mathcal{B}) = \Lambda_t^j(\mathcal{B}^*)$.

We can distinguish between the following cases:

i) If q^* belongs to $\mathcal{A}_6 = \mathcal{R}_t \setminus \bigcup_{j=0}^{5} \Theta_{-j}$ then we may apply Lemma 3.26 to conclude that, $\Lambda_t^j(\mathcal{B}^*) = \mathcal{R}_t$, for some $j \in \mathbb{N}$. Therefore it also holds that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

ii) If q^* belongs to Θ_{-5} we apply Proposition 3.28 to get either some $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^*) = \mathcal{R}_t$ (in this case $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$ also holds) or a ball $\widehat{\mathcal{B}} \subset \mathcal{B}^* \cap \Theta_{-5}$ ($\Lambda_t^j(\widehat{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, \ldots, 5$) such that $\Lambda_t^6(\widehat{\mathcal{B}})$ is a ball with radius greater than r. So, if we define $\widetilde{\mathcal{B}} = \mathbb{S}_2^{-1}(\widehat{\mathcal{B}}) \subset \mathcal{B}$, then from Lemma 3.13, we have $\Lambda_t^j(\widehat{\mathcal{B}}) = \Lambda_t^j(\widetilde{\mathcal{B}})$, for every $j > 2$ and therefore Proposition 3.16 is also proved in this case.

iii) Finally, if q^* belongs to Θ_{-3}, then we apply either Proposition 3.30 (if $K_{-3} \in \mathcal{B}^*$) or Proposition 3.31 (if $K_{-3} \notin \mathcal{B}^*$). If $K_{-3} \in \mathcal{B}^*$, then we get either $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^*) = \Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$ or a ball $\widehat{\mathcal{B}} \subset \mathcal{B}^* \cap \Delta_{-3}$ ($\Lambda_t^j(\widehat{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, \ldots, 5$) such that $\Lambda_t^6(\widehat{\mathcal{B}})$ is a ball with radius greater than r. Defining $\widetilde{\mathcal{B}} = \mathbb{S}_2^{-1}(\widehat{\mathcal{B}})$, we have done.

   If $K_{-3} \notin \mathcal{B}^*$ then we get either $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^*) = \Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$ or a ball $\widehat{\mathcal{B}} \subset \mathcal{B}^* \cap \Theta_{-3}$ ($\Lambda_t^j(\widehat{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, \ldots, 3$) such that $\Lambda_t^4(\widehat{\mathcal{B}})$ is a ball with radius greater than r. Once again, if we define $\widetilde{\mathcal{B}} = \mathbb{S}_2^{-1}(\widehat{\mathcal{B}})$, then we have $\Lambda_t^j(\widehat{\mathcal{B}}) = \Lambda_t^j(\widetilde{\mathcal{B}})$, for every $j > 2$ and therefore Proposition 3.16 is also proved in this case.

Hence, assuming we have proved Lemma 3.27 and Lemma 3.29, we have ended the proof of Proposition 3.16.

3.4.2 Proof of Lemma 3.29

Recall that Δ is the isosceles right triangle contained in Θ_0 with vertices $K = P_{0,3}$, $Z = P_{-2,0}$ and W, being W the intersection between $\mathcal{C}_3$ and $\mathbb{S}_0^{-1}(\mathcal{C}_{-2})$, see Figure 3.9. We consider the point H in $\mathcal{C}_0$ given by $H = \mathbb{S}_2^{-1}(K_{-3})$, see Figure 3.11. This point always exists for $t \in (t_1, 1)$ with $t_1 \approx 0.8326$ (for $t = t_1$ one has that $P_{2,5}$ and $P_{-1,2}$ are symmetric with respect to $\mathcal{C}_0$, see Figure 3.7). In fact, when $t = t_1$ we have $H = K$ (that is K and K_{-3} are symmetric with respect to $\mathcal{C}_{-2}$), and when $t = 1$ it follows that $H = Z$ because the lines $\mathcal{C}_{-2}$ and $\mathcal{C}_{-3}$ intersect at the point $Z = P_{-2,0} = K_{-3}$.

Let us take the straight line passing through H with slope -1 and define F the intersection between this line and $\mathcal{C}_3$. The triangle whose vertices are K, H and F is denoted by Δ'. Observe that $\Delta' \subset \Delta$.

LEMMA **3.32** *For every $t \in (t_1, 1)$ there exists a three-periodic point Q^3 of Λ_t in Δ'.*

Proof. Let us describe how Λ_t^3 acts on the vertices of Δ'. The second image of K is the point $K_2 = P_{2,5}$ given by

$$K_2 = (2t - 2t^2 + 8t^4(1-t), 2t(1-t)),$$

(see Figure 3.5). Therefore, applying the expression of Λ_t given at (3.5) one has that $K_3 = \Lambda_t^3(K)$ is the point in $\mathcal{C}_3$ given by

$$K_3 = (4t^2(1 - t + 2t^3 - 2t^4), 8t^5(1-t)).$$

Now, since $H \in \mathbb{S}_2^{-1}(\mathcal{C}_{-3}) \cap \mathcal{C}_0$ one has $H_3 = \Lambda_t^3(H) \in \mathcal{C}_0 \cap \mathcal{C}_3$. Therefore, $H_3 = K$. Finally, since $F \in \mathbb{S}_0^{-1}(\mathbb{S}_2^{-1}(\mathcal{C}_{-3}))$ we conclude $F_3 = \Lambda_t^3(F) \in \mathcal{C}_0$. But, using that Λ_t^3 is not only linear on Δ' but also preserves angles, we know that $\Lambda_t^3(\Delta')$ must be an isosceles right triangle and therefore the ordinate of F_3 must coincide with the ordinate of K_3. So, $F_3 = \Lambda_t^3(F) = (1, 8t^5(1-t))$. Observe that, for $t = 1$, $\Lambda_t^3(\Delta')$ is the triangle $\mathcal{T}_0$ with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Moreover,

$$\begin{aligned} dist(K, F_3) &= dist(H_3, F_3) = (\sqrt{2}t)^3 dist(H, F) = (\sqrt{2}t)^3 \frac{1}{\sqrt{2}} dist(K, H) > \\ &> dist(K, H) \end{aligned}$$

for every $t \in (\sqrt[3]{\frac{1}{2}}, 1)$. Hence $\Delta' \subset \Lambda_t^3(\Delta')$ and therefore there exists a three-periodic point, that we can denote by Q^3, of Λ_t in Δ'. ■

An easy numerical calculation shows that

$$Q^3 = (x_{Q^3}, y_{Q^3}) = \left(\frac{4t^2(1+2t^3)}{1+4t^3+8t^6}, \frac{8t^5}{1+4t^3+8t^6}\right). \qquad (3.13)$$

Now, Lemma 3.29 follows from the following result:

LEMMA **3.33** *For every $t \in (t_0, 1)$, there exists a set $\mathcal{U} \subset \Delta$ satisfying the following properties:*

i) If $\mathcal{B} = B(q, r)$ is a ball with $q \in \Delta$ such that $K = P_{0,3} \in \mathcal{B}$ and $\mathcal{B} \cap \overline{ZW} = \mathcal{B} \cap \overline{ZW} \neq \emptyset$, then $\mathcal{U} \subset \mathcal{B}$.

ii) There exists $m \in \mathbb{N}$ and a set Δ'_{-m} such that $Q^3 \in \Delta'_{-m} \subset \mathcal{U}$ and $\Lambda_t^m(\Delta'_{-m}) = \Delta'$.

iii) There exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{U}) = \mathcal{R}_t$.

Proof. Let us begin by constructing the set $\mathcal{U}$. Let us take again the triangle Δ with vertices K, Z and W. For the sake of simplicity, we can make a linear change in coordinates $(X, Y) = L_t(x, y)$, depending on the parameter t, in order to transport the point W into the origin, the point K into $(1, 1)$ and the point Z into $(1, 0)$. So, the triangle Δ becomes into the triangle $\mathcal{T}_0$ with vertices $(0, 0)$, $(1, 1)$ and $(1, 0)$. To this end, it is enough to denote by $l = l_t = dist(W, Z)$ and define

$$(X, Y) = L_t(x, y) = (l^{-1}(x - x_W), l^{-1}(y - y_W)), \tag{3.14}$$

being $W = (x_W, y_W)$. In this way, any ball $\mathcal{B} = B(q, r)$ satisfying $q \in \Delta$, $K \in \mathcal{B}$ and $\mathcal{B} \cap \overline{ZW} \neq \emptyset$ is mapping by L_t into a ball $\mathcal{B}' = B(q', r')$, satisfying $q' \in \mathcal{T}_0$, $A' = (1, 1) \in \mathcal{B}'$ and $\mathcal{B}' \cap \widetilde{Y} \neq \emptyset$, being

$$\widetilde{Y} = \{(X, Y) \in \mathbb{R}^2 : Y = 0\}.$$

One may easily deduce that any such ball $\mathcal{B}'$ contains the polygonal domain $\mathcal{U}'$ whose vertices are $G' = (\frac{1}{2}, \frac{1}{2})$, $R' = (\frac{13}{16}, \frac{3}{16})$, $S' = (1, \frac{3}{8})$ and $A' = (1, 1)$. This last claim follows by bearing in mind that the set $\mathcal{U}'$ is convex and therefore it is enough to check that for any such ball $\mathcal{B}'$, $\Omega' \in \mathcal{B}'$ for every $\Omega' \in \{G', R', S', A'\}$. But this fact can be showed by using that the two sets

$$\{(X, Y) : dist((X, Y), (1, 1)) < dist((X, Y), \Omega')\}$$
$$\{(X, Y) : dist((X, Y), \widetilde{Y}) < dist((X, Y), \Omega')\}$$

are disjoint for every $\Omega' \in \{G', R', S', A'\}$. Hence, defining $\mathcal{U} = L_t^{-1}(\mathcal{U}') \subset \Delta$, the first statement of the lemma follows.

The vertices of the domain $\mathcal{U}$ are denoted by $G = L_t^{-1}(G')$ (this is a point in $\mathcal{C}_3$), $R = L_t^{-1}(R')$ (this is a point in the straight segment $\overline{GZ}$), $S = L_t^{-1}(S')$ (this is a point in $\overline{ZK}$) and $K = P_{0,3} = L_t^{-1}(A')$. The shape of this polygonal domain can be seen in Figure 3.11.

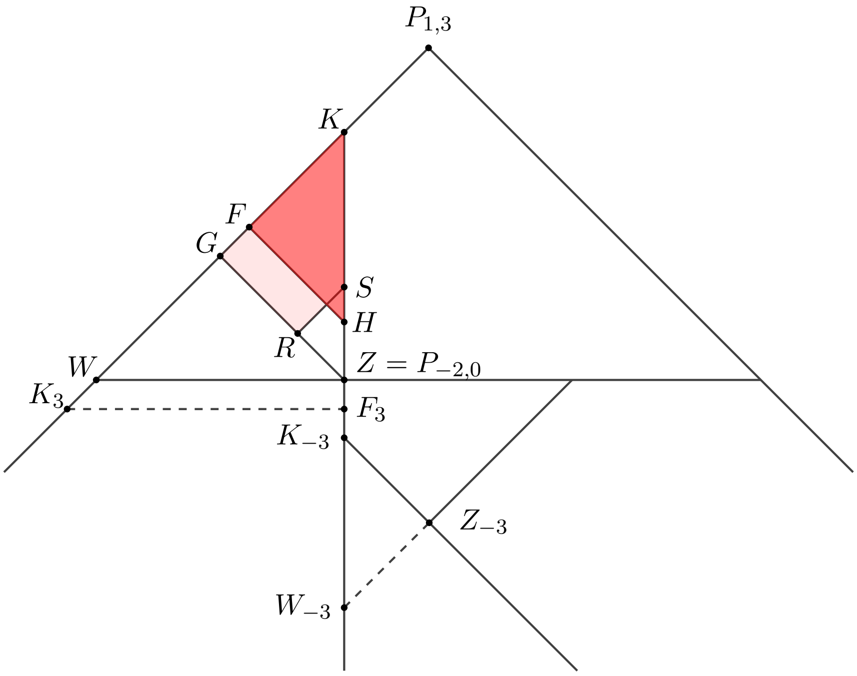


Figure 3.11: The case $F_3 \notin \Delta$

Now, let us show the second statement of the lemma. Recall that $Q^3 \in \Delta'$ and thus in order to prove that $Q^3 \in \mathcal{U}$ it is enough to check that, denoting by $\mathcal{L}_{RS} = \{(x, y) \in \mathbb{R}^2 : y - x = \alpha\}$ the straight line passing through R and S, then $y_{Q^3} - x_{Q^3} > \alpha$. But, taking into account equation (3.13) and the fact that

$$\alpha = \frac{-5 + 10t - 10t^2 - 24t^4 + 24t^5}{16t^2}$$

then, one may check that $y_{Q^3} - x_{Q^3} > \alpha$, for every $t > t_1 \approx 0.8326$ (the value of t for which the periodic point Q^3 arises). Observe that we have proved that $Q^3 \in int(\mathcal{U})$. From the fact that Q^3 is a repelling periodic orbit and $\Lambda^3_{t|\Delta'}$ is linear, there is $m \in \mathbb{N}$ large enough and a neighbourhood Δ'_{-m} of Q^3 such that $\Delta'_{-m} \subset \mathcal{U}$ and $\Lambda^m_t(\Delta'_{-m}) = \Delta'$. Therefore, the second statement of the Lemma is proved.

To prove the third statement, let us recall that Δ is the triangle with vertices K, Z and W and $\Lambda^3_t(\Delta')$ is the triangle with vertices $K = H_3$, $F_3 = (1, 8t^5(1-t))$ and K_3. Let us distinguish between the following cases:

i) Suppose that $F_3 \notin \Delta$ (or, equivalently, that $\Delta \subset \Lambda^3_t(\Delta')$) (this situation is showed in Figure 3.11). This fact takes place when the ordinate of the point F_3 is smaller than the ordinate of the point Z. By using $Z_2 = \Lambda^2_t(Z) = P_{0,2} =$

$(1, 2t(1-t))$ we have $Z = (1, \frac{2t-1}{2t^2})$. Then, $F_3 \notin \Delta$ when

$$8t^5(1-t) < \frac{2t-1}{2t^2}.$$

An easy calculation shows that the above inequality holds if $t \in (t^*, 1)$ with $t^* \approx 0.8894$. Then, if $t \in (t^*, 1)$ third statement easily follows by taking into account that $\Theta_{-2} \subset \Lambda_t^6(\Delta')$. This can be checked by observing that, in fact, $\Theta_{-2} \subset \Lambda_t^3(\Delta)$. This last inclusion can be obtained from the facts that $\Lambda_t^3(Z) = P_{1,3}$ and $\Lambda_t^3(\Delta)$ is an isosceles right triangle ($\Lambda_{t|\Delta}^3$ is linear and preserves angles) with one of its vertices coinciding with $P_{1,3}$ and the ordinate of $K_3 = \Lambda_t^3(K)$ smaller than the ordinate of Z. Since, from the third statement of Lemma 3.13, $P_2^\star \in \Theta_{-2}$ and $\Lambda_t^3(P_2^\star) = P_t$ we have $P \in \Lambda_t^6(\Delta) \subset \Lambda_t^9(\Delta') = \Lambda_t^{9+m}(\Delta'_{-m}) \subset \Lambda_t^{9+m}(\mathcal{U})$. Thus using the basin of neighbourhoods of P_t given at (3.8), $\mathcal{A}_k = \mathcal{R}_t \setminus \bigcup_{j=0}^{k-1} \Theta_{-k}$, we get some $k \in \mathbb{N}$ with $\mathcal{A}_k \subset \Lambda_t^{9+m}(\mathcal{U})$.

Hence, for $j = k + m + 9$ we obtain $\Lambda_t^j(\mathcal{U}) = \mathcal{R}_t$.

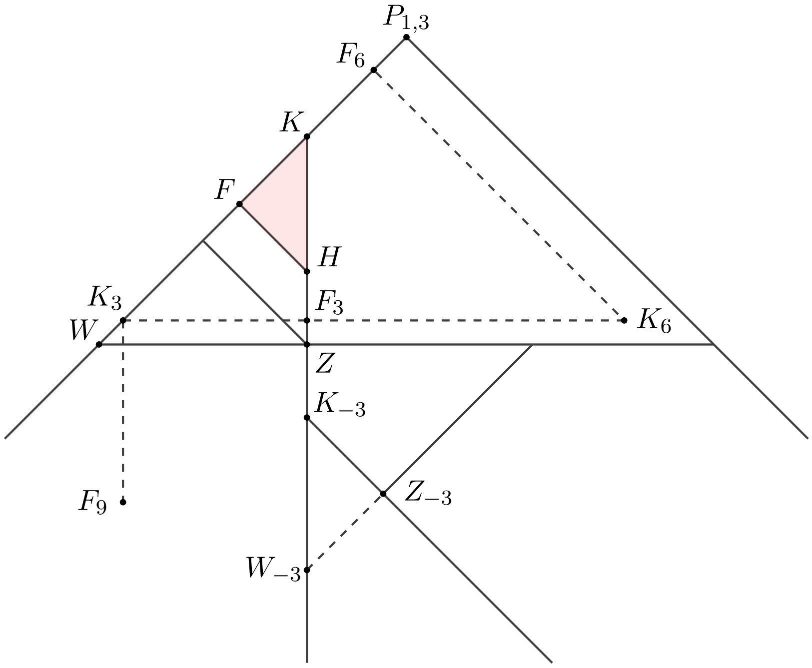


Figure 3.12: The case $F_3 \in \Delta$

ii) Suppose $F_3 \in \Delta$ or, equivalently, $t_0 < t < t^*$ (this case is showed in Figure 3.12). Let us consider $d = dist(H, Z)$. Let us observe that, by definition of Z the distance d coincides with $dist(Z, K_{-3})$. Recall that $Z =$

$(1, \frac{2t-1}{2t^2})$ and, moreover, since the image of K_{-3} belongs to $\mathcal{C}_{-2}$, one easily has $K_{-3} = (1, \frac{2t^3-2t+1}{2t^3})$ and therefore

$$d = \frac{2t-1}{2t^2} - \frac{2t^3-2t+1}{2t^3} = \frac{-2t^3+2t^2+t-1}{2t^3}.$$

Let us also consider $d_1 = dist(H, F_3)$. In order to compute d_1, let us write

$$H = (1, \frac{2t-1}{2t^2} + d) = (1, \frac{-2t^3+4t^2-1}{2t^3}).$$

Since $F_3 = (1, 8t^5(1-t))$, we conclude that

$$d_1 = \frac{-2t^3+4t^2-1}{2t^2} - 8t^5(1-t) = \frac{16t^9-16t^8-2t^3+4t^2-1}{2t^3}.$$

Now, observe that F_6 is a point in $\mathcal{C}_3$ with $dist(F_6, K) = (\sqrt{2}t)^3 dist(F_3, H) = (\sqrt{2}t)^3 d_1$. Moreover, one may check that F_9 is a point in the same vertical line that of K_3 with $dist(F_9, K_3) = (\sqrt{2}t)^3 dist(F_6, K) = (\sqrt{2}t)^6 d_1 = 8t^6 d_1$. Then, the result follows if we prove that $d_1 + 8t^6 d_1 = d_1(1+8t^6) > d$. This holds for every $t \in (t_2, 1)$ with $t_2 \approx 0.85$. This means that for every $t \in (t_0, 1)$, $t_0 \approx 0.882$, the point $Z = P_{-2,0}$ belongs to $\Lambda_t^9(\Delta')$, so $P_{1,3}$ belongs to $\Lambda_t^{12}(\Delta')$ and hence $\Theta_{-2} \subset \Lambda_t^{12}(\Delta')$. Therefore, also using the second statement of this lemma, $P_t \in \Lambda_t^{15}(\Delta') = \Lambda_t^{15+m}(\Delta'_{-m}) \subset \Lambda_t^{15+m}(\mathcal{U})$ and we get again $\Lambda_t^j(\mathcal{U}) = \mathcal{R}_t$, with $j = 15+m+k$ (k large enough so that $\mathcal{A}_k \subset \Lambda_t^{15+m}(\mathcal{U})$). ■

3.4.3 Proof of Lemma 3.27

We will use the three-periodic point $Q^3 \in \Delta$ constructed during the proof of Lemma 3.29. Let us recall that

$$Q^3 = (x_{Q^3}, y_{Q^3}) = (\frac{4t^2(1+2t^3)}{1+4t^3+8t^6}, \frac{8t^5}{1+4t^3+8t^6}).$$

If we denote by $Q_2^3 = \Lambda_t^2(Q^3)$, it is easy to see that $Q_2^3 \in \Theta_0$ and

$$Q_2^3 = (x_{Q_2^3}, y_{Q_2^3}) = (\frac{2t(1+4t^2)}{1+4t^3+8t^6}, \frac{2t}{1+4t^3+8t^6}). \qquad (3.15)$$

We also compute the lines $\mathcal{L}_{\mathcal{C}_k}$ such that $\mathcal{C}_k \subset \mathcal{L}_{\mathcal{C}_k}$ for $k = 2, 3, 5$, which are given by

$$\begin{aligned} \mathcal{L}_{\mathcal{C}_2} &= \{(x,y) \in \mathbb{R}^2 : y = 2t(1-t)\} \\ \mathcal{L}_{\mathcal{C}_3} &= \{(x,y) \in \mathbb{R}^2 : y = x - 4t^2 + 4t^3\} \\ \mathcal{L}_{\mathcal{C}_5} &= \{(x,y) \in \mathbb{R}^2 : y + x = 4t - 4t^2 + 8t^4 - 8t^5\} \qquad (3.16) \end{aligned}$$

First statement of Lemma 3.27 will be proved if we check that for every $t \in (t_0, 1)$, any ball $\mathcal{B} = B(q,r) \subset \mathbb{R}^2$ with $q \in \Theta_0$ and $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$, contains either $P_0^\star$ or Q_2^3. This claim follows by recalling that any sufficiently small neighbourhood of $P_0^\star$ (respectively, Q_2^3) is sent by Λ_t (respectively, a convenient power of Λ_t) into some neighbourhood of the fixed point P_t containing, for large enough k, some $\mathcal{A}_k$ that verifies $\Lambda_t^k(\mathcal{A}_k) = \mathcal{R}_t$ (the sets $\mathcal{A}_k$ where defined in (3.8)).

Let us recall that

$$P_0^\star = (x_{P_0^\star}, y_{P_0^\star}) = \left(\frac{2t+2}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1} \right).$$

We can divide $\Theta_0 = \Theta_0^+ \cup \Theta_0^-$ with

$$\Theta_0^+ = \{(x,y) \in \Theta_0 : y \geq y_{P_0^\star}\} \tag{3.17}$$

and $\Theta_0^- = \Theta_0 \setminus \Theta_0^+$.

Lemma **3.34** *Let $\mathcal{B} = B(q,r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0^+$ and $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$, then $P_0^\star \in \mathcal{B}$ for all $t \in (t_0, 1)$.*

Proof. It will be enough to check that $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_2)$ whenever $q = (x_q, y_q) \in \Theta_0^+$. In fact, it suffices to prove this claim for $y_q = y_{P_0^\star}$. Hence, denote by Q_1 and by Q_2 the intersections between the line $y = y_{P_0^\star}$ and the sets $\mathcal{C}_0$ and $\mathcal{C}_3$ respectively, see Figure 3.14. It suffices to demonstrate that

$$dist(Q_i, P_0^\star) \leq dist(Q_i, \mathcal{C}_2)\ ,\ i = 1, 2.$$

Recall that the segment $\mathcal{C}_2$ is contained in the line $\mathcal{L}_{\mathcal{C}_2}$ (see (3.16)). Since $Q_1 = (1, y_{P_0^\star})$,

$$dist(Q_1, P_0^\star) \leq dist(Q_1, \mathcal{C}_2) \Leftrightarrow 1 - x_{P_0^\star} \leq y_{P_0^\star} - 2t(1-t)$$

and it holds for every $t \in [0,1]$.

On the other hand, since $\mathcal{C}_3$ is contained in the line $\mathcal{L}_{\mathcal{C}_3}$ (see (3.16)) it is easy to see that Q_2 is given by

$$Q_2 = (y_{P_0^\star} + 4t^2 - 4t^3, y_{P_0^\star}).$$

Hence, one has

$$dist(Q_2, P_0^\star) \leq dist(Q_2, \mathcal{C}_2) \Leftrightarrow x_{P_0^\star} - y_{P_0^\star} - 4t^2 + 4t^3 \leq y_{P_0^\star} - 2t(1-t)$$

that holds for every $t \in (\frac{1}{\sqrt{2}}, 1)$. ■

From now on, we will consider $b = \sqrt{2}+1$. It will be very useful to define the following segments in Θ_0 :

$$\begin{aligned} \mathcal{I}_1 &= \{Q \in \Theta_0 : \ dist(Q,\mathcal{C}_0) = dist(Q,\mathcal{C}_3)\} \\ \mathcal{I}_2 &= \{Q \in \Theta_0 : \ dist(Q,\mathcal{C}_0) = dist(Q,\mathcal{C}_2)\} \\ \mathcal{I}_3 &= \{Q \in \Theta_0 : \ dist(Q,\mathcal{C}_2) = dist(Q,\mathcal{C}_5)\} \\ \mathcal{I}_4 &= \{Q \in \Theta_0 : \ dist(Q,\mathcal{C}_3) = dist(Q,\mathcal{C}_5)\} \end{aligned} \tag{3.18}$$

We will also denote by $\mathcal{L}_{\mathcal{I}_k}$ the straight line containing $\mathcal{I}_k$ for $k = 1, 2, 3, 4$. These lines are given by

$$\begin{aligned} \mathcal{L}_{\mathcal{I}_1} &= \{(x,y) \in \mathbb{R}^2 : y = bx - \sqrt{2} - 4t^2 + 4t^3\} \\ \mathcal{L}_{\mathcal{I}_2} &= \{(x,y) \in \mathbb{R}^2 : y + x = 1 + 2t - 2t^2\} \\ \mathcal{L}_{\mathcal{I}_3} &= \{(x,y) \in \mathbb{R}^2 : y - 2t + 2t^2 = b(x-h)\} \\ \mathcal{L}_{\mathcal{I}_4} &= \{(x,y) \in \mathbb{R}^2 : y = 2t - 4t^2 + 2t^3 + 4t^4(1-t)\} \end{aligned} \tag{3.19}$$

where we have introduced

$$h = h(t) = 2t - 2t^2 + 8t^4 - 8t^5.$$

It is evident that $K = P_{0,3}$ is an end point of $\mathcal{I}_1$. If we denote the other one by R_1, one can write $\mathcal{I}_1 = \overline{P_{0,3}R_1}$ and it holds that $R_1 \in \mathcal{C}_2$ for all $t \in (t_0, 1)$ being

$$R_1 = (x_{R_1}, y_{R_1}) = \left(\frac{\sqrt{2} + 2t + 2t^2 - 4t^3}{b}, 2t(1-t) \right)$$

(see Lemma 3.35). In the same way, $\mathcal{I}_2 = \overline{P_{0,2}R_2}$ where R_2 belongs to $\mathcal{C}_3$ and

$$R_2 = (x_{R_2}, y_{R_2}) = \left(\frac{1 + 2t + 2t^2 - 4t^3}{2}, \frac{1 + 2t - 6t^2 + 4t^3}{2} \right). \tag{3.20}$$

On the other hand, we can consider $\mathcal{I}_3 = \overline{P_{2,5}R_3}$. Since $\mathcal{I}_1$ and $\mathcal{I}_3$ have the same slope (equal to b), $R_1 \in \mathcal{C}_2$ implies that $R_3 \in \mathcal{C}_3$ and we conclude that

$$R_3 = (x_{R_3}, y_{R_3}) = \left(\frac{4t^3 - 2t^2 - 2t + bh}{\sqrt{2}}, \frac{4t^3 - 2t^2 - 2t + bh}{\sqrt{2}} + 4t^3 - 4t^2 \right). \tag{3.21}$$

Finally, let us take $\mathcal{I}_4 = \overline{P_{3,5}R_4}$. Since $\mathcal{I}_4$ is horizontal, R_4 is the point in $\mathcal{C}_0$ given by

$$R_4 = \big(1, 2t - 4t^2 + 2t^3 + 4t^4(1-t)\big) \tag{3.22}$$

(recall that $P_{3,5} = M_3 = (2t - 2t^3 + 4t^4(1-t), 2t - 4t^2 + 2t^3 + 4t^4(1-t))$).

All these segments and points are shown in Figure 3.13.

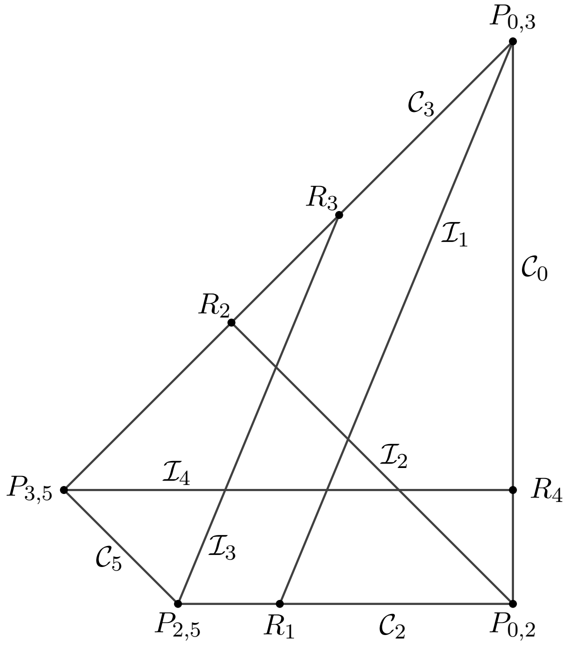


Figure 3.13: The segments $\mathcal{I}_k,\ k = 1, 2, 3, 4$

LEMMA **3.35** *For any* $t \in (t_0, 1)$, $R_1 \in \mathcal{C}_2$.

Proof. By definition, it holds that $\mathcal{L}_{\mathcal{I}_1}$ intersects $\mathcal{L}_{\mathcal{C}_2}$ at the point

$$\left(\frac{\sqrt{2} + 2t + 2t^2 - 4t^3}{b}, 2t(1-t)\right)$$

and $\mathcal{L}_{\mathcal{I}_1}$ intersects $\mathcal{L}_{\mathcal{C}_5}$ at the point

$$\left(\frac{\sqrt{2} + 4t - 4t^3 + 8t^4 - 8t^5}{1+b}, y_0\right)$$

(the value of y_0 is not interesting). Then, for every $t \in (0.8576, 1)$, one may check that

$$\frac{\sqrt{2} + 4t - 4t^3 + 8t^4 - 8t^5}{1+b} < \frac{\sqrt{2} + 2t + 2t^2 - 4t^3}{b}$$

so we conclude that

$$R_1 = (x_{R_1}, y_{R_1}) = \left(\frac{\sqrt{2} + 2t + 2t^2 - 4t^3}{b}, 2t(1-t)\right) \in \mathcal{C}_2$$

for every $t \in (t_0, 1)$. ∎

Now, let us consider the parabolas

$$\mathcal{V}_{0,P_0^\star} = \{Q \in \mathbb{R}^2 : dist(Q, P_0^\star) = dist(Q, \mathcal{C}_0)\}$$
$$\mathcal{V}_{3,P_0^\star} = \{Q \in \mathbb{R}^2 : dist(Q, P_0^\star) = dist(Q, \mathcal{C}_3)\}$$

(see Figure 3.14). Then, $\mathcal{V}_{0,P_0^\star}$ and $\mathcal{V}_{3,P_0^\star}$ intersect in two points and we will denote by $V = (x_V, y_V)$ the one closer to $\mathcal{C}_2$ (the ordinate of V is smaller than the ordinate of $P_0^\star$). In fact, one may check that, since

$$dist(R_1, \mathcal{C}_0) = 1 - x_{R_1} < dist(R_1, P_0^\star)$$

for every $t \in (\frac{1}{\sqrt{2}}, 1)$, then for every $t \in (t_0, 1)$ this point V unfortunately always belongs to Θ_0 and henceforth it also follows that $V \in \mathcal{I}_1$. Moreover, if we define $A_1 = (x_{A_1}, y_{A_1})$ the intersection between $\mathcal{V}_{3,P_0^\star}$ and $\mathcal{C}_2$ (the fact that this point belongs to $\mathcal{C}_2$ follows from Lemma 3.35) and A_2 the point where $\mathcal{V}_{0,P_0^\star}$ intersects with the boundary of Θ_0 (it does not matter if this point belongs to $\mathcal{C}_2$ or $\mathcal{C}_5$) then the points V, A_1 and A_2 delimit a region $\mathcal{Z}$ in Θ_0 (lower shading region in Figure 3.14) such that, if $Q \in \mathcal{Z}$ then

$$dist(Q, P_0^\star) > dist(Q, \mathcal{C}_0)\ ,\ dist(Q, P_0^\star) > dist(Q, \mathcal{C}_3).$$

Of course, there exists another region $\mathcal{Z}' \subset \Theta_0$ (containing $K = P_{0,3}$) formed by points that also satisfy these two conditions. Nevertheless, the first statement of Lemma 3.27 is proved when the center of the ball $\mathcal{B}$ belongs to this region $\mathcal{Z}'$. This claim follows from Lemma 3.34 and the following result.

LEMMA **3.36** *For every $t \in (t_0, 1)$ it holds that $\mathcal{Z}' \subset \Theta_0^+$.*

Proof. We denote by $A_3 = (x_{A_3}, y_{A_3})$ the intersection between $\mathcal{C}_0$ and $\mathcal{V}_{3,P_0^\star}$. Then, in order to prove that $\mathcal{Z}'$ is contained in the set Θ_0^+ (see (3.17)) it is enough to prove that $y_{A_3} > y_{P_0^\star}$. This fact follows if we prove that

$$dist(Q_1, P_0^\star) < dist(Q_1, \mathcal{C}_3),$$

being $Q_1 = (1, y_{P_0^\star})$. Using that $\mathcal{C}_3$ is contained in $\mathcal{L}_{\mathcal{C}_3}$ (see (3.16)) one has

$$dist^2((x, y), \mathcal{C}_3) = \frac{1}{2}(y - x + 4t^2 - 4t^3)^2 \tag{3.23}$$

for every $(x, y) \in \mathbb{R}^2$.

Then, since

$$dist^2(Q_1, P_0^\star) = (x_{P_0^\star} - 1)^2 < \frac{1}{2}(y_{Q_1} - x_{Q_1} + 4t^2 - 4t^3)^2 = dist^2(Q_1, \mathcal{C}_3),$$

holds for every $t \in (0.845, 1)$ the lemma follows. ∎

Therefore, the first statement of Lemma 3.27 is now consequence of the following result.

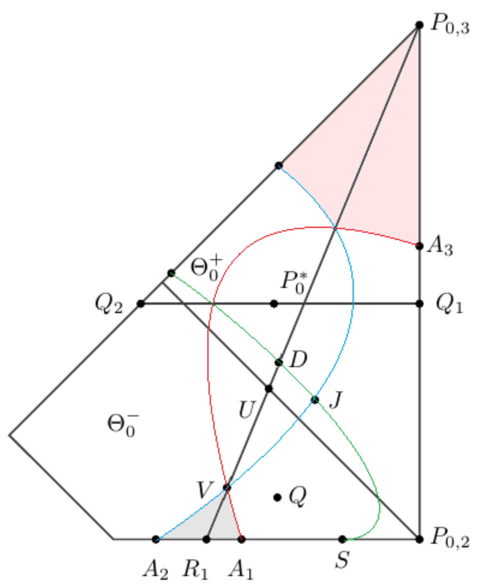


Figure 3.14: Regions $\mathcal{Z}$ and $\mathcal{Z}'$

Lemma **3.37** *Let $\mathcal{B} = B(q,r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0^-$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$. It holds*

1. *If $q \notin \mathcal{Z}$ then $P_0^\star \in \mathcal{B}$.*
2. *If $q \in \mathcal{Z}$ then $Q_2^3 \in \mathcal{B}$.*

Proof. If we assume that $q \notin \mathcal{Z}$, this assumption directly implies that either $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_0)$ or $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_3)$. Then $P_0^\star \in \mathcal{B}$.

Let us now assume that $q \in \mathcal{Z}$. Let us consider the new parabola:

$$\mathcal{V}_{0,Q_2^3} = \{Q \in \mathbb{R}^2 : dist(Q, Q_2^3) = dist(Q, \mathcal{C}_0)\}.$$

We remark that the lemma will be proved if we demonstrate that

$$\mathcal{Z} \subset \{Q \in \mathbb{R}^2 : dist(Q, Q_2^3) < dist(Q, \mathcal{C}_0)\}$$

because in this case any ball in the hypotheses of the lemma (with $q \in \mathcal{Z}$) must contains Q_2^3. Let us therefore consider $S = (x_S, y_S)$ the intersection between $\mathcal{V}_{0,Q_2^3}$ and $\mathcal{C}_2$ and $D = (x_D, y_D)$ the intersection between $\mathcal{V}_{0,Q_2^3}$ and $\mathcal{I}_1$. Then the lemma follows if we prove the following claims (see again Figure 3.14):

i) For every $t \in (t_0, 1)$ one has $x_S > x_{A_1}$.

ii) For every $t \in (t_0, 1)$ one has $y_D > y_V$.

To prove the first claim it is enough to check that, for every $t \in (t_0, 1)$:

$$dist(S, P_0^\star) < dist(S, \mathcal{C}_3), \tag{3.24}$$

and to prove the second claim it is enough to get, for every $t \in (t_0, 1)$, a point $U = (x_U, y_U) \in \mathcal{I}_1$ satisfying

$$\max\{dist(U, P_0^\star), dist(U, Q_2^3)\} < dist(U, \mathcal{C}_0) = dist(U, \mathcal{C}_3). \tag{3.25}$$

REMARK **3.38** *Denoting by J the intersection in Θ_0 between $\mathcal{V}_{0,P_0^\star}$ and $\mathcal{V}_{0,Q_2^3}$ then, if (3.25) holds then J necessarily satisfies $dist(J, \mathcal{C}_0) < dist(J, \mathcal{C}_3)$ and therefore one easily obtains $y_D > y_J > y_V$.*

Let us start by proving (3.24). To this end, we compute the equation of $\mathcal{V}_{0,Q_2^3}$ which is given by

$$2(1 - x_{Q_2^3})x + (y - y_{Q_2^3})^2 + x_Q^2 = 1.$$

Then, also using that any point in $\mathcal{C}_2$ is contained in the line $\mathcal{L}_{\mathcal{C}_2}$ (see (3.16)), one gets

$$S = (x_S, y_S) = \left(\frac{1 - x_Q^2 - (2t(1-t) - y_{Q_2^3})^2}{2(1 - x_{Q_2^3})}, 2t(1-t) \right). \tag{3.26}$$

Now, from (3.23), the inequality given at (3.24) follows by observing that for every $t \in (t_1, 1)$ ($t_1 \approx 0.8326$ the value of t for which the periodic point Q_2^3 arises), it holds that

$$dist^2(S, P_0^\star) = (x_S - 1)^2 < \frac{1}{2}(y_S - x_S + 4t^2 - 4t^3)^2 = dist^2(S, \mathcal{C}_3).$$

To prove (3.25), let $U = (x_U, y_U)$ be the intersection between $\mathcal{L}_{\mathcal{I}_1}$ and $\mathcal{L}_{\mathcal{I}_2}$ (see 3.19)). We obtain (recall that $b = \sqrt{2} + 1$)

$$U = (x_U, y_U) = \left(\frac{b + 2t + 2t^2 - 4t^3}{1 + b}, 1 + 2t - 2t^2 - \frac{b + 2t + 2t^2 - 4t^3}{1 + b} \right).$$

Then the inequality given at (3.25) holds for $t \in (0.833, 1)$.

Therefore the result is proved ■

To demonstrate the second statement of Lemma 3.27, we will need the next result:

LEMMA **3.39** *Let $\mathcal{B} = B(q,r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0$ and $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$. We will consider $\widetilde{t} \approx 0.928$. It holds:*

i) If $t \in (\widetilde{t}, 1)$ then $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$.

ii) If $t \in (t_0, \widetilde{t}\,]$ then either $\mathcal{B} \cap \mathcal{C}_3 \neq \emptyset$ or $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$.

Proof. Along the proof it will be very useful Figure 3.15. Let us recall the segments $\mathcal{I}_1 = \overline{KR_1} \subset \mathcal{L}_{\mathcal{C}_1}$, $\mathcal{I}_2 = \overline{P_{0,2}R_2} \subset \mathcal{L}_{\mathcal{C}_2}$, $\mathcal{I}_3 = \overline{P_{2,5}R_3} \subset \mathcal{L}_{\mathcal{C}_3}$ and $\mathcal{I}_4 = \overline{P_{3,5}R_4} \subset \mathcal{L}_{\mathcal{C}_4}$ defined in (3.18).

From the equation of R_2 given at (3.20), we have that $x_{R_2} > x_{R_3}$ (and also $y_{R_2} > y_{R_3}$) for every $t \in (\widetilde{t}, 1)$ with $\widetilde{t} \approx 0.928$. This means that, if $t \in (\widetilde{t}, 1)$ there is no ball $\mathcal{B} = B(q,r)$ with $q \in \Theta_0$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$ for $i = 0, 5$ and $\mathcal{B} \cap \mathcal{C}_2 = \emptyset$. The first statement if proved.

To prove the second statement, let us assume that $t \in (t_0, \widetilde{t}\,]$. In this case $\mathcal{L}_{\mathcal{I}_2}$ and $\mathcal{L}_{\mathcal{I}_3}$ intersect in one point $R_{2,3} \in \Theta_0$ with

$$R_{2,3} = (x_{R_{2,3}}, y_{R_{2,3}}) = (\frac{1+bh}{b+1}, 1 + 2t - 2t^2 - \frac{1+bh}{b+1}). \tag{3.27}$$

Observe that, since $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$, if $\mathcal{B} \cap \mathcal{C}_2 = \emptyset$ then q must belong to the triangle $\mathcal{Z}_1$ with vertices R_2, R_3 and $R_{2,3}$, see Figure 3.15.

On the other hand, the segment $\mathcal{I}_1$ intersects $\mathcal{I}_4$ in a point $R_{1,4} = (x_{R_{1,4}}, y_{R_{1,4}})$ with

$$y_{R_{1,4}} = 2t - 4t^2 + 2t^3 + 4t^4(1-t).$$

Then, if some ball $\mathcal{B} = B(q,r)$ satisfies $q \in \Theta_0$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$, and $\mathcal{B} \cap \mathcal{C}_3 = \emptyset$ then q must belong to the polygonal region $\mathcal{Z}_2$ with vertices R_1, R_4, $R_{1,4}$ and $P_{0,2}$ (see Figure 3.15).

To conclude the proof of the lemma it is enough to check that $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$. This follows from the fact that the ordinate of the point $R_{2,3}$ is greater than the ordinate of the point $R_{1,4}$ for every $t \in (0.857, 1)$. ∎

Now, we may prove the second statement of Lemma 3.27. We will distinguish between two cases:

A) THE CASE $t \in (\widetilde{t}, 1)$

Let us assume that $t \in (\widetilde{t}, 1)$. Then from Lemma 3.39 we have that $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$. Therefore, we may suppose that $\mathcal{B} \cap \mathcal{C}_3 = \emptyset$, because if not we may apply the first statement of Lemma 3.27 to conclude. Hence, q must belong to $\mathcal{Z}_2$ (recall that this set exists for every $t \in (t_0, 1)$). We will

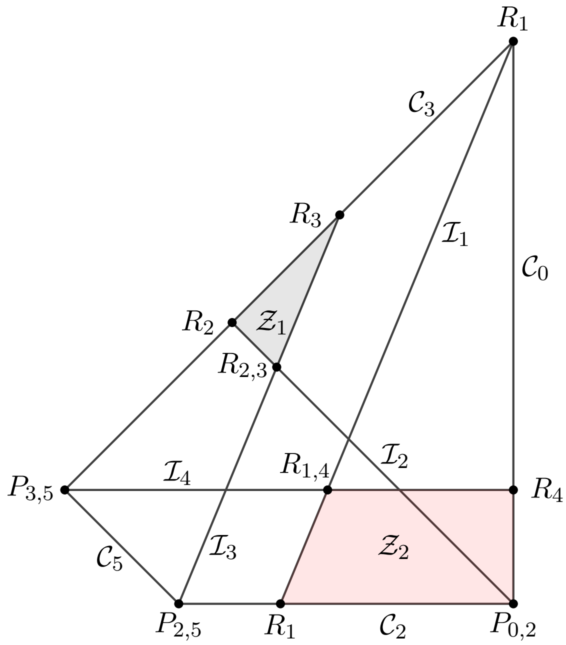


Figure 3.15: The case $t \in (t_0, \widetilde{t}\,]$

demonstrate that the three-periodic orbit Q_2^3 given at (3.15) must belong to $\mathcal{B}$. To this end, let us again use the parabola

$$\mathcal{V}_{0,Q_2^3} = \{Q \in \mathbb{R}^2 : dist(Q, Q_2^3) = dist(Q, \mathcal{C}_0)\}.$$

This parabola intersects $\mathcal{C}_2$ in the point S, see (3.26) and Figures 3.14 and 3.16. We also need to compute $\widetilde{S}$ the intersection between $\mathcal{V}_{0,Q_2^3}$ and $\mathcal{I}_4 \subset \mathcal{L}_{\mathcal{I}_4}$ (see (3.19)):

$$\widetilde{S} = (x_{\widetilde{S}}, y_{\widetilde{S}}) = (\frac{1 - x_Q^2 - (c - y_{Q_2^3})^2}{2(1 - x_{Q_2^3})}, c)$$

with $c = c(t) = 2t - 4t^2 + 2t^3 + 4t^4(1 - t)$.

Now, it is sufficient to demonstrate that if $Q \in \mathcal{Z}_2$ with $dist(Q, Q_2^3) > dist(Q, \mathcal{C}_0)$ then $dist(Q, Q_2^3) < dist(Q, \mathcal{L}_{\mathcal{C}_5})$, being $\mathcal{L}_{\mathcal{C}_5}$ the straight line containing $\mathcal{C}_5$ (see (3.16)). The region

$$\{Q \in \mathcal{Z}_2 : \ dist(Q, Q_2^3) > dist(Q, \mathcal{C}_0)\}$$

is the shading region in Figure 3.16. To this end, it suffices to check that

$$dist(Q, Q_2^3) < dist(Q, \mathcal{L}_{\mathcal{C}_5}) \text{ for } Q \in \{S, \widetilde{S}, P_{0,2}, R_4\} \tag{3.28}$$

This is because the set $\{Q \in \mathbb{R}^2 : \ dist(Q, Q_2^3) < dist(Q, \mathcal{L}_{\mathcal{C}_5})\}$ is convex. Hence, if $dist(Q, Q_2^3) < dist(Q, \mathcal{L}_{\mathcal{C}_5})$ for $Q \in \{S, \widetilde{S}, P_{0,2}, R_4\}$ then $dist(Q, Q_2^3) < dist(Q, \mathcal{L}_{\mathcal{C}_5})$ for every Q in the polygonal domain with vertices $S, \widetilde{S}, P_{0,2}$ and R_4. Since one easily gets

$$dist^2(Q, \mathcal{L}_{\mathcal{C}_5}) = \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_Q - y_Q)^2, \tag{3.29}$$

for every $Q = (x_Q, y_Q) \in \mathbb{R}^2$, we conclude

$$dist^2(S, Q_2^3) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_S - y_S)^2 = dist^2(S, \mathcal{L}_{\mathcal{C}_5}),$$

for every $t \in [0.846, 1)$,

$$dist^2(\widetilde{S}, Q_2^3) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{\widetilde{S}} - y_{\widetilde{S}})^2 = dist^2(\widetilde{S}, \mathcal{L}_{\mathcal{C}_5}),$$

for every $t \in [0.85, 1)$. Moreover, recalling that $P_{0,2} = (1, 2t(1-t))$,

$$dist^2(P_{0,2}, Q_2^3) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - 1 - y_{P_{0,2}})^2 = dist^2(P_{0,2}, \mathcal{L}_{\mathcal{C}_5}),$$

for every $t \in [0.858, 1)$ and finally

$$dist^2(R_4, Q_2^3) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{R_4} - y_{R_4})^2 = dist^2(R_4, \mathcal{L}_{\mathcal{C}_5}),$$

for every $t \in [0.833, 1)$.

So the condition (3.28) holds for all $t \in (\widetilde{t}, 1)$.

REMARK **3.40** *Observe that all the arguments used in the case $t \in (\widetilde{t}, 1)$ still remain valid for $t \in (t_0, 1)$ under the assumption $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$.*

B) THE CASE $t \in (t_0, \widetilde{t}\,]$

Let us assume that $t \in (t_0, \widetilde{t}\,]$. In this case applying Lemma 3.39 we have that either $\mathcal{B} \cap \mathcal{C}_3 \neq \emptyset$ or $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$. If $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$, then we may assume that $\mathcal{B} \cap \mathcal{C}_3 = \emptyset$ (in other case, the result will follow from the first statement of Lemma 3.27). Hence, we may repeat the arguments of the previous case (see Remark 3.40), because they do not depend on the value of $t \in (t_0, 1)$.

Hence let us assume that $t \in (t_0, \widetilde{t}\,]$ and $\mathcal{B} \cap \mathcal{C}_3 \neq \emptyset$. Therefore we may assume that $\mathcal{B} \cap \mathcal{C}_2 = \emptyset$. Then q must belong to the triangle $\mathcal{Z}_1$ with vertices R_2, R_3 and $R_{2,3}$ (see Figure 3.15). To conclude the proof of the lemma we

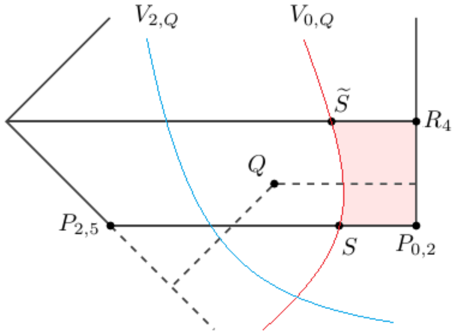


Figure 3.16: The region in $\mathcal{Z}_2$ with $d(Q, Q_2^3) > d(Q, \mathcal{C}_0)$

will prove that any such ball must contain $P_0^\star$, see (3.9). To this end, let us consider the parabola

$$\mathcal{V}_{5,P_0^\star} = \{Q \in \mathbb{R}^2 : dist(Q, P_0^\star) = dist(Q, \mathcal{C}_5)\}.$$

The proof of the lemma ends if we check that the set $\mathcal{Z}_1$ is contained in the set of points satisfying $dist(Q, P_0^\star) < dist(Q, \mathcal{L}_{\mathcal{C}_5})$. However, since both sets are convex it suffices to obtain

$$dist(Q, P_0^\star) < dist(Q, \mathcal{L}_{\mathcal{C}_5}) \text{ for } Q \in \{R_2, R_3, R_{2,3}\}. \tag{3.30}$$

Using the formula (3.29) and the expressions given for $P_0^\star$, R_2, R_3 and $R_{2,3}$ in (3.9), (3.20), (3.21) and (3.22), one may deduce that

$$dist^2(R_2, P_0^\star) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{R_2} - y_{R_2})^2 = dist^2(R_2, \mathcal{L}_{\mathcal{C}_5})$$

for every $t \in (1/\sqrt[3]{2}, 1)$,

$$dist^2(R_3, P_0^\star) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{R_3} - y_{R_3})^2 = dist^2(R_3, \mathcal{L}_{\mathcal{C}_5})$$

for every $t \in [0.759, 0.944]$ (recall that if suffices to suppose $t < \widetilde{t} \approx 0.928$), and

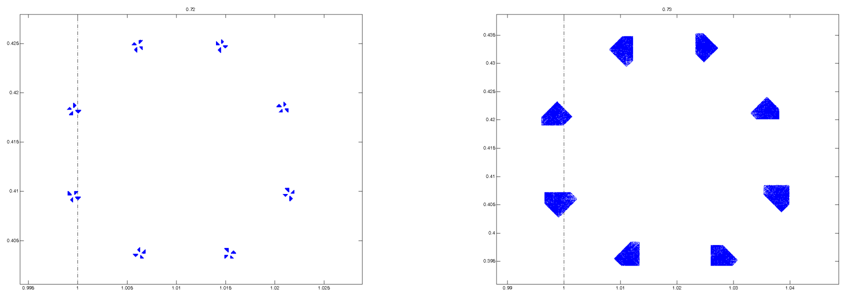
$$dist^2(R_{2,3}, P_0^\star) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{R_{2,3}} - y_{R_{2,3}})^2 = dist^2(R_{2,3}, \mathcal{L}_{\mathcal{C}_5})$$

for every $t \in [0.847, 1)$.

So the condition (3.30) holds for all $t \in (t_0, \widetilde{t}\,]$. Moreover, Lemma 3.27 is proved.

3.5 Appendix: The Fairy Cakes Attractor case

Along this section we start the study of certain dynamical properties for the family $\{\Lambda_t\}_t$ introduced in (3.5) when $t \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[5]{4}})$. As was explained in the Introduction, for this range of parameters the numerically obtained attractors are formed for several pieces (see Figure 3.17).



(a) Attractor for Λ_t, $t = 0.72$ (b) Attractor for Λ_t, $t = 0.73$

Figure 3.17: 32-pieces attractor and 8-pieces attractor

We start by associating an *EBM* to the family of transformations Λ_t^2. To this end, we may firstly compute the straight segments which are mapped into the critical line $x = 1$ for the *EBMs* $\{\Lambda_t\}_t$. These are the intersection between the line $y = x - \frac{2t-1}{t}$ with $\mathcal{T}_1$ and the intersection of $\mathcal{T}_0$ with the line $\mathcal{L} = \mathcal{L}(t)$ given by $x + y = \frac{1}{t}$, see Figure 3.18. These two segments take part of the critical set for Λ_t^2 (of course, the points on the line $x = 1$ too). Hence, see Proposition 3.41, we claim that Λ_t^2 coincides with the Baker maps given by the sequences $(\mathcal{T}, \mathcal{C}, \mathcal{L}(t), \mathcal{O}, A_t^2)$, with

$$A_t^2 = \begin{pmatrix} 2t^2 & 0 \\ 0 & 2t^2 \end{pmatrix}.$$

We recall that, see Remark 3.5, each Λ_t coincides with the the map $EBM(\mathcal{T}, \mathcal{C}, \mathcal{O}, A_t)$ being $\mathcal{C}$ the critical line $\{(x, y) \in \mathcal{T} : \ x = 1\}$, $\mathcal{O} = (0, 0)$ and

$$A_t = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}.$$

Proposition **3.41** *It holds that* $\Lambda_t^2 = EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}(t), \mathcal{O}, A_t^2)$.

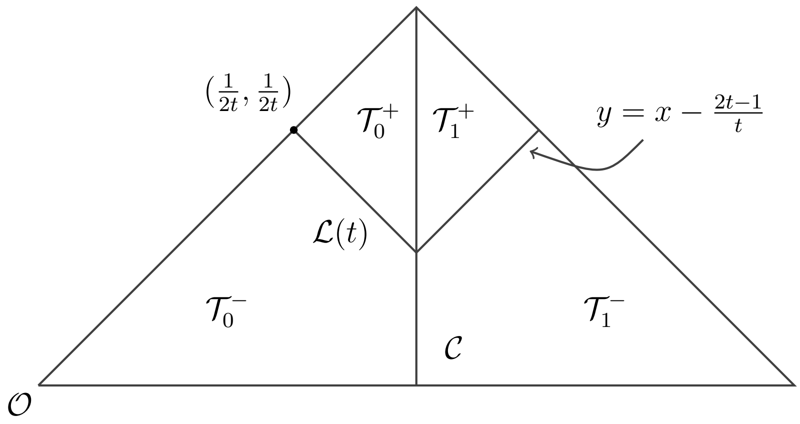


Figure 3.18: The smoothness domains for Λ_t^2

Proof. Let us denote by

$$\begin{aligned}
\mathcal{T}_0^+ &= \{(x,y) \in \mathcal{T}_0 : \ x + y \geq \frac{1}{t}\} \\
\mathcal{T}_0^- &= \mathcal{T}_0 \setminus \mathcal{T}_0^+ \\
\mathcal{T}_1^+ &= \{(x,y) \in \mathcal{T}_1 : \ y \geq x - \frac{2t-1}{t}\} \\
\mathcal{T}_1^- &= \mathcal{T}_1 \setminus \mathcal{T}_1^+
\end{aligned}$$

Let us first check how the map $EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}(t), \mathcal{O}, A_t^2)$ acts on $\mathcal{T}$. To avoid tedious notation, let us write

$$\Lambda_t^* = EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}(t), \mathcal{O}, A_t^2).$$

Observe that Λ_t^* can be seen as the composition of three maps: the first one folds the triangle $\mathcal{T}$ by the critical line $\mathcal{C}$ onto $\mathcal{T}_0$, the second one folds $\mathcal{T}_0$ by the line $\mathcal{L}(t)$ onto $\mathcal{T}_0^-$ and the third one expands $\mathcal{T}_0^-$ by the matrix

$$A_t^2 = \begin{pmatrix} 2t^2 & 0 \\ 0 & 2t^2 \end{pmatrix}.$$

Shortly, see Definition 3.1,

$$\Lambda_t^* = A_t^2 \circ \mathcal{F}_{\mathcal{C},\mathcal{O}} \circ \mathcal{F}_{\mathcal{L}(t),\mathcal{O}}$$

with $\mathcal{F}_{\mathcal{C},\mathcal{O}} : \mathcal{T} \to \mathcal{T}_0$ given by

$$\mathcal{F}_{\mathcal{C},\mathcal{O}}(x,y) = \begin{cases} (x,y) & , \ if (x,y) \in \mathcal{T}_0 \\ (2-x,y) & , \ if (x,y) \in \mathcal{T}_1 \end{cases}$$

and $\mathcal{F}_{\mathcal{L}(t),\mathcal{O}} : \mathcal{T} \to \mathcal{T}_0$ defined by

$$\mathcal{F}_{\mathcal{L}(t),\mathcal{O}}(x,y) = \begin{cases} (x,y) & , \; if (x,y) \in \mathcal{T}_0^- \\ (\frac{1}{t}-y, \frac{1}{t}-x) & , \; if (x,y) \in \mathcal{T}_0^+ \end{cases} .$$

Therefore,

$$\Lambda_t^*(x,y) = \begin{cases} (2t^2x, 2t^2y) & , \; if (x,y) \in \mathcal{T}_0^- \\ (2t(1-ty), 2t(1-tx)) & , \; if (x,y) \in \mathcal{T}_0^+ \\ (2t^2(2-x), 2t^2y) & , \; if (x,y) \in \mathcal{T}_1^- \\ (2t(1-ty), 2t(1-2t+tx)) & , \; if (x,y) \in \mathcal{T}_1^+ \end{cases} .$$

Now, let us compute Λ_t^2. Let us first assume we assume $(x,y) \in \mathcal{T}_0^-$. Then, since $(x,y) \in \mathcal{T}_0$ we have that $\Lambda_t(x,y) = (t(x+y), t(x-y))$ and, from the fact that $(x,y) \in \mathcal{T}_0^-$ we obtain $t(x+y) < 1$ and hence $\Lambda_t(x,y) \in \mathcal{T}_0$. Therefore

$$\begin{aligned} \Lambda_t^2(x,y) &= (t(t(x+y)+t(x-y)), t(t(x+y)-t(x-y))) = \\ &= (2t^2x, 2t^2y). \end{aligned}$$

Now, we assume $(x,y) \in \mathcal{T}_0^+$. We have that $\Lambda_t(x,y) = (t(x+y), t(x-y)) \in \mathcal{T}_1$ and therefore

$$\begin{aligned} \Lambda_t^2(x,y) &= (t(2-t(x+y)+t(x-y)), t(2-t(x+y)-t(x-y))) = \\ &= (2t-2t^2y, 2t-2t^2x) = (2t(1-ty), 2t(1-tx)). \end{aligned}$$

If $(x,y) \in \mathcal{T}_1^-$ we have that $\Lambda_t(x,y) = (t(2-x+y), t(2-x-y)) \in \mathcal{T}_0$ and therefore

$$\begin{aligned} \Lambda_t^2(x,y) &= (t(t(2-x+y)+t(2-x-y)), t(t(2-x+y)-t(2-x-y))) = \\ &= (4t^2-2t^2x, 2t^2y) = (2t^2(2-x), 2t^2y). \end{aligned}$$

Finally, if $(x,y) \in \mathcal{T}_1^+$ we have that $\Lambda_t(x,y) = (t(2-x+y), t(2-x-y)) \in \mathcal{T}_1$ and therefore

$$\begin{aligned} \Lambda_t^2(x,y) &= (t(2-t(2-x+y)+t(2-x-y)), \\ &\quad\; t(2-t(2-x+y)-t(2-x-y))) = \\ &= (2t-2t^2y, 2t-4t^2+2t^2x) = (2t(1-ty), 2t(1-2t+tx)). \end{aligned}$$

Hence, we conclude $\Lambda_t^2 = \Lambda_t^* := EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}(t), \mathcal{O}, A_t^2)$ and the proof is finished. ■

3.5.1 An invariant domain for Λ_t^8

Along this section we will prove, for every $t \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[5]{4}})$, the existence of an invariant triangle for Λ_t^8. The process is represented in Figure 3.19. To this end, it will be useful to denote by $\mathcal{L}(Q, m)$ the line passing through a point $Q \in \mathbb{R}^2$ with slope m. With $\mathcal{L}(Q, \infty)$ we denote the vertical line passing through a point Q. For any point $Q \in \mathcal{T}$ we will also denote, as usual, by $Q_1 = \Lambda_t(Q)$ and, in general, $Q_i = \Lambda_t(Q_{i-1})$. We also denote by $\Omega(Q_1, Q_2, \ldots, Q_n)$ the compact and convex polygonal domain whose consecutive vertices are Q_1, Q_2,..., Q_n.

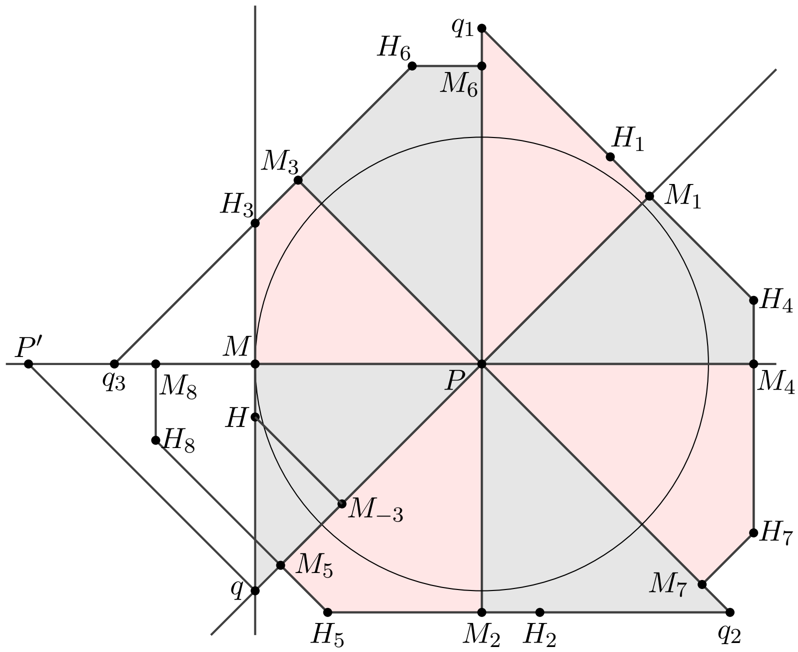


Figure 3.19:

We also recall that, for every $t \in (\frac{1}{\sqrt{2}}, 1]$, the map Λ_t has a fixed point $P_t \in \mathcal{T}_1$ given by

$$P_t = \left(\frac{2t(2t+1)}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1} \right).$$

Let us denote by M the intersection of the critical line $\mathcal{C}$ with the line $\mathcal{L}(P_t, 0)$ in such a way that

$$dist(P_t, M) = \frac{2t^2-1}{2t^2+2t+1}$$

is positive for every $t \in (\frac{1}{\sqrt{2}}, 1]$. Hence, for every $t \in (\frac{1}{\sqrt{2}}, 1]$ we may apply a change in coordinates

$$X = \frac{2t^2+2t+1}{2t^2-1}x - \frac{2t(2t+1)}{2t^2-1}\ ,\ Y = \frac{2t^2+2t+1}{2t^2-1}y - \frac{2t}{2t^2-1}$$

in such a way that, in this new coordinates,

$$P_t = P = (0,0) \quad \text{and} \quad dist(P, M) = 1.$$

Hence, in new coordinates (X, Y), $M = (-1, 0)$ and the critical line is given by $X = -1$.

Remark **3.42** *While the orbit of any point $Q \in \mathcal{T}_1$ does not leave $\mathcal{T}_1$, one may easily calculate the iterates Q_1, $Q_2 \ldots$. In fact, the action of Λ_t on a segment contained in $\mathcal{T}_1$ consists on a rotation (centered at P_t) by an angle $\frac{3\pi}{2}$ and an expansion by a factor $\sqrt{2}t$. In new coordinates (X, Y) this means that*

$$M = (-1, 0)\ ;\ M_1 = (t, t)\ ;\ M_2 = (0, -2t^2)\ ;\ M_3 = (-2t^3, 2t^3).$$

Let us observe that $-2t^3 > 1$ for every $t < \frac{1}{\sqrt[3]{2}}$ and therefore, if $t \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[5]{4}})$ we can also compute

$$M_4 = (4t^4, 0)\ ;\ M_5 = (-4t^5, -4t^5).$$

Now, the relevance of the parameter $\frac{1}{\sqrt[5]{4}}$ becomes patent, because if (and only if) $t \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[5]{4}})$ we may also obtain

$$M_6 = (0, 8t^6)\ ;\ M_7 = (8t^7, -8t^7)\ ;\ M_8 = (-16t^8, 0).$$

Let us consider the point q as the intersection between the critical line with $\mathcal{L}(P, -1)$. This point has coordinates $(-1, -1)$. Let us compute the first three iterates of q. In new coordinates (X, Y) it is easy to write

$$q_1 = (0, 2t),\ q_2 = (2t^2, -2t^2) \text{ and } q_3 = (-4t^3, 0).$$

Hence, for every $t \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[5]{4}})$, the point q_3 always lies between the points $P' = (-2, 0)$ (the symmetric of the fixed point with respect to the critical line) and the point M_8. Let us then consider the triangle

$$\Delta_0 = \Omega((0,0), (-1,-1), (-2,0)) = \Omega(P, q, P') \tag{3.31}$$

Proposition **3.43** *For every $t \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[5]{4}})$, Δ_0 is invariant by Λ_t^8.*

Proof. Of course, if we denote by $\Delta_j = \Lambda_t^j(\Delta_0)$, and recalling that Λ_t is symmetric with respect to the critical line, one has $\Delta_1 = \Lambda_t(\Omega(P,q,M)) = \Omega(P,q_1,M_1)$. This triangle does not intersect the critical line, therefore we may write

$$\Delta_2 = \Lambda_t(\Omega(P,q_1,M_1)) = \Omega(P,q_2,M_2)$$

and, for the same reason,

$$\Delta_3 = \Omega(P,q_3,M_3).$$

Of course, for every $t > \frac{1}{\sqrt{2}}$ the critical line always cuts the interior of Δ_3. Let us denote by H_3 the intersection between the critical line and the line $\mathcal{L}(M_3,-1)$. In coordinates (X,Y) this point is given by $H_3 = (-1, 4t^3 - 1)$. We are specially interest in determine the subset of Δ_0 mapped into the triangle $\Omega(M,H_3,q_3)$. To this end, let us consider the unique point in the region $X > 1$ whose third iterate is $M = (-1,0)$. This point will be denoted by M_{-3}, and coincide with the point in the line $\mathcal{L}(P,-1)$ satisfying $dist(P,M_3) = (2\sqrt{2}t^3)^{-1}$. Hence, we have that

$$M_{-3} = (-\frac{1}{4t^3}, -\frac{1}{4t^3}).$$

Since the fixed point P is a focus and the respective eigenvalues are $-t(1 \pm i)$ it is clear that, if we take a small segment on the line $\mathcal{L}(M_{-3},-1)$ then its image by f_t^3 is a small segment on the line $\mathcal{L}(M,\infty)$. Moreover, if we denote by H the intersection between the line $\mathcal{L}(M_{-3},-1)$ and the critical line, then it is easy to see that $f_t^3(H) = H_3$. Once again, we use new coordinates to write

$$H = (-1, 1 - \frac{1}{2t^3}).$$

Observe that the second coordinate of H is always negative for $t \in (\frac{1}{\sqrt{2}}, t_0]$.

Now, $\Lambda_t^3(\Omega(q,M_{-3},H)) = \Omega(q_3,M,H_3)$ and, moreover, if we take a point $(x,y) \in \Omega(q,M_{-3},H)$ and denote by $(\overline{x},\overline{y})$ its symmetric with respect the line $\mathcal{L}(M_{-3},-1)$, we have that $\Lambda_t^3(x,y)$ and $\Lambda_t^3(\overline{x},\overline{y})$ are symmetric with respect the critical line, and therefore the future images of both points are the same. Hence, the notion of Baker map arises again: Namely, in order to control the first four iterates of Δ_0 it is enough to first fold Δ_0 with respect to the critical line, after that, we have to fold the triangle $\Omega(P,M,q)$ with respect to the line $\mathcal{L}(M_{-3},-1)$ and after that apply Λ_t^3 to the polygonal region

$$\Pi_0 = \Omega(P,M,H,M_{-3}).$$

In this way, we have that $\Omega(P,M_3,H_3,M) = \Lambda_t^3(\Omega(P,M,H,M_{-3}))$ and

$$\Pi_4 = \Omega(P,M_4,H_4,M_1)$$

with

$$H_4 = \Lambda_t(H_3) = (4t^4, 2t - 4t^4)$$

satisfying

$$\Pi_4 = \Omega(P, M_4, H_4, M_1) = \Lambda_t^4(\Pi_0) = \Lambda_t^4(\Delta_0)$$

After that, it is clear that, since Π_4 does not intersects the critical line, then

$$\Pi_5 = \Omega(P, M_5, H_5, M_2) = \Lambda_t^5(\Pi_0) = \Lambda_t^5(\Delta_0)$$

with

$$H_5 = \Lambda_t(H_4) = (2t^2 - 8t^5, -2t^2).$$

Once again, Π_5 does not intersects the critical line, then

$$\Pi_6 = \Omega(P, M_6, H_6, M_3) = \Lambda_t^6(\Pi_0) = \Lambda_t^6(\Delta_0)$$

with

$$H_6 = \Lambda_t(H_5) = (8t^6 - 4t^3, 8t^6).$$

Of course, Π_6 does not intersects the critical line, then

$$\Pi_7 = \Omega(P, M_7, H_7, M_4) = \Lambda_t^7 \Pi_0) = \Lambda_t^7(\Delta_0)$$

with

$$H_7 = \Lambda_t(H_6) = (4t^4, 4t^4 - 16t^7).$$

And, finally, since Π_7 does not intersects the critical line, then

$$\Pi_8 = \Omega(P, M_8, H_8, M_5) = \Lambda_t^8(\Pi_0) = \Lambda_t^8(\Delta_0)$$

with

$$H_8 = \Lambda_t(H_7) = (-16t^8, 16t^8 - 8t^5).$$

We note that Π_8 is contained in Δ_0. Henceforward, Δ_0 is invariant by Λ_t^8 and the proposition is proved. ■

As we have seen in the proof of the above proposition we have

$$\Lambda_t^8(\Delta_0) = \Lambda_t^8(\Pi_0)$$

being $\Delta_0 = \Omega(P, q, P')$ and $\Pi_0 = \Omega(P, M, H, M_{-3})$.

Moreover, since $\Lambda_t^8(\Pi_0)$ does not leave the region $X > -1$ for $j = 0, \ldots, 7$. We have that $\Lambda^8_{t\ |\Pi_0}$ is a linear expansion given by

$$B_t = \begin{pmatrix} 16t^8 & 0 \\ 0 & 16t^8 \end{pmatrix}.$$

Therefore, after a new change in coordinates $x = -X$ $y = -Y$ the invariant domain Δ_0 transforms into our well-know domain $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ with

$$\begin{aligned}\mathcal{T}_0 &= \{(x,y) \in \mathcal{T}:\ 0 \le x \le 1\}\\ \mathcal{T}_1 &= \{(x,y) \in \mathcal{T}:\ 1 \le x \le 2\}\end{aligned}$$

and Λ_t^8 in this new coordinates coincides with the *EBM* given by

$$EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}^*(t), \mathcal{O}, B_t)$$

being

$$\mathcal{L}^*(t) = \{(x,y) \in \mathcal{T}_0:\ x + y = \frac{1}{2t^3}\}.$$

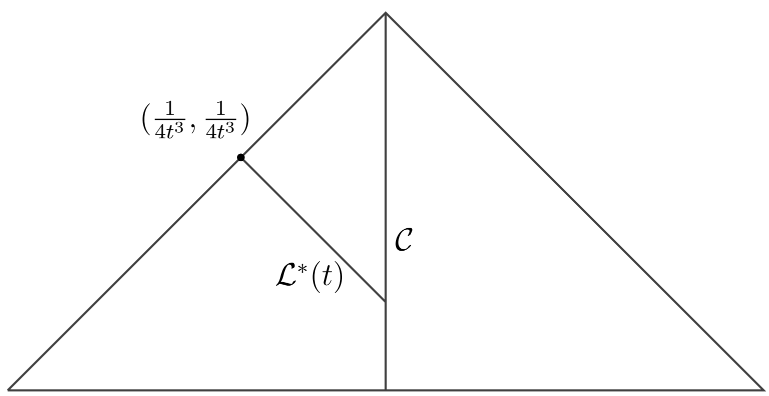


Figure 3.20: The segment $\mathcal{L}^*(t)$

Thus, according to Proposition 3.41 it seems extremely interesting to study the two-parameter family of *EBMs* given by

$$\mathcal{E}_{a,b} = EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}_b, \mathcal{O}, A_a)$$

where $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ and $\mathcal{C} = \{(x,y) \in \mathcal{T}:\ x = 1\}$ as usual and we define

$$\mathcal{L}_b = \{(x,y) \in \mathcal{T}_0:\ x + y = 2b\}$$

for $\frac{1}{2} \le b < 1$ and

$$A_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Of course, in order to get $\mathcal{E}_{a,b}(\mathcal{T}) \subset \mathcal{T}$ we must restrict the variation of the parameters (a,b) to the set

$$\xi = \{(a,b) \in \mathbb{R}^2:\ \frac{1}{2} \le b < 1,\ 1 < a < 2,\ ab < 1\}.$$

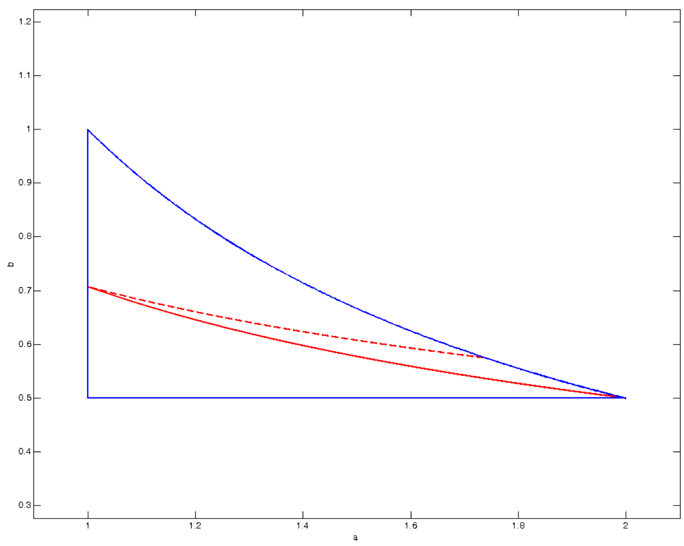


Figure 3.21: The space of parameters $\mathcal{E}$

In Figure 3.21 we also draw two lines. The continuous one represent those parameters representing the map Λ_t^2 for $\frac{1}{\sqrt{2}} < t \leq 1$ (see Proposition 3.41) and the discontinuous one representing Λ_t^8 for $\frac{1}{\sqrt{2}} < t \leq \frac{1}{\sqrt[5]{4}}$. Those curves are respectively given by

$$\{(2t^2, \tfrac{1}{2t}) : \tfrac{1}{\sqrt{2}} < t \leq 1\}$$
$$\{(16t^8, \tfrac{1}{4t^3}) : \tfrac{1}{\sqrt{2}} < t \leq \tfrac{1}{\sqrt[5]{4}}\}$$

Chapter 4

Statistical stability

4.1 Definitions

The aim of this chapter is to prove that our family of *EBMs* $\{\Lambda_t\}_t$ introduced in (3.5) is *statistically stable*: roughly speaking, if we denote by f_t the density associated to Λ_t it holds that f_t converges to $f_{t'}$ when $t \to t'$. To this end, the maps must be strongly transitive, so we will restrict ourselves to the parameter interval $(t_0, 1)$ (see Theorem 3.15). In order to give a rigorous definition for *statistical stability*, we start by giving a brief introduction to *measure theory* containing the definitions that we will need in the rest of this chapter. So let us recall what a *measurable space* and a *measure* are.

Definition **4.1** *Let $\mathcal{X}$ be a set. The set $\mathbb{B} \subset \mathcal{P}(\mathcal{X})$ is a $\sigma-$**algebra** if it satisfies the following three conditions:*

i) $\mathcal{X} \in \mathbb{B}$

ii) if $\mathcal{B} \in \mathbb{B}$ then $\mathcal{X} \setminus \mathcal{B} \in \mathbb{B}$

iii) if $\mathcal{B}_n \in \mathbb{B}$ for $n \in \mathbb{N}$, then $\bigcup_n \mathcal{B}_n \in \mathbb{B}$

*We then call the pair $(\mathcal{X}, \mathbb{B})$ a **measurable space**.*

Definition **4.2** *A **measure** on a measurable space $(\mathcal{X}, \mathbb{B})$ is a function $\mu : \mathbb{B} \to [0, +\infty]$ satisfying $\mu(\emptyset) = 0$ and*

$$\mu(\bigcup_n \mathcal{B}_n) = \sum_n \mu(\mathcal{B}_n)$$

whenever $\{\mathcal{B}_n\}_n$ is a sequence of members of $\mathbb{B}$ which are pairwise disjoint.

*Furthermore, we will say that μ is a **probability measure** if μ is a measure and $\mu(\mathcal{X}) = 1$.*

We are now in conditions to define what a *measure space* is.

Definition **4.3** *We call* ***measure space*** *the triple* $(\mathcal{X}, \mathbb{B}, \mu)$ *where* $(\mathcal{X}, \mathbb{B})$ *is a measurable space and* μ *is a measure on* $(\mathcal{X}, \mathbb{B})$*. If* μ *is a probability measure, we say that* $(\mathcal{X}, \mathbb{B}, \mu)$ *is a* ***probability space****.*

At this point, we can define some properties associated to any measure but first we need to define what a *measurable function* is.

Definition **4.4** *Let* $(\mathcal{X}_1, \mathbb{B}_1)$ *and* $(\mathcal{X}_2, \mathbb{B}_2)$ *be two measurable spaces. We say that* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *is a* ***measurable function*** *if* $f^{-1}(\mathcal{B}) \in \mathbb{B}_1$ *for all set* $\mathcal{B} \in \mathbb{B}_2$*.*

Definition **4.5** *Let* $(\mathcal{X}, \mathbb{B})$ *be a measurable space and let* $f : \mathcal{X} \to \mathcal{X}$ *be a measurable function. We say that a probability measure* μ *on* $\mathbb{B}$ *is* ***f-invariant*** *if*

$$\mu(f^{-1}(\mathcal{B})) = \mu(\mathcal{B})$$

for any every measurable set $\mathcal{B} \in \mathbb{B}$*.*

Definition **4.6** *Let* μ *and* ν *be two measures on a* $\sigma-$*algebra* $\mathbb{B}$*. We say that* μ *is* ***absolutely continuous*** *with respect to* ν $(\mu \ll \nu)$ *if* $\mu(\mathcal{B}) = 0$ *for any* $\mathcal{B} \in \mathbb{B}$ *such that* $\nu(\mathcal{B}) = 0$*.*

If $\nu \ll \mu$ *and* $\mu \ll \nu$*, the measures* ν *and* μ *are called* ***equivalent****.*

Under the assumption $\mu \ll \nu$, the Radon-Nikodym Theorem ensure that there exists a $\nu-$integrable function $h \geq 0$ such that

$$\mu(\mathcal{B}) = \int_{\mathcal{B}} h d\nu$$

for any measurable set $\mathcal{B} \in \mathbb{B}$ (see [57], for example). The function h is usually denoted by $d\mu/d\nu$ and called the **density** of μ with respect to ν or the **Radon-Nikodym derivative** of μ with respect to ν.

Definition **4.7** *Let* $f : \mathcal{X} \to \mathcal{X}$ *be a measurable function and let* μ *be a* $f-$*invariant probability measure. We say that* μ *is* ***ergodic*** *if*

$$\mu(\mathcal{B})\mu(\mathcal{X} \setminus \mathcal{B}) = 0$$

whenever $f^{-1}(\mathcal{B}) = \mathcal{B}$*.*

Now, let us define the concept of *statistical stability* for families of maps. Note that we will work with discrete-time dynamical systems defined in a compact region $\mathcal{K} \subset \mathbb{R}^N$ for some $N \geq 1$ so, in order to simplify the notation, we will restrict the following definitions to this condition. From now on, we will denote by m_N the **Lebesgue measure** on the Borel sets of $\mathbb{R}^N$ and $\mathcal{K}$ will be a compact set in $\mathbb{R}^N$. For each $1 \leq p \leq \infty$ we denote by $L^p(\mathcal{K})$ the Banach space of functions in $L^p(m_N)$ with support contained in $\mathcal{K}$, endowed with the usual norm $\|\cdot\|_p$. We also denote by $\mathrm{int}(\mathcal{A})$ the interior of a set $\mathcal{A}$ and by $\angle(v, w)$ the angle between v and w.

DEFINITION **4.8** *Let $\{\phi_t\}_{t\in I}$ be a family of maps $\phi_t : \mathcal{K} \to \mathcal{K}$ such that each ϕ_t has some absolutely continuous ϕ_t-invariant probability measure. We say that the family $\{\phi_t\}_{t\in I}$ is* ***statistically stable*** *if for any $t' \in I$, any choice of a sequence $(t_n)_{n\in\mathbb{N}} \subset I$ converging to t' and any choice of a sequence of absolutely continuous ϕ_{t_n}-invariant probability measures $\{\mu_{t_n}\}_{n\in\mathbb{N}}$, then any accumulation point of the sequence of densities $d\mu_{t_n}/dm_N$ must converge in the L^1-norm to the density of an absolutely continuous $\phi_{t'}$-invariant probability measure.*

Of course, when each ϕ_t has a unique absolutely continuous invariant probability measure μ_t, then statistical stability means that

$$\frac{d\mu_t}{dm_N} \underset{t\to t'}{\longrightarrow} \frac{d\mu_{t'}}{dm_N}$$

in the L^1-norm. A strictly weaker notion of statistical stability may be given if we assume only weak* convergence of the measures μ_t to $\mu_{t'}$ when $t \to t'$.

Hence, the main objective of this chapter is to demonstrate the following result.

THEOREM **4.9** *The family $\{\Lambda_t\}_{t\in(t_0,1)}$ is statistically stable.*

The strategy for proving 4.9 will be to demonstrate a more general result (see Theorem 4.11). Let us continue with the definitions of *piecewise expanding maps with bounded distortion and long branches.*

DEFINITION **4.10** *Let $\phi : \mathcal{K} \to \mathcal{K}$ be a map for which there is a (Lebesgue mod 0) partition $\{\mathcal{K}_i\}_{i=1}^{\infty}$ of $\mathcal{K}$ such that each $\mathcal{K}_i$ is a closed domain with piecewise C^2 boundary of finite $(N$-1$)$-dimensional measure and $\phi_i = \phi|_{\mathcal{K}_i}$ is a C^2 bijection from $\mathrm{int}(\mathcal{K}_i)$ onto its image with a C^2 extension to $\mathcal{K}_i$.*

(P_1) We say that ϕ is **piecewise expanding** *if there is $0 < \sigma < 1$ such that for every $i \geq 1$ and $x \in \mathrm{int}(\phi(\mathcal{K}_i))$*

$$\|D\phi_i^{-1}(x)\| < \sigma.$$

(P₂) *We say that* ϕ *has* **bounded distortion** *if there is* $D \geq 0$ *such that for every* $i \geq 1$ *and* $x \in \mathrm{int}(\phi(\mathcal{K}_i))$

$$\frac{\left\| D\left(J \circ \phi_i^{-1}\right)(x)\right\|}{\left|J \circ \phi_i^{-1}(x)\right|} \leq D,$$

where J *denotes the Jacobian of* ϕ*.*

(P_3) *We say that* ϕ *has* ***long branches*** *if there are* $\beta, \rho > 0$ *and for each* $i \geq 1$ *there is a* C^1 *unitary vector field* X_i *in* $\partial\phi(\mathcal{K}_i)$ *such that:*

(1) the segments joining each $x \in \partial\phi(\mathcal{K}_i)$ *to* $x + \rho X_i(x)$ *are pairwise disjoint and contained in* $\phi(\mathcal{K}_i)$*, and their union forms a neighbourhood of* $\partial\phi(\mathcal{K}_i)$ *in* $\phi(\mathcal{K}_i)$*.*

(2) for every $x \in \partial\phi(\mathcal{K}_i)$ *and* $v \in T_x\partial\phi(\mathcal{K}_i)\setminus\{0\}$ *the angle* $\angle(v, X_i(x))$ *satisfies* $|\sin\angle(v, X_i(x))| \geq \beta$*.*

Here we assume that at the singular points $x \in \partial\phi(\mathcal{K}_i)$ where $\partial\phi(\mathcal{K}_i)$ is not smooth the vector $X_i(x)$ is a common C^1 extension of X_i restricted to each (N-1)-dimensional smooth component of $\partial\phi(\mathcal{K}_i)$ having x in its boundary. We also assume that the tangent space of any such singular point x is the union of the tangent spaces to the (N-1)-dimensional smooth components it belongs to.

In the one-dimensional case $N = 1$, condition (P_3)(1) is clearly satisfied once we take the sets in the partition of $\mathcal{K}$ as being intervals whose images $\phi(\mathcal{K}_i)$ have sizes uniformly bounded away from zero. Additionally, condition (P_3)(2) always holds in dimension one, since $\partial\phi(\mathcal{K}_i)$ is a 0-dimensional manifold and so $T_x\partial\phi(\mathcal{K}_i) = \{0\}$ for any $x \in \phi(\mathcal{K}_i)$. In this case we can even take the optimal value $\beta = 1$; see Remark 4.15.

Now, we may state the following result, where we denote by $\|\cdot\|_N$ the usual norm in the space $L^N(\mathbb{R})$

THEOREM **4.11** *Let* $\mathcal{K}$ *be a compact set in* $\mathbb{R}^N$ *for some* $N \geq 1$*,* I *a metric space and* $\{\phi_t\}_{t\in I}$ *a family of* C^2 *piecewise expanding maps* $\phi_t : \mathcal{K} \to \mathcal{K}$ *with bounded distortion and long branches. Assume that for each* $t \in I$ *the following assumptions hold:*

(1) for each continuous $f : \mathcal{K} \to \mathbb{R}$ *we have* $\|f \circ \phi_{t'} - f \circ \phi_t\|_N \to 0$ *when* $t' \to t$*;*

(2) there exist $0 < \lambda < 1$ *and* $K > 0$ *for which*

$$\sigma_t\left(1 + \frac{1}{\beta_t}\right) \leq \lambda \text{ and } D_t + \frac{1}{\beta_t\rho_t} + \frac{D_t}{\beta_t} \leq K,$$

where $\sigma_t, D_t, \beta_t, \rho_t$ are constants for which (P_1), (P_2) and (P_3) in Definition 4.10 hold for ϕ_t.

Then $\{\phi_t\}_{t\in I}$ is statistically stable.

It follows from [3, Section 5] that under the assumptions of Theorem 4.11 each ϕ_t has a finite number of ergodic absolutely continuous invariant probability measures. The proof of this theorem uses the space of functions of bounded variation in $\mathbb{R}^N$, which are known to belong to the space $L^p(\mathcal{K})$, with $p = N/(N-1)$; see (4.1) below. Observing that $1/p + 1/N = 1$, this makes the choice of the norm $\|\cdot\|_N$ in condition (1) of Theorem 4.9 less mysterious; see the proof of the theorem. Notice that condition (1) in Theorem 4.11 holds whenever the maps ϕ_t are continuous and ϕ_t depends continuously (in the C^0-norm) on $t \in I$.

We will finish this section by defining the concept of *variation* for functions in multidimensional spaces. We adopt the definition given in [16].

Definition **4.12** *Given $f \in L^1(\mathbb{R}^N)$ with compact support we define the **variation** of f as*

$$V(f) = \sup\left\{\int_{\mathbb{R}^N} f\, div(g) dm_N \ : \ g \in C^1_0(\mathbb{R}^N, \mathbb{R}^N) \text{ and } \|g\| \le 1\right\},$$

where $C^1_0(\mathbb{R}^N, \mathbb{R}^N)$ is the set of C^1 functions from $\mathbb{R}^N$ to $\mathbb{R}^N$ with compact support, $div(g)$ is the divergence of g and $\|\cdot\|$ is the supremum norm in $C^1_0(\mathbb{R}^N, \mathbb{R}^N)$.

Definition **4.13** *Given a bounded set $\mathcal{K} \subset \mathbb{R}^N$ we consider the space of **bounded variation** functions in $L^1(\mathcal{K})$*

$$BV(\mathcal{K}) = \left\{f \in L^1(\mathcal{K}) \ : \ V(f) < +\infty\right\}.$$

Contrarily to the classical one-dimensional definition of bounded variation, a multidimensional bounded variation function need not to be bounded (see [23]). However, as an extension of the Sobolev's Inequality it holds that there exists some constant $c > 0$ (depending on the dimension N) such that for any $f \in BV(\mathcal{K})$

$$\left(\int |f|^p dm_N\right)^{1/p} \le c\, V(f), \tag{4.1}$$

with $p = \frac{N}{N-1}$ (see e.g. [16, Theorem 1.28]). This in particular gives $BV(\mathcal{K}) \subset L^p(\mathcal{K})$.

We shall use the following properties of bounded variation functions whose proofs may be found in [14] or [16]:

(B_1) $BV(\mathcal{K})$ is dense in $L^1(\mathcal{K})$;

(B_2) if $\{f_k\}_k$ is a sequence in $BV(\mathcal{K})$ converging to f in the L^1-norm, then

$$V(f) \leq \liminf_k V(f_k);$$

(B_3) if $\{f_k\}_k$ is a sequence in $BV(\mathcal{K})$ such that $\{\|f_k\|_1\}_k$ and $\{V(f_k)\}_k$ are bounded, then $\{f_k\}_k$ has some subsequence converging in the L^1-norm to a function in $BV(\mathcal{K})$.

4.2 Proof of Theorem 4.11

In this section we will prove Theorem 4.11. Let $\{\mathcal{K}_i^t\}_{i=1}^\infty$ be the domains of smoothness of the family $\{\phi_t\}_{t\in I}$ satisfying the assumptions of Theorem 4.11 and define $\phi_{t,i} = \phi_t|_{\mathcal{K}_i^t}$ for all $i \geq 1$. For each $t \in I$ we consider the *Perron-Frobenius operator*

$$\mathcal{P}_t : L^1(\mathcal{K}) \longrightarrow L^1(\mathcal{K})$$

defined for $f \in L^1(\mathcal{K})$ as

$$\mathcal{P}_t f = \sum_{i=1}^\infty \frac{f \circ \phi_{t,i}^{-1}}{|J \circ \phi_{t,i}^{-1}|} \chi_{\phi_t(\mathcal{K}_i^t)}.$$

It is well known that the following two properties hold for each $\mathcal{P}_t$.

(C_1) $\|\mathcal{P}_t f\|_1 \leq \|f\|_1$ for every $f \in L^1(\mathcal{K})$;

(C_2) $\mathcal{P}_t f = f$ if and only if f is the density of an absolutely continuous ϕ_t-invariant probability measure.

Lemma **4.14** *Let $0 < \lambda < 1$ and $K > 0$ be such that the second statement of Theorem 4.11 holds. Given $t \in I$ and $j \geq 1$ we have for every $f \in BV(\mathcal{K})$*

$$V(\mathcal{P}_t^j f) \leq \lambda^j V(f) + K_1 \|f\|_1.$$

being $K_1 = K \sum_{j=0}^\infty \lambda^j$.

Proof. The proof of this lemma follows immediately from [3, Lemma 5.4], where it is shown that

$$V(\mathcal{P}_t f) \leq \sigma_t \left(1 + \frac{1}{\beta_t}\right) V(f) + \left(D_t + \frac{1}{\beta_t \rho_t} + \frac{D_t}{\beta_t}\right) \|f\|_1.$$

As we assumed that λ and K verify the second statement of Theorem 4.11, it easily follows that

$$V(\mathcal{P}_t f) \leq \lambda V(f) + K\|f\|_1.$$

Furthermore, recalling that $\|\mathcal{P}_t\| \leq \|f\|_1$ (see (C_1)), it follows that

$$\begin{aligned} V(\mathcal{P}_t^j f) &\leq \lambda V(\mathcal{P}_t^{j-1} f) + K\|f\|_1 \\ &\leq \lambda^2 V(\mathcal{P}_t^{j-2} f) + (\lambda + 1)\, K\|f\|_1 \\ &\leq \dots \\ &\leq \lambda^j V(f) + \left(\lambda^{j-1} + \dots + \lambda + 1\right) K\|f\|_1 \end{aligned}$$

So we have prove that

$$V(\mathcal{P}_t^j f) \leq \lambda^j V(f) + K \sum_{n=0}^{j-1} \lambda^n \|f\|_1 \leq \lambda^j V(f) + K_1 \|f\|_1.$$

■

Remark **4.15** *The proof of [3, Lemma 5.4] uses Lemma 3 in [22] applied to the sets $S = \phi(\mathcal{K}_i)$, which gives for a function $f \in C^1(S)$,*

$$\int_{\partial S} |f| dm_N \leq \frac{1}{\beta} \left(\frac{1}{\rho} \int_S |f| dm_N + \int_S \|Df\| dm_N \right). \tag{4.2}$$

In the one-dimensional case we have for any interval S and $x \in S$

$$f(x) \leq \frac{1}{|S|} \int_S |f| dm_1 + \int_S |Df| dm_1,$$

which yields a formula similar to (4.2) with $\beta = 1$.

In the proof of the result below we follow some standard arguments with functions of bounded variation, namely those used in [27] for the one-dimensional case.

Proposition **4.16** *Given $t \in I$ and $f \in L^1(\mathcal{K})$ the sequence $\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{P}_t^j f$ has some accumulation point in the L^1-norm. Moreover, any such accumulation point belongs to $BV(\mathcal{K})$ and has variation bounded by $4K_1\|f\|_1$.*

Proof. Given $f \in L^1(\mathcal{K})$, by property (B_1) we may consider a sequence of functions $\{f_k\}_k$ in $BV(\mathcal{K})$ converging to f in the L^1-norm. With no loss of generality we may assume that $\|f_k\|_1 \le 2\|f\|_1$ for every $k \ge 1$. It follows from Lemma 4.14 that for each $k \ge 1$ and large j we have

$$V(\mathcal{P}_t^j f_k) \le \lambda^j V(f_k) + K_1\|f_k\|_1 \le 3K_1\|f\|_1.$$

So, for large n we have

$$V\left(\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{P}_t^j f_k\right) \le 4K_1\|f\|_1.$$

Using that $\|f_k\|_1 \le 2\|f\|_1$ for every $k \ge 1$, it easily follows from (C_1) that

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{P}_t^j f_k\right\|_1 \le 2\|f\|_1.$$

Then it follows from (B_3) that there exists some $g_k \in BV(\mathcal{K})$ and a sequence $(n_i)_i$ such that $1/n_i\sum_{j=0}^{n_i-1}\mathcal{P}_t^j f_k$ converges in the L^1-norm to g_k as i goes to $+\infty$. Moreover, by (B_2) we have $V(g_k) \le 4K_1\|f\|_1$ for every $k \ge 1$. Hence, we may apply the same argument to the sequence $(g_k)_k$ and obtain a subsequence $(k_i)_i$ such that $(g_{k_i})_i$ converges in the L^1-norm to some $g \in BV(\mathcal{K})$ with $V(g) \le 4K_1\|f\|_1$. Hence, there must be some sequence $(n_\ell)_\ell$ converging to $+\infty$ for which $1/n_\ell\sum_{j=0}^{n_\ell-1}\mathcal{P}_t^j f_{k_\ell}$ converges to g in the L^1-norm as $\ell \to +\infty$. On the other hand,

$$\left\|\frac{1}{n_\ell}\sum_{j=0}^{n_\ell-1}\left(\mathcal{P}_t^j f_{k_\ell} - \mathcal{P}_t^j f\right)\right\|_1 \le \frac{1}{n_\ell}\sum_{j=0}^{n_\ell-1}\|f_{k_\ell} - f\|_1 = \|f_{k_\ell} - f\|_1$$

and this last term goes to 0 as $\ell \to +\infty$. This clearly gives that $1/n_\ell\sum_{j=0}^{n_\ell-1}\mathcal{P}_t^j f$ converges to g in the L^1-norm.

To prove the second part of the lemma, consider some subsequence of $1/n\sum_{j=0}^{n-1}\mathcal{P}_t^j f$ converging to f_0 in the L^1-norm. Taking that subsequence playing the role of the whole sequence in the argument above we easily see that f_0 satisfies the conclusion by uniqueness of the limit. ■

COROLLARY **4.17** *If h_t is the density of an absolutely continuous ϕ_t-invariant probability measure, then $h_t \in BV(\mathcal{K})$ and $V(h_t) \le 4K_1$.*

Proof. Take h_t the density of an absolutely continuous ϕ_t-invariant probability measure. We have from property (C_2) that $\mathcal{P}_t^j h_t = h_t$ for all $j \ge 1$. This implies that the sequence $1/n\sum_{j=0}^{n-1}\mathcal{P}_t^j h_t$ is constant and equal to h_t, and so the result follows. ■

Now we are in conditions to conclude the proof of Theorem 4.11. To this end, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in I converging to some $t'\in I$. Assume that for each $n\in\mathbb{N}$ we have an absolutely continuous ϕ_{t_n}-invariant probability measure μ_n and consider

$$h_n=\frac{d\mu_n}{dm_N}.$$

Using the fact that each h_n is the density of a probability measure and Corollary 4.17 we have for all $n\geq 1$

$$\|h_n\|_1=1 \text{ and } V(h_n)\leq 4K_1.$$

Hence, by (B_2) and (B_3) there exists $h_0\in BV$ with $V(h_0)\leq 4K_1$ such that the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges to h_0 in the L^1-norm. Let μ_0 be the probability measure in $\mathcal{K}$ whose density with respect to m_N is h_0. It still remains to prove that μ_0 is a $\phi_{t'}$-invariant measure.

Since the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges to h_0 in the L^1-norm, it easily follows that $\{\mu_n\}_{n\in\mathbb{N}}$ converges to μ_0 in the weak* topology. Thus, given any $f\colon\mathcal{K}\to\mathbb{R}$ continuous we have

$$\int f d\mu_n \xrightarrow[n\to\infty]{} \int f d\mu_0.$$

On the other hand, since μ_n is ϕ_{t_n}-invariant we have

$$\int f d\mu_n=\int (f\circ\phi_{t_n})d\mu_n$$

for every n. It is enough to prove that

$$\int (f\circ\phi_{t_n})d\mu_n \xrightarrow[n\to\infty]{} \int (f\circ\phi_{t'})d\mu_0.$$

We have

$$\begin{aligned}
\left|\int (f\circ\phi_{t_n})d\mu_n-\int (f\circ\phi_{t'})d\mu_0\right| &\leq \\
&\leq \left|\int (f\circ\phi_{t_n})d\mu_n-\int (f\circ\phi_{t'})d\mu_n\right| + \\
&+\left|\int (f\circ\phi_{t'})d\mu_n-\int (f\circ\phi_{t'})d\mu_0\right| \\
&\leq \int |f\circ\phi_{t_n}-f\circ\phi_{t'}|\,d\mu_n+\left|\int (f\circ\phi_{t'})d\mu_n-\int (f\circ\phi_{t'})d\mu_0\right| \\
&= \int |f\circ\phi_{t_n}-f\circ\phi_{t'}|\,h_n\,dm_N+\left|\int (f\circ\phi_{t'})(h_n-h_0)\,dm_N\right|.
\end{aligned}$$

Now using (4.1) we easily get that each $h_n \in L^p(\mathcal{K})$ with $p = N/(N-1)$ and

$$\|h_n\|_p \leq C \cdot V(h_n) \leq 4CK_1.$$

Observing that $1/p + 1/N = 1$, then by Hölder's Inequality we get

$$\int |f \circ \phi_{t_n} - f \circ \phi_{t'}|\, h_n \, dm_N \leq 4CK_1 \|f \circ \phi_{t_n} - f \circ \phi_{t'}\|_N,$$

and this clearly converges to zero when $n \to +\infty$ by assumption (1) in Theorem 4.11. On the other hand, as f is bounded we have

$$\left| \int (f \circ \phi_{t'})(h_n - h_0)\, dm_N \right| \leq \|f \circ \phi_{t'}\|_\infty \cdot \|h_n - h_0\|_1$$

and this clearly converges to 0 when $n \to +\infty$ as well.

4.3 Proof of Theorem 4.9

To conclude this chapter we have to prove Theorem 4.9. The idea is to obtain it as a corollary of Theorem 4.11. As observed before, each Λ_t is *strongly transitive*: any open set becomes the whole space under a finite number of iterations by Λ_t. This implies that the absolutely continuous Λ_t-invariant ergodic probability measure μ_t must be unique. Moreover, any power of Λ_t has a unique absolutely continuous invariant ergodic probability measure as well, which must necessarily coincide with μ_t. Thus, it is enough to obtain the statistical stability for some power of the maps in our family: in this case, $\{\Lambda_t^6\}_{t\in(t_0,1)}$.

Note that from the definition of Λ_t in (3.5) we obviously have that $\mathcal{T}_0$ and $\mathcal{T}_1$ are the domains of smoothness of Λ_t. The map Λ_t is piecewise linear with

$$D\Lambda_t(Q) = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}$$

for $Q \in \mathcal{T}_0 \setminus \mathcal{C}$, and

$$D\Lambda_t(Q) = \begin{pmatrix} -t & t \\ -t & -t \end{pmatrix}$$

for $Q \in \mathcal{T}_1 \setminus \mathcal{C}$. From here we deduce that for all $Q \in \mathcal{T} \setminus \mathcal{C}$ we have

$$\|D\Lambda_t^{-1}(Q)\| = \frac{1}{\sqrt{2}t}. \tag{4.3}$$

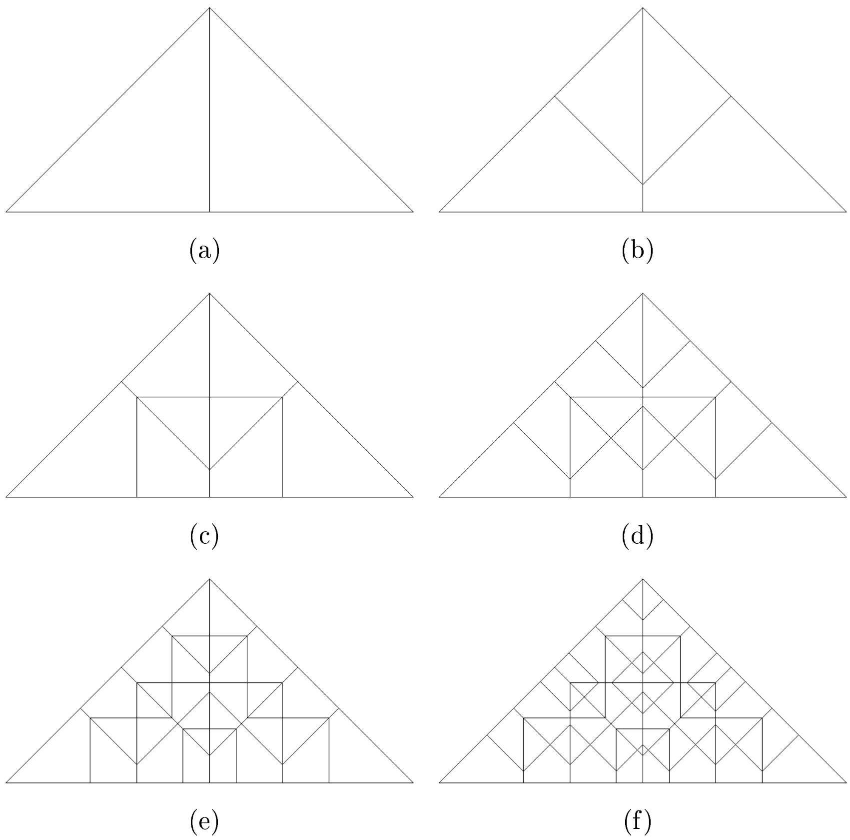


Figure 4.1: *Smoothness domains: (a) for Λ_t (b) for Λ_t^2 (c) for Λ_t^3 (d) for Λ_t^4 (e) for Λ_t^5 (f) for Λ_t^6*

Now take $\mathcal{K} = \mathcal{T}$ and $\{\mathcal{K}_i^t\}_{i=1}^{64}$ the (Lebesgue mod 0) partition of $\mathcal{K}$ given by the domains of smoothness of Λ_t^6. From (4.3) we easily deduce that

$$\|(D\Lambda_t^6)^{-1}(Q)\| = \frac{1}{(\sqrt{2}t)^6} = \frac{1}{8t^6} \tag{4.4}$$

As we have seen, in order to prove Theorem 4.9 it is enough to see that the family $\{\Lambda_t^6\}_{t\in(t_0,1)}$ satisfies the conditions in Theorem 4.11, i.e.

(i) each $\Lambda_t^6 : \mathcal{T} \to \mathcal{T}$ is a C^2 piecewise expanding map with bounded distortion and long branches;

(ii) for each continuous $f : \mathcal{T} \to \mathbb{R}$ we have $\|f \circ \Lambda_{t'}^6 - f \circ \Lambda_t^6\|_2 \to 0$ when $t' \to t$;

(iii) there exist $0 < \lambda < 1$ and $K > 0$ such that

$$\sigma_t\left(1+\frac{1}{\beta_t}\right) \leq \lambda \text{ and } D_t + \frac{1}{\beta_t \rho_t} + \frac{D_t}{\beta_t} \leq K,$$

where $\sigma_t, D_t, \beta_t, \rho_t$ are constants for which (i) holds.

First of all, observe that as the maps $\{\Lambda_t^6\}_{t\in(t_0,1)}$ are continuous then condition (ii) is trivially satisfied.

On the other hand, if we define $\sigma_t = \frac{1}{8t^6}$ it is evident that

$$\|(D\Lambda_t^6)^{-1}(Q)\| = \sigma_t < 1$$

and so property (P_1) holds for each $t \in (t_0, 1)$ (see (4.4) and recall that $t_0 \approx 0.882$). Furthermore, since Λ_t^6 is linear on each domain of smoothness, then it has 0 distortion and we obtain property (P_2) with $D_t = 0$ for each $t \in (t_0, 1)$.

We have already proved that $\Lambda_t^6 : \mathcal{T} \to \mathcal{T}$ is a C^2 piecewise expanding map with bounded distortion. Let us now check that it has long branches. As each Λ_t^6 is linear on each $\mathcal{K}_i^t$ and preserves angles, it is enough to obtain the geometric property (P_3) for the domains $\{\mathcal{K}_i^t\}_i$ instead of their images. Since the pre-image of the critical set $\mathcal{C}$ delimits the boundary of the domains of smoothness, it easily follows that the boundary of each $\mathcal{K}_i^t$ is formed by at most five straight line segments with slope -1, 0, 1 or ∞ meeting at an angle at least $\pi/4$; see Figure 4.1(f). Then, it is not hard to check that for every $t \in (t_0, 1)$ and $i = 1, \dots, 64$ there is a piecewise C^1 unitary vector filed X_i^t in $\partial\mathcal{K}_i^t$ such that

$$|\sin\angle(v, X_i^t(Q))| \geq \sin\frac{\pi}{8} := \beta_t$$

for every $Q \in \partial\mathcal{K}_i^t$ and $v \in T_Q\partial\mathcal{K}_i^t \setminus \{0\}$; see Figure 4.2. To prove the existence of ρ_t is is enough to observe that the domains of smoothness of Λ_t^6 depend continuously on the parameter t as illustrated in Figure 4.3, and so it is possible to choose an uniform value of ρ such that (P_3) holds for each $t \in (t_0, 1)$.

It still remains to prove that *(iii)* holds. Note that

$$\sigma_t\left(1+\frac{1}{\beta_t}\right) \leq \frac{1}{8t^6}\left(1+\frac{1}{\sin(\pi/8)}\right) < 1$$

for all $t \in (0.875922, 1]$ so we can take $0 < \lambda < 1$ such that

$$\sigma_t\left(1+\frac{1}{\beta_t}\right) \leq \lambda$$

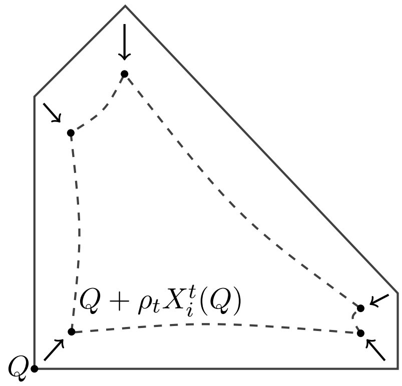


Figure 4.2: *A long branch for* Λ_t^6

for every $t \in (t_0, 1)$ (recall that $t_0 \approx 0.882$). On the other hand, we can define $K > 0$ as

$$D_t + \frac{1}{\beta_t \rho_t} + \frac{D_t}{\beta_t} = \frac{1}{\rho \sin(\pi/8)} := K,$$

Altogether, we have proved *(iii)* and hence Theorem 4.9.

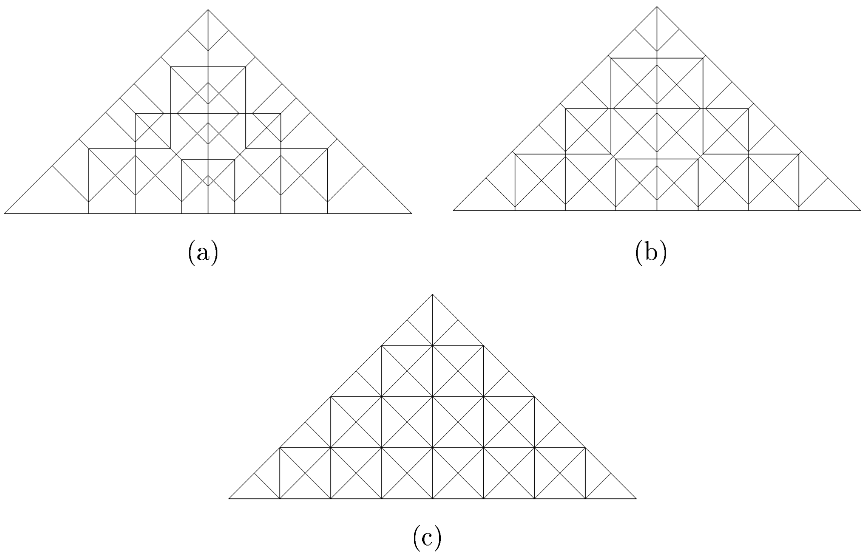


Figure 4.3: *Domains of smoothness for* Λ_t^6*: (a)* $t = t_0$ *(b)* $t = 0.95$ *(c)* $t = 1$

Final remarks

As a consequence of the results obtained in this thesis, we have started the study of the complex dynamics emerging when a homoclinic tangency is unfolded in the three-dimensional setting.

We could highlight the definition of the maps that we have called ***Expanding Baker Maps***, not only for their relationship with the unfolding of homoclinic tangencies, but also to emphasise the intricate dynamics emerging when we iterate a piecewise linear transformation defined on a triangle. This kind of transformations are certainly the equivalent, in the two-dimensional scenario, to the wellknown and wellstudied *one-dimensional tent map*.

In the same way, the Main Theorem of this thesis give an example of a family of two-dimensional transformations exhibiting persistent two-dimensional strange attractors. This fact, outside the context of triangular maps, is an important milestone in the field of Dynamical Systems.

In addition, it is shown the first example (outside the context of the complex analysis) of a family of two-dimensional transformations for which a ***renormalization scheme*** could be applied. These ideas have given fruitful results in dimension one along the last four decades. However, they had not been discovered in dimension two (except in exceptional complex variable settings). Roughly speaking, the idea is to find subsets in the phase space where the dynamics of a particular power of the map behaves like the dynamics of an equivalent map defined in the whole phase space. Returning to the usual example, the idea translate into thinking that each one of the fairy cakes arising in certain *Expanding Baker Maps* behave like country breads arising in other *Expanding Baker Maps*. At the end of chapter three of the thesis, these ideas are exposed in order to be explored in a near future. So we hope that our examples and results will be very helpful in the future.

We point out that the family of *Expanding Baker Maps* defined in this thesis gives a first example of several results published in the last two decades in which the authors study the existence of absolutely continuous invariant ergodic measures associated to families of piecewise expansive transformations defined on a 2-D domain.

Finally, we prove that our family of *Expanding Baker Maps* is statistically stable.

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Appendix A

Introducción

En el estudio de las dinámicas disipativas, se entiende por *caos* la existencia de atractores extraños en el espacio de fases. En este trabajo se tratará la existencia de atractores extraños bidimensionales asociados a una familia uniparamétrica de aplicaciones bidimensionales y lineales a trozos definidas en cierto triángulo. Es más, los atractores extraños serán persistentes en un conjunto abierto de parámetros. Comencemos introduciendo una serie de definiciones.

Definición **A.1** *Dada una aplicación f definida en una variedad $\mathcal{M} \subset \mathbb{R}^N$, se dice que un conjunto $f-$invariante $\mathcal{K}$ es* ***transitivo*** *si existe alguna $f-$órbita densa en $\mathcal{K}$. Equivalentemente, para todo par de abiertos $\mathcal{U}$ y $\mathcal{V}$ contenidos en $\mathcal{K}$ existe un número natural n tal que $f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$.*

Definición **A.2** *Un* ***atractor*** *para una aplicación f definida en una variedad $\mathcal{M}$ es un conjunto $\mathcal{K}$ compacto, $f-$invariante y transitivo cuyo conjunto estable*

$$W^s(\mathcal{K}) = \{Q \in \mathcal{M} : d(f^n(Q), \mathcal{K}) \to 0 \text{ as } n \to \infty\}$$

tiene interior no vacío. Se dice que un atractor es ***extraño*** *si contiene una órbita densa $\{f^n(\widetilde{Q}) : \ n \geq 0\}$ con crecimiento exponencial en la derivada, es decir, existe una constante $c > 0$ tal que, para todo $n \geq 0$,*

$$\|Df^n(\widetilde{Q})\| \geq \exp(cn).$$

Entenderemos por *atractores extraños bidimensionales* aquellos atractores extraños para los cuales la suma de los exponentes de Lyapounov sobre la órbita densa es positiva. En este trabajo, de hecho, los atractores extraños serán aún “más extraños” puesto que el exponente de Lyapounov será positivo en cualquier dirección para un conjunto de condiciones iniciales con medida de Lebesgue positiva.

El ejemplo más sencillo de atractor extraño viene dado por la familia uniparamétrica de aplicaciones tienda unidimensionales $\{\lambda_\mu\}_{\mu\in(1,2]}$, siendo $\lambda_\mu : [0,1] \to [0,1]$ la aplicación definida por

$$\lambda_\mu(x) = \begin{cases} \mu x & , \text{ si } x \in [0, \frac{1}{2}] \\ \mu(1-x) & , \text{ si } x \in [\frac{1}{2}, 1]. \end{cases}$$

El intervalo $[\mu(1-\frac{\mu}{2}), \frac{\mu}{2})]$ es un atractor extraño de λ_μ para $\sqrt{2} < \mu \leq 2$.

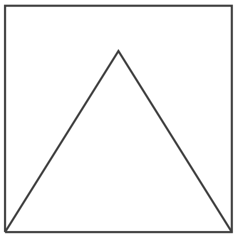


Figure A.1: Aplicación tienda

Como en nuestro caso, estos atractores extraños son persistentes (en un conjunto abierto de parámetros) y el número de exponentes de Lyapounov positivos coincide con la dimensión de la variedad ambiente.

No es difícil comprobar que cada una de las aplicaciones tienda que acabamos de definir puede expresarse como la composición de dos aplicaciones. La primera dobla (por la mitad) el intervalo $[0,1]$ sobre el intervalo $[0, \frac{1}{2}]$ y la segunda es una expansión lineal por un factor μ del intervalo $[0, \frac{1}{2}]$ en el intervalo $[0, \frac{\mu}{2}]$.

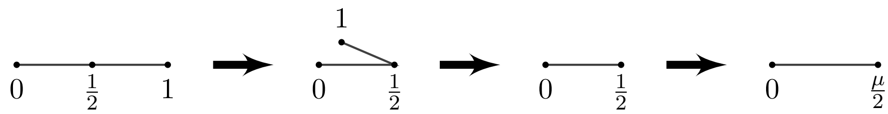


Figure A.2: Dinámica de la aplicación tienda

Uno de los principales objetivos de esta memoria es trasladar estas ideas geométricas a dimensión dos.

Introducimos la familia $\{\Lambda_t\}_{t\in[0,1]}$ siendo $\Lambda_t : \mathcal{T} \to \mathcal{T}$ la aplicación bidimensional dada por

$$\Lambda_t(x,y) = \begin{cases} (t(x+y), t(x-y)) & , \text{ si } (x,y) \in \mathcal{T}_0 \\ (t(2-x+y), t(2-x-y)) & , \text{ si } (x,y) \in \mathcal{T}_1 \end{cases} \tag{A.1}$$

donde $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ y

$$\begin{aligned} \mathcal{T}_0 &= \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, 0 \leq y \leq x\}, \\ \mathcal{T}_1 &= \{(x,y) \in \mathbb{R}^2 : \ 1 \leq x \leq 2, 0 \leq y \leq 2-x\}. \end{aligned} \tag{A.2}$$

En primer lugar, cabe señalar que esta familia de aplicaciones bidimensionales y lineales a trozos está íntimamente relacionada con las aplicaciones retorno que surgen en el estudio de ciertas bifurcaciones homoclínicas desplegadas por familias de difeomorfismos tridimensionales, como veremos a lo largo del trabajo. En cualquier caso, como recordaremos en distintos momentos de esta Introducción, esto mismo ocurre en el caso de las aplicaciones tienda unidimensionales cuando las bifurcaciones homoclínicas se producen en el escenario bidimensional.

Cada una de estas aplicaciones Λ_t puede verse como la composición de dos aplicaciones. La primera de ellas viene dada por $\mathcal{F}_{\mathcal{C},O} : (x,y) \in \mathcal{T} \to \mathcal{T}_0$, siendo

$$\mathcal{F}_{\mathcal{C},O}(x,y) = \begin{cases} (x,y) & \text{, si } (x,y) \in \mathcal{T}_0 \\ (2-x,y) & \text{, si } (x,y) \in \mathcal{T}_1, \end{cases}$$

y pliega el triángulo $\mathcal{T}$ sobre $\mathcal{T}_0$. La segunda, dada por $A_t : \mathcal{T}_0 \to \mathcal{T}$ donde

$$A_t(x,y) = \begin{pmatrix} t & t \\ t & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{T},$$

es una transformación lineal de $\mathcal{T}_0$ en $\mathcal{T}$.

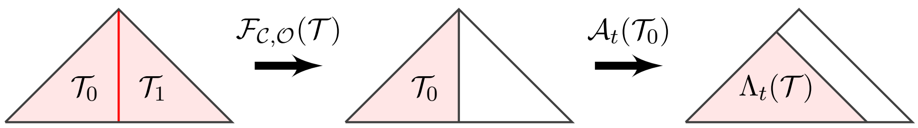


Figure A.3: Dinámica de Λ_t

Antes de entrar en detalles, es necesario presentar el que será el resultado principal de esta memoria (ver Teorema A.6). Con este objetivo, introducimos las siguientes definiciones.

DEFINICIÓN **A.3** *Sea $f : \mathcal{M} \to \mathcal{M}$ una aplicación, $\mathcal{M} \subset \mathbb{R}^N$. Se dice que f es* ***fuertemente topológicamente mezclante*** *en un conjunto invariante $\mathcal{A}$ si para todo abierto $\mathcal{U} \subset \mathcal{A}$ existe un número natural k tal que $f^k(\mathcal{U}) = \mathcal{A}$.*

Es fácil comprobar que si f es fuertemente topológicamente mezclante en un conjunto $\mathcal{A}$, entonces $\mathcal{A}$ es transitivo.

DEFINICIÓN **A.4** *Sea $f : \mathcal{M} \to \mathcal{M}$ una aplicación medible, $\mathcal{M} \subset \mathbb{R}^N$.*

a) Una medida de probabilidad μ sobre los borelianos de $\mathcal{M}$ se dice ***f-invariante*** *si*

$$\mu(f^{-1}(\mathcal{B})) = \mu(\mathcal{B})$$

para todo boreliano $\mathcal{B} \subset \mathcal{M}$.

*b) Si $\mu(\mathcal{B}) = 0$ cuando $m_N(\mathcal{B}) = 0$, siendo m_N la medida de Lebesgue sobre los borelianos de $\mathbb{R}^N$, diremos que f es **absolutamente continua**.*

*c) Una medida de probabilidad f-invariante μ se dice **ergódica** si*

$$\mu(\mathcal{B})\mu(\mathcal{M} \setminus \mathcal{B}) = 0$$

cuando $f^{-1}(\mathcal{B}) = \mathcal{B}$.

Notemos que si μ es una medida de probabilidad absolutamente continua entonces existe una función m_N-integrable $h \geq 0$ que se denomina ***densidad*** de μ con respecto a m_N.

DEFINICIÓN **A.5** *Sea I un espacio métrico y sea $\{f_t\}_{t \in I}$ una familia de aplicaciones, $f_t : \mathcal{M} \to \mathcal{M}$, tal que cada f_t tiene asociada una única medida de probabilidad absolutamente continua y f_t-invariante cuya densidad es h_t. La familia $\{f_t\}_{t \in I}$ se dice **estadísticamente estable** si h_t converge a $h_{t'}$ con respecto a la norma L^1 cuando $t \to t'$.*

TEOREMA **A.6 (Teorema Principal)** *Para todo $t \in (t_0, 1]$ siendo $t_0 = \frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}} \approx 0.882$, la aplicación Λ_t posee un atractor extraño $\mathcal{R}_t \subset \mathcal{T}$. Además, la aplicación Λ_t es fuertemente topológicamente mezclante en $\mathcal{R}_t$, las órbitas periódicas son densas en $\mathcal{R}_t$ y $\mathcal{R}_t$ es un atractor extraño bidimensional: existe una Λ_t-órbita densa en $\mathcal{R}_t$ con dos exponentes de Lyapounov positivos. Además, $\mathcal{R}_t$ tiene asociada una única medida de probabilidad ergódica, absolutamente continua e invariante y la familia $\{\Lambda_t\}_{t \in (t_0,1]}$ es estadísticamente estable.*

Veamos entonces la relación entre la familia Λ_t y las dinámicas que surgen cuando ciertas tangencias homoclínicas son desplegadas en el escenario tridimensional.

A.1 DINÁMICAS HOMOCLÍNICAS

Desde los estudios de Poincaré, los escenarios homoclínicos suelen asociarse a la presencia de una importante cantidad de dinámicas complejas. En especial, desde la década de los 60 hasta el día de hoy se ha realizado un gran esfuerzo para clarificar todos los comportamientos caóticos que surgen cuando se despliega una tangencia homoclínica en el escenario bidimensional. Entendemos por *despliegue* la creación de órbitas homoclínicas asociadas a puntos de silla periódicos (en el Capítulo 1 se dará una explicación más detallada). Más concretamente, sea $f : \mathcal{M} \to \mathcal{M}$ un difeomorfismo definido sobre una superficie $\mathcal{M}$ y asumamos que existe un punto periódico de tipo silla p cuyas variedades invariantes se intersecan en un punto homoclínico q. Una pregunta natural en este escenario es la siguiente: ¿cómo afecta a la dinámica la existencia de este punto homoclínico? O, de forma más precisa: ¿qué cambios se producen en la dinámica cuando se crea dicha órbita homoclínica?

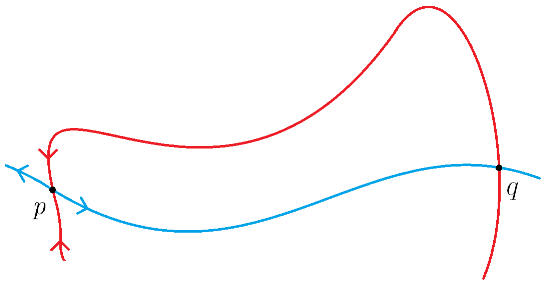


Figure A.4: Tangencia homoclínica bidimensional

La mayor parte de los resultados obtenidos para este problema comparten un mismo punto de partida: el uso de las llamadas *aplicaciones límite retorno* que surgen asociadas al despliegue de tangencias homoclínicas bidimensionales. En pocas palabras, las ***aplicaciones límite retorno*** explican el comportamiento asintótico de las iteradas de orden alto del difeomorfismo restringido a ciertos entornos de la órbita homoclínica. Bajo hipótesis genéricas, en el caso disipativo bidimensional las aplicaciones límite retorno vienen dadas por

$$F_a(x,y) = (1 - ax^2, 0)$$

y, por tanto, el núcleo de las respectivas dinámicas se corresponde con la más que conocida familia cuadrática bidimensional $f_a(x) = 1 - ax^2$.

Puesto que estamos interesados en atractores, hemos de recordar un resultado obtenido por L. Mora y M. Viana en [33] (véase también el Teorema A.7). Este resultado garantiza la existencia de atractores extraños físicamente persistentes cuando se despliega una tangencia homoclínica genérica asociada a un punto periódico disipativo Q en dimensión dos. En este sentido, cabe señalar que en [7] y [25] se realiza un profundo estudio del comportamiento "caótico" de la familia cuadrática f_a para un conjunto de parámetros con medida de Lebesgue positiva y, además, las técnicas empleadas y los resultados obtenidos en estos artículos fueron utilizados en [33]. Entendemos por *atractores extraños físicamente persistentes* aquellos atractores extraños persistentes en el sentido de la medida de probabilidad. Véase, de nuevo, el Teorema A.7.

Es interesante señalar que los resultados obtenidos en [33] han sido establecidos en el escenario bidimensional, por lo que es natural preguntarse si es posible obtener resultados similares en dimensión superior. Como puede verse en [47] y [48], existen evidencias numéricas de la aparición de diferentes tipos de atractores extraños cuando se despliegan tangencias homoclínicas asociadas a cierta clase de difeomorfismos tridimensionales. Como en el caso bidimensional, el punto de partida para estudiar estos fenómenos vuelve a ser la existencia de aplicaciones límite retorno asociadas al despliegue de dichas tangencias en dimensión tres. Remitimos al lector al artículo [51], donde se construyen estas familias de aplicaciones límite retorno. Además, en el Capítulo 1 se hará un breve resumen de este trabajo.

Como es de esperar, al igual que en el despliegue de tangencias en dimensión dos, estas aplicaciones límite retorno no son lineales. De hecho, las aplicaciones límite retorno vienen dadas por

$$T_{a,b}(\widetilde{x},\widetilde{y}) = (a + \widetilde{y}^2, \widetilde{x} + b\widetilde{y}). \tag{A.3}$$

(véase la Sección A.3 para una explicación detallada).

Sin embargo, para los valores $a = -4$ y $b = -2$, la correspondiente aplicación límite retorno es conjugada a una aplicación lineal a trozos definida en cierto triángulo (véase Capítulo 2, Sección 2.1, o [47]). Esto mismo se cumple en el caso de dimensión inferior, es decir, cuando $\dim \mathcal{M} = 2$ la aplicación límite retorno $f_2(x) = 1 - 2x^2$ es conjugada a la aplicación lineal a trozos (*aplicación tienda*) $\lambda_2(x) = 1 - 2|x|$ y este hecho es muy útil de cara a obtener el *Main Theorem* en [33]. Ambas aplicaciones, f_2 y λ_2, presentan sensibilidad con respecto a las condiciones iniciales: una aplicación $f : I \to I$ definida sobre el intervalo I se dice que presenta ***sensibilidad con respecto a las condiciones iniciales*** si existe un valor $\epsilon > 0$ tal que para todo $x \in I$ y para todo entorno U de x, existe un punto $y \in U$ y un natural $n \in \mathbb{N}$ con $|f^n(x) - f^n(y)| > \epsilon$.

Como se pone de manifiesto en [12] (página 145), el deseo de comparar aplicaciones que presentan sensibilidad con respecto a las condiciones iniciales con aplicaciones sencillas (lineales a trozos) tiene una larga historia para los difeomorfismos del círculo, véase [24]. En el caso de aplicaciones definidas en intervalos, cabe citar en primer lugar un resultado de Milnor-Thurston, véase [31], que asegura que toda función continua y monótona a trozos con entropía topológica positiva es semiconjugada a una aplicación continua y lineal a trozos con pendiente constante y la misma entropía. Puede consultarse un resultado previo de Parry en [41] y el Capítulo II, Sección 8, de [30] donde este tipo de resultados se tratan de forma más extensa. Además, a partir del Teorema II.7.12 de [12] sabemos que si $f_a(x) = 1 - ax^2$ no tiene órbitas periódicas estables ni puntos centrales restrictivos entonces existe un valor $a' \in (\sqrt{2}, 2]$ tal que f_a y $\lambda_{a'}$ son conjugadas, siendo $\lambda_{a'}(x) = 1 - a'|x|$. En la Sección II.7 de [12] puede verse esto de forma más detallada.

Por tanto, uno de los principales propósitos de esta memoria será diseñar una familia de aplicaciones tienda bidimensionales que desempeñen el mismo papel que la aplicación λ_a en dimensión dos. Esta familia de aplicaciones tienda bidimensionales será la ya definida $\{\Lambda_t\}_t$ cuya construcción puede verse en el Capítulo 3, Sección 3.2. Todas las funciones de esta familia serán *Expanding Baker Maps*.

La definición de *Expanding Baker Map* se expone en el Capítulo 3, Sección 3.1. En pocas palabras, una *Expanding Baker Map* es una aplicación que pliega un dominio de $\mathbb{R}^2$ y expande la región resultante, al igual que hace un panadero cuando amasa. Como hemos visto al principio de esta Introducción, esto es exactamente lo que la aplicación Λ_t introducida en (A.1) hace sobre el triángulo $\mathcal{T}$.

Para finalizar esta sección es preciso señalar que las aplicaciones límite retorno definidas en (A.3) fueron introducidas previamente por Gonchenko, Shilnikov y Turaev en una serie de artículos como [18] y [19], o [20] y [21]. En estos trabajos, los autores estudiaron familias uniparamétricas $\{X_\mu\}_\mu$ de sistemas dinámicos que despliegan una tangencia homoclínica en dimensión mayor que dos obteniendo familias biparamétricas de aplicaciones límite retorno en las que uno de los parámetros depende de los autovalores de la diferencial en la órbita periódica de tipo silla correspondiente. Sin embargo, nosotros trataremos con familias biparamétricas de aplicaciones límite retorno definidas en una variedad de dimensión tres y desplegando una tangencia homoclínica generalizada (véase la definición en el Capítulo 1) y, como consecuencia, en nuestro caso ambos parámetros (a y b en la definición de $T_{a,b}$) dependen de la geometría de la tangencia.

Notar además que bajo el cambio de variable

$$X = \widetilde{y} \ , \ Y = \widetilde{x} + b\widetilde{y}$$

la aplicación $T_{a,b}$ definida en (A.3) toma la forma

$$T_{a,b}(X, Y) = (Y, a + X^2 + bY).$$

Estas aplicaciones también han sido introducidas en un libro de Mira y otros donde los autores desarrollan un extenso estudio de las aplicaciones cuadráticas no invertibles (véase [32]).

De hecho, puede comprobarse en el Teorema 1.18 que si $f : \mathbb{R}^2 \to \mathbb{R}^2$ es una aplicación cuadrática tal que

i) $f \circ f$ es cuadrática

ii) f no tiene foliaciones lineales invariantes

entonces f es linealmente conjugada a la aplicación $(x, y) \mapsto (a + bx - y^2, x)$ para ciertos valores a y b (véase Teorema 2 en [51] para más detalle).

A.2 Revisión del caso bidimensional

Sea $\mathcal{M}$ una variedad bidimensional y consideremos una familia uniparamétrica de difeomorfismos $\{f_\mu\}_\mu$, $f_\mu : \mathcal{M} \to \mathcal{M}$, tal que f_{μ_0} presenta un punto homoclínico q_{μ_0} asociado a un punto hiperbólico de tipo silla p_{μ_0}. Más aún, asumimos que para $\mu = \mu_0$ las variedades estable e inestable de p_{μ_0} presentan una tangencia cuadrática en q_{μ_0}. Se dice que la familia $\{f_\mu\}_\mu$ ***despliega genéricamente la tangencia homoclínica*** si para $\mu < \mu_0$ las variedades invariantes de p_μ (la continuación analítica de p_{μ_0}) no se cortan y para $\mu > \mu_0$ existe un punto homoclínico transversal asociado a p_μ. Véase como ejemplo la Figura A.5.

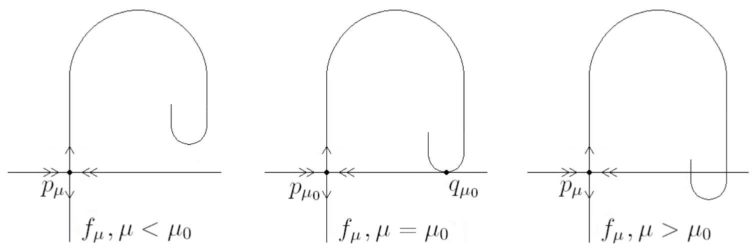


Figure A.5: Despliegue de una tangencia homoclínica en dimensión dos

Uno de los propósitos durante las últimas cuatro décadas ha sido determinar si las dinámicas hiperbólicas prevalecen para valores del parámetro $\mu > \mu_0$. Entre los principales resultados en este sentido cabe mencionar el demostrado por L. Mora y M. Viana en el siguiente teorema.

Teorema **A.7** *Sea $\{f_\mu\}_\mu$ una familia de difeomorfismos definidos en una superficie y que despliega una tangencia homoclínica de f_{μ_0}. Supongamos que f_{μ_0} es disipativa en el punto de silla involucrado en la tangencia. En estas condiciones, existe un conjunto de parámetros con medida positiva para los cuales f_μ presenta atractores extraños de tipo Hénon.*

Antes de definir qué es un *atractor de tipo Hénon*, es necesario remarcar que tanto el resultado que acabamos de ver como otros resultados interesantes, véase por ejemplo los resultados de Colli ([13]), Newhouse ([36]), Yorke-Alligood ([58]) o Palis-Yoccoz ([40]) entre otros, utilizan la existencia de *aplicaciones límite retorno* asociadas a un despliegue de tangencias homoclínicas.

Describamos brevemente el ***esquema de renormalización*** en el caso bidimensional (véase [52] y [38]). Para una visión más completa de esto puede consultarse [56]. Sea $\{f_\mu\}_\mu$ una familia uniparamétrica de difeomorfismos disipativos $f_\mu : \mathcal{M} \to \mathcal{M}$ tales que f_{μ_0} presenta una tangencia homoclínica genérica (la tangencia es cuadrática y se despliega genéricamente). Entonces, para valores $n \in \mathbb{N}$ suficientemente grandes, existen entornos U_n próximos al punto homoclínico q que verifican:

i) $f_\mu^n(U_n) \cap U_n \neq \varnothing$.

ii) f_μ^n restringido a U_n es conjugada a una aplicación $F_{a,n} = \Phi_n \circ f_\mu^n \circ \Phi_n^{-1}$, $a = a(\mu)$, definida en cierto dominio de $\mathbb{R}^2$.

iii) $F_{a,n} \to F_a$ cuando $n \to \infty$, siendo F_a la llamada ***aplicación límite retorno***.

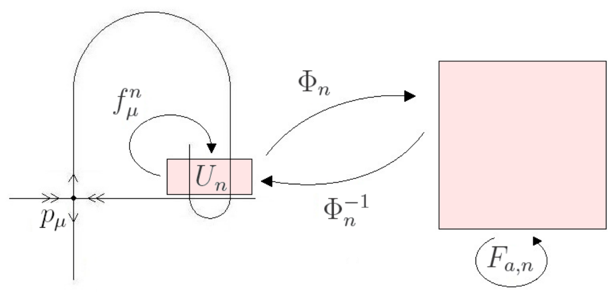


Figure A.6: Esquema de renormalización en dimensión dos

Más aún, en el caso bidimensional la familia de aplicaciones límite retorno puede escribirse de la forma (véase [52])

$$(x, y) \mapsto (a - x^2, 0).$$

Para $a > 1$ esta familia de endomorfismos es equivalente a la siguiente

$$F_a(x, y) = (1 - ax^2, 0). \qquad \text{(A.4)}$$

Esto significa que una iterada suficientemente alta de f_μ restringida a cierto entorno U_n próximo al punto homoclínico es conjugada a una pequeña perturbación de una aplicación unidimensional.

La familia de aplicaciones Hénon $h_{a,b}(x,y) = (1 - ax^2 + y, bx)$ para $b > 0$ puede escribirse como

$$h_{a,b}(x,y) = (1 - ax^2 + \sqrt{b}y, \sqrt{b}x)$$

de forma que para valores pequeños de b la propia familia Hénon es una pequeña perturbación de (A.4). Este es el principal motivo por el que los atractores extraños que surgen en [33] se llaman *atractores de tipo Hénon*: aparecen en la dinámica de pequeñas perturbaciones de (A.4).

Recordemos que (A.4) también aparece como familia de aplicaciones límite retorno en dimensión superior. Por ejemplo, bajo la hipótesis de *disipatividad seccional*, es decir, si $\dim \mathcal{M} = n$ y $f_\mu : \mathcal{M} \to \mathcal{M}$ despliega una tangencia homoclínica asociada a un punto periódico de tipo silla p cuyos autovalores $\lambda_1, \lambda_2, \ldots, \lambda_n$ verifican

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_{n-1}| < 1 < |\lambda_n| \ , \ |\lambda_{n-1}\lambda_n| < 1.$$

En ese caso, en [55] queda demostrado que la familia de aplicaciones límite retorno que surge está definida por

$$F_a(x_1, x_2, \ldots, x_n) = (1 - ax_1^2, 0, \ldots, 0).$$

Este hecho también se utiliza en [39] y [49] para extender los resultados clásicos de Newhouse en dimensión dos, véase [34] y [35], a cualquier dimensión. Es más, en [19] se prueba que los conjuntos hiperbólicos salvajes de Newhouse (véase [34]) existen para sistemas arbitrarios que están C^2-próximos a algún sistema con una tangencia homoclínica.

En vista de los resultados anteriores, parece claro que conocer las dinámicas de la familia cuadrática $f_a(x) = 1 - ax^2$ ha sido esencial para obtener los resultados de [8] y [33].

De hecho, muchos de los argumentos utilizados para demostrar la existencia de atractores extraños persistentes en el caso bidimensional (véase [33]) están basados en los utilizados para probar la existencia de un conjunto de parámetros con medida de Lebesgue positiva para los cuales la órbita del punto crítico de $f_a(x) = 1 - ax^2$ presenta crecimiento exponencial en la derivada, véase [25] y [7].

Por último, cabe señalar que un buen acercamiento para entender las dinámicas de la famosa familia cuadrática unidimensional es estudiar primero las dinámicas de la familia de aplicaciones tienda dadas por $\lambda_a(x) = 1 - a|x|$. Como ya vimos, es conocido que para algunos valores del parámetro (por ejemplo $a = 2$) ambas familias muestran el mismo tipo de comportamiento dinámico. Más concretamente, el hecho de que para $a = 2$ ambos sistemas dinámicos sean conjugados es muy útil para obtener los resultados en [7].

A.3 Aplicaciones retorno para cierta clase de difeomorfismos 3−D

En [51] se obtienen las familias de aplicaciones límite retorno asociadas a familias biparamétricas genéricas de difeomorfismos tridimensionales que despliegan tangencias homoclínicas generalizadas. Más concretamente, se consideran familias biparamétricas $\{f_{a,b}\}_{a,b}$ de difeomorfismos tridimensionales con un punto fijo hiperbólico de tipo silla p_0 para $(a,b)=(0,0)$ tales que:

1.- La familia $\{f_{a,b}\}_{a,b}$ satisface la condición de linealización (una condición genérica para familias de difeomorfismos con puntos fijos de tipo silla).

2.- Los autovalores λ_1, λ_2 y λ_3 son reales y satisfacen $0<|\lambda_1|<1<|\lambda_2|<|\lambda_3|$. Nótese que debido a la condición de linealización el correspondiente punto fijo no puede ser seccionalmente disipativo.

3.- Las variedades invariantes de p_0 presentan una tangencia homoclínica generalizada que se despliega genéricamente (nótese que en el conjunto de familias biparamétricas de difeomorfismos tridimensionales que despliegan tangencias homoclínicas aquellas que despliegan una tangencia homoclínica generalizada son genéricas).

Puede encontrarse la definición de *tangencia homoclínica generalizada* y la *condición de linealización* en el Capítulo 1 de la presente memoria o en [51].

En [51] (véase *Theorem* 1) se prueba que, bajo las hipótesis anteriores, puede construirse una familia biparamétrica de aplicaciones límite retorno $\{\widetilde{f}_{\widetilde{a},\widetilde{b}}\}$ (asociada a una órbita homoclínica) dada por

$$\lim_{n\to\infty} F_{\widetilde{a},\widetilde{b},n}(\widetilde{x},\widetilde{y},\widetilde{z}) = \widetilde{f}_{\widetilde{a},\widetilde{b}}(\widetilde{x},\widetilde{y},\widetilde{z}) = (\widetilde{z},\widetilde{a}+\widetilde{b}\widetilde{y}+\widetilde{z}^2,\widetilde{y}).$$

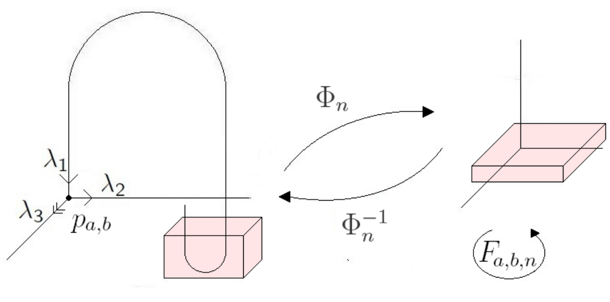


Figure A.7: Esquema de renormalización en dimensión tres

Notar que para cada $\left(\widetilde{a},\widetilde{b}\right) \in \mathbb{R}^2$, todo punto de $\mathbb{R}^3$ "cae" tras una iterada por la aplicación $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ en la superficie

$$C_{\widetilde{a},\widetilde{b}} = \left\{(\widetilde{x},\widetilde{y},\widetilde{z}) : \widetilde{y} = \widetilde{a} + \widetilde{b}\widetilde{z} + \widetilde{x}^2\right\}.$$

Entonces, $C_{\widetilde{a},\widetilde{b}}$ es invariante por $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ y será suficiente estudiar la dinámica de $\widetilde{f}_{\widetilde{a},\widetilde{b}}$ sobre $C_{\widetilde{a},\widetilde{b}}$. No es difícil ver que la aplicación restringida a $C_{\widetilde{a},\widetilde{b}}$ es conjugada a la familia de aplicaciones bidimensionales definida en $\mathbb{R}^2$ por

$$T_{a,b}(\widetilde{x},\widetilde{y}) = (a + \widetilde{y}^2, \widetilde{x} + b\widetilde{y})$$

(hemos escrito $a = \widetilde{a}$ y $b = \widetilde{b}$ para simplificar la notación).

NOTA **A.8** *Existe otro tipo de tangencia homoclínica generalizada en el caso en el que la variedad inestable del punto fijo es unidimensional pero el punto fijo no es seccionalmente disipativo (véase [51] y [17]).*

Como puede verse numéricamente en [48], el comportamiento dinámico de esta familia de aplicaciones límite retorno es bastante complicado y, en particular, los atractores exhibidos por $T_{a,b}$ para un amplio conjunto de parámetros parecen ser atractores extraños bidimensionales (la suma de los exponentes de Lyapounov es positiva). Además, los autores construyen un amplio conjunto de parámetros (a, b) para los cuales $T_{a,b}$ posee una región invariante. En particular, en [47] se construye una curva de parámetros $(a(t), b(t))$ de forma que la correspondiente transformación $T_{a(t),b(t)}$ presenta una región invariante en $\mathbb{R}^2$ homeomorfa al triángulo $\mathcal{T}$. Esta curva de parámetros viene dada por

$$\mathcal{G} = \{(a(t), b(t)) = (-\frac{1}{4}t^3(t^3 - 2t^2 + 2t - 2), -t^2 + t) : t \in \mathbb{R}\}. \qquad \text{(A.5)}$$

De hecho, la curva de parámetros anterior se corresponde con aquellos parámetros para los cuales existe una recta en $\mathbb{R}^2$ que es invariante por $T^2_{a,b}$. Este hecho es utilizado en [47] para demostrar la existencia del dominio invariante $\mathcal{D}_t$ por $T_{a,b}$ (la forma de $\mathcal{D}_t$ se muestra en la Figura A.8). Cabe señalar que la curva de parámetros definida en (A.5) contiene al punto $(-4, -2) = (a(2), b(2))$ y, para este valor, la aplicación $T_{-4,-2}$ es conjugada a la aplicación lineal a trozos y no invertible

$$\widetilde{\Lambda}(\Phi, \Psi) = (\pi - |\pi - \Phi - \Psi|, \Phi - \Psi)$$

definida en el triángulo

$$\widetilde{\mathcal{T}} = \{(\Phi, \Psi) : 0 \leq \Phi \leq \pi, 0 \leq \Psi \leq \Phi\}.$$

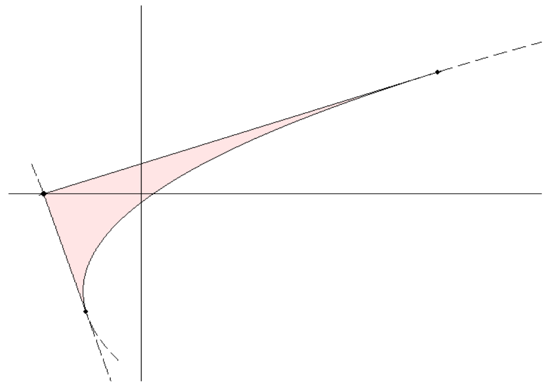


Figure A.8: Conjunto $\mathcal{D}_t$ para $t = 1.9$.

(véase [47]). Como se puede comprobar en el Capítulo 2, Sección 2.1, o en [47], la conjugación entre $T_{-4,-2}$ y $\tilde{\Lambda}$ viene dada por la aplicación $\Gamma_2 : \mathcal{D}_2 \to \widetilde{\mathcal{T}}$ definida por

$$\Gamma_2(\tilde{x},\tilde{y}) = \left(\arccos\left(\frac{\tilde{y}-\sqrt{\tilde{y}^2+8\tilde{y}-4\tilde{x}}}{4}\right), \arccos\left(\frac{\tilde{y}+\sqrt{\tilde{y}^2+8\tilde{y}-4\tilde{x}}}{4}\right)\right). \tag{A.6}$$

Tras un nuevo cambio de coordenadas

$$(x,y) = \Xi(\Phi,\Psi) = (\frac{1}{\pi}(\Phi+\Psi), \frac{1}{\pi}(\Phi-\Psi)) \tag{A.7}$$

es fácil ver que la aplicación $\tilde{\Lambda}$ puede escribirse de la forma

$$\Lambda(x,y) = \begin{cases} (x+y, x-y) & , \text{si } (x,y) \in \mathcal{T}_0 \\ (2-x+y, 2-x-y) & , \text{si } (x,y) \in \mathcal{T}_1 \end{cases} \tag{A.8}$$

definida sobre el triángulo $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ descrito en (A.2).

Obsérvese que la aplicación Λ dada en (A.8) coincide con la aplicación Λ_t definida en (A.1) cuando $t = 1$.

Como se indica en [47], la aplicación Λ muestra las mismas buenas propiedades que la aplicación tienda unidimensional $\lambda_2(x) = 1-2|x|$. En particular, las sucesivas preimágenes de la recta crítica $\mathcal{C} = \{(x,y) \in \mathcal{T} = 1\}$, denotadas por $\{\Lambda^{-n}(\mathcal{C})\}_{n\in\mathbb{N}}$, definen una secuencia de particiones de $\mathcal{T}$ cuyo diámetro tiende a cero cuando n tiende a infinito. Esta secuencia de particiones nos permite conjugar Λ (y, por tanto, $T_{-4,-2}$) con una aplicación *shift* de dos símbolos (véase [47] para una explicación detallada). Además, para toda condición inicial $Q_0 \in \mathcal{T}$ cuya órbita nunca visite la recta crítica, el exponente de Lyapounov a lo largo de la órbita de Q_0 es positivo en toda dirección no nula (de hecho, coincide con el valor $\frac{1}{2}\log 2$) y lo mismo se mantiene para la aplicación límite retorno $T_{-4,-2}$. Por último, no es difícil definir una medida ergódica invariante y absolutamente continua para Λ y, por tanto, para $T_{-4,-2}$ (véase [47]).

Estas son las razones por las que los autores denominaron a Λ la ***aplicación tienda bidimensional***.

Un primer acercamiento para simplificar las dinámicas de las aplicaciones límite retorno $T_{a,b}$ podría ser aplicar el cambio de coordenadas Γ_2 dado en (A.6) a las aplicaciones $T_{a(t),b(t)}$, siendo $(a(t), b(t)) \in \mathcal{G}$ (véase (A.5)). Recordemos que este cambio de coordenadas conjuga $T_{-4,-2}$ con Λ tras el cambio de variables universal Ξ definido en (A.7).

Con este objetivo, definimos la aplicación

$$\Gamma_{1,t} : (\widetilde{x}, \widetilde{y}) \mapsto \Gamma_{1,t}(\widetilde{x}, \widetilde{y}) = \left(\frac{16}{t^4}x + \frac{16(2-t)}{t^3}y + 4 - \frac{8}{t}, \frac{8}{t^3}y \right)$$

que transforma el conjunto invariante $\mathcal{D}_t$ asociado a $T_{a(t),b(t)}$ en el dominio invariante $\mathcal{D}_2$ (véase [48] para más detalles). Así, denotando por

$$\widetilde{F}_t = \Xi \circ \Gamma_2 \circ \Gamma_{1,t} \circ T_{a(t),b(t)} \circ \Gamma_{1,t}^{-1} \circ \Gamma_2^{-1} \circ \Xi^{-1}$$

(véase *Proposition 3.4* en el Capítulo 3, Sección 3.2, se prueban ciertas propiedades dinámicas de $\widetilde{F}_t$ y, además, en la Proposición 3.6 del Capítulo 3, Sección 3.2), se demuestra que si $G : \mathcal{T} \to \mathcal{T}$ es una aplicación lineal a trozos que satisface todas estas propiedades dinámicas, entonces $G = \Lambda_t$ para algún t, siendo $\{\Lambda_t\}_t$ la familia de aplicaciones definida en (A.1).

Estas son las principales razones por las que trabajaremos con Λ_t a lo largo de esta memoria (véase *Remark* 3.7).

A.4 Diferentes tipos de atractores (panes)

Es fácil ver que para valores del parámetro $t \in [0, 1/\sqrt{2})$ el origen es un atractor global de Λ_t (véase Lema 3.8 en el Capítulo 3, Sección 3.2). Por otro lado, si $t \in (1/\sqrt{2}, 1]$ se puede comprobar fácilmente que Λ_t tiene un punto fijo en $\mathcal{T}_1$ dado por

$$P_t = \left(\frac{2t(2t+1)}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1} \right),$$

Distinguimos los tres casos siguientes.

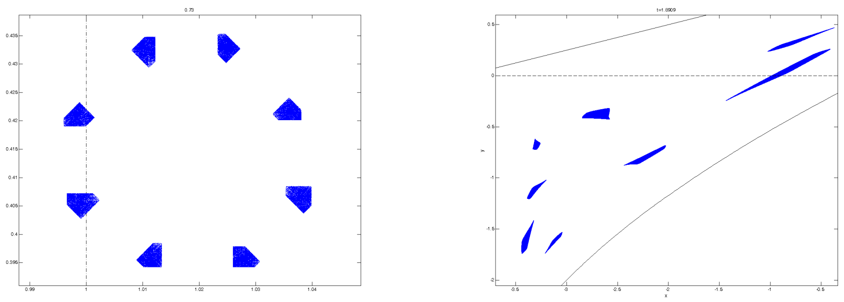
A) ATRACTOR DE TIPO MAGDALENAS.-

Asumamos que $\frac{1}{\sqrt{2}} < t < (\frac{1}{4})^{\frac{1}{5}}$. En este intervalo de parámetros se obtienen atractores formados por varias piezas, como puede verse en la Figura A.9(a). Cuando $t = \frac{1}{\sqrt{2}}$ aparecen órbitas periódicas repulsoras Q_t de periodo ocho. No podemos hablar de bifurcaciones de Hopf puesto que no tenemos regularidad, pero es preciso señalar que para todo $\frac{1}{\sqrt{2}} < t \leq 1$ los autovalores λ_1 y λ_2 del punto fijo P_t satisfacen

$$\lambda_1 = -t + ti \qquad \lambda_2 = -t - ti$$

y por tanto $\lambda_1^8 = \lambda_2^8$ es un número real positivo.

Como se indica en [48], para $T_{a,b}$ también se observan atractores formados por ocho piezas. Por ejemplo, si nos restringimos a la curva de parámetros $(a(t), b(t))$ definida en (A.5), para $t = 1.8909$ se obtiene el atractor que se muestra en la Figura A.9(b). Cabe señalar que estos atractores de ocho piezas también contienen órbitas de periodo ocho que surgen debido a una bifurcación de Hopf (véase [48] para más detalles).

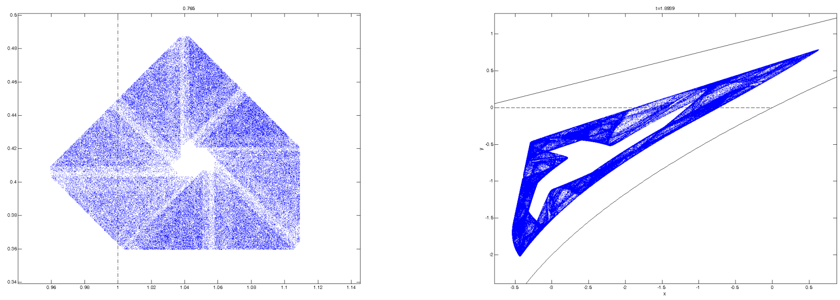


(a) Atractor para Λ_t, $t = 0.73$ (b) Atractor para $T_{a(t),b(t)}$, $t = 1.8909$

Figure A.9: Atractores de tipo magdalenas

B) ATRACTORES DE TIPO ROSCA.-

Si $(\frac{1}{4})^{\frac{1}{5}} < t < (\frac{1}{2})^{\frac{1}{3}}$, entonces Λ_t presenta atractores formados por una sola pieza con un agujero central, como puede verse en la Figura A.10(a). Este tipo de atractores también se observan en la familia $T_{a(t),b(t)}$ (véase [48]). Por ejemplo, para $t = 1.88817$ se obtiene numéricamente el atractor que se muestra en la Figura A.10(b).

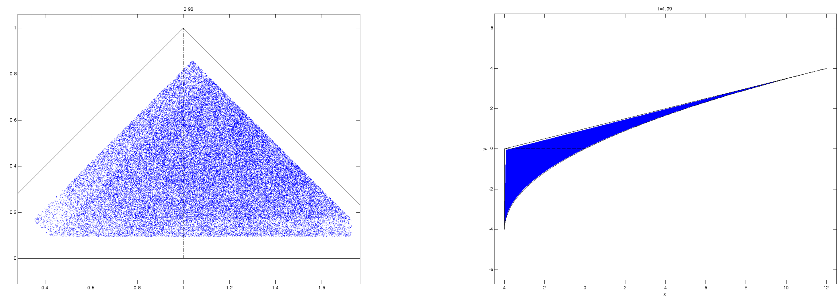


(a) Atractor para $\Lambda_t,\ t = 0.765$ (b) Atractor para $T_{a(t),b(t)},\ t = 1.88817$

Figure A.10: Atractores de tipo rosca

c) ATRACTORES DE TIPO HOGAZA.-

Por último, si $(\frac{1}{2})^{\frac{1}{3}} < t \leq 1$, entonces Λ_t presenta un atractor formado por una pieza sin agujero (véase la Figura A.11(a)) y este tipo de atractores se observan también para $T_{a(t),b(t)}$. Por ejemplo, para $t = 1.88904$ se obtiene el atractor que se muestra en la Figura A.11(b).



(a) Atractor para $\Lambda_t,\ t = 0.80$ (b) Atractor para $T_{a(t),b(t)},\ t = 1.88904$

Figure A.11: Atractores de tipo hogaza

Con todo esto, es claro que para ciertos valores de los parámetros las aplicaciones $T_{a(t),b(t)}$ y Λ_t exhiben el mismo "tipo" de atractores.

Esta memoria se organiza como sigue. En el Capítulo 1 recordamos los resultados de [51] que nos llevan a la construcción de la familia de aplicaciones límite retorno dada en (A.3).

El Capítulo 2 está orientado a describir propiedades básicas de las aplicaciones $T_{a,b}$ haciendo especial hincapié en la dinámica para el caso $a = -4$

y $b = -2$. Estos resultados pueden encontrarse en [47] y [48]. Terminaremos este capítulo estableciendo dos conjeturas "difíciles?" que probablemente nos tendrán ocupados durante las próximas décadas.

En el Capítulo 3 demostraremos una parte del Teorema Principal (véase Teorema A.6). Para valores $t \in (t_0, 1]$ se construye el conjunto invariante maximal $\mathcal{R}_t$ para Λ_t y se demuestra que Λ_t es fuertemente topológicamente mezclante sobre $\mathcal{R}_t$. Es más, demostramos la existencia de una única medida ergódica invariante y absolutamente continua μ_t para Λ_t. Estos resultados pueden encontrarse en [44], [45] y [46]. La existencia de μ_t está íntimamente relacionada con los resultados dados en [10], [11], [50], [53] y [54]. De esta forma, la familia $\{\Lambda_t\}_{t\in(t_0,1]}$ es un ejemplo para los resultados obtenidos en dichos artículos.

Por último, en el Capítulo 4 se demuestra que la familia $\{\Lambda_t\}_{t\in(t_0,1]}$ es estadísticamente estable. Todos los argumentos utilizados pueden encontrarse en [5].

Para finalizar, hemos de mencionar los contenidos del Apéndice del Capítulo 3 (véase Capítulo 3, Sección 3.5). Aunque los resultados del Teorema Principal están restringidos al caso de *atractores de tipo hogaza*, tenemos ciertos resultados parciales ("sensaciones") para el caso de *atractores de tipo magdalenas*. Estamos buscando resultados en la línea de las técnicas de renormalización usadas en dimensión uno y, en este sentido, esperamos alcanzarlos en un futuro próximo.

En especial, cabe señalar que la Proposición 3.41 nos muestra que Λ_t^2 puede ser vista como una nueva *Expanding Baker Map* pero con dos pliegues previos a la expansión. Esto significa que el panadero puede obtener el mismo pan amasando de dos maneras distintas: o bien dobla la masa, la estira, dobla en el otro sentido y vuelve a estirarla o bien realiza los dos pliegues sucesivamente antes de estirar la masa.

Para el caso de *atractores de tipo rosca* aún no hemos obtenido resultados interesantes, pero el tiempo dirá...

Appendix B

Conclusiones

Como consecuencia de los resultados de la tesis, se ha iniciado el estudio de las complejas dinámicas que surgen en el despliegue de tangencias homoclínicas para difeomorfismos definidos en variedades tridimensionales.

Cabe destacar la definición de lo que hemos bautizado como ***Expanding Baker Maps***, no solamente por su relación con el fenómeno homoclínico antes descrito, sino por poner de manifiesto la complicada dinámica que surge al iterar una aplicación lineal a trozos definida en un triángulo. Sin lugar a duda, este tipo de aplicaciones son el equivalente en dimensión dos a las bien conocidas, y extensamente estudiadas, aplicaciones tienda en dimensión uno.

De igual manera, el teorema principal de la tesis ofrece un ejemplo de familia de aplicaciones bidimensionales exhibiendo atractores extraños bidimensionales persistentes. Este hecho, fuera del contexto de las aplicaciones triangulares, supone un hito en el campo de los Sistemas Dinámicos.

Asimismo, se ofrece por vez primera (fuera del contexto del análisis complejo) una familia de transformaciones bidimensionales susceptibles de presentar el llamado ***fenómeno de renormalización***. Estas ideas han dado fructuosos resultados en dimensión uno a lo largo de las últimas cuatro décadas y, sin embargo, no habían sido descubiertas en dimensión dos (exceptuando algún caso excepcional en variable compleja). Hablando de manera poco científica, se trata de encontrar subconjuntos en el espacio de fases donde la dinámica de una determinada iterada de la transformación se comporte de igual manera que la dinámica de otra transformación equivalente definida en todo el espacio de fases. Volviendo al ejemplo cotidiano, esta idea equivale a pensar que cada una de las magdalenas que surgen en determinadas *Expanding Baker Maps* se comportan como las hogazas que surgen en otras *Expanding Baker Maps*. Al final del capítulo tres de la memoria se recogen estas ideas para ser exploradas en un futuro. Por todo ello, esperamos que los ejemplos y resultados contenidos en la tesis sean de gran utilidad para el futuro.

Cabe destacar que la familia de *Expanding Baker Maps* objeto de estudio en la tesis constituye un primer ejemplo para diversos artículos publicados en las últimas dos décadas en relación a la existencia de medidas invariantes ergódicas y absolutamente continuas para familias de transformaciones expansoras y lineales a trozos definidas en un dominio 2-D.

Por último, se prueba en la tesis que nuestra familia de *Expanding Baker Maps* es además estadísticamente estable.

Appendix C

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