

Localisation on Sasaki-Einstein manifolds from holomorphic functions on the cone

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ABSTRACT: We study super Yang-Mills theories on five-dimensional Sasaki-Einstein manifolds. Using localisation techniques, we find that the contribution from the vector multiplet to the perturbative partition function can be calculated by counting holomorphic functions on the associated Calabi-Yau cone. This observation allows us to use standard techniques developed in the context of quiver gauge theories to obtain explicit results for a number of examples; namely S^5 , $T^{1,1}$, $Y^{7,3}$, $Y^{2,1}$, $Y^{2,0}$, and $Y^{4,0}$. We find complete agreement with previous results obtained by Qiu and Zabzine using equivariant indices except for the orbifold limits $Y^{p,0}$ with $p > 1$.

KEYWORDS: Supersymmetric gauge theory, Nonperturbative Effects, Field Theories in Higher Dimensions, Differential and Algebraic Geometry

ARXIV EPRINT: [1401.3266](https://arxiv.org/abs/1401.3266)

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1 Introduction

Localisation allows for exact evaluation of path integrals and expectation values of supersymmetric operators [1–3]. Following the work of Pestun [4], the method has been applied to a large number of theories in two [5–8], three [9–15] four [16, 17], and five dimensions, [18–24]. This development went hand-in-hand with an increased interest in theories with rigid supersymmetry on curved manifolds [25–29]. Eventually, Källen, Qiu, and Zabzine (KQZ) realised, that the construction of the S^5 theory can be directly generalized to generic five-dimensional Sasaki-Einstein manifolds as it only depends on the existence of the conformal Killing spinors. Subsequently, the perturbative partition functions of $Y^{p,q}$ and $L^{a,b,c}$ were calculated in [23, 24]. The recent work [24] conjectures the full partition functions using factorization [30–32].

The focus of this paper is the perturbative partition function of vector multiplets on an arbitrary Sasaki-Einstein manifold Y . Building on the work of KQZ we will argue that

the one-loop super determinant can be expressed in terms of the so-called Kohn-Rossi cohomology groups $H_{\bar{\partial}_b}^{p,q}(Y)$ and the Lie-derivative along the Reeb vector ξ . Previously, the $H_{\bar{\partial}_b}^{p,q}(Y)$ have appeared in the context of holographic calculations of superconformal indices of three- and four-dimensional SCFTs [33–35]. Together with the isomorphism $H_{\bar{\partial}_b}^{0,0}(Y) \cong H^0(\mathcal{O}_{C(Y)})$, our result allows for a easy evaluation of the perturbative partition function, as the whole calculation reduces to the counting of holomorphic functions on the Calabi-Yau cone $C(Y)$, weighted by their charge along the Reeb. This problem is very well known in the context of AdS/CFT duality on $\text{AdS} \times Y$. Here, the holomorphic functions on $C(Y)$ correspond to supersymmetric operators in the chiral ring with R -charge determined by \mathcal{L}_ξ . Following [36] we will use the methods developed in this context to evaluate the partition function. To verify our result, we will do explicit calculations for $S^5, T^{1,1}, Y^{7,3}, Y^{2,1}, Y^{2,0}, Y^{4,0}$. This choice of examples is motivated by the fact that $Y^{7,3}$ and $Y^{2,1}$ are simple examples of quasi-regular and irregular Sasaki-Einstein manifolds. We will find full agreement with previous results except for the last two cases, which arise as \mathbb{Z}_2 and \mathbb{Z}_4 orbifolds of the conifold. We will comment on this in the conclusions.

While the bulk of this paper uses the methods developed in the context of AdS/CFT duality to evaluate partition functions, we will reverse this logic in the final section. There, we will use the partition function on $Y^{p,q}$ as well as the insights won from the examples evaluated to guess the general form of the generating function for $Y^{p,q}$ written as a series.

The paper is organised as follows. In section 2, we review some essentials of Sasaki-Einstein geometry as well as of super Yang-Mills theories defined on them. Section 3 contains the main argument of this paper, relating the super determinant appearing in the perturbative partition function to Kohn-Rossi cohomology groups. Explicit calculation of a number of examples is done in section 4 which is complemented by an evaluation of the same examples using the results of Qiu and Zabzine for comparison in appendix B. The short section 5 concerns the general form of the generating function for the quiver gauge theories dual to $\text{AdS}_5 \times Y^{p,q}$. Further appendices complement the discussion.

2 Localisation on Sasaki-Einstein manifolds

2.1 Aspects of Sasaki-Einstein geometry

We begin with a review of the relevant aspects of Sasaki-Einstein geometry. For a more detailed introduction, we refer to the review articles [37, 38]; further material on the tangential Cauchy-Riemann operator and Kohn-Rossi cohomology groups can be found in [33–35] and references therein.

Let Y be a five-dimensional Sasaki-Einstein manifold, $C(Y)$ it’s metric cone. Y inherits a number of differential forms from $C(Y)$; namely the contact form η , the associated Reeb vector field¹ ξ satisfying $\xi^\mu \eta_\mu = 1$, a two-form $2J = d\eta$, and another two-form Ω . Out of these, only Ω is charged under the Reeb:

$$\mathcal{L}_\xi \Omega = 3\iota \Omega. \tag{2.1}$$

¹In contrast to [23, 24], we only consider the Reeb that admits a Sasaki-Einstein metric. We will return to this restriction in the conclusions.

The tangent bundle TY can be decomposed as $TY = D \oplus L_\xi$, with L_ξ the line tangent to the Reeb. Moreover, J defines an endomorphism on TY which satisfies $J^2 = -1 + \xi \otimes \eta$. It follows that the complexified tangent bundle can be decomposed as $T_{\mathbb{C}}Y = (\mathbb{C} \otimes D)^{1,0} \oplus (\mathbb{C} \otimes D)^{0,1} \oplus (\mathbb{C} \otimes \xi)$. The same holds for the cotangent bundle

$$T_{\mathbb{C}}^*Y = \Omega^{1,0} \oplus \Omega^{0,1} \oplus \mathbb{C}\eta. \tag{2.2}$$

This decomposition extends to the exterior algebra $\Omega^\bullet = \bigoplus_{p,q} \Omega^{p,q} \wedge (1 \oplus \eta)$ and to the exterior derivative: $d = \partial_b + \bar{\partial}_b + \eta \wedge \mathcal{L}_\xi$. $\bar{\partial}_b$ is the tangential Cauchy-Riemann operator. Elements of $\bigoplus \Omega^{p,q}$ are sometimes referred to as horizontal and we will indicate this with a subscript H where appropriate. In terms of the decomposition (2.2), the forms J, Ω are of degree $(1, 1)$ and $(2, 0)$, while η is naturally transverse to $\Omega^{p,q}$. The complex

$$\dots \xrightarrow{\bar{\partial}_b} \Omega^{p,q-1} \xrightarrow{\bar{\partial}_b} \Omega^{p,q} \xrightarrow{\bar{\partial}_b} \Omega^{p,q+1} \xrightarrow{\bar{\partial}_b} \dots \tag{2.3}$$

defines the Kohn-Rossi cohomology groups $H_{\bar{\partial}_b}^{p,q}(Y)$.

2.2 The super Yang-Mills theories

We summarize the aspects of [21–23] that are relevant to our discussion. See also [39, 40] for details regarding the localization argument. Starting point for the construction by Qiu and Zabzine is the S^5 theory of [26]. The field content is given by a vector and a hyper multiplet. In this paper, we will only consider the former. Thus, we are dealing with a gauge field A_m , a scalar σ , an $SU(2)$ doublet of scalars D_{IJ} , and a symplectic Majorana gaugino λ_I . The Lagrangian is

$$L_{\text{vec}} = \frac{1}{g_{\text{YM}}^2} \text{tr} \left(\frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + \frac{2}{r} \sigma t^{IJ} D_{IJ} - \frac{10}{r^2} t^{IJ} t_{IJ} \sigma^2 + \iota \lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] - \frac{\iota}{r} t^{IJ} \lambda_I \lambda_J \right). \tag{2.4}$$

The crucial observation is that both the closure of the supersymmetry algebra, as well as the invariance of the action depend only on the existence of the conformal Killing spinors and the dimension of the space. It follows that the theory as defined on S^5 can be used to define a super Yang-Mills theory on any simply connected² Sasaki-Einstein manifold.

The gaugino is mapped to a 1-form Ψ_m and a 2-form χ_{mn} , the latter of which satisfies $\iota_\xi \chi = 0$ and $\iota_\xi \star \chi = \chi$. Using equations (A.3) and (A.6) one can show that this is equivalent to the decomposition

$$\chi = \chi^{2,0} + \chi^{0,2} + J\chi^{0,0} \quad \text{with} \quad \chi^{p,q} \in \Omega^{p,q}. \tag{2.5}$$

Similarly, the D_{IJ} are mapped to a two-form H with an identical decomposition (2.5). In terms of these, the Localisation term is

$$V_{\text{vec}} = \text{tr} \left[\frac{1}{2} \Psi \wedge \star (-\iota_\xi F - D\sigma) - \chi \wedge \star H + 2\chi \wedge \star F \right]. \tag{2.6}$$

²This criterium is necessary to ensure the existence of the spinors.

In the large- t limit the theory localizes to contact instantons [41, 42],

$$F_H^\pm = 0, \quad \iota_\xi F = 0, \quad D\sigma = 0. \quad (2.7)$$

Here, F_H^\pm is defined as $\frac{1}{2}(1 \pm \iota_\xi \star)F_H = \frac{1}{2}(1 \pm \bullet)F_H$, where \bullet is a restriction of the Hodge dual to horizontal forms and defined in the appendix.

On inclusion of the ghost sector, the perturbative partition function is

$$Z_{\text{pert}} = \frac{1}{|W|} \frac{\text{vol}(G)}{\text{vol}(T)} \int_{\mathfrak{t}} dx \left(\prod_{\beta > 0} \langle \beta, x \rangle \right) \exp \left(-\frac{8 \text{vol}_{\text{SE}}}{g_{\text{YM}}^2 r^2} \text{tr}(x^2) \right) \text{sdet}'_{\text{vec}}(-\iota \mathcal{L}_\xi - x)^{1/2}. \quad (2.8)$$

Here, the domain of integration has been reduced from \mathfrak{g} to the Cartan subalgebra \mathfrak{t} using the Weyl integration formula, $|W|$ is the order of the Weyl group, and sdet' indicates the exclusion of zero-modes. The modes contributing to the superdeterminant are as follows:

$$\begin{aligned} \text{Bosonic modes} & \quad \Omega^1(Y, \mathfrak{g}) \oplus H^0(Y, \mathfrak{g}) \oplus H^0(Y, \mathfrak{g}), \\ \text{Fermionic modes} & \quad (\Omega^{2,0}(Y, \mathfrak{g}) \oplus \Omega^{0,2}(Y, \mathfrak{g}) \oplus \Omega^0(Y, \mathfrak{g})) \oplus \Omega^0(Y, \mathfrak{g}) \oplus \Omega^0(Y, \mathfrak{g}). \end{aligned} \quad (2.9)$$

The three terms in parantheses are identical to $\Omega^{2+}(Y, \mathfrak{g})$ in [21].

3 One-loop contributions and holomorphic functions

In this section, we will study the superdeterminant appearing in (2.8). To simplify the discussion, we drop the contribution from the Lie algebra from all expressions. The argument holds however whether forms are valued in \mathbb{C} or \mathfrak{g} , so one can simply reinstate them later. We will do so at the end of this section.

Proceeding as in [21], we recall that $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1} \oplus \mathbb{C}\eta$. The Lie derivative \mathcal{L}_ξ respects this decomposition. As an aside, note that the tangential Cauchy-Riemann operator $\bar{\partial}_b$ does not. Indeed, since $\bar{\partial}_b \eta = J$, it follows that $\bar{\partial}_b : \mathbb{C}\eta \rightarrow \Omega^{1,1} \oplus \Omega^{0,1} \wedge \eta$. In either case, since the Lie derivative respects the decomposition — and since the contact form is not charged under it — it follows that when it comes to calculating the determinant, all one-forms can be decomposed into the sum of a $(1, 0)$ -, a $(0, 1)$ -form, and a scalar function. In total,

$$\text{sdet}'(-\iota \mathcal{L}_\xi) = \left(\frac{\det_{\Omega^0}(-\iota \mathcal{L}_\xi) \det_{\Omega^{2,0}}(-\iota \mathcal{L}_\xi) \det_{\Omega^0}(-\iota \mathcal{L}_\xi) \det_{\Omega^{0,2}}(-\iota \mathcal{L}_\xi)}{\det_{\Omega^{1,0}}(-\iota \mathcal{L}_\xi) \det_{\Omega^{0,1}}(-\iota \mathcal{L}_\xi)} \right)^{\frac{1}{2}} \frac{1}{\det_{H^0}(-\iota \mathcal{L}_\xi)}. \quad (3.1)$$

In [21–23] equation (3.1) is evaluated using index theorems. We will follow a different route, which is inspired by the supergravity calculations of [33, 34]. For specificity, we focus on the second factor, including determinants over $\Omega^{0,q}$. For all $f \in \Omega^{0,0}$, we can consider

$$\partial_b f \lrcorner \bar{\Omega}, \bar{\partial}_b f \in \Omega^{0,1}, \quad f \bar{\Omega} \in \Omega^{0,2}. \quad (3.2)$$

Any $(0, 1)$ -form α not included in this list cannot be $\bar{\partial}_b$ -exact and has to be co-closed. For all such forms, we consider in addition

$$\bar{\partial}_b \alpha \in \Omega^{0,2}. \quad (3.3)$$

One can construct further forms as $\alpha \lrcorner \Omega \in \Omega^{1,0}$, yet this will be covered by the equivalent discussion of the factor involving the $\Omega^{p,q}$. Now, schematically,

$$\frac{\det_{\Omega^0}(-\imath \mathcal{L}_\xi) \det_{\Omega^{0,2}}(-\imath \mathcal{L}_\xi)}{\det_{\Omega^{0,1}}(-\imath \mathcal{L}_\xi)} \supset (-\imath \mathcal{L}_\xi) \left| \frac{f}{\bar{\partial}_b f} \frac{f \bar{\Omega}}{\partial_b f \lrcorner \bar{\Omega}} \frac{\bar{\partial}_b \alpha}{\alpha} \right. . \quad (3.4)$$

Since $f, \bar{\partial}_b f$ carry the same charge under \mathcal{L}_ξ , their contributions cancel unless f is holomorphic with respect to $\bar{\partial}_b$. Identical considerations hold for the other two factors — recall that $\mathcal{L}_\xi \Omega = 3\imath \Omega$ — as well as for the $\Omega^{p,0}$ terms. In the end, we are left with determinants over Kohn-Rossi cohomology groups

$$\text{sdet}'(-\imath \mathcal{L}_\xi) = \left(\det'_{H_{\bar{\partial}_b}^{0,0}}(-\imath \mathcal{L}_\xi) \det'_{H_{\bar{\partial}_b}^{0,0}}(-\imath \mathcal{L}_\xi) \det_{H_{\bar{\partial}_b}^{0,0}}(-\imath \mathcal{L}_\xi - 3) \det_{H_{\bar{\partial}_b}^{0,0}}(-\imath \mathcal{L}_\xi + 3) \right)^{\frac{1}{2}} . \quad (3.5)$$

The forms α would contribute a determinant over $H_{\bar{\partial}_b}^{0,1}(Y)$, yet this cohomology group vanishes (as does $H_{\bar{\partial}_b}^{1,0}(Y)$). Similarly, all harmonic scalar functions on Sasaki-Einstein manifolds are constants, carry thus zero charge, and are excluded from the superdeterminant sdet' ; so there is no contribution from $H^0(Y)$ either. The latter follows from the inequality for the Laplacian, $\Delta \geq -\mathcal{L}_\xi^2 - 4\imath \mathcal{L}_\xi$ proved in [33, 35]. Alternatively one can simply follow the considerations in [43] in the context of the discussion of the Lichnerowicz obstruction. There is an isomorphism $H_{\bar{\partial}_b}^{0,0}(Y) \cong H_{\bar{\partial}_b}^{2,0}(Y)$, made explicit by the map $f \mapsto f\Omega$. Since Ω carries charge, this absorbs the factors of 3 in (3.5).

$$\text{sdet}'(-\imath \mathcal{L}_\xi) = \left(\det'_{H_{\bar{\partial}_b}^{0,0}}(-\imath \mathcal{L}_\xi) \det'_{H_{\bar{\partial}_b}^{0,0}}(-\imath \mathcal{L}_\xi) \det_{H_{\bar{\partial}_b}^{2,0}}(-\imath \mathcal{L}_\xi) \det_{H_{\bar{\partial}_b}^{0,2}}(-\imath \mathcal{L}_\xi) \right)^{\frac{1}{2}} . \quad (3.6)$$

In theory, elements of Kohn-Rossi cohomology groups could be obtained by restriction of corresponding cohomologies on the cone [33]. In the case of $H_{\bar{\partial}_b}^{0,0}(Y)$ however, we simply need to count holomorphic functions on the cone:

$$H_{\bar{\partial}_b}^{0,0}(Y) \cong H^0(\mathcal{O}_{C(Y)}) . \quad (3.7)$$

Thus, equations (3.5) and (3.7) show that the one-loop contribution to the partition function (2.8) can be calculated solely in terms of the holomorphic functions on $C(Y)$.

For the non-abelian case, we follow [21, 44] and decompose the Lie algebra into root spaces, $\mathfrak{g} = \bigoplus_{\beta} \mathfrak{g}_{\beta}$, which includes the Cartan as $\mathfrak{g}_0 = \mathfrak{t}$. The decomposition extends to the exterior algebra, $\Omega^{p,q}(Y, \mathfrak{g}) = \bigoplus_{\beta} \Omega^{p,q}(Y, \mathfrak{g}_{\beta})$. By definition $\forall g \in \mathfrak{g}_{\beta} : [x, g] = \imath \langle \beta, x \rangle g$. Rewriting $\Omega^{p,q}(Y, \mathfrak{g}_{\beta})$ as $\Omega^{p,q}(Y) \otimes \mathfrak{g}_{\beta}$, the Lie derivative acts on the first factor while x acts only on the second. So, in the non-Abelian case we have

$$\text{sdet}'(-\imath \mathcal{L}_\xi - x) = \prod_{\beta} \left(\det'_{H_{\bar{\partial}_b}^{0,0}}(-\imath O_{\beta,x}) \det'_{H_{\bar{\partial}_b}^{0,0}}(-\imath O_{\beta,x}) \det_{H_{\bar{\partial}_b}^{0,0}}(-\imath O_{\beta,x} - 3) \det_{H_{\bar{\partial}_b}^{0,0}}(-\imath O_{\beta,x} + 3) \right)^{\frac{1}{2}} \quad (3.8)$$

with $O_{\beta,x} = \mathcal{L}_\xi + \langle \beta, x \rangle$.

4 Examples

In this section, we will evaluate (3.5) explicitly for a number of examples and compare our results to those in [23]. We will find complete agreement for $S^5, T^{1,1}, Y^{7,3}, Y^{2,1}$. Curiously, our results disagree for $Y^{2,0}$ and $Y^{4,0}$. Again we restrict to the Abelian case, keeping in mind that one can always incorporate the effect of a non-trivial gauge group as in equation (3.8).

Essentially, we will be counting holomorphic functions on $C(Y)$ with fixed charge under \mathcal{L}_ξ . Fortunately, this is a very well understood problem in AdS/CFT duality, due to the following fact: *Given a Sasaki-Einstein manifold Y and a SCFT dual to $\text{AdS} \times Y$, the holomorphic functions on $C(Y)$ correspond to single trace BPS operators in the chiral ring.* Note that we are talking about entirely unrelated theories — four dimensional SCFTs dual to $\text{AdS} \times Y$ with four supercharges and the five dimensional sYM theories on Y with two supercharges. However, it should always be entirely clear from the context which theory we are referring to. Since all our examples are toric, we will be using [36, 45, 46] to solve the counting problem. In what follows, we will be looking at generating functions [36]

$$P(\{t_i\}) = \sum_{i_1, \dots, i_k} c_{k_1, \dots, k_n} t_1^{k_1} \dots t_n^{k_n}. \quad (4.1)$$

Here, each t_i corresponds to a $U(1)$ symmetry of the SCFT and the multiplicities c_{k_1, \dots, k_n} count the number of operators with charge (k_1, \dots, k_n) . Of course, we are only interested in the charge under the R-symmetry, so we will set the t_i to the relevant linear combination as obtained from a -maximization [46].

Given a generating function of the form

$$P(t; C(Y)) = \sum_{n=0} b_n (t^\alpha)^n, \quad (4.2)$$

one finds

$$\begin{aligned} \det'_{H_{\partial_b}^{0,0}}(-\imath \mathcal{L}_\xi) &= \prod_{n \geq 1} (\alpha n)^{b_n}, & \det'_{H_{\partial_b}^{0,0}}(-\imath \mathcal{L}_\xi) &= \prod_{n \geq 1} (-\alpha n)^{b_n}, \\ \det_{H_{\partial_b}^{0,0}}(-\imath \mathcal{L}_\xi - 3) &= \prod_{n \geq 0} [\alpha(n - 3/\alpha)]^{b_n}, & \det_{H_{\partial_b}^{0,0}}(-\imath \mathcal{L}_\xi + 3) &= \prod_{n \geq 0} [-\alpha(n - 3/\alpha)]^{b_n}. \end{aligned} \quad (4.3)$$

The different bounds on the products on the right hand side arise from the exclusion of zero modes. Thinking in terms of $(0,0)$ - and $(2,0)$ -forms, we are excluding constants c , yet keeping the form $c\Omega$. The above can be rewritten by shifting the charge,

$$\det_{H_{\partial_b}^{0,0}}(-\imath \mathcal{L}_\xi - 3) = \prod_{n \geq 3/\alpha} (\alpha n)^{b_{n-3/\alpha}}, \quad \det_{H_{\partial_b}^{0,0}}(-\imath \mathcal{L}_\xi + 3) = \prod_{n \geq 3/\alpha} (-\alpha n)^{b_{n-3/\alpha}}. \quad (4.4)$$

When we put everything together, the minus signs will cancel in all abelian examples and so the overall result for the simpler cases is

$$\text{sdet}'(-\imath \mathcal{L}_\xi) = \prod_{n \geq 1} (\alpha n)^{b_n} \prod_{n \geq 3/\alpha} (\alpha n)^{b_{n-3/\alpha}}. \quad (4.5)$$

In our examples, we will follow the conventions from [46]. For $Y^{p,q}$ [47], the toric diagram is given by

$$v_1 = [1, 0, 0], \quad v_2 = [1, p - q - 1, p - q], \quad v_3 = [1, p, p], \quad v_4 = [1, 1, 0]; \quad (4.6)$$

the Reeb vector is

$$b_{\min} = \left(3, \frac{3p - 3q + \ell^{-1}}{2}, \frac{3p - 3q + \ell^{-1}}{2} \right); \quad (4.7)$$

and

$$\ell^{-1} = \frac{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}{q}. \quad (4.8)$$

The generating function of $Y^{p,q}$ is thus

$$P(z, x, y; Y^{p,q}) = \sum_{a=1}^p \frac{1}{(1 - yx^{-1})(1 - x^{1-a+p-q}y^{a-p+q}z^{1-a})(1 - x^{a-p+1}y^{-1-a+p-q}z^a)} + \frac{1}{(1 - xy^{-1})(1 - x^{a-1}y^{2-a}z^{1-a})(1 - x^{-a}y^{a-1}z^a)}. \quad (4.9)$$

4.1 S^5

The generating function for S^5 can be found in [36]:

$$P(t; S^5) = \sum_{n \geq 0} \left(\frac{n^2}{2} + \frac{3n}{2} + 1 \right) t^n. \quad (4.10)$$

Upon substitution into (4.5), the super determinant is

$$\text{sdet}'(-\iota\mathcal{L}_\xi) = \prod_{n \geq 1} n^{n^2+2}, \quad (4.11)$$

which agrees with [21]. Since S^5 is a S^1 bundle over $\mathbb{C}\mathbb{P}^2$, one can also use the Borel-Weil-Bott theorem together with the Weyl dimension formula to obtain the same result. See appendix C.

4.2 $T^{1,1}$

We proceed by considering the next canonical example — the base of the conifold ($T^{1,1} = Y^{1,0}$). Here, the generating function is

$$P(t; T^{1,1}) = \sum_{n \geq 0} (n + 1)^2 \left(t^{3/2} \right)^n. \quad (4.12)$$

Therefore

$$\text{sdet}'(-\iota\mathcal{L}_\xi) = \prod_{n \geq 1} \left(\frac{3}{2}n \right)^{2(n^2+1)}. \quad (4.13)$$

As shown in appendix B, this agrees with [23]. Both S^5 and $T^{1,1}$ are regular Sasaki-Einstein manifolds.

4.3 $Y^{7,3}$

In contrast to the previous two examples, $Y^{7,3}$ is a quasi-regular Sasaki-Einstein manifold. The condition for quasi-regularity is that $4p^2 - 3q^2 = n^2$ with $n \in \mathbb{Z}$. $(7, 3)$ is the simplest example, followed by $\{(7, 5); (13, 7); (13, 8); (14, 6); (14, 10); \dots\}$. Using (4.7) and (4.9), we substitute $z \mapsto t^3$, $x, y \mapsto t^{\frac{28}{3}}$. Since (4.9) contains terms of order $(1 - x/y)^{-1}$ and our substitution sets $x = y$, one has to take some care when taking the limit. After doing so, one obtains a series expansion with integer coefficients in terms of $\tau = t^{\frac{1}{3}}$:

$$P(\tau; Y^{7,3}) = 1 + 3\tau^9 + 5\tau^{18} + 7\tau^{27} + 5\tau^{28} + 11\tau^{35} + 9\tau^{36} + 7\tau^{37} + \mathcal{O}(\tau^{44}). \quad (4.14)$$

There are different ways of rewriting this in the form of equation (4.2). In order to be able to compare our result with appendix B, we define

$$\begin{aligned} I_7 &= \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid i - j = 0 \pmod{7}\}, \\ m_{ij} &= \frac{10i + 4j}{7} + 1, \\ P_\Sigma(\tau; Y^{7,3}) &= \sum_{I_7} m_{ij} \tau^{5i+4j}. \end{aligned} \quad (4.15)$$

Using Mathematica, one sees that $P(\tau; Y^{7,3}) - P_\Sigma(\tau; Y^{7,3}) = \mathcal{O}(\tau^{4001})$ which seems sufficient to assume that equality holds to all orders and that both series have the same limit. After some further algebra one finds

$$\text{sdet}'(-i\mathcal{L}_\xi) = \prod_{I_7 \mid i, j > 0} \left(\frac{5i + 4j}{3}\right)^{2\frac{10i+4j}{7}} \prod_{I_7 \mid i=0 \vee j=0} \left(\frac{5i + 4j}{3}\right)^{\frac{10i+4j}{7} + 1}. \quad (4.16)$$

Again, this agrees with [23].

4.4 $Y^{2,1}$

Finally, we turn to an example of an irregular Sasaki-Einstein manifold, $Y^{2,1}$. The necessary steps are in principle the same as for $Y^{7,3}$, yet the series expansion is naively a bit more difficult due to the appearance of irrational exponents. We proceed by calculating $P(z, x, y; Y^{2,1})$, substituting $y \mapsto x$ and then performing a double series expansion in x, z . In detail,

$$P(z, x, x; Y^{2,1}) = \frac{x \{x - z [-2z + (-3 + z(3 + z))x + 2x^2]\}}{(z^2 - x)^2 (1 - x)^2}. \quad (4.17)$$

In analogy to section 4.3, we define

$$\begin{aligned} I_2 &= \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid i - j = 0 \pmod{2}\}, \\ m_{ij} &= \frac{3i + j}{2} + 1, \\ P_\Sigma(z, x, x; Y^{2,1}) &= \sum_{I_2} m_{ij} x^{\frac{j-i}{2}} z^i. \end{aligned} \quad (4.18)$$

Again, one can check agreement between $P_\Sigma(z, x, x; Y^{2,1})$ and $P(z, x, x; Y^{2,1})$ using Mathematica; one finds $P(z, x, x; Y^{2,1}) - P_\Sigma(z, x, x; Y^{2,1}) = \mathcal{O}(z^{3 \times 150}) \mathcal{O}(x^{(\sqrt{13}-1) \times 150})$. Now,

we substitute $z \mapsto t^3, x \mapsto t^{\sqrt{13}-1}$ using (4.7) and find the generating function of $Y^{2,1}$ in terms of the Reeb

$$P_{\Sigma}(t; Y^{2,1}) = \sum_{I_2} m_{ij} t^{\frac{(7-\sqrt{13})i + (\sqrt{13}-1)j}{2}}. \quad (4.19)$$

Again, this allows us to calculate the one-loop contribution to the partition function,

$$\begin{aligned} \text{sdet}'(-\iota \mathcal{L}_{\xi}) &= \prod_{I_2 | i, j > 0} \left(\frac{(7-\sqrt{13})i + (\sqrt{13}-1)j}{2} \right)^{3i+j} \\ &\quad \prod_{I_2 | i=0 \vee j=0} \left(\frac{(7-\sqrt{13})i + (\sqrt{13}-1)j}{2} \right)^{\frac{3i+j}{2}+1}. \end{aligned} \quad (4.20)$$

Once again, appendix B shows that this agrees with [23]. For a detailed discussion of $Y^{2,1}$ in the context of quiver gauge theories see [48]. These theories themselves were first constructed in [49].

4.5 $Y^{p,0}$

Recall that $Y^{p,0} = (\text{conifold})/\mathbb{Z}_p$ while $Y^{p,p} = (\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C})/\mathbb{Z}_p$ [45]. In regards to what follows, one should keep in mind that it is not clear whether the super Yang-Mills theory is well defined on orbifolds. Nevertheless, one can use identical methods as in the previous paragraphs to evaluate the super determinant for $Y^{2,0}$. One finds

$$\begin{aligned} P(t; Y^{2,0}) &= \sum_{n=0}^{\infty} (2n+1)^2 (t^3)^n, \\ \text{sdet}'(-\iota \mathcal{L}_{\xi}) &= \prod_{n \geq 1} (3n)^{2[(2n)^2+1]} = \prod_{n \in 2\mathbb{Z}_{>0}} \left(\frac{3}{2}n \right)^{2(n^2+1)}. \end{aligned} \quad (4.21)$$

As to $Y^{4,0}$,

$$\begin{aligned} P(t; Y^{4,0}) &= \sum_{n=0}^{\infty} \frac{(2n+1)(2n+1-(-1)^{n+1})}{2} (t^3)^n, \\ \text{sdet}'(-\iota \mathcal{L}_{\xi}) &= \prod_{n \geq 1} (3n)^{4n^2+(-1)^{n+1}}. \end{aligned} \quad (4.22)$$

As we argue in appendix B, if one naively applies the results of [23] for the one-loop contribution on $Y^{p,0}$, the result is always (4.13), independent of p . Clearly, both our results for $Y^{2,0}$ and $Y^{4,0}$ do not show this behavior. While in the former case the result has the same overall form with the product being taken over a different lattice, this is not the case for $Y^{4,0}$. Since the result for $Y^{2,0}$ differs from the $Y^{1,0}$ one by a factor two in the lattice spacing, one can speculate whether the two will agree after renormalization. Naive application of zeta function regularization does not yield agreement.

5 Generating functions for $Y^{p,q}$

So far, we have used [36] and [46] in order to compute (3.5) and compare the result with that of [23]. In this section, we simply invert this process and use the general form of the contribution to the one-loop partition function from [23] in order to guess the generating function for generic $Y^{p,q}$ manifolds in terms of the Reeb; i.e. as in equation (4.2). While [36] gives a prescription for the calculation of generating functions that is very straightforward to implement, rewriting them in the form (4.2) can be a bit of a nuisance, as our calculations for $Y^{7,3}$ and $Y^{2,1}$ show. Thus, comparing our results (4.15) and (4.18) with the material in appendix B suggests that

$$\begin{aligned}
 m_{ij} &= \frac{(p+q)i + (p-q)j}{p} + 1, \\
 I_p &= \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid i - j = 0 \pmod{p}\}, \\
 P(t; Y^{p,q}) &= \sum_{I_p} m_{ij} t^{\frac{[3(p+q)-\ell^{-1}]i + [3(p-q)+\ell^{-1}]j}{2p}}.
 \end{aligned} \tag{5.1}$$

Of course, it would be interesting to verify this starting from [36].

6 Conclusions

In this paper, we have studied the perturbative partition function of super Yang-Mills theories on five-dimensional Sasaki-Einstein manifolds Y following the work of Qiu, Zabzine, and collaborators. Using the intrinsic structure of Y , we argued that the contribution from the vector multiplet can be calculated in terms of Kohn-Rossi cohomology groups (3.5). Thus, the calculation can be reduced to a counting problem on the Calabi-Yau cone $C(Y)$ which is very well understood in the context of AdS/CFT duality. This gives an alternative approach to that via index theorems previously used in the literature.

Of course, the disagreement of our results for $Y^{2,0}$ and $Y^{4,0}$ with [23] is puzzling; yet this has to be taken in light of the question whether it is possible to define the theory on an orbifold in the first place. As we argue in appendix B, one can see quickly that the result of [23] for the super determinant (denoted there as P_{vec}) is independent of p for $Y^{p,0}$, which holds not in our case. However, if one restricts to the case $p > q > 0$, our examples in sections 4.3 and 4.4 suggest full agreement with [23]. Indeed, when performing the necessary calculations for various examples, the relevant steps take on a somewhat mechanical nature that simply needs adapting some parameters. This goes hand in hand with our guess for the generating function in section 5. Assuming that the theory might be well-defined as it is, it is interesting to note that our results for $Y^{2,0}$ and $Y^{1,0}$ take an identical form, with half the modes contributing to the latter having been modded out.

Independently of this, note that for regular Sasaki-Einstein manifolds Y , the tangential Cauchy-Riemann operator can be thought of as an ordinary Dolbeault operator twisted by a suitable line bundle over the Kähler-Einstein base. I.e. $\bar{\partial}_b$ is now related to some $\bar{\partial}_V$ when acting on forms of fixed charge. The latter was used in [21] to evaluate the super determinant with an index theorem. Considering this comparison in the context of generic

Sasaki-Einstein manifolds, one sees that it should be possible to calculate the perturbative partition function in terms of the equivariant index $\text{ind}_{\mathcal{L}_\xi}(\bar{\partial}_b)$. This can also be seen by considering (3.6).

There are some immediate directions of possible future research, such as the inclusion of hypermultiplets or the calculation of additional examples such as del Pezzo surfaces. Furthermore, as highlighted earlier, we only considered the Reeb vector that admits a Sasaki-Einstein metric while the results of [23, 24] hold for generic choices of ξ . Assuming the validity of our construction in this general case, one might achieve this generalization by choosing a different diagonal $U(1)$ in the generating functions $P(t; Y)$. The choice of Reeb and equivariant parameters features strongly in the latter of the above references, where the authors used factorization to conjecture the full, non-perturbative form of the partition function. In general, any use of the methods employed here towards a better understanding of contact instantons and the full, non-perturbative partition function is of obvious great interest.

Acknowledgments

I would like to thank Diego Rodríguez-Gómez for the many discussions without which the completion of this project would have taken considerably longer. Furthermore, I would like to thank Andrés Viña Escalar, Eoin Ó Colgáin, Yolanda Lozano, and Maxim Zabzine for various discussions, comments on the manuscript, and very helpful correspondence respectively.

A Hodge duals on Sasaki-Einstein manifolds

We review some notation from [35] that is quite useful when manipulating expressions involving the Hodge star operator. The material is a straightforward generalization of identical ideas on Kähler manifolds to the Sasaki-Einstein case. To begin, we define the adjoint of the Lefschetz operator $L \equiv J \wedge$ as well as an adjoint for the action of the Reeb $L_\eta \equiv \eta \wedge$:

$$\Lambda = L^* = J \lrcorner, \quad \Lambda_\eta = L_\eta^* = \iota_\xi. \tag{A.1}$$

The space of horizontal forms can be denoted as $\bigoplus \Omega^{p,q} = \bigwedge^* D^*$. For elements of this space, we introduce the operator

$$I = \sum_{p,q} \iota^{p-q} \Pi^{p,q}, \tag{A.2}$$

which uses the projection $\Pi^{p,q} : \Omega_{\mathbb{C}}^* \rightarrow \Omega^{p,q}$. Finally, we can introduce a restricted Hodge dual \bullet that acts only on $\bigwedge^* D^*$. The first useful relation we find is

$$\star |_{\bigwedge^* D^*} = L_\eta \bullet, \quad \star |_{\bigwedge^* D^* \wedge \eta} = \bullet (-1)^{d^0} \Lambda_\eta. \tag{A.3}$$

Where

$$d^0 |_{\bigwedge^k D^* \wedge (1 \oplus \eta)} = k \cdot \text{id} \tag{A.4}$$

yields the horizontal degree of a form. We also introduce $P^k = \{\alpha \in \wedge^k D^* | \Lambda\alpha = 0\}$, the set of primitive k -forms. With all this notation, one can introduce Lefschetz decomposition. Given any $\alpha \in \wedge^k D^*$, there is a unique decomposition

$$\alpha = \sum_r L^r \alpha_r, \quad \alpha_r \in P^{k-2r}. \quad (\text{A.5})$$

Moreover, one can prove the identity

$$\forall \alpha \in P^k, \quad \bullet L^j \alpha = (-1)^{\frac{k(k-1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \mathbf{I}(\alpha), \quad (\text{A.6})$$

where $d = 2n + 1$ is the dimension of the Sasaki-Einstein manifold. Together with (A.3) this allows for an efficient evaluation of Hodge duals. The complete algebra involving $\partial_b, \bar{\partial}_b, L, L_\eta, \mathcal{L}_\xi$ and their adjoints was derived in [35].

B The super determinant as computed by Qiu and Zabzine

We summarize the result for the one-loop contribution to the partition function on $Y^{p,q}$ from [23]. Again, we restrict to the abelian case

$$\text{sdet}'(-iL_\xi) = \prod_{\Lambda_0^+} (i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4)^2 \prod_{\Lambda_1^+} (i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4). \quad (\text{B.1})$$

The integers i, j, k, l lie in the lattices

$$\begin{aligned} \Lambda^+ &= \{i, j, k, l \in \mathbb{Z}_{\geq 0} | (p+q)i + (p-q)j = p(k+l)\}, \\ \Lambda_0^+ &= \{i, j, k, l \in \mathbb{Z}_{> 0} | (p+q)i + (p-q)j = p(k+l)\}, \\ \Lambda_1^+ &= \Lambda^+ \setminus (\Lambda_0^+ \cup \{0, 0, 0, 0\}). \end{aligned} \quad (\text{B.2})$$

The ω_i depend on the choice of Reeb with the supersymmetric choice being

$$\omega_1 = 0, \quad \omega_2 = \frac{\ell^{-1}}{p+q}, \quad \omega_3 = \omega_4 = \frac{3}{2} - \frac{\ell^{-1}}{2(p+q)}. \quad (\text{B.3})$$

ℓ^{-1} was defined in equation (4.8).

For $Y^{1,0} = T^{1,1}$, we define $n \equiv k+l$ and note that the number of lattice points for fixed n is

$$\#\Lambda^+|_{n \geq 0} = (n+1)^2, \quad \#\Lambda_0^+|_{n > 0} = (n-1)^2, \quad \#\Lambda_1^+|_{n \geq 0} = 4n. \quad (\text{B.4})$$

Upon substitution, this confirms (4.13). As a matter of fact, the lattices (B.4) are identical for all $Y^{p,0}$ since p simply drops out. The same holds for the ω_i .

For $Y^{7,3}$, we note that the integers i, j, k, l are subject to the constraint

$$10i + 4j = 7(k+l). \quad (\text{B.5})$$

We introduce the set

$$I_7 = \{(i, j) \in \mathbb{Z}_{\geq 0}^2 | i - j = 0 \pmod{7}\}. \quad (\text{B.6})$$

For a pair $(i, j) \in I_7$, we find that $\Lambda_0^+|_{(i,j)}$ contains $\frac{10i+4j}{7} - 1$ and $\Lambda^+|_{(i,j)}$ $\frac{10i+4j}{7} + 1$ lattice points. If $i = 0$ or $j = 0$, $\Lambda_1^+|_{(i,j)}$ consists also of $\frac{10i+4j}{7} + 1$ points, yet if $ij \neq 0$, there are only two points in $\Lambda_1^+|_{(i,j)}$. To calculate the super-determinant, we eliminate $k + l$ and find

$$\text{sdet}'(-\iota\mathcal{L}_\xi) = \prod_{I_7|i,j>0} \left(\frac{5i+4j}{3}\right)^{2\frac{10i+4j}{7}} \cdot \prod_{I_7|i=0\vee j=0} \left(\frac{5i+4j}{3}\right)^{\frac{10i+4j}{7}+1}. \tag{B.7}$$

For $Y^{2,1}$ the situation is almost identical. Here we define

$$I_2 = \{(i, j) \in \mathbb{Z}_{\geq 0}^2 | i - j = 0 \pmod{2}\} \tag{B.8}$$

to parametrize the lattices. Things work out in a way identical to $Y^{7,3}$ and one finds

$$\begin{aligned} \text{sdet}(-\iota\mathcal{L}_\xi) &= \prod_{I_2|i,j>0} \left(\frac{(7-\sqrt{13})i + (\sqrt{13}-1)j}{2}\right)^{3i+j} \\ &\quad \prod_{I_2|i=0\vee j=0} \left(\frac{(7-\sqrt{13})i + (\sqrt{13}-1)j}{2}\right)^{\frac{3i+j}{2}+1}. \end{aligned} \tag{B.9}$$

C S^5 and the Borel-Weil-Bott theorem

Since S^5 is a regular Sasaki-Einstein manifold, the orbits of the Reeb close and yield a principal bundle over $\mathbb{C}\mathbb{P}^2$. It follows that the Kohn-Rossi cohomology groups with fixed charge n are isomorphic to the cohomology groups of the base twisted by a suitable line bundle. Then, the Borel-Weil-Bott theorem³ allows us to relate these to representations of A_2 .

$$\begin{aligned} H_{\bar{\partial}_b}^{0,0}(S^5)|_n &\cong H^0(\mathbb{C}\mathbb{P}^2, \mathcal{L}^n) \cong V_{[n,0,0]}^{A_2}, \\ H_{\bar{\partial}_b}^{2,0}(S^5)|_n &\cong H^0(\mathbb{C}\mathbb{P}^2, \Omega^2 \otimes \mathcal{L}^n) \cong V_{[n-3,0,0]}^{A_2}. \end{aligned} \tag{C.1}$$

Finally, we use the Weyl dimension formula⁴

$$\dim V_\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{C.2}$$

to calculate the dimension of the cohomology groups:

$$\begin{aligned} \dim V_{[n,0,0]} &= 1 + \frac{3}{2}n + \frac{1}{2}n^2, & (= \text{ind } \bar{\partial}_V), \\ \dim V_{[n-3,0,0]} &= 1 - \frac{3}{2}n + \frac{1}{2}n^2, & (= \text{ind } \partial_V). \end{aligned} \tag{C.3}$$

The indices $\text{ind } \bar{\partial}_V$ and $\text{ind } \partial_V$ were calculated in [21].

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³See the appendix of [34] for further examples of this.

⁴See e.g. equation (7.18) in [50].

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