EVALUATION OF FATIGUE LIFE DATA BY NORMALISING PROCEDURES

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ABSTRACT

The generally scarce and sparse number of fatigue results available from experimental programmes emphasizes the necessity of efficient procedures for evaluating the statistical parameters defining the S-N field. This, in turn, has an influence on the cumulative damage calculation and prediction, since damage indices use the S-N field as basic information. In this paper a consistent and compatible regression model for defining the S-N field will be considered and normalising procedures are suggested for evaluating the model parameters. Different normalizing approaches are subsequently used for deriving damage indices for fatigue life prediction under variable loading and their comparative suitabilities are then discussed.

1.- INTRODUCTION

Structures and mechanical components are frequently submitted to loads of variable amplitudes and of a random nature. The corresponding fatigue life prediction has to be analysed by means of damage accumulation models which utilise as basic information the S-N field of the material, determined from fatigue life tests conducted at various different constant stress ranges. Thus, the reliability of the life prediction under variable amplitude loading depends to a great extent on the quality of the estimation of the parameters related to the S-N field. Accordingly, a statistical non-linear regression analysis of the fatigue results is needed on account of the limited number of fatigue results spread over several stress ranges and of the considerable scatter of the results within each stress range. Since two random variables have to be considered – the stress range $\Delta\sigma$ or the stress level σ, depending on the material tested, and the number of cycles to failure N – two different statistical distributions, $F(N; \Delta\sigma)$, representing the number of cycles to failure given the stress range $\Delta\sigma$, or else $E(\Delta\sigma; N)$, representing the stress range given the number of cycles to failure, could be envisaged. Both distributions must fulfil physical and statistical conditions for the statistical model to be valid.

In this paper, a consistent statistical model for analysing the S-N field is presented as well as methods for estimating the model parameters, based on normalising test data. Additionally, damage indices identified as the normalised variables are defined and their interpretation discussed.

2.- A MODEL FOR ANALYZING FATIGUE LIFE DATA

In what follows the statistical model for analysing fatigue life data developed by Castillo et al. [1] will be considered. This model, based on the weakest link principle, arises from a functional equation after setting physical and statistical requirements (stability, compatibility and limit conditions) to the distributions $F(N; \Delta\sigma)$ and $E(\Delta\sigma; N)$ which prove to be three-parameter Weibull distribution families for minima [1, 2].

Thus, the cumulative distribution function (c.d.f.) of the logarithm of the lifetime N, given the stress range $\Delta\sigma_i$, is given by:

$$F(\log N; \Delta\sigma_i) = 1 - \exp\left[\left(-\frac{(\log N - B)(\log \Delta\sigma_i - C)}{D} + E\right)^A\right], \tag{2}$$

from which the percentile curves (see Figure 1) can be derived:

$$(\log N - B)\,(\log \Delta\sigma - C) = D\left[\left[-\log(1-P)\right]^{1/A} - E\right], \tag{1}$$

where N is the lifetime measured in number of cycles to failure, $\Delta\sigma_i$ is the stress range and A, B, C, D and E are the model parameters to be determined, with the following meanings:

A = Weibull shape parameter;
B = Threshold value or limit lifetime;
C = Endurance limit;
D = Scale parameter;
E = Parameter determining the position of the zero-percentile curve.

As soon as the five parameters are determined, the analytical expression of the whole S-N field is known, which allows for the probabilistic prediction of the fatigue failure under constant amplitude loading. As can be observed, the percentile curves are represented by equilateral hyperbolas.

The Weibull parameters corresponding to the lifetime at a given stress range $\Delta\sigma_i$ can be related to the model parameters A, B, C, D and E through the expressions:

$$\lambda(\Delta\sigma_i) = B - \frac{ED}{\log \Delta\sigma_i - C}, \tag{3}$$

$$\delta(\Delta\sigma_i) = \frac{D}{\log \Delta\sigma_i - C}, \tag{4}$$

$$\beta(\Delta\sigma_i) = A\,. \tag{5}$$

from which the mean and standard deviation values at each level can be derived:

$$\mu_i = B + \frac{K_1}{\log \Delta\sigma_i - C}, \tag{6}$$

$$\sigma_i = \frac{K_2}{\log \Delta\sigma_i - C}, \tag{7}$$

where $K_1 = D\left[\Gamma\left(1+\frac{1}{A}\right) - E\right]$ and $K_2 = D\sqrt{\Gamma\left(1+\frac{2}{A}\right) - \Gamma^2\left(1+\frac{1}{A}\right)}$. As a consequence, the mean and standard deviation curves are also represented by equilateral hyperbolas, though latter, logically, not identifiable with the percentile curves.

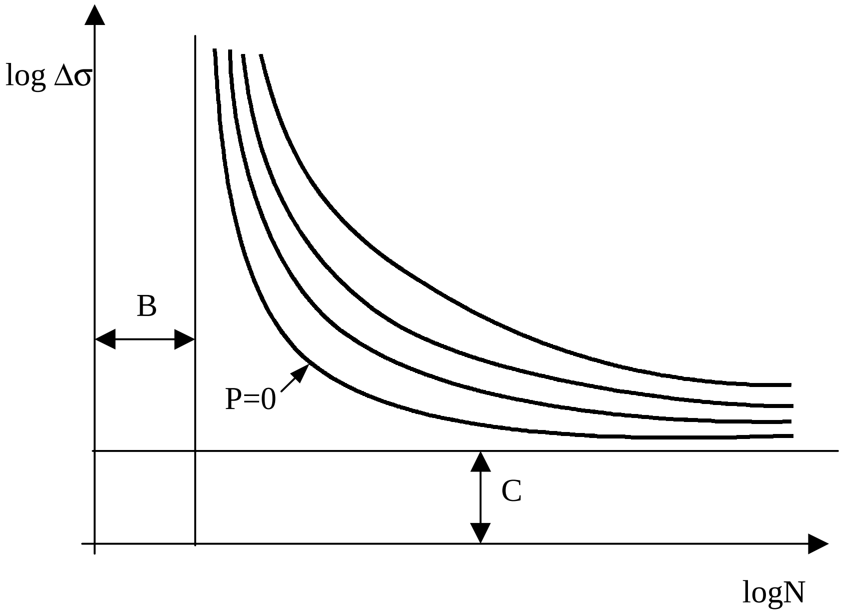

Figure 1: S-N field with percentiles curves in the fatigue model of Castillo et al. [1].

3.- ESTIMATION OF MODEL PARAMETERS USING NORMALIZED VARIABLES

To estimate the five parameters of the model, the analyst has pairs of values (N_i,$\Delta\sigma_i$) obtained in experimental tests carried out at several stress ranges. It is possible to estimate the five parameters simultaneously through maximization of the likelihood function, but this procedure generally encounters convergence and precision problems due to the presence of multiple relative maxima in the likelihood function. Alternatively, a more advantageous two-step method for estimating the model parameters has been proposed by Castillo et al. [3,4,5]: Equation (1) indicates that if the parameters B and C were known the random variable $(\log N - B)(\log \Delta\sigma - C)$ would follow a Weibull distribution with three parameters λ', δ' and β', depending only on A, D and E. This fact suggests estimating the five parameters in two steps, first B and C, then A, D and E. Once B and C have been estimated, the problem becomes a standard estimation of the three parameters of the Weibull distribution. However, while parameters A, D and E are constant over the whole S-N field, the parameters λ_i, δ_i and β_i are related to the stress range, $\Delta\sigma_i$, so that their estimation is conditioned by the fatigue model proposed.

Since all the data related to all participating stress levels affect the evaluation of the model parameters, a statistical normalisation seems to be a desirable procedure. This permits data pertaining to Weibull distributions with the same shape parameter β, but with different location and scale parameters λ and δ respectively (in the present case, the lifetime distributions for different stress ranges) to be pooled in a single distribution in order to evaluate subsequently its parameters with increased reliability**.**

This procedure is based on the fact that a Weibull distribution remains stable with respect to location and scale transformations. Thus, if the variable X follows a Weibull distribution for minima $W(\lambda, \delta, \beta)$, expressed as $X \sim W(\lambda, \delta, \beta)$, the normalised variable Z, defined as $Z = \frac{X - a}{b}$, will also follow a Weibull distribution for minima (see Figure 2):

$$Z \sim W\left(\frac{\lambda - a}{b}, \frac{\delta}{b}, \beta\right) \tag{8}$$

In the fatigue analysis, log N is the original variable identifiable with X. Depending on the location and the scale transformation, i.e. on the “a” and “b” values, Z can result in different expressions which have to satisfy the fatigue model requirements given by Equations (6) and (7). A suitable choice for “a” and “b” will be crucial in the analysis of the reliability of the evaluation and in its interpretation. In this paper three different normalised variables V, D_M and N* will be studied. Table 1 shows the final expression of the normalised variables together with the location and scale transformation parameters, "a" and "b" respectively, while their resulting Weibull parameters are shown in Table 2. The normalisation process has to be applied to the whole data taking into consideration the different stress range levels.

In what follows three different normalised variables, called N*, M_D and V, are presented. The first two are defined irrespective of the related fatigue model, whereas the latter is based on the fatigue model proposed here so that its validity is restricted to the acceptance of the model.

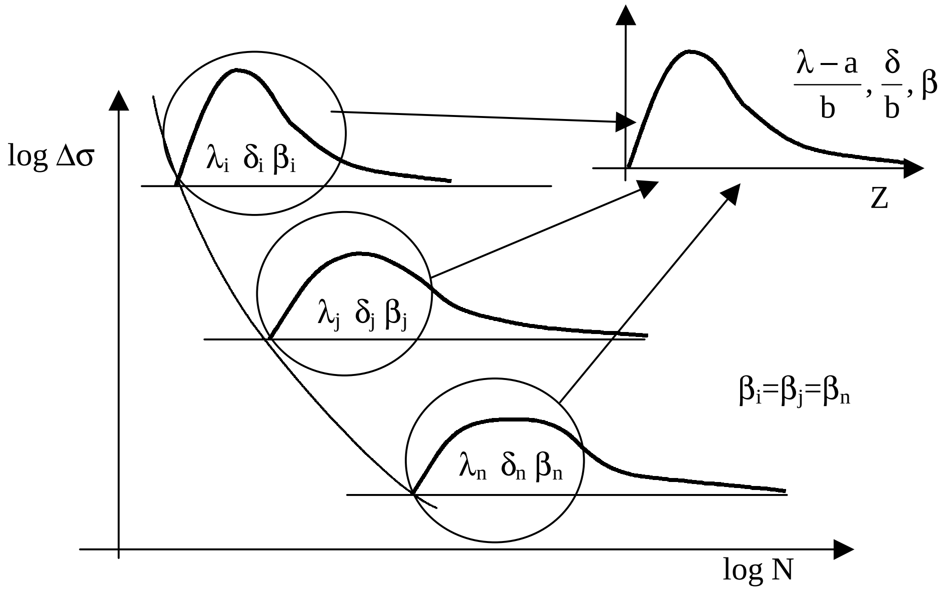

Figure 2: S-N field with pdfs of log N at different stress ranges and corresponding pdf of the normalized lifetime, Z.

3.1.- The variable N*.

The most characteristic normalization consists in the transformation

$$N^* = \frac{N_{ji} - \mu_i}{\sigma_i}, \tag{9}$$

which uses the mean μ_i and standard deviation σ_i values of the experimental results at every stress level as location and scale transformation magnitudes respectively. Since the quality of the estimation of the standard deviation depends largerly on the number of results available at the level studied, this normalising procedure begins with an initial estimation of μ_i and σ_i , followed by iterations in order to fulfil the additional requirements of the fatigue model. According to Expressions (6) and (7) this normalisation requires previous knowledge of four parameters, namely, B, C, K_1 and K_2, which could have an influence on the reliability of the final estimation. On the other hand, the mean value and the standard deviation of N* are always 0 and 1 respectively and the range of variation extends from $-\infty$ to $+\infty$, which facilitates its interpretation at least in part (see Figure 3).

As can be observed in Table 2, the location and scale parameters of N*, i.e. λ^* and δ^*, are only a function of the shape parameter β^*. Thus, the standard methods, which assume independence between the Weibull parameters, do not apply here. A simple computer program, based on a linear regression on Weibull probability paper, can then be developed to estimate β^*.

A parallel procedure could be applied making use of the Weibull location and scale parameters λ_i and δ_i at the stress range $\Delta\sigma_i$ as the magnitudes in the transformation, that is:

$$N^* = \frac{\log N_{ji} - \lambda_i}{\delta_i} \tag{10}$$

As in the previous case, the same considerations apply since λ_i and δ_i can be expressed as simple relations to the mean and standard deviation.

3.2.- The variable D_M.

The variable D_M is defined as:

$$D_M = \frac{\log N_{ji}}{\mu_i}, \tag{11}$$

i.e. as the ratio of the logarithm of the number of cycles to failure to its mean value at the stress level "i", and can be recognised as a variant of the Miner number measured in a logarithmic scale [6] . The mean values for the different stress ranges μ_i are calculated from the data but they are not independent, D_M depends on three parameters B, C and K_1. It can be easily shown that the mean value of D_M is always equal to 1 whereas the standard deviation depends on B, C, K_1 and K_2 thus varying in every experimental fatigue programme (Figure 3). These parameters are independent so that they can be estimated by standard methods.

As can be seen, D_M follows a Weibull distribution for minima at every stress range, but it does neither succeed to constitute a unique variable distribution independent of the stress range, nor fulfil the requirements of the fatigue model. Consequently, it has to be rejected as a candidate to normalising variable.

3.3.- The variable V

Expression (2) reveals that for the proposed fatigue model the probability of failure for a piece subjected to a stress level $\Delta\sigma_i$ during N cycles solely depends on the product $(\log N - B)(\log \Delta\sigma_i - C)$. Thus, the transformation $V_{ij} = (\log N_{ij} - B)(\log \Delta\sigma_i - C)$ provides a useful normalized variable of the number of cycles resulting in a test as soon as the model parameters B and C are known. As mentioned above, different methods for estimating B and C have been proposed, as for instance the one suggested by Castillo et al. [5] which considers estimating B and C by minimizing the function

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left[\log N_{ij} - B - \frac{K_1}{\log \Delta\sigma_{i-C}} \right]^2 \tag{11}$$

with respect to B, C and K_1 using standard procedures. Initial estimates of the parameters seem to be advisable in order to avoid convergence problems.

Once B and C have been estimated, all the normalised data can be pooled together through the transformation $V_{ij} = (\log N_{ij} - B)(\log \Delta\sigma_i - C)$ and the model parameters A, D and E or, equivalently, the normalized Weibull parameters can be estimated using standard techniques.

The mean and standard deviation of V are found to be K_1 and K_2 respectively, using Expressions (6) and (7). Since K_1 and K_2 are functions of the model parameters A, D and E, the mean and standard deviation of V change with each S-N field of the experimental programmes (see Figure 3). This renders not obvious the interpretation of V since this normalized variable can, generally, adopt different variation ranges, depending on the case at hand.

TABLE 1
NORMALISED VARIABLES.

Variable	a	b
$V=(\log N - B)(\log \Delta\sigma_i - C)$	B	$\dfrac{1}{\log \Delta\sigma_i - C}$
$D_M = \dfrac{\log N}{\mu_i}$	0	$\mu_i = B + \dfrac{K_1}{\log \Delta\sigma_i - C}$
$N^* = \dfrac{\log N - \mu_i}{\sigma_i}$	$\mu_i = B + \dfrac{K_1}{\log \Delta\sigma_i - C}$	$\sigma_i = \dfrac{K_2}{\log \Delta\sigma_i - C}$

TABLE 2
PARAMETERS OF NORMALISED VARIABLES.

Variable	λ	δ	β
$V \sim W(\lambda', \delta', \beta')$	$\lambda' = -E\,D$	$\delta' = D$	$\beta' = A$
$D_M \sim W(\lambda_D, \delta_D, \beta_D)$	$\lambda_D = \dfrac{-E\,D + B(\log \Delta\sigma_i - C)}{K_1 + B(\log \Delta\sigma_i - C)}$	$\delta_D = \dfrac{D}{K_1 + B(\log \Delta\sigma_i - C)}$	$\beta_D = A$
$N^* \sim W(\lambda^*, \delta^*, \beta^*)$	$\lambda^* = \dfrac{-\Gamma\left(1+\frac{1}{A}\right)}{\sqrt{\Gamma\left(1+\frac{2}{A}\right) - \Gamma^2\left(1+\frac{1}{A}\right)}}$	$\delta^* = \dfrac{1}{\sqrt{\Gamma\left(1+\frac{2}{A}\right) - \Gamma^2\left(1+\frac{1}{A}\right)}}$	$\beta^* = A$

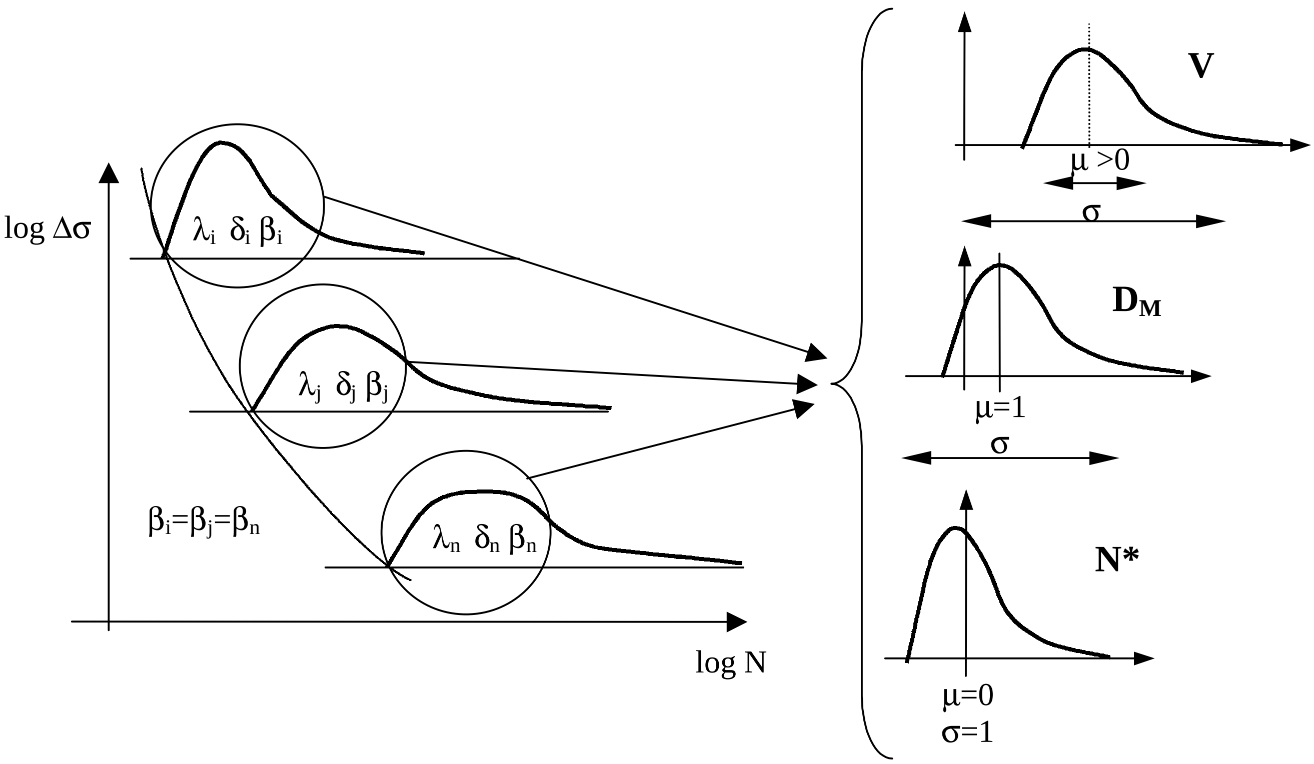

Figure 3: Normalised variables V, D_M and N*

3.4.- Relation between N*, D_M and V.

Since all the normalised variables have been obtained after applying location and scale transformations to the original variable log N, there will be a linear relation between them and also between the corresponding parameters too. Thus, V can be expressed as a function of N* and D_M:

$$V = K_2 \cdot N^* + K_1, \qquad V = K_1 D_M + B\,(\log \Delta\sigma_i - C)\,(D_M - 1) \tag{11}$$

Consequently, the analyst has the choice of using the most suitable variable in each case or some of them simultaneously, although the use of V is recommended since the parameter estimation process is presumably the most convenient and reliable.

A new statistical normalisation of V or D_M is possible and would lead to the variable N* according to:

$$V^* = \frac{V - \mu_V}{\sigma_V} = N^*, \tag{12}$$

$$D^* = \frac{D_M - \mu_D}{\sigma_D} = N^*. \tag{13}$$

4.- DAMAGE ACCUMULATION INDICES

The S-N curves discussed in previous sections were obtained by assuming a constant stress range. For fatigue lifetime prediction and analysis of data from experimental programs under varying loading, a damage accumulation model is required.

Damage accumulation models are established to predict the fatigue life of an element or component without the necessity for experimental observations of the damage process. In order to achieve this, they define a magnitude, usually termed the damage index, which allows the damage state to be transferred from the current stress range to the next one, at which the loading sequence continues, under the hypothesis that in this conversion process (isodamage hypothesis) the damage remains unchangeable [7]. Fatigue failure occurs when the damage index exceeds a critical value depending on the damage index considered. It is apparent that the only way to master this problem is to define the damage index in a probabilistic manner.

The most appropriate way to analyse the real damage accumulation process would be to consider the only variable that seems to possess a real physical meaning: the crack size. Since it is practically impossible to measure microcracks sizes in the material, this magnitude must be defined only from a probabilistic view, that is to say, calculating the probability that a determined crack size would be present in the material. The normalising concept can be advantageously applied here to define the equivalence, in terms of damage, between two fatigue states pertaining to different stress levels. As long as the value of the normalised lifetime variable, identified as damage index, is preserved in each stress range conversion, the damage equivalence is ensured. This is ensured if the damage index, in our case defined as one normalised lifetime variable, is maintained in each conversion since the value of the normalised variable can be associated with a probability of failure. Thus, the case of multistep loading can be considered as a simple extension of the one-step constant stress range case simply assuming the number of cycles at a certain stress range can be replaced by an equivalent number of cycles at the onset of the subsequent stress range by means of the normalised variable. The generalisation to random loading is straightforward. The conversion between levels is identical in blocks and in random loading. The only difference is that in blocks the stress amplitude is known in each cycle, while in random loading, a representative load history of the process has to be generated. Thus, the fundamentals for the statistical interpretation of the damage accumulation indices for variable loading have been established.

A probabilistic conversion from one level $\Delta\sigma_i$ to another $\Delta\sigma_j$ using the variable V is shown in Figure 4.

It can be shown that the damage conversion resulting from V, that is:

$$(\log N_{ik} - B)(\log \Delta\sigma_i - C) = (\log N_{jk} - B)(\log \Delta\sigma_j - C) \tag{14}$$

is totally equivalent to that resulting from N* in both variants, that is:

$$\frac{\log N_{ik} - \lambda_i}{\delta_i} = \frac{\log N_{jk} - \lambda_j}{\delta_j} \quad \text{or} \quad \frac{\log N_{ik} - \mu_i}{\sigma_i} = \frac{\log N_{jk} - \mu_j}{\sigma_j} \tag{15}$$

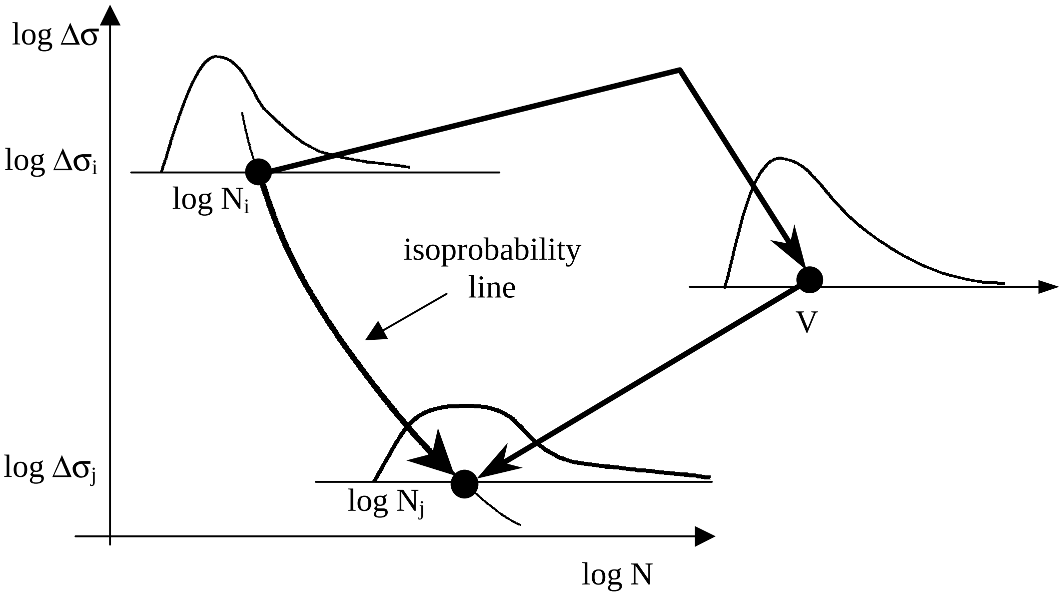

Figure 4: Conversion process with variable V.

5.- CONCLUSIONS.

From the sections above we can conclude:

1. Normalising data leads to an improvement in the estimation of the S-N field model parameters since it allow to poole in a single distribution all the data obtained for different stress ranges thus increasing the reliability of the estimation procedure.

2. Among the normalising variables considered here under acceptance of the S-N model proposed, the one defined through the transformation V_{ij} = (log N_{ij} –B) (log $\Delta\sigma_i$ – C) seems to be the most suitable to define the S-N field for it provides the simplest and more reliable procedure to estimate the model parameters.

3. Normalising variables are related to probabilities of failure and in this way they represent damage states independently of the stress range considered. They allow the conversion between stress ranges and can therefore be adopted as damage accumulation indices.

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