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SCALING FACTOR ESTIMATION USING AN OPTIMIZED MASS CHANGE STRATEGY.

PART 1: THEORY

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Abstract

In natural input modal analysis, only un-scaled mode shapes can be obtained. The mass change method is, in many cases, the simplest way to estimate the scaling factors, which involves repeated modal testing after changing the mass in different points of the structure where the mode shapes are known. The scaling factors are determined using the natural frequencies and mode shapes of both the modified and the unmodified structure. However, the uncertainty on the scaling factor estimation depends on the modal analysis and the mass change strategy (number, magnitude and location of the masses) used to modify the dynamic behavior of the structure. In this paper, a procedure to optimize the mass change strategy is proposed, which uses the modal parameters (natural frequencies and mode shapes) of the original structure as the basic information.

1 Introduction

When natural input modal analysis is performed, only the un-scaled mode shapes $\{\psi\}$ can be obtained for each mode. The scaled and un-scaled mode shapes are related by the equation:

$$\{\phi\} = \alpha \cdot \{\psi\} \tag{1}$$

where α is the scaling factor.

The scaling factors are needed when a modal model, identified by modal analysis, is going to be used for structural response simulation, structural modification or health monitoring.

A way to estimate the scaling factors is to modify the dynamic behavior of the structure changing the stiffness or the mass and perform a new natural modal testing and analysis. The methods based on dynamic modification use both the modal parameters of the unmodified and modified structure. Therefore, a more extensive experimental testing procedure has to be carried out to estimate the

scaling factors. The mass change method consists of attaching several masses in the points where the mode shapes are known.

The accuracy obtained in the scaling factor estimation depends on both the accuracy obtained in the modal parameter identification and the mass change strategy used to modify the dynamic behavior of the structure. Mass change strategy means to define the magnitude, the location and the number of masses to be attached to the structure.

In this paper, several simple rules are proposed that can be used to define, advantageously, the mass change strategy to be followed in the mass change method.

2   The mass change method

The mass change method consists on performing natural input modal analysis on both the original and the modified structure [1, 2, 3, 4, and 5]. The modification is carried out attaching masses to the points of the structure where the mode shapes of the unmodified structure are known. The user selects the number, the magnitude and the location of the masses. The process is, schematically, shown in Figure 1.In order to facilitate the mass modification and the calculation of the scaling factors, lumped masses are often used, so that the mass change matrix $[\Delta m]$ becomes, in general, diagonal.

Simple formulas to determine the scaling factors can be derived from the eigenvalue equations of the original and the modified structure. The classical eigenvalue equation in case of no-damping or proportional damping is:

$$[m] \cdot \{\phi_0\} \cdot \omega_0^2 = [k] \cdot \{\phi_0\}, \tag{2}$$

where $\{\phi_0\}$ is the mode shape, ω_0 the natural frequency, $[m]$ the mass matrix and $[k]$ the stiffness matrix. If we make a mass change so that the new mass matrix is $[m] + [\Delta m]$, then the eigenvalue equation becomes:

$$\Big([m] + [\Delta m] \Big) \cdot \{\phi_1\} \cdot \omega_1^2 = [k] \cdot \{\phi_1\}, \tag{3}$$

where $\{\phi_1\}$ and ω_1 are the new modal parameters of the modified problem.

Assuming that the mass change is so small that the mode shapes does not change significantly, i.e., $\{\phi_1\} \cong \{\phi_0\} \cong \{\phi\}$ and combining equations (1), (2) and (3), the unknown scaling factor can be derived by means of the equation (see [3, 4, 5]):

$$\alpha = \sqrt{\frac{\left(\omega_0^2 - \omega_1^2\right)}{\omega_1^2 \cdot \{\psi\}^T \cdot [\Delta m] \cdot \{\psi\}}} \tag{4}$$

In Equation (4), both the modified or unmodified mode shapes may be used. However, according to [4], the best results are obtained using the equation:

$$\alpha_{01} = \sqrt{\frac{\left(\omega_0^2 - \omega_1^2\right)}{\omega_1^2 \cdot \{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_1\}}} , \tag{5}$$

or, alternatively (see [3]):

$$\alpha^* = \frac{\sqrt{\dfrac{\left(\omega_0^2 - \omega_1^2\right)}{\omega_1^2 \cdot \{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}}} + \sqrt{\dfrac{\left(\omega_0^2 - \omega_1^2\right)}{\omega_1^2 \cdot \{\psi_1\}^T \cdot [\Delta m] \cdot \{\psi_1\}}}}{2}.$$ (6)

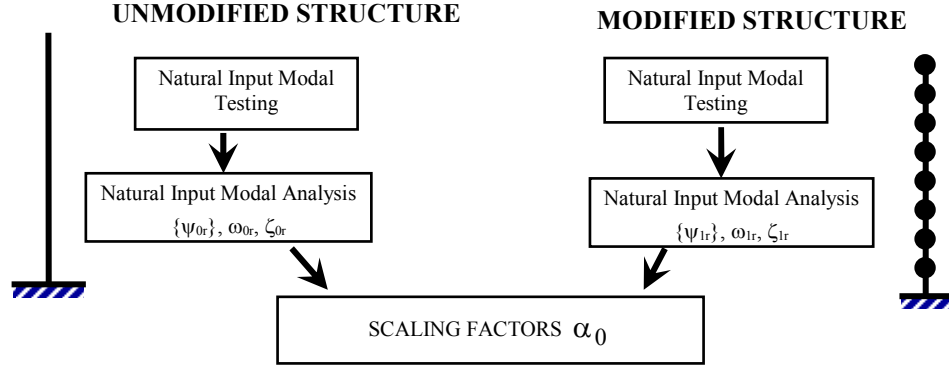**UNMODIFIED STRUCTURE**   **MODIFIED STRUCTURE**

Figure 1. The mass change method, schematically.

3   Uncertainty

An idea of the sensitivity on the scaling factors to errors in the experimental frequency shifts can be obtained by differentiating equation (4) with respect to the frequency ratio $\eta_\omega = \left(\omega_0 / \omega_1\right)$, i.e.:

$$\frac{\delta\alpha}{\alpha} = \frac{\eta_\omega^2}{\eta_\omega^2 - 1} \cdot \frac{\delta\eta_\omega}{\eta_\omega} = \frac{\eta_\omega}{\eta_\omega^2 - 1} \cdot \left(\frac{\delta\omega_0}{\omega_0} - \frac{\delta\omega_1}{\omega_1}\right).$$ (7)

From this equation it is clear that a relative error $\varepsilon_{\eta_\omega}$ on the frequency ratio will induce a relative error in the scaling factor given by:

$$\varepsilon_\alpha = \frac{\eta_\omega^2}{\eta_\omega^2 - 1} \cdot \varepsilon_{\eta_\omega} = \frac{\eta_\omega^2}{\eta_\omega^2 - 1} \cdot \left(\varepsilon_{\omega_0} - \varepsilon_{\omega_1}\right)$$ (8)

This equation, shown in Figure 2, emphasizes the importance of:

- Maximizing the frequency ratio, η_ω, which implies to perform large enough mass changes to allow for a reasonably frequency ratio.

- Performing good modal parameter identification, in order to keep the relative uncertainty of the frequency ratio down to a reasonable value.

From equation (7) can be also inferred that the effect of the same absolute error $\delta\omega$ becomes higher for low frequencies.

On the other hand, the sensitivity to errors in the kth component of the experimental mode shape $\{\psi_0\}$ is (see [2]):

$$\frac{\delta\alpha}{\alpha} = -\sum_{k=1}^{Nmasses} \frac{\psi_{0k}^2 \cdot \Delta m_k}{\{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}} \cdot \frac{\delta\psi_{0k}}{\psi_{0k}} \qquad (9)$$

This proves that a relative error ε_ψ on the k-th component of the mode shape will induce a relative error in the scaling factor of the same order. Consequently, it is recommended:

Figure 2. Influence of the frequency ratio in the scaling factor uncertainty

- To keep the relative uncertainty of the mode shape values $\frac{\Delta\psi_{0k}}{\psi_{0k}}$ down to a reasonable value, i.e., to perform a good modal parameter identification.

- To use as many points as possible for the mass change in order to maximize $\{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}$.

- To perform small enough mass changes in order the approximations given by equation (5) and (6) to be valid without modifying the mode shapes significantly.

From equation (4) can be derived how the frequency shift can be maximized. The relation between the natural frequencies of the original and the modified structure is:

$$\eta_\omega^2 = 1 + \alpha_0^2 \cdot \{\psi\}^T \cdot [\Delta m] \cdot \{\psi\} \qquad (10)$$

which means that the frequency ratio is maximized when $\{\psi\}^T \cdot [\Delta m] \cdot \{\psi\}$ is maximum.

On the other hand, the sensitivity equations can be used to show how to minimize the change in mode shapes. The sensitivity of the ith mode shape, corresponding to a local change in the mass at the kth degree of freedom, is given by (see [2] and [6]):

$$\frac{\partial\{\phi_i\}}{\partial m_k} \cong -\frac{\phi_{ki}^2}{2}\{\phi_i\} + \phi_{ki}\sum_{t=1, t\neq l}^{Nm} \frac{\omega_i^2}{\omega_t^2 - \omega_i^2}\phi_{kt}\{\phi_t\} \qquad (11)$$

If the mass change is performed simultaneously in several degrees of freedom, and finite difference approximation is used, the equation (11) becomes:

$$\Delta\{\phi_i\} \cong -\frac{\{\phi_i\}^T[\Delta m]\cdot\{\phi_i\}}{2}\{\phi_i\} + \sum_{t=1, t\neq i}^{Nm}\left(\frac{\omega_i^2}{\omega_t^2 - \omega_i^2}\{\phi_i\}^T[\Delta m]\cdot\{\phi_t\}\right)\cdot\{\phi_t\} \qquad (12)$$

which can also be expressed as:

$$\Delta\{\psi_i\} \cong \frac{\Delta\omega_i}{\omega_i}\{\psi_i\} - \sum_{t=1, t\neq i}^{Nm}\left(\frac{\omega_i^2}{\omega_t^2 - \omega_i^2}\cdot\frac{2\Delta\omega_t}{\omega_t}\cdot\frac{\{\psi_i\}^T[\Delta m]\cdot\{\psi_t\}}{\{\psi_t\}^T[\Delta m]\cdot\{\psi_t\}}\right)\cdot\{\psi_t\} \qquad (13)$$

Equation (13) shows that the mode shape modification $\Delta\{\psi_i\}$ is expressed as linear a combination of the contributions of each mode. It is concluded that:
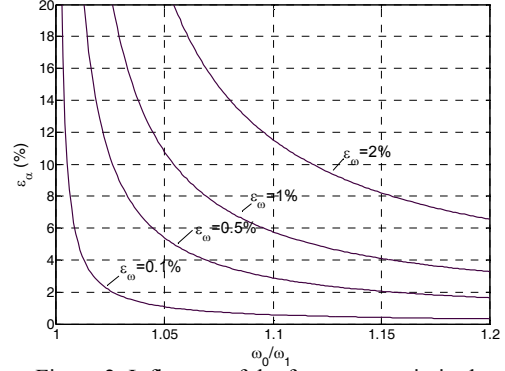
- The mode $\{\psi_i\}$ contribute to modify the scale of the mode but not to modify the shape.

- The mode shape modification increases with the frequency shift.

- The main contributions to modify the mode shapes come from the near modes.

- The mode shapes do not change at all if the matrix $[\Delta m]$ is proportional to the mass matrix $[m]$, because in this case the mode shapes are orthogonal with respect to the matrix $[\Delta m]$, i.e., $\{\psi_i\}^T[\Delta m]\cdot\{\psi_t\}=0$ and the other modes do not contribute to modify the mode shapes.

Therefore, we must try to perform proportional and small mass changes in order to minimize the frequency shift.

4 Mass change strategy

The discussions in the previous paragraph have shown that several opposite objectives have to be fulfilled simultaneously. On the one hand, the mass change must be high in order to maximize the frequency shift but on the other hand, the mass change must be low in order to minimize the changes in mode shapes.

We should also take into consideration that the mass modification can be difficult and expensive or that there can be degree of freedoms where it is not possible to attach masses, etc.

Thus, a mass change strategy must be carefully studied before applying the mass change method. The mass change strategy involves not only decide about the magnitude, but also about the location and the number of the masses to be attached to the structure

4.1 Mass magnitude

From the modal parameters $\{\psi_0\}$ and ω_0 of the original structure it is not possible to estimate the frequency shift and the mode shape modification. To determine the magnitude of the masses to be attached it is needed, at least, an estimation of the mass distribution in the original structure in order to construct a mass matrix (for example a diagonal matrix).

A formula that relates the frequency shift and the mass change magnitude can be derived from the natural frequencies of both the original structure and the modified structure. Assuming that the stiffness of the structure is not modified, the following equation is obtained:

$$\frac{\omega_1}{\omega_0}=\sqrt{\frac{1}{1+\frac{\Delta m}{m}}}\,. \tag{14}$$

If the frequency shift is expressed as $\Delta\omega=\omega_0-\omega_1$, the former equation becomes:

$$\frac{\Delta\omega}{\omega_0}=1-\sqrt{\frac{1}{1+\frac{\Delta m}{m}}} \tag{15}$$

It can be observed that for small mass changes $\dfrac{\Delta\omega}{\omega_0}\approx\dfrac{1}{2}\dfrac{\Delta m}{m}$.

Equation (15) is only valid for a one-degree-of-freedom system. In the general case, the modal masses $\Delta M=\{\psi_0\}^T\cdot[\Delta m]\cdot\{\psi_0\}$ and $M=\{\psi_0\}^T\cdot[m]\cdot\{\psi_0\}$ must be used, i.e.:

$$\frac{\Delta\omega}{\omega_0} = 1 - \sqrt{\frac{1}{1 + \frac{\Delta M}{M}}} \tag{16}$$

The procedure to be followed is:

- Firstly, to choose the ratio $\frac{\omega_0}{\omega_1}$ that we want to obtain from figure 2, depending on the accuracy obtained in the modal analysis stage and the accuracy wanted in the scaling factor.

- Secondly, to estimate the magnitude of the mass to be attached from equation (15).

4.2 Number of masses

The number of masses to be attached depends on the number of modes going to be estimated simultaneously. Theoretically, we should attach as many masses as possible.

For each mode, the number of masses to be attached should be equal or higher than the number of peaks and valleys of the mode shape.

The simultaneous calculation of scaling factors for several modes means that is not possible to optimize the mass location for all modes. In these cases, we should study other possibilities such as:

- To attach more masses to the structure.

- To perform several modal tests changing the position of some masses. Note that changing the location of one mass can be a relatively easy task but it can improve significantly the accuracy in the estimation of the scaling factor.

4.3 Location of masses

As can be derived from the equation (10), the masses located in peaks and valleys of mode shapes contribute most to the frequency shift whereas those masses located at nodal positions do not contribute to modify the frequency. Thus, the best location for the masses, in order to modify the natural frequencies, is that for which the components of the mode shape are maximum.

To select the location of the masses we can construct a table for each mode, such as table 1 including the values ψ_{0jk}^2. Each term of the table 1 indicates the contribution of a unit mass, located in the j degree of freedom, to the modification of the natural frequency corresponding to the k mode.

The row TOTAL MODES provides an idea of the contribution of a unit mass located at the j degree of freedom to all modes considered in the analysis, which can be useful when the location

Table 1. Contribution of a unity mass to the frequency shift.

Mode	DOF			
	1	…	j	…
1	ψ_{011}^2	…	ψ_{0j1}^2	…
…	…	…	…	…
k	ψ_{01k}^2	…	ψ_{0jk}^2	…
…	…	…	…	…

TOTAL MODES	$\sum_{k=1}^{n} \psi_{01k}^2$	…	$\sum_{k=1}^{n} \psi_{0jk}^2$	…

of the masses has to be optimized for several modes simultaneously.

4.4   Comparison of strategies

Once the location, the magnitude and the number of masses has been selected, the matrix $[\Delta m]$ is already defined and equation (17) can be used to estimate the frequency shift for each mode (in case we have an estimation of the mass matrix).

In order to have an idea of the effectiveness of the mass change for each mode, table 2 is supplied. It provides information of the relative frequency shift that we are going to obtain, compared with the frequency shift that we would obtain locating masses at positions allowing for a maximum frequency shift (maximum values of the mode shapes).

On the other hand, if two mass changes, defined by the mass change matrices $[\Delta m]_a$ and $[\Delta m]_b$, respectively, are performed, the expression:

$$\left(1 - \frac{\omega_{1a}^2}{\omega_0^2}\right) \cdot \{\psi_0\}^T \cdot [\Delta m]_a \cdot \{\psi_0\} = \left(1 - \frac{\omega_{1b}^2}{\omega_0^2}\right) \cdot \{\psi_0\}^T \cdot [\Delta m]_b \cdot \{\psi_0\} \tag{17}$$

is obtained from equation (4).

Taking the approximation $\dfrac{\omega_1^2}{\omega_0^2} = \dfrac{(\omega_0 - \Delta\omega)^2}{\omega_0^2} = 1 - \dfrac{2\Delta\omega}{\omega_0} + \dfrac{\Delta\omega^2}{\omega_0^2} \cong 1 - \dfrac{2\Delta\omega}{\omega_0}$, equation (17) becomes:

$$\frac{(\Delta\omega)_a}{(\Delta\omega)_b} \cong \frac{\{\psi_0\}^T \cdot [\Delta m]_b \cdot \{\psi_0\}}{\{\psi_0\}^T \cdot [\Delta m]_a \cdot \{\psi_0\}} \tag{18}$$

from which it can be concluded that the frequency shift is, approximately, proportional to $\{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}$. Therefore, a comparison between the frequency shifts calculated for two different mass changes can also be obtained from table 2.

Finally, to minimize the changes in mode shapes, it follows from equation (12) that the term $\left(\dfrac{\omega_i^2}{\omega_t^2 - \omega_i^2} \cdot \dfrac{2\Delta\omega_t}{\omega_t} \cdot \dfrac{\{\psi_i\}^T [\Delta m] \cdot \{\psi_t\}}{\{\psi_t\}^T [\Delta m] \cdot \{\psi_t\}}\right)$, which provides information of the orthogonality of the modes with respect to the matrix $[\Delta m]$, should be minimized for each mode. However, only the near modes contribute significantly to modify the mode shape.

Using the equation (15) for estimating the relative frequency shift, table 3 can be prepared including the contribution of each mode to change the mode shape. We can also compare the values in table 3 for two different mass changes, in order to know which one is the best.

Table 2. Comparison of frequency shifts corresponding to two different mass changes

Mode	$\dfrac{\text{OPTION1}}{\text{OPTIMAL}}$	$\dfrac{\text{OPTION2}}{\text{OPTIMAL}}$	$\dfrac{\text{OPTION1}}{\text{OPTION2}}$
…	…	…	…
k	$\dfrac{\{\psi_{0k}\}^T \cdot [\Delta m]_1 \cdot \{\psi_{0k}\}}{\{\psi_{0k}\}^T \cdot [\Delta m]_{opt} \cdot \{\psi_{0k}\}}$	$\dfrac{\{\psi_{0k}\}^T \cdot [\Delta m]_1 \cdot \{\psi_{0k}\}}{\{\psi_{0k}\}^T \cdot [\Delta m]_{opt} \cdot \{\psi_{0k}\}}$	$\dfrac{\{\psi_{0k}\}^T \cdot [\Delta m]_1 \cdot \{\psi_{0k}\}}{\{\psi_{0k}\}^T \cdot [\Delta m]_2 \cdot \{\psi_{0k}\}}$
…	…	…	…

Table 3. Contribution of each mode to the mode shape modification

Mode	Mode			
	1	...	I	...
1	-	...	$\left(\dfrac{\omega_i^2}{\omega_1^2 - \omega_i^2} \cdot \dfrac{2\Delta\omega_1}{\omega_1} \cdot \dfrac{\{\psi_i\}^T [\Delta m] \cdot \{\psi_1\}}{\{\psi_1\}^T [\Delta m] \cdot \{\psi_1\}} \right)$	...
...	...	...	...	...
t	$\left(\dfrac{\omega_1^2}{\omega_t^2 - \omega_1^2} \cdot \dfrac{2\Delta\omega_t}{\omega_t} \cdot \dfrac{\{\psi_1\}^T [\Delta m] \cdot \{\psi_t\}}{\{\psi_t\}^T [\Delta m] \cdot \{\psi_t\}} \right)$	...	$\left(\dfrac{\omega_i^2}{\omega_t^2 - \omega_i^2} \cdot \dfrac{2\Delta\omega_t}{\omega_t} \cdot \dfrac{\{\psi_i\}^T [\Delta m] \cdot \{\psi_t\}}{\{\psi_t\}^T [\Delta m] \cdot \{\psi_t\}} \right)$	...
...	...	...	...	...

5   Conclusions

- The uncertainty in the scaling factor estimation by the mass change method depends mainly on both the uncertainty in the modal analysis and the mass change strategy used to modify the dynamic behaviour of the structure.

- Simple rules have been proposed to optimize the mass change strategy, i.e., the number, the magnitude and the location of the masses.

6   Acknowledgements

7   References

[1]   R. Brincker, C. Ventura and P. Andersen: Why Output-Only Modal Testing is a Desirable Tool for a Wide Range of Practical Applications. In Proc. Of the International Modal Analysis Conference (IMAC) XXI, paper 265, February, 2003

[2]   E. Parloo, P. Verboven, P. Guillaume, M. Van Overmeire. "Sensitivity- Based Operational Mode Shape Normalization" In Mech. Systems and Signal Proc., 16(5), pp. 757-767, 2002

[3]   M. López Aenlle, R. Brincker, A. Fernández Canteli. L. M. Villa. "Scaling Factor Estimation by the Mass Change Method". Proc. Of the International Operational Modal Analysis Conferencee (IOMAC), Copenhagen, 2005.

[4]   R. Brincker and P. Andersen. "A Way of Getting Scaled Mode Shapes in Output Only Modal Analysis". In Proc. Of the International Modal Analysis Conference (IMAC) XXI, paper 141, February 2003.

[5]   M. López Aenlle, R. Brincker, A. Fernández Canteli. "Some Methods to Determine Scaled Mode Shapes in Natural Input Modal Analysis". In Proc. Of the International Modal Analysis Conference (IMAC) XXIII, paper, February 2005.

[6]   W. Heylen, S.,   Lammens and P. Sas. "Modal Analysis: Theory and Testing". Katholieke Universiteit Leuve, 1997.