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ANALYSIS OF AN AUGMENTED FULLY-MIXED FINITE ELEMENT METHOD FOR A THREE-DIMENSIONAL FLUID-SOLID INTERACTION PROBLEM

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Abstract. We introduce and analyze an augmented fully-mixed finite element method for a fluid-solid interaction problem in 3D. The media are governed by the acoustic and elastodynamic equations in time-harmonic regime, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. We first employ dual-mixed variational formulations in both domains, which yields the Cauchy stress tensor and the rotation of the solid, together with the gradient of the pressure of the fluid, as the preliminary unknowns. This approach allows us to extend an idea from a recent own work in such a way that both transmission conditions are incorporated now into the definitions of the continuous spaces, and therefore no unknowns on the coupling boundary appear. As a consequence, the pressure of the fluid and the displacement of the solid become explicit unknowns of the coupled problem, and hence two redundant variational terms arising from the constitutive equations, both of them multiplied by stabilization parameters, need to be added for well-posedness. In fact, we show that explicit choices of the above mentioned parameters and a suitable decomposition of the spaces allow the application of the Babuška-Brezzi theory and the Fredholm alternative for concluding the solvability of the resulting augmented formulation. The unknowns of the fluid and the solid are then approximated by a conforming Galerkin scheme defined in terms of Arnold-Falk-Winther and Lagrange finite element subspaces of order 1. The analysis of the discrete method relies on a stable decomposition of the finite element spaces and also on a classical result on projection methods for Fredholm operators of index zero. Finally, numerical results illustrating the theory are also presented..

Key words. mixed finite elements, Arnold-Falk-Winther elements, Helmholtz, elastodynamic.

1. Introduction

In this paper we focus again on the three-dimensional fluid-solid interaction problem studied recently in [20] (see also [17] for the corresponding 2D version). More precisely, our physical model of interest consists of a bounded elastic body (obstacle) Ω_s in $\mathbf{R}^3$ with Lipschitz-continuous boundary Σ, subject to a volume force $\mathbf{F}$, that is fully surrounded by a fluid. Then, given an incident acoustic wave $\mathbf{P}_i$ upon Ω_s, we are interested in determining both the response of the body and the scattered wave. We assume that $\mathbf{P}_i$ and $\mathbf{F}$ exhibit a time-harmonic behaviour with frequency ω and amplitudes p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbf{R}^3\backslash\Omega_s$. Hence, we may consider that this interaction problem is posed in the frequency domain. In addition, in what follows we let $\boldsymbol{\sigma}_s : \Omega_s \to \mathbf{C}^{3\times 3}$, $\mathbf{u} : \Omega_s \to \mathbf{C}^3$, and $p : \mathbf{R}^3\backslash\Omega_s \to \mathbf{C}$ be the amplitudes of the Cauchy stress tensor,

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the displacement field, and the total (incident + scattered) pressure, respectively, where $\mathbf{C}$ stands for the set of complex numbers.

The fluid is assumed to be perfect, compressible, and homogeneous, with density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid, whereas the solid is supposed to be isotropic and linearly elastic with density ρ_s and Lamé constants μ and λ. The latter means, in particular, that the corresponding constitutive equation is given by Hooke's law, that is

$$\boldsymbol{\sigma}_s \,=\, \lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{I} \,+\, 2\,\mu\,\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega_s\,, \tag{1}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, ∇ is the gradient tensor, tr denotes the matrix trace, $^{\mathtt{t}}$ stands for the transpose of a matrix, and $\mathbf{I}$ is the identity matrix of $\mathbf{C}^{3\times3}$. Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}_s$, $\mathbf{u}$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned}
\mathbf{div}\,\boldsymbol{\sigma}_s \,+\, \kappa_s^2\,\mathbf{u} \quad &= \quad -\,\mathbf{f} \qquad \text{in} \quad \Omega_s\,,\\
\Delta p \,+\, \kappa_f^2\,p \quad &= \quad 0 \qquad\;\; \text{in} \quad \mathbf{R}^3\backslash\Omega_s\,,
\end{aligned}$$

where the wave number κ_s of the solid is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned}
\boldsymbol{\sigma}_s\,\boldsymbol{\nu} \quad &= \quad -\,p\,\boldsymbol{\nu} \qquad \text{on} \quad \Sigma\,,\\
\rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} \quad &= \quad \frac{\partial p}{\partial\boldsymbol{\nu}} \qquad \text{on} \quad \Sigma\,,
\end{aligned} \tag{2}$$

and the behaviour at infinity given by

$$p - p_i \,=\, O(\mathtt{r}^{-1}) \tag{3}$$

and

$$\frac{\partial(p-p_i)}{\partial\mathtt{r}} \,-\, \imath\,\kappa_f\,(p-p_i) \,=\, o(\mathtt{r}^{-1})\,, \tag{4}$$

as $\mathtt{r} := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \mathbf{R}^3$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward $\mathbf{R}^3\backslash\Omega_s$. The transmission conditions given in (2) constitute the equilibrium of forces and the equality of the normal displacements of the solid and fluid, whereas the equation (4) is known as the Sommerfeld radiation condition.

In the recent work [20] we introduce and analyze a new finite element method for the above interaction problem. Actually, we initially proceed as in [17] and simplify a bit the original model by assuming that the fluid occupies a bounded annular region Ω_f. Hence, a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on the exterior boundary of Ω_f, which is located far from the obstacle. Then, we employ a dual-mixed formulation for plane elasticity in the solid, in which the elastodynamic equation is used to eliminate the displacement unknown, and keep the usual primal method in the fluid region. Needless to say, avoiding the locking phenomenon that arises in the nearly incompressible case or obtaining direct finite element approximations of the stresses constitute the main reasons for using the dual-mixed method in the solid. Now, the main novelty of our approach in [20] with respect to [17] is the incorporation of the first equation of (2) into the definition of the product space to which the pair $(\boldsymbol{\sigma}_s, p)$ belongs. In this way, we avoid the introduction of further unknowns (Lagrange multipliers) on the boundary of the solid, which otherwise would yield later on a more complicated discrete analysis and a bit more expensive Galerkin scheme. Indeed, the strategy involving a Lagrange multiplier on the transmission boundary and the consequent need of an associated discrete inf-sup condition would require the use of finite element meshes satisfying a stability condition between their corresponding mesh sizes, which certainly constitutes a very cumbersome restriction in 3D computations. Therefore, thanks to the availability of the new stable mixed finite elements for 3D linear elasticity with weak symmetry (see, e.g. [3], [5], [10]), the unknowns of the solid and the fluid are approximated in [20] by the corresponding components of the Arnold-Falk-Winther (AFW) and Lagrange finite element subspaces of order 1, respectively. The resulting AFW element, which involves the lowest polynomial degrees, consists of piecewise linear approximations for the stress and piecewise constants functions for both the displacement and rotation unknowns. Thus, because of the coincidence between the polynomial shape-functions approximating $\boldsymbol{\sigma}_s \boldsymbol{\nu}$ and $-p\boldsymbol{\nu}$, we are able to generate a conforming finite element subspace for $(\boldsymbol{\sigma}_s, p)$. In other words, the equilibrium of forces on Σ is exactly satisfied at the discrete level, and hence a natural coupling of the Lagrange and AFW elements of lowest order with respect to that transmission condition is obtained.

The purpose of the present paper is to further extend the approach from [20] by employing now dual-mixed formulations in both media. This means that, besides $\boldsymbol{\sigma}_s$, we now set the additional unknown

$$\boldsymbol{\sigma}_f := \nabla p \quad \text{in} \quad \mathbf{R}^3 \backslash \Omega_s \,, \tag{5}$$

so that the Helmholtz equation and the second condition in (2) are rewritten, respectively, as

$$\operatorname{div} \boldsymbol{\sigma}_f + \kappa_f^2 \, p = 0 \quad \text{in} \quad \mathbf{R}^3 \backslash \Omega_s \,, \tag{6}$$

and

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f \, \omega^2 \, \mathbf{u} \cdot \boldsymbol{\nu} \quad \text{on} \quad \Sigma \,. \tag{7}$$

The main motivation for introducing $\boldsymbol{\sigma}_f$ and the resulting equation (6) lies on the eventual need of obtaining direct and more accurate finite element approximations for the pressure gradient $\boldsymbol{\sigma}_f := \nabla p$ (instead of applying numerical differentiation, with the consequent loss of accuracy, to the approximation of p arising from the usual primal formulation). In particular, the above is required for solving the inverse problem related to the Helmholtz equation, in which the boundary integral representation of the far field pattern, a crucial variable in an associated iterative algorithm, depends not only on the trace of p but also on the normal trace of $\boldsymbol{\sigma}_f$ (see, e.g. [11, Chapter 2, Theorem 2.5]). Certainly, a $H(\text{div})$-type approximation of $\boldsymbol{\sigma}_f$ is better suited for this purpose. The usefulness of the mixed formulation for the pressure p is also justified by the fact that it is locally mass conservative. Furthermore, the fact that the second transmission condition on Σ is given now by (7) allows to incorporate it into the definition of the space to which the pair $(\boldsymbol{\sigma}_f, \mathbf{u})$ belongs, thus providing another advantage of utilizing a dual-mixed approach in the fluid. More precisely, besides the absence of unknowns on the interface, the fact that the verification of related discrete inf-sup conditions is no longer needed, constitutes the main advantage of incorporating both transmission conditions into the definition of the spaces. In turn, since the pressure of the fluid and the displacement of the solid become explicit unknowns of the coupled problem, two redundant variational equations taken from (1) and (5), both of them multiplied by stabilization parameters, need to be added for the corresponding solvability analysis. In this way, we arrive at what we call the augmented fully-mixed formulation of our fluid-solid interaction problem. The rest of this work is organized as follows. In Section 2 we redefine the fluid-solid interaction problem on a bounded domain of $\mathbf{R}^3$ (as in [17] and [20]), and derive the associated continuous variational formulation. Then, in Section 3 we analyze the resulting saddle point problem and provide sufficient conditions for its well-posedness. The corresponding Galerkin scheme is studied in Section 4. Finally, numerical results illustrating the analysis are reported in Section 5.

We end this section with further notations to be used below. Since in the sequel we deal with complex valued functions, we use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbf{C}$. Also, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbf{C}^{3\times 3}$, we define the deviator tensor $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$, the tensor product $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}} := (\overline{\tau}_{ij})$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbf{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^3, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{3\times 3}, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^3 .$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^3 and $H^{3\times 3}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \operatorname{div}\mathbf{w} \in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see [8], [22]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The continuous variational formulation

We begin by remarking, as a consequence of (3) and (4), that the outgoing waves are absorbed by the far field. According to this fact, and in order to obtain a suitable simplification of our model problem, we now proceed similarly as in [17] and [20] and introduce a sufficiently large polyhedral surface Γ, whose interior contains Ω_s. Then, we define Ω_f as the annular region bounded by Σ and Γ, and consider, for simplicity, the Dirichlet boundary condition:

$$p = p_i \quad \text{on} \quad \Gamma . \tag{8}$$

Therefore, given $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $p_i \in H^{1/2}(\Gamma)$, we are now interested in the following fluid-solid interaction problem: Find $\boldsymbol{\sigma}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, and $p \in H^1(\Omega_f)$, such that there hold in the distributional sense:

$$
\begin{aligned}
\boldsymbol{\sigma}_s &= \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in} \quad \Omega_s\,,\\
\mathbf{div}\,\boldsymbol{\sigma}_s + \kappa_s^2\,\mathbf{u} &= -\,\mathbf{f} && \text{in} \quad \Omega_s\,,\\
\boldsymbol{\sigma}_f &= \nabla p && \text{in} \quad \Omega_f\,,\\
\operatorname{div}\boldsymbol{\sigma}_f + \kappa_f^2\,p &= 0 && \text{in} \quad \Omega_f\,,\\
\boldsymbol{\sigma}_s\,\boldsymbol{\nu} &= -\,p\,\boldsymbol{\nu} && \text{on} \quad \Sigma\,,\\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} &= \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} && \text{on} \quad \Sigma\,,\\
p &= p_i && \text{on} \quad \Gamma\,,
\end{aligned}
\tag{9}
$$

where $\mathcal{C}$ is the elasticity operator given by Hooke's law, that is

$$
\mathcal{C}\,\boldsymbol{\zeta} \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \;+\; 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega_s)\,. \tag{10}
$$

It is clear from (10) that $\mathcal{C}$ is bounded and invertible and that the operator $\mathcal{C}^{-1}$ reduces to

$$
\mathcal{C}^{-1}\,\boldsymbol{\zeta} \;:=\; \frac{1}{2\,\mu}\,\boldsymbol{\zeta} \;-\; \frac{\lambda}{2\,\mu\,(3\lambda+2\mu)}\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega_s)\,. \tag{11}
$$

In addition, the above identity and simple algebraic manipulations imply that

$$
\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta} : \boldsymbol{\tau} \;=\; \frac{1}{2\,\mu}\int_{\Omega_s} \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \;+\; \frac{1}{3\,(3\lambda+2\mu)}\int_{\Omega_s} \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\,\in\,\mathbb{L}^2(\Omega_s)\,,
$$

which yields

$$
\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta} : \overline{\boldsymbol{\zeta}} \;\geq\; \frac{1}{2\mu}\,\|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,\Omega_s}^2 \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega_s)\,. \tag{12}
$$

This estimate will be useful for our analysis below in Sections 3 and 4.

In what follows we apply dual-mixed approaches in the solid Ω_s and the fluid Ω_f to derive the fully-mixed variational formulation of (9). To this end, we first apply the usual procedure from linear elasticity (see [1], [17] and [31]) and introduce the rotation

$$
\boldsymbol{\gamma} \;:=\; \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}) \,\in\, \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)
$$

as a further unknown, where $\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$ denotes the space of asymmetric tensors with entries in $L^2(\Omega_s)$. In this way, the constitutive equation can be rewritten in the form

$$
\mathcal{C}^{-1}\,\boldsymbol{\sigma}_s \,=\, \boldsymbol{\varepsilon}(\mathbf{u}) \;=\; \nabla\mathbf{u} \,-\, \boldsymbol{\gamma}\,,
$$

which, multiplying by a function $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ and integrating by parts, yields

$$
\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\sigma}_s : \boldsymbol{\tau}_s \,+\, \int_{\Omega_s} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \,-\, \langle\boldsymbol{\tau}_s\,\boldsymbol{\nu},\mathbf{u}\rangle_\Sigma \,+\, \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\gamma} \,=\, 0\,. \tag{13}
$$

Then, replacing from the elastodynamic equation in Ω_s (cf. second equation of (9))

$$
\mathbf{u} \,=\, -\,\frac{1}{\kappa_s^2}\left(\mathbf{f} \,+\, \mathbf{div}\,\boldsymbol{\sigma}_s\right),
$$

we find that (13) becomes

$$
\int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\sigma}_s : \boldsymbol{\tau}_s - \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\sigma}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s - \langle\boldsymbol{\tau}_s\,\boldsymbol{\nu},\mathbf{u}\rangle_\Sigma + \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\gamma} = \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s. \tag{14}
$$

In turn, the symmetry of $\boldsymbol{\sigma}_s$ is imposed weakly through the relation

$$\int_{\Omega_s} \boldsymbol{\sigma}_s : \boldsymbol{\eta} \,=\, 0 \qquad \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\,. \tag{15}$$

On the other hand, proceeding similarly in the fluid region Ω_f, that is multiplying the constitutive equation $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f by $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, integrating by parts, noting that the normal vector points inward Ω_f on Σ, using the Dirichlet boundary condition on Γ, and replacing from the Helmholtz equation $p = -\frac{1}{\kappa_f^2}\,\mathrm{div}\,\boldsymbol{\sigma}_f$ in Ω_f, we arrive at

$$\int_{\Omega_f} \boldsymbol{\sigma}_f \cdot \boldsymbol{\tau}_f \,-\, \frac{1}{\kappa_f^2}\int_{\Omega_f} \mathrm{div}\,\boldsymbol{\sigma}_f\,\mathrm{div}\,\boldsymbol{\tau}_f \,+\, \langle \boldsymbol{\tau}_f\cdot\boldsymbol{\nu}, p\rangle_\Sigma \,=\, \langle \boldsymbol{\tau}_f\cdot\boldsymbol{\nu}, p_i\rangle_\Gamma\,. \tag{16}$$

It remains to incorporate the transmission conditions on Σ (cf. fifth and sixth equations in (9)) into our continuous formulation. For this purpose, as already announced in Section 1, we now introduce the closed subspaces of $\mathbb{H}(\mathbf{div};\Omega_s)\times H^1(\Omega_f)$ and $\mathbf{H}(\mathrm{div};\Omega_f)\times \mathbf{H}^1(\Omega_s)$, respectively, given by

$$\mathbb{X}_1 \,:=\, \Big\{ \widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \,\in\, \mathbb{H}(\mathbf{div};\Omega_s)\times H^1(\Omega_f) : \quad \boldsymbol{\tau}_s\,\boldsymbol{\nu} \,=\, -\,q\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma \Big\}$$

and

$$\mathbb{X}_2 \,:=\, \Big\{ \widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)\times \mathbf{H}^1(\Omega_s) : \quad \boldsymbol{\tau}_f\cdot\boldsymbol{\nu} \,=\, \rho_f\,\omega^2\,\mathbf{v}\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma \Big\}.$$

In this way, replacing $\boldsymbol{\tau}_s\,\boldsymbol{\nu}$ by $-\,q\,\boldsymbol{\nu}$ in (14) for each $\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1$, and $\boldsymbol{\tau}_f\cdot\boldsymbol{\nu}$ by $\rho_f\,\omega^2\,\mathbf{v}\cdot\boldsymbol{\nu}$ in (16) for each $\widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2$, the equations (14) and (16) become

$$\begin{aligned} &\int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\sigma}_s : \boldsymbol{\tau}_s \,-\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\sigma}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s \,+\, \langle q\,\boldsymbol{\nu}, \mathbf{u}\rangle_\Sigma \,+\, \int_{\Omega_s}\boldsymbol{\tau}_s : \boldsymbol{\gamma} \\ &\qquad =\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \qquad \forall\, \widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \,\in\, \mathbb{X}_1\,, \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\int_{\Omega_f} \boldsymbol{\sigma}_f\cdot\boldsymbol{\tau}_f \,-\, \frac{1}{\kappa_f^2}\int_{\Omega_f} \mathrm{div}\,\boldsymbol{\sigma}_f\,\mathrm{div}\,\boldsymbol{\tau}_f \,+\, \rho_f\,\omega^2\,\langle p\,\boldsymbol{\nu}, \mathbf{v}\rangle_\Sigma \\ &\qquad =\, \langle \boldsymbol{\tau}_f\cdot\boldsymbol{\nu}, p_i\rangle_\Gamma \qquad \forall\, \widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \,\in\, \mathbb{X}_2\,, \end{aligned} \tag{18}$$

where the unknowns $\widehat{\boldsymbol{\sigma}}_s := (\boldsymbol{\sigma}_s, p)$ and $\widehat{\boldsymbol{\sigma}}_f := (\boldsymbol{\sigma}_f, \mathbf{u})$ are now sought in $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively.

It is important to remark at this point that, instead of replacing $\mathbf{u}$ and p by the expressions obtained from the elastodynamic and Helmholtz equations, the usual dual-mixed method would simply test these equations against $\mathbf{v} \in \mathbf{L}^2(\Omega_s)$ and $q \in L^2(\Omega_f)$, respectively, obtaining

$$\int_{\Omega_s} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}_s \,+\, \kappa_s^2\int_{\Omega_s}\mathbf{u}\cdot\mathbf{v} \,=\, -\int_{\Omega_s}\mathbf{f}\cdot\mathbf{v}\,,$$

and

$$\int_{\Omega_f} q\,\mathrm{div}\,\boldsymbol{\sigma}_f \,+\, \kappa_f^2\int_{\Omega_f} p\,q \,=\, 0\,.$$

However, it is easy to see, mainly because of the lack of coerciveness of the underlying operators, that the resulting continuous formulation does not fit into the framework of any of the available theories for proving well-posedness, and hence a different procedure must be adopted. To this respect, we recall that the above mentioned replacement of $\mathbf{u}$ was sufficient for the corresponding analyses of the elastodynamic equation in [17] and [20] since it basically allowed to remove this unknown. The same would hold for the dual-mixed formulation of the Helmholtz equation if the pressure p were replaced as indicated. Nevertheless, it is quite clear that the fully-mixed approach that we are applying to the present fluid-solid interaction does not actually eliminate $\mathbf{u}$ and p (in spite of the above described replacements), but on the contrary it does confirm their incorporation as explicit unknowns of the coupled problem. According to this, and keeping in mind that $\mathbf{u}$ and p belong to $\mathbf{H}^1(\Omega_s)$ and $H^1(\Omega_f)$, respectively, we now make redundant use of the constitutive relations given by the first and third equations in (9), and propose to enrich our variational formulation with the identities

$$-\kappa_1 \int_{\Omega_s} \left\{\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{C}^{-1}\boldsymbol{\sigma}_s\right\} : \boldsymbol{\varepsilon}(\mathbf{v}) = 0 \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega_s)\,, \tag{19}$$

and

$$-\kappa_2 \int_{\Omega_f} \left\{\nabla p - \boldsymbol{\sigma}_f\right\} \cdot \nabla q = 0 \qquad \forall\, q \in H^1(\Omega_f)\,, \tag{20}$$

where κ_1 and κ_2 are positive stabilization parameters to be chosen conveniently later on. In particular, the reason for choosing minus signs multiplying (19) and (20) will become clear from the analysis in Section 3.

Consequently, adding (15), (17), (18), (19), and (20), and defining the spaces

$$\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \qquad \text{and} \qquad \mathbb{Y} := \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\,,$$

we arrive at the following augmented fully-mixed formulation of the fluid-solid interaction problem: Find $\widehat{\boldsymbol{\sigma}} := (\widehat{\boldsymbol{\sigma}}_s, \widehat{\boldsymbol{\sigma}}_f) := ((\boldsymbol{\sigma}_s, p), (\boldsymbol{\sigma}_f, \mathbf{u})) \in \mathbb{X}$ and $\boldsymbol{\gamma} \in \mathbb{Y}$ such that

$$\begin{aligned} \mathbb{A}(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) + \mathbb{B}(\widehat{\boldsymbol{\tau}}, \boldsymbol{\gamma}) &= \mathbb{F}(\widehat{\boldsymbol{\tau}}) \qquad \forall\, \widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s, \widehat{\boldsymbol{\tau}}_f) := ((\boldsymbol{\tau}_s, q), (\boldsymbol{\tau}_f, \mathbf{v})) \in \mathbb{X}\,, \\ \mathbb{B}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\eta}) &= 0 \qquad \forall\, \boldsymbol{\eta} \in \mathbb{Y}\,, \end{aligned} \tag{21}$$

where $\mathbb{F} : \mathbb{X} \to \mathbf{C}$ is the linear functional

$$\mathbb{F}(\widehat{\boldsymbol{\tau}}) := \frac{1}{\kappa_s^2} \int_{\Omega_s} \mathbf{f} \cdot \mathbf{div}\, \boldsymbol{\tau}_s + \langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, p_i \rangle_\Gamma \qquad \forall\, \widehat{\boldsymbol{\tau}} \in \mathbb{X}\,,$$

and $\mathbb{A} : \mathbb{X} \times \mathbb{X} \to \mathbf{C}$, and $\mathbb{B} : \mathbb{X} \times \mathbb{Y} \to \mathbf{C}$ are the bilinear forms defined by

$$\begin{aligned} \mathbb{A}(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) = & \int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\zeta}_s : \boldsymbol{\tau}_s - \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\zeta}_s \cdot \mathbf{div}\,\boldsymbol{\tau}_s + \langle q\,\boldsymbol{\nu}, \mathbf{w}\rangle_\Sigma \\ & + \int_{\Omega_f} \boldsymbol{\zeta}_f \cdot \boldsymbol{\tau}_f - \frac{1}{\kappa_f^2}\int_{\Omega_f} \operatorname{div}\boldsymbol{\zeta}_f \operatorname{div}\boldsymbol{\tau}_f + \rho_f\,\omega^2 \langle r\,\boldsymbol{\nu}, \mathbf{v}\rangle_\Sigma \\ & - \kappa_1 \int_{\Omega_s} \left\{\boldsymbol{\varepsilon}(\mathbf{w}) - \mathcal{C}^{-1}\boldsymbol{\zeta}_s\right\} : \boldsymbol{\varepsilon}(\mathbf{v}) - \kappa_2 \int_{\Omega_f} \left\{\nabla r - \boldsymbol{\zeta}_f\right\} \cdot \nabla q \\ & \forall\, \widehat{\boldsymbol{\zeta}} := (\widehat{\boldsymbol{\zeta}}_s, \widehat{\boldsymbol{\zeta}}_f) := ((\boldsymbol{\zeta}_s, r), (\boldsymbol{\zeta}_f, \mathbf{w})) \in \mathbb{X}\,, \\ & \forall\, \widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s, \widehat{\boldsymbol{\tau}}_f) := ((\boldsymbol{\tau}_s, q), (\boldsymbol{\tau}_f, \mathbf{v})) \in \mathbb{X}\,, \end{aligned} \tag{22}$$

and

$$\mathbb{B}(\widehat{\boldsymbol{\tau}}, \boldsymbol{\eta}) := \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\eta} \qquad \forall\, (\widehat{\boldsymbol{\tau}}, \boldsymbol{\eta}) \in \mathbb{X} \times \mathbb{Y}\,. \tag{23}$$

It is straightforward to see, applying the Cauchy-Schwarz inequality and the usual trace theorems in $\mathbf{H}(\mathrm{div};\Omega_f)$, $\mathbf{H}^1(\Omega_s)$ and $H^1(\Omega_f)$, that $\mathbb{F}$, $\mathbb{A}$, and $\mathbb{B}$ are all bounded with constants depending on ω, ρ_f, ρ_s, κ_f, κ_s, μ, κ_1 and κ_2, in the case of $\mathbb{F}$ and $\mathbb{A}$, and constants independent of these parameters in the case of $\mathbb{B}$.

We end this section by commenting additionally on the choice of either (3) or (4) to define the boundary condition on the artificial boundary Γ. In fact, while we could certainly consider any of those behaviours at infinity, the fact that we are employing a mixed formulation in the fluid region Ω_f makes it more suitable the incorporation of the Dirichlet boundary condition (8) (which arises from (3)) since, as observed in (16), it becomes a natural boundary condition. If instead of (3) we consider (4), we arrive at the Robin boundary condition

$$\frac{\partial p}{\partial \boldsymbol{\nu}} - \imath\,\kappa_f\, p = \frac{\partial p_i}{\partial \boldsymbol{\nu}} - \imath\,\kappa_f\, p_i =: g_i \in H^{-1/2}(\Gamma) \quad \text{on} \quad \Gamma\,,$$

which, according to the additional unknown $\boldsymbol{\sigma}_f$ (cf. (5)), can be rewritten as

$$\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} - \imath\,\kappa_f\, p = g_i \quad \text{on} \quad \Gamma\,. \tag{24}$$

In this case, and instead of (16), the integration by parts procedure and the replacement of p by $-\frac{1}{\kappa_f^2}\,\mathrm{div}\,\boldsymbol{\sigma}_f$ in Ω_f, yield

$$\int_{\Omega_f}\boldsymbol{\sigma}_f\cdot\boldsymbol{\tau}_f - \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\sigma}_f\,\mathrm{div}\,\boldsymbol{\tau}_f + \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},p\rangle_\Sigma + \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi\rangle_\Gamma = 0\,,$$

where $\varphi := -p|_\Gamma \in H^{1/2}(\Gamma)$ is another auxiliary unknown. In turn, since the Robin boundary condition (24) becomes essential, it is imposed weakly through the equation

$$\langle\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu},\psi\rangle_\Gamma - \imath\,\kappa_f\int_\Gamma\varphi\,\psi = \langle g_i,\psi\rangle_\Gamma \quad \forall\,\psi\in H^{1/2}(\Gamma)\,.$$

Consequently, the only change of the analysis below would be caused by the extra term $\langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi\rangle_\Gamma$ (and its dual one $\langle\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu},\psi\rangle_\Gamma$), which would imply the necessity of verifying the corresponding continuous and discrete inf-sup conditions for it. On the other hand, the expression $\imath\,\kappa_f\displaystyle\int_\Gamma\varphi\,\psi$ does not cause any further change since this term dissapears when taking the real part of it with $\psi = \overline{\varphi}$.

3. Analysis of the continuous variational formulation

In this section we proceed analogously as in [17] and [20], and employ suitable decompositions of $\mathbb{X}_1$ and $\mathbb{X}_2$ (and hence of $\mathbb{X}$) to show that (21) becomes a compact perturbation of a well-posed problem. For this purpose, we now need to introduce two projectors defined in terms of auxiliary boundary value problems posed in Ω_s and Ω_f, respectively.

3.1. The associated projectors. We begin by recalling from the analysis in [20, Section 4.1] the definition of the projector in Ω_s. In fact, let us first denote by $\mathbb{RM}(\Omega_s)$ the space of rigid body motions in Ω_s, that is

$$\mathbb{RM}(\Omega_s) := \Big\{\mathbf{v}:\Omega_s\to\mathbf{C}^3:\quad \mathbf{v}(\mathbf{x}) = \mathbf{a}+\mathbf{b}\times\mathbf{x}\quad \forall\,\mathbf{x}\in\Omega_s\,,\ \mathbf{a},\,\mathbf{b}\in\mathbf{C}^3\Big\}\,,$$

and let $\mathbf{M}:\mathbf{L}^2(\Omega_s)\to\mathbb{RM}(\Omega_s)$ be the associated orthogonal projector. Then, given $\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s,q)\in\mathbb{X}_1$, we consider the boundary value problem with unknown

$\tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega_s)$:

$$
(25) \qquad \begin{aligned} \tilde{\boldsymbol{\sigma}}_s &= \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega_s\,, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_s = \mathbf{div}\,\boldsymbol{\tau}_s + \mathbf{r}(\widehat{\boldsymbol{\tau}}_s) \quad \text{in} \quad \Omega_s\,, \\ \tilde{\boldsymbol{\sigma}}_s\,\boldsymbol{\nu} &= -q\boldsymbol{\nu} \quad \text{on} \quad \Sigma, \quad \tilde{\mathbf{u}} \in (\mathbf{I} - \mathbf{M})(\mathbf{L}^2(\Omega_s))\,, \end{aligned}
$$

where $\mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ is defined according to (10) and $\mathbf{r}(\widehat{\boldsymbol{\tau}}_s) \in \mathbb{RM}(\Omega_s)$ is characterized by

$$
\int_{\Omega_s} \mathbf{r}(\widehat{\boldsymbol{\tau}}_s)\cdot\mathbf{w} \;=\; -\,\langle\, q\boldsymbol{\nu}, \mathbf{w}\,\rangle_\Sigma - \int_{\Omega_s} \mathbf{div}\,\boldsymbol{\tau}_s\cdot\mathbf{w} \qquad \forall\, \mathbf{w} \in \mathbb{RM}(\Omega_s).
$$

Hereafter, $\mathbf{I}$ denotes also a generic identity operator. Note that $\mathbf{r}(\widehat{\boldsymbol{\tau}}_s)$ is just an auxiliary rigid motion that is needed to guarantee the usual compatibility condition for the Neumann problem (25) (cf. [7, Theorem 9.2.30]), and that the orthogonality condition on $\tilde{\mathbf{u}}$ is required for uniqueness. Indeed, it is well known (see, e.g. [4, Section 11.7, Theorem 11.7] or [18, Section 3, Theorem 3.1]) that (25) is well-posed. In addition, owing to the regularity result for the elasticity problem with Neumann boundary conditions (see, e.g. [13]), we know that $(\tilde{\boldsymbol{\sigma}}_s, \tilde{\mathbf{u}}) \in \mathbb{H}^\epsilon(\Omega_s) \times \mathbf{H}^{1+\epsilon}(\Omega_s)$, for some $\epsilon > 0$, and there holds

$$
(26) \qquad \|\tilde{\boldsymbol{\sigma}}_s\|_{\epsilon,\Omega_s} + \|\tilde{\mathbf{u}}\|_{1+\epsilon,\Omega_s} \;\leq\; C\left\{ \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} + \|q\|_{1,\Omega_f} \right\}.
$$

We now introduce the linear operators $P_1 : \mathbb{X}_1 \to \mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_1 : \mathbb{X}_1 \to \mathbb{X}_1$ defined by

$$
(27) \quad P_1(\widehat{\boldsymbol{\tau}}_s) := \tilde{\boldsymbol{\sigma}}_s \qquad \text{and} \qquad \mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s) := (P_1(\widehat{\boldsymbol{\tau}}_s), q) \qquad \forall\,\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1\,,
$$

where $\tilde{\boldsymbol{\sigma}}_s := \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of (25). It is clear from (25) that

$$
(28) \qquad P_1(\widehat{\boldsymbol{\tau}}_s)^{\mathtt{t}} = P_1(\widehat{\boldsymbol{\tau}}_s) \quad \text{in} \quad \Omega_s\,, \quad \mathbf{div}\,P_1(\widehat{\boldsymbol{\tau}}_s) = \mathbf{div}\,\boldsymbol{\tau}_s + \mathbf{r}(\widehat{\boldsymbol{\tau}}_s) \quad \text{in} \quad \Omega_s\,,
$$

and

$$
(29) \qquad P_1(\widehat{\boldsymbol{\tau}}_s)\,\boldsymbol{\nu} \;=\; -q\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,,
$$

which confirms that $\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s)$ belongs to $\mathbb{X}_1$. Then, the continuous dependence result for (25) gives

$$
\|P_1(\widehat{\boldsymbol{\tau}}_s)\|_{\mathbf{div};\Omega_s} \;\leq\; C\left\{ \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} + \|q\|_{1,\Omega_f} \right\} \qquad \forall\,\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1\,,
$$

which shows that $\mathbf{P}_1$ is bounded. Moreover, it is easy to see from (25), (27), (28), and (29) that $\mathbf{P}_1$ is actually a projector, and hence there holds

$$
(30) \qquad \mathbb{X}_1 \;=\; \mathbf{P}_1(\mathbb{X}_1) \,\oplus\, (\mathbf{I} - \mathbf{P}_1)(\mathbb{X}_1)\,.
$$

Finally, it is clear from (26) that $P_1(\widehat{\boldsymbol{\tau}}_s) \in \mathbb{H}^\epsilon(\Omega_s)$ and

$$
(31) \quad \|P_1(\widehat{\boldsymbol{\tau}}_s)\|_{\epsilon,\Omega_s} \;\leq\; C\left\{ \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} + \|q\|_{1,\Omega_f} \right\} \qquad \forall\,\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1\,.
$$

On the other hand, given $\widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2$, we consider the boundary value problem

$$
(32) \qquad \begin{aligned} \tilde{\boldsymbol{\sigma}}_f &= \nabla\tilde{p} \quad \text{in} \quad \Omega_f\,, \quad \operatorname{div}\tilde{\boldsymbol{\sigma}}_f = \operatorname{div}\boldsymbol{\tau}_f \quad \text{in} \quad \Omega_f\,, \\ \tilde{\boldsymbol{\sigma}}_f\cdot\boldsymbol{\nu} &= \rho_f\,\omega^2\,\mathbf{v}\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma, \quad \tilde{p} = 0 \quad \text{on} \quad \Gamma\,. \end{aligned}
$$

It is not difficult to see that (32) is well-posed. In addition, the classical regularity result for the Poisson problem with mixed boundary conditions (see, e.g. [23], [24]) implies that $(\tilde{\boldsymbol{\sigma}}_f, \tilde{p}) \in \mathbf{H}^\epsilon(\Omega_f) \times H^{1+\epsilon}(\Omega_f)$, for some $\epsilon > 0$ (which can be assumed to be the same of (26)), and that

$$\|\tilde{\boldsymbol{\sigma}}_f\|_{\epsilon,\Omega_f} + \|\tilde{p}\|_{1+\epsilon,\Omega_f} \leq C \Big\{ \|\operatorname{div} \boldsymbol{\tau}_f\|_{0,\Omega_f} + \|\mathbf{v}\|_{1,\Omega_s} \Big\} . \tag{33}$$

We now define the linear operators $P_2 : \mathbb{X}_2 \to \mathbf{H}(\operatorname{div};\Omega_f)$ and $\mathbf{P}_2 : \mathbb{X}_2 \to \mathbb{X}_2$ by

$$P_2(\widehat{\boldsymbol{\tau}}_f) := \tilde{\boldsymbol{\sigma}}_f \quad \text{and} \quad \mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f) := (P_2(\widehat{\boldsymbol{\tau}}_f), \mathbf{v}) \quad \forall\, \widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2 , \tag{34}$$

where $\tilde{\boldsymbol{\sigma}}_f := \nabla \tilde{p}$ and $\tilde{p}$ is the unique solution of (32). It follows that

$$\operatorname{div} P_2(\widehat{\boldsymbol{\tau}}_f) = \operatorname{div} \boldsymbol{\tau}_f \quad \text{in} \quad \Omega_f \quad \text{and} \quad P_2(\widehat{\boldsymbol{\tau}}_f)\cdot \boldsymbol{\nu} = \rho_f\,\omega^2\,\mathbf{v}\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma , \tag{35}$$

which confirms that $\mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f)$ belongs to $\mathbb{X}_2$. In addition, thanks to the continuous dependence result for (32), there holds

$$\|P_2(\widehat{\boldsymbol{\tau}}_f)\|_{\operatorname{div};\Omega_f} \leq C \Big\{ \|\operatorname{div} \boldsymbol{\tau}_f\|_{0,\Omega_f} + \|\mathbf{v}\|_{1,\Omega_s} \Big\} \qquad \forall\, \widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2 ,$$

which shows that $\mathbf{P}_2$ is bounded. Furthermore, it is straightforward from (32), (34), and (35) that $\mathbf{P}_2$ is a projector, and therefore

$$\mathbb{X}_2 = \mathbf{P}_2(\mathbb{X}_2) \oplus (\mathbf{I} - \mathbf{P}_2)(\mathbb{X}_2) . \tag{36}$$

Also, it is clear from (33) that $P_2(\widehat{\boldsymbol{\tau}}_f) \in \mathbf{H}^\epsilon(\Omega_f)$ and

$$\|P_2(\widehat{\boldsymbol{\tau}}_f)\|_{\epsilon,\Omega_f} \leq C \Big\{ \|\operatorname{div} \boldsymbol{\tau}_f\|_{0,\Omega_f} + \|\mathbf{v}\|_{1,\Omega_s} \Big\} \qquad \forall\, \widehat{\boldsymbol{\tau}}_f := (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2 . \tag{37}$$

We now observe, according to (30) and (36), that the space $\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2$ can be certainly decomposed as

$$\mathbb{X} = \mathbf{P}(\mathbb{X}) \oplus (\mathbf{I} - \mathbf{P})(\mathbb{X}) , \tag{38}$$

where $\mathbf{P} : \mathbb{X} \to \mathbb{X}$ is the projector defined by

$$\mathbf{P}(\widehat{\boldsymbol{\tau}}) := (\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s), \mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f)) \qquad \forall\, \widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s, \widehat{\boldsymbol{\tau}}_f) \in \mathbb{X} . \tag{39}$$

In order to show that our augmented fully-mixed formulation (21) is well-posed, we now employ the stable decompositions (30) and (36) (equivalently (38)) to reformulate (21) in a more suitable form.

3.2. Decomposition of the bilinear form $\mathbb{A}$. Let us begin by introducing the bilinear forms $\mathbb{A}_s : \mathbb{X}_1 \times \mathbb{X}_1 \to \mathbf{C}$ and $\mathbb{A}_f : \mathbb{X}_2 \times \mathbb{X}_2 \to \mathbf{C}$ given by

$$\begin{gathered}\mathbb{A}_s(\widehat{\boldsymbol{\zeta}}_s, \widehat{\boldsymbol{\tau}}_s) := \int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\zeta}_s : \boldsymbol{\tau}_s + \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\zeta}_s \cdot \mathbf{div}\,\boldsymbol{\tau}_s + \kappa_2 \int_{\Omega_f} \Big\{ \nabla r \cdot \nabla q + r\,q \Big\} \\ \forall\,(\widehat{\boldsymbol{\zeta}}_s, \widehat{\boldsymbol{\tau}}_s) := ((\boldsymbol{\zeta}_s, r), (\boldsymbol{\tau}_s, q)) \in \mathbb{X}_1 \times \mathbb{X}_1 ,\end{gathered} \tag{40}$$

and

$$\begin{gathered}\mathbb{A}_f(\widehat{\boldsymbol{\zeta}}_f, \widehat{\boldsymbol{\tau}}_f) := \int_{\Omega_f} \boldsymbol{\zeta}_f : \boldsymbol{\tau}_f + \frac{1}{\kappa_f^2}\int_{\Omega_f} \operatorname{div}\boldsymbol{\zeta}_f \cdot \operatorname{div}\boldsymbol{\tau}_f \\ + \kappa_1 \int_{\Omega_s} \Big\{ \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \mathbf{w}\cdot\mathbf{v} \Big\} \\ \forall\,(\widehat{\boldsymbol{\zeta}}_f, \widehat{\boldsymbol{\tau}}_f) := ((\boldsymbol{\zeta}_f, \mathbf{w}), (\boldsymbol{\tau}_f, \mathbf{v})) \in \mathbb{X}_2 \times \mathbb{X}_2 ,\end{gathered} \tag{41}$$

which are clearly bounded, symmetric, and positive semi-definite. Actually, it is established below in Section 3.3 that they are positive definite (cf. Lemmata 3.3 and 3.4).

In what follows, we plan to utilize the decomposition (38) in (21) so that the unknown $\widehat{\boldsymbol{\sigma}} := (\widehat{\boldsymbol{\sigma}}_s, \widehat{\boldsymbol{\sigma}}_f)$ and the corresponding test function $\widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s, \widehat{\boldsymbol{\tau}}_f)$, both in $\mathbb{X}$, are replaced, respectively, by the expressions

$$\begin{aligned} \widehat{\boldsymbol{\sigma}}_s &= \mathbf{P}_1(\widehat{\boldsymbol{\sigma}}_s) + (\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\sigma}}_s) = (P_1(\widehat{\boldsymbol{\sigma}}_s), p) + (\boldsymbol{\sigma}_s - P_1(\widehat{\boldsymbol{\sigma}}_s), 0)\,, \\ \widehat{\boldsymbol{\sigma}}_f &= \mathbf{P}_2(\widehat{\boldsymbol{\sigma}}_f) + (\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\sigma}}_f) = (P_2(\widehat{\boldsymbol{\sigma}}_f), \mathbf{u}) + (\boldsymbol{\sigma}_f - P_2(\widehat{\boldsymbol{\sigma}}_f), \mathbf{0})\,, \end{aligned} \tag{42}$$

and

$$\begin{aligned} \widehat{\boldsymbol{\tau}}_s &= \mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s) + (\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s) = (P_1(\widehat{\boldsymbol{\tau}}_s), q) + (\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s), 0)\,, \\ \widehat{\boldsymbol{\tau}}_f &= \mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f) + (\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f) = (P_2(\widehat{\boldsymbol{\tau}}_f), \mathbf{v}) + (\boldsymbol{\tau}_f - P_2(\widehat{\boldsymbol{\tau}}_f), \mathbf{0})\,. \end{aligned} \tag{43}$$

To this respect, we now recall from (28) and (29) that for each $\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1$ there hold $\operatorname{div}(\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s)) = -\mathbf{r}(\widehat{\boldsymbol{\tau}}_s) \in \mathbb{RM}(\Omega_s)$, $P_1(\widehat{\boldsymbol{\tau}}_s)$ is symmetric, and $P_1(\widehat{\boldsymbol{\tau}}_s)\,\boldsymbol{\nu} = -q\,\boldsymbol{\nu}$ on Σ, whence, noting also that $\nabla\mathbf{r}(\widehat{\boldsymbol{\tau}}_s) \in \mathbb{Y}$, we find that

$$\begin{aligned} &\int_{\Omega_s} \mathbf{div}\,(\boldsymbol{\zeta}_s - P_1(\widehat{\boldsymbol{\zeta}}_s)) \cdot \mathbf{div}\, P_1(\widehat{\boldsymbol{\tau}}_s) = -\int_{\Omega_s} \mathbf{r}(\widehat{\boldsymbol{\zeta}}_s) \cdot \mathbf{div}\, P_1(\widehat{\boldsymbol{\tau}}_s) \\ &\quad = \int_{\Omega_s} \nabla\mathbf{r}(\widehat{\boldsymbol{\zeta}}_s) : P_1(\widehat{\boldsymbol{\tau}}_s) - \langle P_1(\widehat{\boldsymbol{\tau}}_s)\,\boldsymbol{\nu}, \mathbf{r}(\widehat{\boldsymbol{\zeta}}_s)\rangle_\Sigma = \int_\Sigma \mathbf{r}(\widehat{\boldsymbol{\zeta}}_s)\cdot\boldsymbol{\nu}\, q \end{aligned} \tag{44}$$

for all $\widehat{\boldsymbol{\zeta}}_s := (\boldsymbol{\zeta}_s, r) \in \mathbb{X}_1$.

Next, using the decomposition (38) and the identity (44), and adding and substracting suitable terms, we find that the bilinear form $\mathbb{A}$ (cf. (22)) can be decomposed as

$$\mathbb{A}(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) = \mathbb{A}_0(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) + \mathbb{K}(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}})\,,$$

$$\forall\,\widehat{\boldsymbol{\zeta}} := (\widehat{\boldsymbol{\zeta}}_s, \widehat{\boldsymbol{\zeta}}_f) := ((\boldsymbol{\zeta}_s, r), (\boldsymbol{\zeta}_f, \mathbf{w})) \in \mathbb{X}\,,$$

$$\forall\,\widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s, \widehat{\boldsymbol{\tau}}_f) := ((\boldsymbol{\tau}_s, q), (\boldsymbol{\tau}_f, \mathbf{v})) \in \mathbb{X}\,,$$

where $\mathbb{A}_0 : \mathbb{X}\times\mathbb{X} \to \mathbf{C}$ and $\mathbb{K} : \mathbb{X}\times\mathbb{X}\to\mathbf{C}$ are given by

$$\begin{aligned} \mathbb{A}_0(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) &= -\mathbb{A}_s(\mathbf{P}_1(\widehat{\boldsymbol{\zeta}}_s), \mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s)) + \mathbb{A}_s((\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\zeta}}_s), (\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s)) \\ &\quad - \mathbb{A}_f(\mathbf{P}_2(\widehat{\boldsymbol{\zeta}}_f), \mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f)) + \mathbb{A}_f((\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\zeta}}_f), (\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f)) \\ &\quad + \kappa_1 \int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta}_s : \boldsymbol{\varepsilon}(\mathbf{v}) + \kappa_2 \int_{\Omega_f} \boldsymbol{\zeta}_f \cdot \nabla q\,, \end{aligned} \tag{45}$$

and

$$\mathbb{K}(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) = \mathbb{K}_s(\widehat{\boldsymbol{\zeta}}_s, \widehat{\boldsymbol{\tau}}_s) + \mathbb{K}_f(\widehat{\boldsymbol{\zeta}}_f, \widehat{\boldsymbol{\tau}}_f) + \langle q\,\boldsymbol{\nu}, \mathbf{w}\rangle_\Sigma + \rho_f\,\omega^2\,\langle r\,\boldsymbol{\nu}, \mathbf{v}\rangle_\Sigma\,, \tag{46}$$

with

(47)

$$\begin{aligned} \mathbb{K}_s(\widehat{\boldsymbol{\zeta}}_s, \widehat{\boldsymbol{\tau}}_s) &:= 2\int_{\Omega_s} \mathcal{C}^{-1}\,P_1(\widehat{\boldsymbol{\zeta}}_s) : P_1(\widehat{\boldsymbol{\tau}}_s) + \int_{\Omega_s} \mathcal{C}^{-1}\,P_1(\widehat{\boldsymbol{\zeta}}_s) : (\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s)) \\ &+ \int_{\Omega_s} \mathcal{C}^{-1}\,(\boldsymbol{\zeta}_s - P_1(\widehat{\boldsymbol{\zeta}}_s)) : P_1(\widehat{\boldsymbol{\tau}}_s) - \frac{2}{\kappa_s^2}\int_{\Omega_s} \mathbf{r}(\widehat{\boldsymbol{\zeta}}_s)\cdot\mathbf{r}(\widehat{\boldsymbol{\tau}}_s) \\ &- \frac{1}{\kappa_s^2}\int_\Sigma \mathbf{r}(\widehat{\boldsymbol{\tau}}_s)\cdot r\,\boldsymbol{\nu} - \frac{1}{\kappa_s^2}\int_\Sigma \mathbf{r}(\widehat{\boldsymbol{\zeta}}_s)\cdot q\,\boldsymbol{\nu} + \kappa_2\int_{\Omega_f} r\,q\,, \end{aligned}$$

and

$$
\begin{aligned}
\mathbb{K}_f(\widehat{\boldsymbol{\zeta}}_f, \widehat{\boldsymbol{\tau}}_f) \,:=\, & 2 \int_{\Omega_f} P_2(\widehat{\boldsymbol{\zeta}}_f) \cdot P_2(\widehat{\boldsymbol{\tau}}_f) \,+\, \int_{\Omega_f} P_2(\widehat{\boldsymbol{\zeta}}_f) \cdot (\boldsymbol{\tau}_f - P_2(\widehat{\boldsymbol{\tau}}_f)) \\
& + \int_{\Omega_f} (\boldsymbol{\zeta}_f - P_2(\widehat{\boldsymbol{\zeta}}_f)) \cdot P_2(\widehat{\boldsymbol{\tau}}_f) \,+\, \kappa_1 \int_{\Omega_s} \mathbf{w} \cdot \mathbf{v}\,.
\end{aligned} \tag{48}
$$

Note that, in spite of the symmetry of $\mathbb{A}_s$ and $\mathbb{A}_f$, and because of the last two terms defining $\mathbb{A}_0$ (cf. (45)), this latter bilinear form is not symmetric. We also remark at this point that the minus signs multiplying the stabilization parameters in the definition of $\mathbb{A}$ (cf. (22)) are needed for obtaining the minus signs in front of the first and third terms in the resulting expression for $\mathbb{A}_0$ (cf. (45)). In turn, these minus signs are required later on to show that $\mathbb{A}_0$ satisfy certain inf-sup conditions (see (53) and the proof of Lemma 3.6 below).

We now let $\mathbf{A}_0 : \mathbb{X} \to \mathbb{X}$, $\mathbf{K} : \mathbb{X} \to \mathbb{X}$, and $\mathbf{B} : \mathbb{X} \to \mathbb{Y}$ be the linear and bounded operators induced by the bilinear forms $\mathbb{A}_0$, $\mathbb{K}$, and $\mathbb{B}$, respectively. In addition, we let $\mathcal{F} \in \mathbb{X}$ be the Riesz representant of $\mathbb{F}$. Hence, using these notations and taking into account the decompositions (42) and (43), the augmented-fully-mixed variational formulation (21) can be rewritten as the following operator equation: Find $(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) \in \mathbb{X} \times \mathbb{Y}$ such that

$$
\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \gamma \end{pmatrix} + \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \gamma \end{pmatrix} = \begin{pmatrix} \mathcal{F} \\ \mathbf{0} \end{pmatrix}. \tag{49}
$$

In the following section we show that the matrix operators on the left hand side of (49) become invertible and compact, respectively. More precisely, since the one involving $\mathbf{A}_0$ and $\mathbf{B}$ shows a saddle point structure, the well known Babuška-Brezzi theory will be applied for the respective analysis.

3.3. Application of the Babuška-Brezzi and Fredholm theories. We first recall from [20, Section 4.2] the ocurrence of the continuous inf-sup condition for the bilinear form $\mathbb{B}$, which is equivalent to the surjectivity of the operator $\mathbf{B}$. To this respect, note from the definition of $\mathbb{B}$ (cf. (23)) that

$$
\mathbf{B}(\widehat{\boldsymbol{\tau}}) \,:=\, \frac{1}{2}\big(\boldsymbol{\tau}_s - \boldsymbol{\tau}_s^{\mathtt{t}}\big) \qquad \forall\, \widehat{\boldsymbol{\tau}} \,:=\, (\widehat{\boldsymbol{\tau}}_s, \widehat{\boldsymbol{\tau}}_f) \,:=\, ((\boldsymbol{\tau}_s, q), (\boldsymbol{\tau}_f, \mathbf{v})) \,\in\, \mathbb{X}\,.
$$

Lemma 3.1. *There exists $C_1 > 0$ such that*

$$
\sup_{\substack{\widehat{\boldsymbol{\tau}} \in \mathbb{X} \\ \widehat{\boldsymbol{\tau}} \neq \mathbf{0}}} \frac{|\,\mathbb{B}(\widehat{\boldsymbol{\tau}}, \boldsymbol{\eta})\,|}{\|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}}} \,\geq\, C_1\, \|\boldsymbol{\eta}\|_{\mathbb{Y}} \qquad \forall\, \boldsymbol{\eta} \,\in\, \mathbb{Y}\,.
$$

Proof. See the proof of [20, Lemma 4.1]. □

We now aim to establish that the operator induced by the restriction of $\mathbb{A}_0$ to $\mathbb{V} \times \mathbb{V}$, where $\mathbb{V}$ is the kernel of $\mathbf{B}$, is bijective. The corresponding analysis is based on the series of technical inequalities given in what follows.

We begin by introducing the decomposition

$$
\mathbb{H}(\mathbf{div}; \Omega_s) \,=\, \mathbb{H}_0(\mathbf{div}; \Omega_s) \,\oplus\, \mathbf{C}\,\mathbf{I}\,,
$$

with

$$
\mathbb{H}_0(\mathbf{div}; \Omega_s) \,:=\, \Big\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div}; \Omega_s) : \quad \int_{\Omega_s} \operatorname{tr} \boldsymbol{\tau}_s \,=\, 0 \Big\},
$$

which means that for any $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ there exist unique $\boldsymbol{\tau}_{s,0} \in \mathbb{H}_0(\mathbf{div};\Omega_s)$ and $d \in \mathbf{C}$ given by $d := \dfrac{1}{3\,|\Omega_s|}\displaystyle\int_{\Omega_s} \operatorname{tr}\boldsymbol{\tau}_s$, where $|\Omega_s|$ denotes the measure of Ω_s, such that $\boldsymbol{\tau}_s = \boldsymbol{\tau}_{s,0} + d\,\mathbf{I}$. Then we have the following result.

Lemma 3.2. *There exists $c_1 > 0$, depending only on Ω_s, such that*

$$\|\boldsymbol{\tau}_s^{\mathrm{d}}\|_{0,\Omega_s}^2 + \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \;\geq\; c_1\,\|\boldsymbol{\tau}_{s,0}\|_{0,\Omega_s}^2 \qquad \forall\,\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)\,. \tag{50}$$

Proof. See [2, Lemma 3.1] or [8, Proposition 3.1, Chapter IV]. □

The already announced positive definiteness of the bilinear forms $\mathbb{A}_s$ and $\mathbb{A}_f$ are shown now.

Lemma 3.3. *There exists $c_s > 0$, directly depending on* $\min\left\{1, \dfrac{1}{\kappa_s^2}\right\}$, *such that*

$$\mathbb{A}_s(\widehat{\boldsymbol{\tau}}_s, \overline{\widehat{\boldsymbol{\tau}}}_s) \;\geq\; c_s\,\|\widehat{\boldsymbol{\tau}}_s\|_{\mathbb{X}_1}^2 \qquad \forall\,\widehat{\boldsymbol{\tau}}_s \,\in\, \mathbb{X}_1\,.$$

Proof. Given $\widehat{\boldsymbol{\tau}}_s := (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1$, we obtain directly from the definition of $\mathbb{A}_s$ (cf. (40)) that

$$\mathbb{A}_s(\widehat{\boldsymbol{\tau}}_s, \overline{\widehat{\boldsymbol{\tau}}}_s) \;=\; \int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\tau}_s : \overline{\boldsymbol{\tau}}_s \,+\, \frac{1}{\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \,+\, \kappa_2\,\|q\|_{1,\Omega_f}^2\,,$$

which, applying inequalities (12) and (50) (cf. Lemma 3.2), gives,

$$\mathbb{A}_s(\widehat{\boldsymbol{\tau}}_s, \overline{\widehat{\boldsymbol{\tau}}}_s) \;\geq\; C\left\{\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}^2 \,+\, \|q\|_{1,\Omega_f}^2\right\}, \tag{51}$$

where $\boldsymbol{\tau}_s = \boldsymbol{\tau}_{s,0} + d\,\mathbf{I}$, and C is a positive constant depending on μ, $\min\left\{1, \dfrac{1}{\kappa_s^2}\right\}$, c_1 (cf. Lemma 3.2), and κ_2. Next, we proceed exactly as in the proof of [20, Lemma 4.3] (see also [16, Lemma 2.2]). In fact, since $\boldsymbol{\tau}_s\,\boldsymbol{\nu} = -\,q\,\boldsymbol{\nu}$ on Σ, we get $-\,q\,\boldsymbol{\nu} = \boldsymbol{\tau}_{s,0}\,\boldsymbol{\nu} + d\,\boldsymbol{\nu}$ on Σ, which, applying the trace theorems in $\mathbb{H}(\mathbf{div};\Omega_s)$ and $H^1(\Omega_f)$, yields

$$|d|\,\|\boldsymbol{\nu}\|_{-1/2,\Sigma} \;=\; \|\boldsymbol{\tau}_{s,0}\,\boldsymbol{\nu} \,+\, q\,\boldsymbol{\nu}\|_{-1/2,\Sigma} \;\leq\; C\left\{\,\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s} \,+\, \|q\|_{1,\Omega_f}\,\right\}.$$

Hence, noting that $\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}^2 \;=\; \|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}^2 \,+\, 3\,d^2\,|\Omega_s|$, we easily find, with a constant c depending on $\|\boldsymbol{\nu}\|_{-1/2,\Sigma}$, that

$$\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}^2 \;\leq\; c\left\{\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}^2 \,+\, \|q\|_{1,\Omega_f}^2\right\},$$

which, together with (51), finishes the proof. □

Lemma 3.4. *There exists $c_f > 0$, directly depending on* $\min\left\{1, \dfrac{1}{\kappa_f^2}\right\}$, *such that*

$$\mathbb{A}_f(\widehat{\boldsymbol{\tau}}_f, \overline{\widehat{\boldsymbol{\tau}}}_f) \;\geq\; c_f\,\|\widehat{\boldsymbol{\tau}}_f\|_{\mathbb{X}_2}^2 \qquad \forall\,\widehat{\boldsymbol{\tau}}_f \,\in\, \mathbb{X}_2\,.$$

Proof. It follows straightforwardly from the definition of $\mathbb{A}_f$ (cf. (41)) and the usual Korn inequality (cf. [7, Theorem 9.2.16]). We omit further details. □

A very basic inequality related to the inverse of the Hooke operator (cf. (11)) is provided next.

Lemma 3.5. *There holds*

$$\|\mathcal{C}^{-1}\,\boldsymbol{\zeta}\|^2_{0,\Omega_s} \;\le\; \frac{1}{2\,\mu}\int_{\Omega_s}\mathcal{C}^{-1}\,\boldsymbol{\zeta}:\overline{\boldsymbol{\zeta}} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega_s)\,.$$

Proof. Given $\boldsymbol{\zeta}\in\mathbb{L}^2(\Omega_s)$, simple algebraic computations yield, using (11), that

$$\int_{\Omega_s}\mathcal{C}^{-1}\,\boldsymbol{\zeta}:\overline{\boldsymbol{\zeta}} \;=\; \frac{1}{2\,\mu}\,\|\boldsymbol{\zeta}\|^2_{0,\Omega_s} \;-\; \frac{\lambda}{2\,\mu\,(3\,\lambda+2\,\mu)}\,\|\mathrm{tr}\,\boldsymbol{\zeta}\|^2_{0,\Omega_s}$$

and

$$\|\mathcal{C}^{-1}\,\boldsymbol{\zeta}\|^2_{0,\Omega_s} \;=\; \frac{1}{2\,\mu}\left\{\frac{1}{2\,\mu}\,\|\boldsymbol{\zeta}\|^2_{0,\Omega_s} \;-\; \frac{\lambda}{2\,\mu\,(3\,\lambda+2\,\mu)}\,\frac{(3\,\lambda+4\,\mu)}{(3\,\lambda+2\,\mu)}\,\|\mathrm{tr}\,\boldsymbol{\zeta}\|^2_{0,\Omega_s}\right\}.$$

Hence, the proofs follows directly from the fact that $\frac{(3\,\lambda+4\,\mu)}{(3\,\lambda+2\,\mu)} \,>\, 1$.

□

We are now in a position to prove the necessary estimates for concluding later on the inf-sup conditions required by the Babuška-Brezzi theory for the bijectivity of the operator induced by $\mathbb{A}_0|_{\mathbb{V}\times\mathbb{V}}$. To this end, we need to introduce the linear and bounded operator $\mathbf{T} := (\mathbf{I}\,-\,2\,\mathbf{P}):\mathbb{X}\to\mathbb{X}$, that is, according to (39),

$$\mathbf{T}(\widehat{\boldsymbol{\tau}}) \;:=\; \big((\mathbf{I}-2\,\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s),(\mathbf{I}-2\,\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f)\big) \qquad \forall\,\widehat{\boldsymbol{\tau}} \;:=\; (\widehat{\boldsymbol{\tau}}_s,\widehat{\boldsymbol{\tau}}_f)\,\in\,\mathbb{X}\,. \tag{52}$$

It is important to observe here, since $\mathbf{P}_1$ and $\mathbf{P}_2$ are projectors, that

$$\mathbf{P}_i\,(\mathbf{I}\,-\,2\,\mathbf{P}_i) \;=\; -\,\mathbf{P}_i \quad\text{and}\quad (\mathbf{I}\,-\,\mathbf{P}_i)\,(\mathbf{I}\,-\,2\,\mathbf{P}_i) \;=\; (\mathbf{I}\,-\,\mathbf{P}_i) \quad \forall\,i\in\{1,2\}\,, \tag{53}$$

or equivalently,

$$\mathbf{P}\,\mathbf{T} \;=\; -\,\mathbf{P} \quad\text{and}\quad (\mathbf{I}\,-\,\mathbf{P})\,\mathbf{T} \;=\; (\mathbf{I}\,-\,\mathbf{P})\,.$$

Lemma 3.6. *Assume that the stabilization parameters κ_1 and κ_2 are chosen independently of λ and such that $0\,<\,\kappa_1\,<\,2\,\mu$ and $0\,<\,\kappa_2\,<\,1$. Then, there exist C_1, $C_2\,>\,0$, depending only on μ, c_s, c_f, κ_1, and κ_2, and hence directly depending on $\min\left\{1,\dfrac{1}{\kappa_s^2}\right\}$ and $\min\left\{1,\dfrac{1}{\kappa_f^2}\right\}$, such that for each $\widehat{\boldsymbol{\tau}}\,\in\,\mathbb{X}$ there hold*

$$\mathrm{Re}\left\{\mathbb{A}_0(\widehat{\boldsymbol{\tau}},\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}))\right\} \;\ge\; C_1\,\|\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}} \tag{54}$$

and

$$\mathrm{Re}\left\{\mathbb{A}_0(\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}),\widehat{\boldsymbol{\tau}})\right\} \;\ge\; C_2\,\|\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}}\,. \tag{55}$$

Proof. Given $\widehat{\boldsymbol{\tau}} \,:=\, (\widehat{\boldsymbol{\tau}}_s,\widehat{\boldsymbol{\tau}}_f) \,:=\, ((\boldsymbol{\tau}_s,q),(\boldsymbol{\tau}_f,\mathbf{v}))\,\in\,\mathbb{X}$, we easily find from the definition of $\mathbb{A}_0$ (cf. (45)), utilizing the identities given by (53), that

$$\begin{aligned}\mathbb{A}_0(\widehat{\boldsymbol{\tau}},\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}})) \;&=\; \mathbb{A}_s(\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s),\mathbf{P}_1(\overline{\widehat{\boldsymbol{\tau}}}_s)) \,+\, \mathbb{A}_s((\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s),(\mathbf{I}-\mathbf{P}_1)(\overline{\widehat{\boldsymbol{\tau}}}_s))\\ &+\mathbb{A}_f(\mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f),\mathbf{P}_2(\overline{\widehat{\boldsymbol{\tau}}}_f)) \,+\, \mathbb{A}_f((\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f),(\mathbf{I}-\mathbf{P}_2)(\overline{\widehat{\boldsymbol{\tau}}}_f))\\ &-\kappa_1\int_{\Omega_s}\mathcal{C}^{-1}\,\boldsymbol{\tau}_s:\boldsymbol{\varepsilon}(\overline{\mathbf{v}}) \,-\, \kappa_2\int_{\Omega_f}\boldsymbol{\tau}_f\cdot\nabla\overline{q}\,,\end{aligned}$$

which, applying Cauchy-Schwarz's inequality and the trivial estimate $ab \le \frac{1}{2}(a^2 + b^2)$, gives
(56)

$$\begin{aligned}
\mathrm{Re}\left\{ \mathbb{A}_0(\widehat{\boldsymbol{\tau}}, \mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}})) \right\} \ge\ & \mathbb{A}_s(\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s), \mathbf{P}_1(\overline{\widehat{\boldsymbol{\tau}}}_s)) + \mathbb{A}_s((\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s), (\mathbf{I}-\mathbf{P}_1)(\overline{\widehat{\boldsymbol{\tau}}}_s)) \\
+\ & \mathbb{A}_f(\mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f), \mathbf{P}_2(\overline{\widehat{\boldsymbol{\tau}}}_f)) + \mathbb{A}_f((\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f), (\mathbf{I}-\mathbf{P}_2)(\overline{\widehat{\boldsymbol{\tau}}}_f)) \\
-\ & \frac{\kappa_1}{2} \|\mathcal{C}^{-1} \boldsymbol{\tau}_s\|^2_{0,\Omega_s} - \frac{\kappa_1}{2} \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,\Omega_s} - \frac{\kappa_2}{2} \|\boldsymbol{\tau}_f\|^2_{0,\Omega_f} - \frac{\kappa_2}{2} \|\nabla q\|^2_{0,\Omega_s} .
\end{aligned}$$

In turn, using the basic Lemma 3.5 we obtain

$$\begin{aligned}
\|\mathcal{C}^{-1} \boldsymbol{\tau}_s\|^2_{0,\Omega_s} &\le 2 \left\{ \|\mathcal{C}^{-1} P_1(\widehat{\boldsymbol{\tau}}_s)\|^2_{0,\Omega_s} + \|C^{-1}(\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s))\|^2_{0,\Omega_s} \right\} \\
&\le \frac{1}{\mu} \left\{ \int_{\Omega_s} \mathcal{C}^{-1} P_1(\widehat{\boldsymbol{\tau}}_s) : P_1(\overline{\widehat{\boldsymbol{\tau}}}_s) + \int_{\Omega_s} \mathcal{C}^{-1}(\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s)) : (\overline{\boldsymbol{\tau}}_s - P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)) \right\},
\end{aligned}$$

and it is clear that

$$\|\boldsymbol{\tau}_f\|^2_{0,\Omega_f} \le 2 \left\{ \|P_2(\widehat{\boldsymbol{\tau}}_f)\|^2_{0,\Omega_f} + \|(\boldsymbol{\tau}_f - P_2(\widehat{\boldsymbol{\tau}}_f))\|^2_{0,\Omega_f} \right\}.$$

Then, substituting the last two inequalities into (56), and having in mind the definitions of $\mathbb{A}_s$ and $\mathbb{A}_f$ (cf. (40), (41)), we arrive at

$$\begin{aligned}
&\mathrm{Re}\left\{ \mathbb{A}_0(\widehat{\boldsymbol{\tau}}, \mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}})) \right\} \\
&\ge \left(1 - \frac{\kappa_1}{2\mu}\right) \int_{\Omega_s} \mathcal{C}^{-1} P_1(\widehat{\boldsymbol{\tau}}_s) : P_1(\overline{\widehat{\boldsymbol{\tau}}}_s) + \frac{1}{\kappa_s^2} \int_{\Omega_s} \mathbf{div}\, P_1(\widehat{\boldsymbol{\tau}}_s) \cdot \mathbf{div}\, P_1(\overline{\widehat{\boldsymbol{\tau}}}_s) \\
&+ \frac{\kappa_2}{2} \|\nabla q\|^2_{0,\Omega_f} + \kappa_2 \|q\|^2_{0,\Omega_f} + \left(1 - \frac{\kappa_1}{2\mu}\right) \int_{\Omega_s} \mathcal{C}^{-1} \big(\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s)\big) : \big(\overline{\boldsymbol{\tau}}_s - P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)\big) \\
&+ \frac{1}{\kappa_s^2} \int_{\Omega_s} \mathbf{div}\big(\boldsymbol{\tau}_s - P_1(\widehat{\boldsymbol{\tau}}_s)\big) \cdot \mathbf{div}\big(\overline{\boldsymbol{\tau}}_s - P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)\big) + (1-\kappa_2) \int_{\Omega_f} P_2(\widehat{\boldsymbol{\tau}}_f) \cdot P_2(\overline{\widehat{\boldsymbol{\tau}}}_f) \\
&+ \frac{1}{\kappa_f^2} \int_{\Omega_f} \mathrm{div}\, P_2(\widehat{\boldsymbol{\tau}}_f)\, \mathrm{div}\, P_2(\overline{\widehat{\boldsymbol{\tau}}}_f) + \frac{\kappa_1}{2} \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,\Omega_s} + \kappa_1 \|\mathbf{v}\|^2_{0,\Omega_s} \\
&+ (1-\kappa_2) \int_{\Omega_f} \big(\boldsymbol{\tau}_f - P_2(\widehat{\boldsymbol{\tau}}_f)\big) \cdot \big(\overline{\boldsymbol{\tau}}_f - P_2(\overline{\widehat{\boldsymbol{\tau}}}_f)\big) \\
&+ \frac{1}{\kappa_f^2} \int_{\Omega_f} \mathrm{div}\big(\boldsymbol{\tau}_f - P_2(\widehat{\boldsymbol{\tau}}_f)\big)\, \mathrm{div}\big(\overline{\boldsymbol{\tau}}_f - P_2(\overline{\widehat{\boldsymbol{\tau}}}_f)\big),
\end{aligned}$$

from which we conclude, defining $\tilde{\kappa}_1 := 1 - \dfrac{\kappa_1}{2\mu}$ and $\tilde{\kappa}_2 := 1 - \kappa_2$, that

$$\begin{aligned}
&\mathrm{Re}\left\{ \mathbb{A}_0(\widehat{\boldsymbol{\tau}}, \mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}})) \right\} \\
&\quad \ge \min\left\{\tilde{\kappa}_1, \frac{1}{2}\right\} \mathbb{A}_s(\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s), \mathbf{P}_1(\overline{\widehat{\boldsymbol{\tau}}}_s)) + \tilde{\kappa}_1\, \mathbb{A}_s((\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s), (\mathbf{I}-\mathbf{P}_1)(\overline{\widehat{\boldsymbol{\tau}}}_s)) \\
&\quad + \min\left\{\tilde{\kappa}_2, \frac{1}{2}\right\} \mathbb{A}_f(\mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f), \mathbf{P}_2(\overline{\widehat{\boldsymbol{\tau}}}_f)) + \tilde{\kappa}_2\, \mathbb{A}_f((\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f), (\mathbf{I}-\mathbf{P}_2)(\overline{\widehat{\boldsymbol{\tau}}}_f)) .
\end{aligned}$$

In this way, the above inequality together with Lemmas 3.3 and 3.4, and the stability of the decomposition (38) imply the estimate (54).

For the proof of (55) we first observe, similarly to the derivation of (56), that

$$\begin{aligned}\mathrm{Re}\left\{\mathbb{A}_0(\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}),\widehat{\boldsymbol{\tau}})\right\} &\geq \mathbb{A}_s(\mathbf{P}_1(\overline{\widehat{\boldsymbol{\tau}}}_s),\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_s)) + \mathbb{A}_s((\mathbf{I}-\mathbf{P}_1)(\overline{\widehat{\boldsymbol{\tau}}}_s),(\mathbf{I}-\mathbf{P}_1)(\widehat{\boldsymbol{\tau}}_s))\\ &+ \mathbb{A}_f(\mathbf{P}_2(\overline{\widehat{\boldsymbol{\tau}}}_f),\mathbf{P}_2(\widehat{\boldsymbol{\tau}}_f)) + \mathbb{A}_f((\mathbf{I}-\mathbf{P}_2)(\overline{\widehat{\boldsymbol{\tau}}}_f),(\mathbf{I}-\mathbf{P}_2)(\widehat{\boldsymbol{\tau}}_f))\\ &- \frac{\kappa_1}{2}\,\|\mathcal{C}^{-1}\left(\overline{\boldsymbol{\tau}}_s - 2\,P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)\right)\|^2_{0,\Omega_s} - \frac{\kappa_1}{2}\,\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,\Omega_s}\\ &- \frac{\kappa_2}{2}\,\|\overline{\boldsymbol{\tau}}_f - 2\,P_2(\overline{\widehat{\boldsymbol{\tau}}}_f)\|^2_{0,\Omega_f} - \frac{\kappa_2}{2}\,\|\nabla q\|^2_{0,\Omega_s}\,.\end{aligned}$$

The rest proceeds as before, by noting now that

$$\|\mathcal{C}^{-1}\left(\overline{\boldsymbol{\tau}}_s - 2\,P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)\right)\|^2_{0,\Omega_s} \leq 2\left\{\|C^{-1}\left(\overline{\boldsymbol{\tau}}_s - P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)\right)\|^2_{0,\Omega_s} + \|\mathcal{C}^{-1}\,P_1(\overline{\widehat{\boldsymbol{\tau}}}_s)\|^2_{0,\Omega_s}\right\},$$

and

$$\|\overline{\boldsymbol{\tau}}_f - 2\,P_2(\overline{\widehat{\boldsymbol{\tau}}}_f)\|^2_{0,\Omega_f} \leq 2\left\{\|(\overline{\boldsymbol{\tau}}_f - P_2(\overline{\widehat{\boldsymbol{\tau}}}_f))\|^2_{0,\Omega_f} + \|P_2(\overline{\widehat{\boldsymbol{\tau}}}_f)\|^2_{0,\Omega_f}\right\},$$

and then using again the definitions of $\mathbb{A}_s$ and $\mathbb{A}_f$. We omit further details. □

At this point we observe that the estimate (55) would certainly be a straightforward consequence of (54) if the original bilinear form $\mathbb{A}$ (cf. (22)), and hence $\mathbb{A}_0$, were symmetric. While this could have been possible by redefining $\mathbb{A}$ with the incorporation of extra terms in the identities (19) and (20), we preferred to keep it in this way because the resulting Galerkin scheme becomes less expensive.

Throughout the rest of the paper we assume that the stabilization parameters κ_1 and κ_2 are chosen independently of λ and such that $0 < \kappa_1 < 2\,\mu$ and $0 < \kappa_2 < 1$.

We now let $\mathbb{V}$ be the kernel of $\mathbf{B}$, that is $\mathbb{V} := \left\{\widehat{\boldsymbol{\tau}} \in \mathbb{X}: \quad \mathbf{B}(\widehat{\boldsymbol{\tau}}) = 0\right\}$, which, recalling that $\mathbf{B}(\widehat{\boldsymbol{\tau}}) := \frac{1}{2}\left(\boldsymbol{\tau}_s - \boldsymbol{\tau}_s^{\mathtt{t}}\right) \quad \forall\,\widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s,\widehat{\boldsymbol{\tau}}_f) = ((\boldsymbol{\tau}_s,q),(\boldsymbol{\tau}_f,\mathbf{v})) \in \mathbb{X}$, becomes

$$\mathbb{V} = \left\{\widehat{\boldsymbol{\tau}} \in \mathbb{X}: \quad \boldsymbol{\tau}_s^{\mathtt{t}} = \boldsymbol{\tau}_s\right\}.$$

The weak coercivity of $\mathbb{A}_0$ on $\mathbb{V}$ is established now.

Lemma 3.7. *There exists $\alpha > 0$, directly depending on $\min\left\{1,\frac{1}{\kappa_s^2}\right\}$ and $\min\left\{1,\frac{1}{\kappa_f^2}\right\}$, such that*

$$\sup_{\substack{\widehat{\boldsymbol{\zeta}}\in\mathbb{V}\\ \widehat{\boldsymbol{\zeta}}\neq\mathbf{0}}} \frac{|\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\zeta}})\,|}{\|\widehat{\boldsymbol{\zeta}}\|_{\mathbb{X}}} \geq \alpha\,\|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}} \qquad \forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{V}\,. \tag{57}$$

In addition, there holds

$$\sup_{\widehat{\boldsymbol{\zeta}}\in\mathbb{V}} |\,\mathbb{A}_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}})\,| > 0 \qquad \forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{V}\,,\ \widehat{\boldsymbol{\tau}} \neq \mathbf{0}\,. \tag{58}$$

Proof. Since $P_1(\widehat{\boldsymbol{\tau}}_s)^{\mathtt{t}} = P_1(\widehat{\boldsymbol{\tau}}_s) \quad \forall\,\widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s,\widehat{\boldsymbol{\tau}}_f) = ((\boldsymbol{\tau}_s,q),(\boldsymbol{\tau}_f,\mathbf{v})) \in \mathbb{X}$, we find that $\mathbf{P}(\widehat{\boldsymbol{\tau}})$ (cf. (39)), and hence $\mathbf{T}(\widehat{\boldsymbol{\tau}})$, belong to $\mathbb{V}$ for each $\widehat{\boldsymbol{\tau}} \in \mathbb{V}$. In addition, it is clear from (54) that $\mathbf{T}(\widehat{\boldsymbol{\tau}}) \neq \mathbf{0}$ for each $\widehat{\boldsymbol{\tau}} \in \mathbb{X}$, $\widehat{\boldsymbol{\tau}} \neq \mathbf{0}$. In particular, for each $\widehat{\boldsymbol{\tau}} \in \mathbb{V}$, $\widehat{\boldsymbol{\tau}} \neq \mathbf{0}$, there holds

$$\sup_{\substack{\widehat{\boldsymbol{\zeta}}\in\mathbb{V}\\ \widehat{\boldsymbol{\zeta}}\neq\mathbf{0}}} \frac{|\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\zeta}})\,|}{\|\widehat{\boldsymbol{\zeta}}\|_{\mathbb{X}}} \geq \frac{|\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}},\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}))\,|}{\|\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}})\|_{\mathbb{X}}} \geq \frac{\mathrm{Re}\left\{\mathbb{A}_0(\widehat{\boldsymbol{\tau}},\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}))\right\}}{\|\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}})\|_{\mathbb{X}}}\,,$$

which, applying (54), the boundedness of $\mathbf{T}$, and the fact that $\|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}} = \|\overline{\widehat{\boldsymbol{\tau}}}\|_{\mathbb{X}}$, yields (57) with $\alpha = C_1/\|\mathbf{T}\|$. Similarly, given $\widehat{\boldsymbol{\tau}} \in \mathbb{V}$, there holds

$$\sup_{\widehat{\boldsymbol{\zeta}}\in\mathbb{V}} |\,\mathbb{A}_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}})\,| \;\geq\; |\,\mathbb{A}_0(\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}),\widehat{\boldsymbol{\tau}})\,| \;\geq\; \mathrm{Re}\left\{\mathbb{A}_0(\mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}),\widehat{\boldsymbol{\tau}})\right\},$$

which, together with (55), implies (58).

□

We now aim to show that the bilinear form $\mathbb{K}$ induces a compact operator $\mathbf{K}$.

Lemma 3.8. *The operator* $\mathbf{K} : \mathbb{X} \to \mathbb{X}$ *is compact.*

Proof. We first observe that the last two terms defining $\mathbb{K}$ (cf. (46)), that is $\langle q\,\boldsymbol{\nu},\mathbf{w}\rangle_\Sigma$ and $\rho_f\,\omega^2\,\langle r\,\boldsymbol{\nu},\mathbf{v}\rangle_\Sigma$, yield compact operators because of the compactness of the composition defined by the following diagram

$$\begin{array}{ccccccccc} H^1(\Omega_f) & \overset{continuous}{\longrightarrow} & H^{1/2}(\Sigma) & \overset{compact}{\longrightarrow} & L^2(\Sigma) & \overset{continuous}{\longrightarrow} & \mathbf{L}^2(\Sigma) & \overset{compact}{\longrightarrow} & \mathbf{H}^{-1/2}(\Sigma) \\ q & \longrightarrow & q|_\Sigma & \longrightarrow & q|_\Sigma & \longrightarrow & q\,\boldsymbol{\nu} & \longrightarrow & q\,\boldsymbol{\nu} \end{array}$$

Next, because of the compact imbeddings $H^1(\Omega_f) \hookrightarrow L^2(\Omega_f)$ and $\mathbf{H}^1(\Omega_s) \hookrightarrow \mathbf{L}^2(\Omega_s)$, the last terms defining $\mathbb{K}_s$ and $\mathbb{K}_f$ (cf. (47) and (48)), that is $\int_{\Omega_f} r\,q$ and $\int_{\Omega_s} \mathbf{w}\cdot\mathbf{v}$, induce compact operators, as well. In addition, it is clear that the three terms containing rigid motions in the definition of $\mathbb{K}_s$ yield a finite rank operator. It remains to check the compactness of the operators induced by the terms involving P_1 and P_2 in the definitions of $\mathbb{K}_s$ and $\mathbb{K}_f$. In fact, we recall from Section 3.1 (cf. (31) and (37)) that there exists $\epsilon > 0$ such that $P_1(\widehat{\boldsymbol{\tau}}_s) \in \mathbb{H}^\epsilon(\Omega_s)$ and $P_2(\widehat{\boldsymbol{\tau}}_f) \in \mathbf{H}^\epsilon(\Omega_f)$ for each $\widehat{\boldsymbol{\tau}} := (\widehat{\boldsymbol{\tau}}_s,\widehat{\boldsymbol{\tau}}_f) \in \mathbb{X}$, which, thanks to the compact imbeddings $\mathbb{H}^\epsilon(\Omega_s) \hookrightarrow \mathbb{L}^2(\Omega_s)$ and $\mathbf{H}^\epsilon(\Omega_f) \hookrightarrow \mathbf{L}^2(\Omega_f)$, imply the compactness of $P_1 : \mathbb{X}_1 \to \mathbb{L}^2(\Omega_s)$ and $P_2 : \mathbb{X}_2 \to \mathbf{L}^2(\Omega_f)$. It follows that the adjoints $P_1^* : \mathbb{L}^2(\Omega_s) \to \mathbb{X}_1$ and $P_2^* : \mathbf{L}^2(\Omega_f) \to \mathbb{X}_2$, and hence the operators $P_1^*\,\mathcal{C}^{-1}\,P_1$, $(\mathbf{I}-P_1)^*\,\mathcal{C}^{-1}\,P_1$, $P_1^*\,\mathcal{C}^{-1}\,(\mathbf{I}-P_1)$, $P_2^*\,P_2$, $(\mathbf{I}-P_2)^*\,P_2$, and $P_2^*\,(\mathbf{I}-P_2)$ are all compact, which completes the proof.

□

The main result of this section is established next.

Theorem 3.1. *Assume that the homogeneous problem associated to* (21) *has only the trivial solution. Then, given* $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ *and* $p_i \in H^{1/2}(\Gamma)$, *there exists a unique solution* $(\widehat{\boldsymbol{\sigma}},\boldsymbol{\gamma}) \in \mathbb{X}\times\mathbb{Y}$ *to* (21) (*equivalently* (49)). *In addition, there exists* $C > 0$, *depending on* $\min\left\{1,\dfrac{1}{\kappa_s^2}\right\}$, $\min\left\{1,\dfrac{1}{\kappa_f^2}\right\}$, $\max\left\{1,\dfrac{1}{\kappa_s^2}\right\}$, *and* $\max\left\{1,\dfrac{1}{\kappa_f^2}\right\}$, *such that*

$$\|(\widehat{\boldsymbol{\sigma}},\boldsymbol{\gamma})\|_{\mathbb{X}\times\mathbb{Y}} \;\leq\; C\left\{\,\|\mathbf{f}\|_{0,\Omega_s} \,+\, \|p_i\|_{1/2,\Gamma}\,\right\}.$$

Proof. It follows straightforwardly from Lemma 3.1, Lemma 3.7, the boundedness of $\mathbf{A}_0$ and $\mathbf{B}$, and the classical Babuška-Brezzi theory, that $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is an isomorphism. Note here that precisely the boundedness constant of $\mathbf{A}_0$ (cf. (40), (41), (45)) depends on $\max\left\{1,\dfrac{1}{\kappa_s^2}\right\}$, and $\max\left\{1,\dfrac{1}{\kappa_f^2}\right\}$. In addition, it is clear from Lemma 3.8 that $\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ is a compact operator. Consequently, the left hand side of (49) becomes a Fredholm operator of index zero, which finishes the proof. □

4. Analysis of the Galerkin scheme

In this section we introduce a Galerkin approximation of (21) and show, under the assumptions of Theorem 3.1, that it is well-posed.

4.1. Preliminaries. We first let $\{\mathcal{T}_h\}_{h>0} := \{\mathcal{T}_{h_s}\}_{h_s>0} \cup \{\mathcal{T}_{h_f}\}_{h_f>0}$, where $\{\mathcal{T}_{h_s}\}_{h_s>0}$ and $\{\mathcal{T}_{h_f}\}_{h_f>0}$ are shape-regular families of triangulations of the polyhedral regions $\bar{\Omega}_s$ and $\bar{\Omega}_f$, respectively, by tetrahedra T of diameter h_T with mesh sizes $h_s := \max\{ h_T : \quad T \in \mathcal{T}_{h_s} \}$, $h_f := \max\{ h_T : \quad T \in \mathcal{T}_{h_f} \}$, and $h := \max\{h_s, h_f\}$, and such that the vertices of $\{\mathcal{T}_{h_s}\}_{h_s>0}$ and $\{\mathcal{T}_{h_f}\}_{h_f>0}$ coincide on Σ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbf{R}^3$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. In addition, following the same terminology described at the end of the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^3$ and $\mathbb{P}_\ell(S) := [P_\ell(S)]^{3\times 3}$. Then, we define

$$\mathbb{H}_h^s := \left\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div}; \Omega_s) : \quad \boldsymbol{\tau}_{s,h}|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_{h_s} \right\},$$

$$\mathbf{H}_h^f := \left\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}(\mathrm{div}; \Omega_f) : \quad \boldsymbol{\tau}_{f,h}|_T \in \mathbf{P}_1(T) \quad \forall T \in \mathcal{T}_{h_f} \right\},$$

$$W_h^f := \left\{ q_h \in C(\bar{\Omega}_f) : \quad q_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_{h_f} \right\},$$

$$\mathbf{W}_h^s := \left\{ \mathbf{v}_h \in \mathbf{C}(\bar{\Omega}_s) : \quad \mathbf{v}_h|_T \in \mathbf{P}_1(T) \quad \forall T \in \mathcal{T}_{h_s} \right\},$$

and introduce the finite element subspaces of $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}$, and $\mathbb{Y}$, given, respectively, by

$$\mathbb{X}_{1,h} := \left\{ \widehat{\boldsymbol{\tau}}_{s,h} = (\boldsymbol{\tau}_{s,h}, q_h) \in \mathbb{H}_h^s \times W_h^f : \quad \boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu} = -\,q_h\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma \right\}, \tag{59}$$

$$\mathbb{X}_{2,h} := \left\{ \widehat{\boldsymbol{\tau}}_{f,h} = (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h) \in \mathbf{H}_h^f \times \mathbf{W}_h^s : \quad \boldsymbol{\tau}_{f,h}\cdot\boldsymbol{\nu} = \rho_f\,\omega^2\,\mathbf{v}_h\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma \right\}, \tag{60}$$

$$\mathbb{X}_h := \mathbb{X}_{1,h} \times \mathbb{X}_{2,h},$$

and

$$\mathbb{Y}_h := \left\{ \boldsymbol{\eta}_h \in \mathbb{Y} : \quad \boldsymbol{\eta}_h|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_{h_s} \right\}.$$

In addition, the analysis below will also require the subspaces

$$\mathbf{U}_h^s := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega_s) : \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall T \in \mathcal{T}_{h_s} \right\}$$

and

$$U_h^f := \left\{ v_h \in L^2(\Omega_f) : \quad v_h|_T \in P_0(T) \quad \forall T \in \mathcal{T}_{h_f} \right\}.$$

We recall here that $\mathbb{H}_h^s \times \mathbf{U}_h^s \times \mathbb{Y}_h$ constitutes the lowest order mixed finite element approximation of the linear elasticity problem introduced recently by Arnold, Falk and Winther (AFW) (see [5], [4]). In turn, $\mathbf{H}_h^f \times U_h^f$ is the lowest order mixed finite element approximation of the Poisson problem for the Laplace equation, introduced by Brezzi, Douglas, Durán, and Fortin (BDDF) in 1987 (see [9]). Furthermore, it is important to remark that, thanks to the natural matchings between the polynomial degrees involved in the definitions of $\mathbb{H}_h^s$ and W_h^f (resp. $\mathbf{H}_h^f$ and $\mathbf{W}_h^s$), it is possible to incorporate the transmission conditions exactly at the discrete level, which actually allows the introduction of the conforming finite element subspaces $\mathbb{X}_{1,h}$ and $\mathbb{X}_{2,h}$.

The Galerkin scheme associated to our continuous problem (21) is then defined as follows: Find $\widehat{\boldsymbol{\sigma}}_h := (\widehat{\boldsymbol{\sigma}}_{s,h}, \widehat{\boldsymbol{\sigma}}_{f,h}) := ((\boldsymbol{\sigma}_{s,h}, p_h), (\boldsymbol{\sigma}_{f,h}, \mathbf{u}_h)) \in \mathbb{X}_h$ and $\boldsymbol{\gamma}_h \in \mathbb{Y}_h$ such that

$$\begin{aligned} \mathbb{A}(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\tau}}_h) + \mathbb{B}(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\gamma}_h) &= \mathbb{F}(\widehat{\boldsymbol{\tau}}_h), \\ \mathbb{B}(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\eta}_h) &= 0. \end{aligned} \tag{61}$$

for all $\widehat{\boldsymbol{\tau}}_h := (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) := ((\boldsymbol{\tau}_{s,h}, q_h), (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h)) \in \mathbb{X}_h$, for all $\boldsymbol{\eta}_h \in \mathbb{Y}_h$.

Before analyzing the well-posedness of (61), we now compare the number of unknowns defining $\mathbb{X}_h \times \mathbb{Y}_h$ with that of the finite element subspace employed by an alternative method. We begin by recalling from [5] (see also [15]) that $\mathbb{H}_h^s$ has 36 local degrees of freedom (9 per face of each tetrahedron), and hence, since each interior face belongs to two tetrahedra, the number of degrees of freedom defining $\mathbb{H}_h^s$ is given, approximately, by

$$\left(\frac{36}{2}\right) \times m_s = 18 \times m_s .$$

where m_s is the total number of tetrahedra of $\mathcal{T}_{h_s}$. However, this number decreases to $12 \times m_s$ when the corresponding AFW reduced element (see [5], [15]) is employed. In turn, it is well known that each component of $\mathbf{W}_h^s$ has 4 local degrees of freedom (one per vertex of each tetrahedron), and hence, since each interior vertex of a uniform triangulation $\mathcal{T}_h$ belongs to 24 tetrahedrons (see [19, Section 4] for details), we find that the number of degrees of freedom defining $\mathbf{W}_h^s$ is given, approximately, by

$$\left(\frac{3 \times 4}{24}\right) \times m_s = 0.5 \times m_s .$$

Finally, it is straightforward to see that the number of degrees of freedom defining $\mathbb{Y}_h$ is given by $3 \times m_s$, and hence the total number of unknowns in the solid Ω_s, which corresponds to the degrees of freedom defining $\mathbb{H}_h^s \times \mathbf{W}_h^s \times \mathbb{Y}_h$, behaves approximately as $15.5 \times m_s$. Proceeding similarly, we find that the number of unknowns in the fluid Ω_f, which corresponds to the degrees of freedom defining $\mathbf{H}_h^f \times W_h^f$, behaves approximately as $(4 + \frac{1}{6}) \times m_f = \frac{25}{6} \times m_f$, where m_f is the total number of tetrahedra of $\mathcal{T}_{h_f}$.

Alternatively, and for the above mentioned purpose of comparison, we now consider the 3D version of the fully-mixed finite element method suggested in [14]. This means that, while $\mathbf{u}$ and p are eliminated from the formulation, the traces of them on the respective boundaries are incorporated as additional unknowns, and the spaces $\mathbb{H}_h^s \times \mathbb{Y}_h$ and $\mathbf{H}_h^f$ are replaced by the corresponding components of the classical *PEERS* and Raviart-Thomas spaces in 3D (see, e.g. [27, Definition 3.1]). In this case, it is easy to see that the number of degrees of freedom defining the resulting finite element subspaces in Ω_s and Ω_f behave approximately as $9.5 \times m_s$ and $(2 + \frac{1}{6}) \times m_f = \frac{13}{6} \times m_f$, respectively. In particular, it is well known that each row of a local Raviart-Thomas tensor of order 0 is uniquely determined by its normal components on the 4 faces of T, and hence, using again that each interior face belongs to 2 tetrahedra, we find that the number of degrees of freedom defining the associated global space is given, approximately, by:

$$\left(\frac{3 \times 4}{2}\right) \times m_s = 6 \times m_s .$$

The factor 9.5 indicated above is achieved after adding the quantities $3 \times m_s$ and $0.5 \times m_s$, which arise from the bubble functions on each tetrahedra and the continuous piecewise linear elements approximating the rotation, respectively.

Certainly, one could use static condensation to eliminate the $3 \times m_s$ degrees of freedom representing the rotation living in $\mathbb{Y}_h$. However, this reduction by a factor of 3 is also applicable to the local degrees of freedom associated with the bubble functions of each tetrahedron. In any case, it becomes clear from the above paragraphs that the alternative method suggested by the 3D version of [14] involves a lesser amount of degrees of freedom (determined by the factors $15.5 - 9.5 = 6 \times m_s$ and $\frac{25}{6} - \frac{13}{6} = 2 \times m_f$) than the present approach. Assuming similar sizes for Ω_s and Ω_f, averaging the above factors, and denoting by m the total number of tetrahedra of $\mathcal{T}_h$, we could summarize this comparison by saying that the method proposed in this paper yields $4 \times m$ more degrees of freedom than the alternative one. Nevertheless, it is also clear from the above analysis that this increase is not mainly caused by the introduction of the new internal fields belonging to the spaces $\mathbf{W}_h^s$ and W_h^f, which actually contributes with $0.5 \times m_s$ and $\frac{1}{6} \times m_f$ additional degrees of freedom, respectively, but by the properties of the polynomial spaces defining $\mathbb{H}_h^s$ and $\mathbf{H}_h^f$. In addition, irrespective of the number of unknowns, it is important to emphasize that the main advantages of our present approach have to do with three main features: it provides direct finite element approximations for $\mathbf{u}$ and p in the whole domains $\overline{\Omega}_s$ and $\overline{\Omega}_f$, respectively, it satisfies both discrete transmission conditions exactly, and, because of the absence of boundary unknowns, no discrete inf-sup conditions imposing particular relationships between the meshsizes on the boundaries are required for the stability and solvability analysis of the corresponding discrete schemes. The first feature avoids the eventual application of postprocessing formulae, the second one increases the accuracy of the method, and, as already mentioned in the Introduction, the third one frees us of having to impose unverifiable cumbersome restrictions. Moreover, these advantages should be clearly strengthened by the incorporation of a hybridization technique handling the continuity of the normal components of the tensor and vector fields belonging to $\mathbb{H}_h^s$ and $\mathbf{H}_h^f$, respectively. We plan to discuss this issue in a forthcoming paper.

4.2. Approximation properties of the subspaces. In what follows we collect the approximation properties of the finite element subspaces introduced above. They are all known, except the one of $\mathbb{X}_{1,h}$, which was derived in our previous work [20], and the one of $\mathbb{X}_{2,h}$, which will be provided below. We begin with the subspaces $\mathbb{H}_h^s$ and $\mathbf{H}_h^f$. Hence, given $\delta \in (0,1]$, we let

$$\mathcal{E}_h^s : \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div}; \Omega_s) \to \mathbb{H}_h^s \quad \text{and} \quad \mathcal{E}_h^f : \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\text{div}; \Omega_f) \to \mathbf{H}_h^f$$

be the usual interpolation operators (see [5], [8]), which, given $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div}; \Omega_s)$ and $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\text{div}; \Omega_f)$, are characterized by the identities

$$\int_F \mathcal{E}_h^s(\boldsymbol{\tau}_s)\boldsymbol{\nu} \cdot \mathbf{q} = \int_F \boldsymbol{\tau}_s \boldsymbol{\nu} \cdot \mathbf{q} \quad \forall\, \mathbf{q} \in \mathbf{P}_1(F)\,, \quad \forall \text{ face } F \text{ of } \mathcal{T}_{h_s}\,, \tag{62}$$

and

$$\int_F \mathcal{E}_h^f(\boldsymbol{\tau}_f) \cdot \boldsymbol{\nu}\, q = \int_F \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}\, q \quad \forall\, q \in P_1(F)\,, \quad \forall \text{ face } F \text{ of } \mathcal{T}_{h_f}\,. \tag{63}$$

Moreover, the corresponding commuting diagram properties yield

$$\mathbf{div}(\mathcal{E}_h^s(\boldsymbol{\tau}_s)) = \mathcal{P}_h^s(\mathbf{div}\,\boldsymbol{\tau}_s) \qquad \forall\, \boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div}; \Omega_s)\,, \tag{64}$$

and

$$\text{div}(\mathcal{E}_h^f(\boldsymbol{\tau}_f)) = \mathcal{P}_h^f(\text{div}\,\boldsymbol{\tau}_f) \qquad \forall\, \boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\text{div}; \Omega_f)\,, \tag{65}$$

where $\mathcal{P}_h^s : \mathbf{L}^2(\Omega_s) \to \mathbf{U}_h^s$ and $\mathcal{P}_h^f : L^2(\Omega_f) \to U_h^f$ are the orthogonal projectors. In addition, it is easy to show, using the well-known Bramble-Hilbert Lemma and the boundedness of the local interpolation operators on the reference element $\widehat{T}$ (see, e.g. [25, equation (3.39)]), that there exist C_s, $C_f > 0$, independent of h, such that for each $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div};\, \Omega_s)$ and for each $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div};\, \Omega_f)$, there hold

$$\|\boldsymbol{\tau}_s - \mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{0,T} \,\le\, C_s\, h_T^\delta \left\{ |\boldsymbol{\tau}_s|_{\delta,T} \,+\, \|\mathbf{div}\, \boldsymbol{\tau}_s\|_{0,T} \right\} \quad \forall\, T \in \mathcal{T}_{h_s}\,, \tag{66}$$

and

$$\|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{0,T} \,\le\, C_f\, h_T^\delta \left\{ |\boldsymbol{\tau}_f|_{\delta,T} \,+\, \|\mathrm{div}\, \boldsymbol{\tau}_f\|_{0,T} \right\} \quad \forall\, T \in \mathcal{T}_{h_f}\,. \tag{67}$$

We now let $\Pi_h^s : \mathbf{H}^1(\Omega_s) \to \mathbf{W}_h^s$, $\Pi_h^f : H^1(\Omega_f) \to W_h^f$, and $\mathcal{R}_h : \mathbb{L}^2(\Omega_s) \to \mathbb{Y}_h$ be the corresponding orthogonal projectors with respect to the natural norms of each space. Then, we have (see [6], [8], [30]):

($\mathrm{AP}_h^{\boldsymbol{\sigma}_s}$) For each $\delta \in (0,1]$ and for each $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s)$, with $\mathbf{div}\, \boldsymbol{\tau}_s \in \mathbf{H}^\delta(\Omega_s)$, there holds

$$\|\boldsymbol{\tau}_s - \mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{\mathbf{div};\Omega_s} \,\le\, C\, h^\delta \left\{ \|\boldsymbol{\tau}_s\|_{\delta,\Omega_s} \,+\, \|\mathbf{div}\, \boldsymbol{\tau}_s\|_{\delta,\Omega_s} \right\}.$$

($\mathrm{AP}_h^{\boldsymbol{\sigma}_f}$) For each $\delta \in (0,1]$ and for each $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f)$, with $\mathrm{div}\, \boldsymbol{\tau}_f \in H^\delta(\Omega_f)$, there holds

$$\|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f} \,\le\, C\, h^\delta \left\{ \|\boldsymbol{\tau}_f\|_{\delta,\Omega_f} \,+\, \|\mathrm{div}\, \boldsymbol{\tau}_f\|_{\delta,\Omega_f} \right\}.$$

(AP_h^p) For each $t \in (1,2]$ and for each $q \in H^t(\Omega_f)$, there holds

$$\|q - \Pi_h^f(q)\|_{1,\Omega_f} \,\le\, C\, h^{t-1}\, \|q\|_{t,\Omega_f}\,.$$

($\mathrm{AP}_h^{\mathbf{u}}$) For each $t \in (1,2]$ and for each $\mathbf{v} \in \mathbf{H}^t(\Omega_s)$, there holds

$$\|\mathbf{v} - \Pi_h^s(\mathbf{v})\|_{1,\Omega_s} \,\le\, C\, h^{t-1}\, \|\mathbf{v}\|_{t,\Omega_s}\,.$$

($\mathrm{AP}_h^{\boldsymbol{\gamma}}$) For each $t \in (0,1]$ and for each $\boldsymbol{\eta} \in \mathbb{H}^t(\Omega_s) \cap \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$, there holds

$$\|\boldsymbol{\eta} - \mathcal{R}_h(\boldsymbol{\eta})\|_{0,\Omega_s} \,\le\, C\, h^t\, \|\boldsymbol{\eta}\|_{t,\Omega_s}\,.$$

(AP_h^s) For each $t \in (0,1]$ and for each $\mathbf{v} \in \mathbf{H}^t(\Omega_s)$, there holds

$$\|\mathbf{v} - \mathcal{P}_h^s(\mathbf{v})\|_{0,\Omega_s} \,\le\, C\, h^t\, \|\mathbf{v}\|_{t,\Omega_s}\,.$$

(AP_h^f) For each $t \in (0,1]$ and for each $v \in H^t(\Omega_f)$, there holds

$$\|v - \mathcal{P}_h^f(v)\|_{0,\Omega_f} \,\le\, C\, h^t\, \|v\|_{t,\Omega_f}\,.$$

Note here that ($\mathrm{AP}_h^{\boldsymbol{\sigma}_s}$) is actually a straightforward consequence of (64), (66), and (AP_h^s). Similarly, ($\mathrm{AP}_h^{\boldsymbol{\sigma}_f}$) follows directly from (65), (67), and (AP_h^f).

We now provide the approximation properties of the coupled finite element subspaces $\mathbb{X}_{1,h}$ (cf. (59)) and $\mathbb{X}_{2,h}$ (cf. (60)). To this end, we proceed as in [21, Section 5.2, Lemma 5.1] and assume from now on that $\{\mathcal{T}_{h_s}\}_{h_s>0}$ and $\{\mathcal{T}_{h_f}\}_{h_f>0}$ are quasi-uniform around Σ. This means that there exists an open neighborhood of Σ, say Ω_Σ, with Lipschitz boundary, and such that the elements of $\mathcal{T}_{h_s}$ and $\mathcal{T}_{h_f}$ intersecting that region are more or less of the same size. Equivalently, we define

$$\mathcal{T}_{\Sigma,h} \,:=\, \left\{ T \in \mathcal{T}_{h_s} \cup \mathcal{T}_{h_f} : \quad T \cap \Omega_\Sigma \neq \emptyset \right\},$$

and assume that there exists $c > 0$, independent of h, such that

$$\max_{T\in\mathcal{T}_{\Sigma,h}} h_T \;\le\; c \min_{T\in\mathcal{T}_{\Sigma,h}} h_T \qquad \forall\, h > 0\,. \tag{68}$$

Note that this assumption and the shape-regularity property of the meshes imply that Σ_h, the partition on Σ inherited from $\mathcal{T}_{h_s}$ (or from $\mathcal{T}_{h_f}$), is also quasi-uniform, which means that there exists $C > 0$, independent of h, such that

$$h_\Sigma \,:=\, \max\Big\{ \operatorname{diam}\{F\} : \quad F \text{ face of } \Sigma_h \Big\} \,\le\, C \min\Big\{ \operatorname{diam}\{F\} : \quad F \text{ face of } \Sigma_h \Big\}\,.$$

In addition, the quasi-uniformity of Σ_h guarantees the inverse inequality on $\Phi_h(\Sigma)$, the subspace of $L^2(\Sigma)$ given by the piecewise polynomials of degree ≤ 1, that is, in particular,

$$\|\boldsymbol{\phi}_h\|_{0,\Sigma} \;\le\; C\, h_\Sigma^{-1/2}\, \|\boldsymbol{\phi}_h\|_{-1/2,\Sigma} \qquad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)\,. \tag{69}$$

The derivation of (69), which is rather technical, can be found as a particular case of [12, Theorems 4.1 and 4.6], where it is shown, using scaling and interpolation arguments, that for each $t \in [-1, 0]$ there holds

$$\|\boldsymbol{\phi}_h\|_{0,\Sigma} \;\le\; C\, h_\Sigma^{t}\, \|\boldsymbol{\phi}_h\|_{t,\Sigma} \qquad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)\,. \tag{70}$$

Actually, the way of proving (70) is establishing first for $t = -1$, for instance using some duality arguments, which gives

$$\|\boldsymbol{\phi}_h\|_{0,\Sigma} \;\le\; C\, h_\Sigma^{-1}\, \|\boldsymbol{\phi}_h\|_{-1,\Sigma} \qquad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)\,, \tag{71}$$

and then applying the classical result on interpolation of spaces and operators (see, e.g. [29, Appendix B, Theorem B.2]) to (71) and the obvious inequality $\|\boldsymbol{\phi}_h\|_{0,\Sigma} \le \|\boldsymbol{\phi}_h\|_{0,\Sigma} \quad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)$. Similarly, applying now the same interpolation result to (69) and the other obvious inequality

$$\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma} \;\le\; \|\boldsymbol{\phi}_h\|_{-1/2,\Sigma} \quad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)\,,$$

using in this case the interpolation parameter $\theta = 1/2$ (cf. [29, Theorem B.2]), we obtain

$$\|\boldsymbol{\phi}_h\|_{-1/4,\Sigma} \;\le\; C\, h_\Sigma^{-1/4}\, \|\boldsymbol{\phi}_h\|_{-1/2,\Sigma} \qquad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)\,. \tag{72}$$

The estimate (72) will be used below at the end of the proof of Lemma 4.2.

The approximation property of $\mathbb{X}_{1,h}$, whose proof makes use of the quasi-uniformity of $\{\mathcal{T}_{h_s}\}_{h_s>0}$ around Σ, the characterization (62), and the inverse inequality on $\Phi_h(\Sigma)$, was proved in [20, Lemma 5.1]. The corresponding result is stated as follows.

Lemma 4.1. *Given $\epsilon \in (0,1]$, define $\mathbb{X}_{1,\epsilon} := \Big\{\mathbb{H}(\mathbf{div};\Omega_s) \cap \mathbb{H}^\epsilon(\Omega_s)\Big\} \times H^{1+\epsilon}(\Omega_f)$. Then, there exists a linear operator $\mathbb{I}_{1,h} : \mathbb{X}_{1,\epsilon} \longrightarrow \mathbb{X}_{1,h}$, such that for each $\widehat{\boldsymbol{\tau}}_s = (\boldsymbol{\tau}_s, q) \in \mathbb{X}_1 \cap \mathbb{X}_{1,\epsilon}$ there holds*

$$\|\widehat{\boldsymbol{\tau}}_s - \mathbb{I}_{1,h}(\widehat{\boldsymbol{\tau}}_s)\|_{\mathbb{X}_1} \;\le\; C \Big\{ \|\boldsymbol{\tau}_s - \mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{\mathbf{div};\Omega_s} \,+\, \|q - \Pi_h^f(q)\|_{1,\Omega_f} \Big\}\,. \tag{73}$$

In turn, the approximation property of $\mathbb{X}_{2,h}$ follows the same lines and it is proved next.

Lemma 4.2. *Given $\epsilon \in (0,1]$, define $\mathbb{X}_{2,\epsilon} := \Big\{\mathbf{H}(\mathrm{div};\Omega_f) \cap \mathbf{H}^\epsilon(\Omega_f)\Big\} \times \mathbf{H}^{1+\epsilon}(\Omega_s)$. Then, there exists a linear operator $\mathbb{I}_{2,h} : \mathbb{X}_{2,\epsilon} \longrightarrow \mathbb{X}_{2,h}$, such that for each $\widehat{\boldsymbol{\tau}}_f = (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2 \cap \mathbb{X}_{2,\epsilon}$ there holds*

$$\|\widehat{\boldsymbol{\tau}}_f - \mathbb{I}_{2,h}(\widehat{\boldsymbol{\tau}}_f)\|_{\mathbb{X}_2} \;\le\; C \Big\{ \|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f} \,+\, \|\mathbf{v} - \Pi_h^s(\mathbf{v})\|_{1,\Omega_s} \Big\}\,. \tag{74}$$

Proof. Let us first increase the region Ω_f across the external surface Γ to a new annular region $\widetilde{\Omega}_f$ with Lipschitz-continuous boundary $\partial\widetilde{\Omega}_f = \Sigma \cup \widetilde{\Gamma}$. Then, given $\widehat{\boldsymbol{\tau}}_f = (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_{2,\epsilon}$, we let $\varphi \in H^1(\widetilde{\Omega}_f)$ be the unique solution (guaranteed by the Lax-Milgram Lemma) of the boundary value problem with mixed boundary conditions:

(75)

$$\Delta\varphi = 0 \quad \text{in} \quad \widetilde{\Omega}_f\,, \quad \frac{\partial\varphi}{\partial\boldsymbol{\nu}} = \mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \quad \varphi = 0 \quad \text{on} \quad \widetilde{\Gamma}\,,$$

whose corresponding continuous dependence result states that

$$\|\varphi\|_{1,\widetilde{\Omega}_f} \le C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}\,. \tag{76}$$

In addition, since the Neumann datum $\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}$, being an element of $\Phi_h(\Sigma)$, belongs to $\mathbf{H}^\delta(\Sigma)$ for any $\delta \in [-1/2, 1/2)$, the classical regularity result for mixed boundary value problems on polyhedral domains (see, e.g. [24]) implies that $\varphi \in H^{5/4}(\widetilde{\Omega}_f)$ and

$$\|\varphi\|_{5/4,\widetilde{\Omega}_f} \le C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/4,\Sigma}\,. \tag{77}$$

Furthermore, since $\Omega_f^{\mathtt{int}} := \Omega_f\backslash\Omega_\Sigma$ is an interior region of $\widetilde{\Omega}_f$, the interior elliptic regularity estimate (see, e.g. [29, Theorem 4.16]) insures that

$$\|\varphi\|_{2,\Omega_f^{\mathtt{int}}} \le C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}\,. \tag{78}$$

We now let $\boldsymbol{\zeta}_f := \nabla\varphi$ in Ω_f, whence $\boldsymbol{\zeta}_f$ belongs to $\mathbf{H}^{1/4}(\Omega_f)$, and notice from (75) that

$$\operatorname{div}\boldsymbol{\zeta}_f = 0 \quad \text{in} \quad \Omega_f\,, \quad \text{and} \quad \boldsymbol{\zeta}_f\cdot\boldsymbol{\nu} = \mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \tag{79}$$

which shows, in particular, that $\boldsymbol{\zeta}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$. According to the above, we now define

$$\mathbb{I}_{2,h}(\widehat{\boldsymbol{\tau}}_f) := (\mathcal{E}_h^f(\boldsymbol{\tau}_f - \boldsymbol{\zeta}_f), \Pi_h^s(\mathbf{v})) \in \mathbf{H}_h^f \times \mathbf{W}_h^s\,.$$

It follows, using the characterization (63) and the second identity in (79), similarly as in the proof of [20, Lemma 5.1], that there holds $\mathcal{E}_h^f(\boldsymbol{\tau}_f - \boldsymbol{\zeta}_f)\cdot\boldsymbol{\nu} = \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}$ on Σ, which proves that $\mathbb{I}_{2,h}(\widehat{\boldsymbol{\tau}}_f)$ belongs to $\mathbb{X}_{2,h}$.

We now assume additionally that $\widehat{\boldsymbol{\tau}}_f = (\boldsymbol{\tau}_f, \mathbf{v}) \in \mathbb{X}_2$, which means that $\boldsymbol{\tau}_f\cdot\boldsymbol{\nu} = \rho_f\,\omega^2\,\mathbf{v}\cdot\boldsymbol{\nu}$ on Σ, and aim to prove (74). We first observe, applying the triangle inequality, that

(80)

$$\|\widehat{\boldsymbol{\tau}}_f - \mathbb{I}_{2,h}(\widehat{\boldsymbol{\tau}}_f)\|_{\mathbb{X}_2}^2 \le 2\,\|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f}^2 + 2\,\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_f}^2 + \|\mathbf{v} - \Pi_h^s(\mathbf{v})\|_{1,\Omega_s}^2\,,$$

where we have also used, thanks to (65) and (79), that $\operatorname{div}\mathcal{E}_h^f(\boldsymbol{\zeta}_f) = \mathcal{P}_h^f(\operatorname{div}\boldsymbol{\zeta}_f) = 0$ in Ω_f. Next, in order to estimate the remaining term $\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_f}^2$, we now let

$$\mathcal{T}_{\Sigma,h}^f := \Big\{ T \in \mathcal{T}_{h_f} : \quad T \cap \Omega_\Sigma \neq \emptyset \Big\}, \quad \Omega_{\Sigma,h}^f := \cup\Big\{ T : \quad T \in \mathcal{T}_{\Sigma,h}^f \Big\},$$

and

$$\Omega_{f,h}^{\mathtt{int}} := \Omega_f\backslash\Omega_{\Sigma,h}^f\,.$$

It follows, using the stability of $\mathcal{E}_h^f$ in $\mathbf{H}^1(\Omega_{f,h}^{\mathtt{int}})$, the fact that $\boldsymbol{\zeta}_f|_{\Omega_{f,h}^{\mathtt{int}}} \in \mathbf{H}^1(\Omega_{f,h}^{\mathtt{int}})$, the inclusion $\Omega_{f,h}^{\mathtt{int}} \subseteq \Omega_f^{\mathtt{int}}$, and the estimate (78), that

$$\begin{aligned}
\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_f} &\leq \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{f,h}^{\mathtt{int}}} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} \\
&\leq C\,\|\varphi\|_{2,\Omega_f^{\mathtt{int}}} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} \\
&\leq C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f}\,.
\end{aligned} \tag{81}$$

Now, adding and substracting $\boldsymbol{\tau}_f\cdot\boldsymbol{\nu} = \rho_f\,\omega^2\,\mathbf{v}\cdot\boldsymbol{\nu}$ on Σ, and applying the trace theorems in $\mathbf{H}(\mathrm{div};\Omega_f)$ and $\mathbf{H}^1(\Omega_s)$, we find that

$$\begin{aligned}
&\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma} \\
&\quad\leq \|\big(\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\big)\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma} + \rho_f\,\omega^2\,\|\big(\mathbf{v} - \Pi_h^s(\mathbf{v})\big)\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma} \\
&\quad\leq C\Big\{\|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f} + \|\mathbf{v} - \Pi_h^s(\mathbf{v})\|_{1,\Omega_s}\Big\}.
\end{aligned} \tag{82}$$

Finally, we estimate $\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f}$ from (81). In fact, adding and substracting $\boldsymbol{\zeta}_f = \nabla\varphi$ in $\Omega_{\Sigma,h}^f \subseteq \Omega_f$, noting that $\|\boldsymbol{\zeta}_f\|_{0,\Omega_{\Sigma,h}^f} \leq \|\varphi\|_{1,\Omega_f} \leq \|\varphi\|_{1,\widetilde{\Omega}_f}$, and employing the estimates (76), (67) (with $\delta = 1/4$) and (77), the fact that $\mathrm{div}\,\boldsymbol{\zeta}_f = 0$ in Ω_f, the quasi-uniformity bound (68), and the inverse inequality (72), we arrive at

$$\begin{aligned}
\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f}^2 &\leq C\Big\{\|\boldsymbol{\zeta}_f - \mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f}^2 + \|\boldsymbol{\zeta}_f\|_{0,\Omega_{\Sigma,h}^f}^2\Big\} \\
&\leq C\sum_{T\in\mathcal{T}_{\Sigma,h}^f} h_T^{1/2}\,\|\varphi\|_{5/4,T}^2 + C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}^2 \\
&\leq C\,h_\Sigma^{1/2}\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/4,\Sigma}^2 \\
&\quad + C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}^2 \\
&\leq C\,\|\mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\Pi_h^s(\mathbf{v})\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}^2\,.
\end{aligned} \tag{83}$$

Consequently, (80), (81), (82), and (83) yield the estimate (74) and complete the proof.

□

4.3. A discrete approximation of $\mathbf{P}|_{\mathbb{X}_h}$. In what follows we introduce uniformly bounded linear operators $\mathbf{P}_{1,h} : \mathbb{X}_{1,h} \to \mathbb{X}_{1,h}$ and $\mathbf{P}_{2,h} : \mathbb{X}_{2,h} \to \mathbb{X}_{2,h}$ so that $\mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h) := (\mathbf{P}_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h}), \mathbf{P}_{2,h}(\widehat{\boldsymbol{\tau}}_{f,s}))$ becomes a suitable discrete approximation of $\mathbf{P}(\widehat{\boldsymbol{\tau}}_h) := (\mathbf{P}_1(\widehat{\boldsymbol{\tau}}_{s,h}), \mathbf{P}_2(\widehat{\boldsymbol{\tau}}_{f,s}))$ for each $\widehat{\boldsymbol{\tau}}_h := (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) \in \mathbb{X}_h$, and then estimate the corresponding error $\|\mathbf{P}(\widehat{\boldsymbol{\tau}}_h) - \mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbb{X}}$.

Indeed, given $\widehat{\boldsymbol{\tau}}_h := (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) := ((\boldsymbol{\tau}_{s,h}, q_h), (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h)) \in \mathbb{X}_{1,h}\times\mathbb{X}_{2,h} =: \mathbb{X}_h$, we first recall from (27) and (25) that

$$P_1(\widehat{\boldsymbol{\tau}}_{s,h}) := \tilde{\boldsymbol{\sigma}}_s \qquad\text{and}\qquad \mathbf{P}_1(\widehat{\boldsymbol{\tau}}_{s,h}) := (P_1(\widehat{\boldsymbol{\tau}}_{s,h}), q_h)\,, \tag{84}$$

where $\tilde{\boldsymbol{\sigma}}_s := \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of the problem

$$\begin{gathered}
\tilde{\boldsymbol{\sigma}}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad\text{in}\quad \Omega_s\,, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_s = \mathbf{div}\,\boldsymbol{\tau}_{s,h} + \mathbf{r}(\widehat{\boldsymbol{\tau}}_{s,h}) \quad\text{in}\quad \Omega_s\,, \\
\tilde{\boldsymbol{\sigma}}_s\,\boldsymbol{\nu} = -q_h\,\boldsymbol{\nu} \quad\text{on}\quad \Sigma\,, \quad \tilde{\mathbf{u}} \in (\mathbf{I} - \mathbf{M})(\mathbf{L}^2(\Omega_s))\,.
\end{gathered} \tag{85}$$

In turn, we recall from (34) and (32) that

$$P_2(\widehat{\boldsymbol{\tau}}_{f,h}) := \tilde{\boldsymbol{\sigma}}_f \quad \text{and} \quad \mathbf{P}_2(\widehat{\boldsymbol{\tau}}_{f,h}) := (P_2(\widehat{\boldsymbol{\tau}}_{f,h}), \mathbf{v}_h)\,, \tag{86}$$

where $\tilde{\boldsymbol{\sigma}}_f := \nabla\tilde{p}$ and $\tilde{p}$ is the unique solution of the problem

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}_f &= \nabla\tilde{p} \quad \text{in} \quad \Omega_f\,, \quad \operatorname{div}\tilde{\boldsymbol{\sigma}}_f = \operatorname{div}\boldsymbol{\tau}_{f,h} \quad \text{in} \quad \Omega_f\,, \\ \tilde{\boldsymbol{\sigma}}_f\cdot\boldsymbol{\nu} &= \rho_f\,\omega^2\,\mathbf{v}_h\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma, \quad \tilde{p} = 0 \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{87}$$

Next, we let $(\tilde{\boldsymbol{\sigma}}_{s,h}, \tilde{\mathbf{u}}_h, \tilde{\gamma}_h) \in \mathbb{H}_h^s \times (\mathbf{I}-\mathbf{M})(\mathbf{U}_h^s)\times\mathbb{Y}_h$ be the mixed finite element approximation of the solution of (85), which was introduced and analyzed in [20, Section 5.2], and define

$$P_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h}) := \tilde{\boldsymbol{\sigma}}_{s,h} \quad \text{and} \quad \mathbf{P}_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h}) := (P_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h}), q_h)\,. \tag{88}$$

In particular, we know from [20, Section 5.2, Theorem 5.1 and eq. (5.27)] that there hold

$$\|P_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h})\|_{\mathbf{div};\Omega_s} \le C\left\{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s} + \|q_h\|_{1,\Omega_f}\right\}, \tag{89}$$

(90)

$$P_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h})\,\boldsymbol{\nu} = -\,q_h\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma \qquad \text{and} \qquad \int_{\Omega_s} P_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h}) : \tilde{\boldsymbol{\eta}}_h = 0 \quad \forall\,\tilde{\boldsymbol{\eta}}_h \in \mathbb{Y}_h\,.$$

It is clear that (89) yields the uniform boundedness of $\mathbf{P}_{1,h}$, while the first equation of (90) guarantees that $\mathbf{P}_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h})$ belongs to $\mathbb{X}_{1,h}$. In addition, according to [20, Lemma 5.4], whose proof makes use of the definition (84), the commuting diagram identity (64), the approximation properties (66), (AP_h^s), and ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$), and the regularity estimate for (85) (cf. (26), (31)), we have the following error estimate.

Lemma 4.3. *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (85). *Then, there exists $C > 0$, independent of h, such that for each $\widehat{\boldsymbol{\tau}}_{s,h} := (\boldsymbol{\tau}_{s,h}, q_h) \in \mathbb{X}_{1,h}$ there holds*

$$\|P_1(\widehat{\boldsymbol{\tau}}_{s,h}) - P_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h})\|_{\mathbf{div};\Omega_s} \le C\,h^\epsilon\left\{\|\mathbf{div}\,\boldsymbol{\tau}_{s,h}\|_{0,\Omega_s} + \|q_h\|_{1,\Omega_f}\right\}. \tag{91}$$

On the other hand, we know from (33) and (37) that $P_2(\widehat{\boldsymbol{\tau}}_{f,h})$ belongs to $\mathbf{H}^\epsilon(\Omega_f)$ and

$$\|P_2(\widehat{\boldsymbol{\tau}}_{f,h})\|_{\epsilon,\Omega_f} \le C\left\{\|\operatorname{div}\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f} + \|\mathbf{v}_h\|_{1,\Omega_s}\right\}, \tag{92}$$

whence we can define

$$P_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h}) := \mathcal{E}_h^f\left(P_2(\widehat{\boldsymbol{\tau}}_{f,h})\right) \quad \text{and} \quad \mathbf{P}_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h}) := (P_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h}), \mathbf{v}_h)\,. \tag{93}$$

It follows from (86) and (87) that

$$\operatorname{div} P_2(\widehat{\boldsymbol{\tau}}_{f,h}) = \operatorname{div}\boldsymbol{\tau}_{f,h} \quad \text{in} \quad \Omega_f \quad \text{and} \quad P_2(\widehat{\boldsymbol{\tau}}_{f,h})\cdot\boldsymbol{\nu} = \rho_f\,\omega^2\,\mathbf{v}_h\cdot\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,.$$

Therefore, employing the conmuting diagram property (65) and the fact that $\operatorname{div}\boldsymbol{\tau}_{f,h}$ is piecewise constant, we deduce that

(94)

$$\operatorname{div} P_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h}) = \mathcal{P}_h^f\big(\operatorname{div} P_2(\widehat{\boldsymbol{\tau}}_{f,h})\big) = \mathcal{P}_h^f\big(\operatorname{div}\boldsymbol{\tau}_{f,h}\big) = \operatorname{div}\boldsymbol{\tau}_{f,h} = \operatorname{div} P_2(\widehat{\boldsymbol{\tau}}_{f,h})\,.$$

Furthermore, the uniform boundedness of $\mathcal{E}_h^f : \mathbf{H}^\epsilon(\Omega_f)\cap\mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}_h^f$ (which follows from (67) and (65)), the estimate (92), and the identity (94), imply that $\mathbf{P}_{2,h}$ is uniformly bounded. In addition, using the characterization property (63) and the fact that $\rho_f\,\omega^2\,\mathbf{v}_h\cdot\boldsymbol{\nu}$ is piecewise polynomial of degree 1, we easily find that $\mathcal{E}_h^f\left(P_2(\widehat{\boldsymbol{\tau}}_{f,h})\right)\cdot\boldsymbol{\nu} = \rho_f\,\omega^2\,\mathbf{v}_h\cdot\boldsymbol{\nu}$ on Σ, which proves that $\mathbf{P}_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h})$ belongs to $\mathbb{X}_{2,h}$. Moreover, we have the following error estimate.

Lemma 4.4. *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (87). *Then, there exists $C > 0$, independent of h, such that for each $\widehat{\boldsymbol{\tau}}_{f,h} := (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h) \in \mathbb{X}_{2,h}$ there holds*

$$\|P_2(\widehat{\boldsymbol{\tau}}_{f,h}) - P_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h})\|_{\mathrm{div};\Omega_f} \le C\,h^\epsilon \left\{\|\mathrm{div}\,\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f} + \|\mathbf{v}_h\|_{1,\Omega_s}\right\}. \tag{95}$$

Proof. We first observe, according to (93) and (94), that

$$\begin{aligned}\|P_2(\widehat{\boldsymbol{\tau}}_{f,h}) - P_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h})\|_{\mathrm{div};\Omega_f} &= \|P_2(\widehat{\boldsymbol{\tau}}_{f,h}) - P_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h})\|_{0,\Omega_f}\\ &= \|(\mathbf{I} - \mathcal{E}_h^f)\big(P_2(\widehat{\boldsymbol{\tau}}_{f,h})\big)\|_{0,\Omega_f}.\end{aligned}$$

Then, applying the approximation property (67) and the identity (94), we obtain

$$\begin{aligned}\|(\mathbf{I} - \mathcal{E}_h^f)\big(P_2(\widehat{\boldsymbol{\tau}}_{f,h})\big)\|_{0,\Omega_f}^2 &= \sum_{T\in\mathcal{T}_{h_f}} \|(\mathbf{I} - \mathcal{E}_h^f)\big(P_2(\widehat{\boldsymbol{\tau}}_{f,h})\big)\|_{0,T}^2\\ &\le C \sum_{T\in\mathcal{T}_{h_f}} h_T^{2\epsilon}\left\{|P_2(\widehat{\boldsymbol{\tau}}_{f,h})|_{\epsilon,T}^2 + \|\mathrm{div}\,P_2(\widehat{\boldsymbol{\tau}}_{f,h})\|_{0,T}^2\right\}\\ &\le C\,h^{2\epsilon}\left\{\|P_2(\widehat{\boldsymbol{\tau}}_{f,h})\|_{\epsilon,\Omega_f}^2 + \|\mathrm{div}\,\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f}^2\right\},\end{aligned}$$

which, together with the estimate (92), completes the proof. □

We now formally let $\mathbf{P}_h : \mathbb{X}_h \longrightarrow \mathbb{X}_h$ be the discrete approximation of $\mathbf{P}|_{\mathbb{X}_h}$ given by

$$\mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h) := (\mathbf{P}_{1,h}(\widehat{\boldsymbol{\tau}}_{s,h}), \mathbf{P}_{2,h}(\widehat{\boldsymbol{\tau}}_{f,h})) \qquad \forall\,\widehat{\boldsymbol{\tau}}_h := (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) \in \mathbb{X}_h,$$

where $\mathbf{P}_{1,h}$ and $\mathbf{P}_{2,h}$ are defined by (88) and (93), respectively. Note that $\mathbf{P}_h$ is certainly uniformly bounded, as well. Then, as a direct consequence of Lemmas 4.3 and 4.4, we obtain the following error estimate.

Lemma 4.5. *Let $\epsilon > 0$ be the parameter defining the regularity of the solutions of* (85) *and* (87). *Then, there exists $C > 0$, independent of h, such that for each $\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h$ there holds*

$$\|\mathbf{P}(\widehat{\boldsymbol{\tau}}_h) - \mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbb{X}} \le C\,h^\epsilon\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}.$$

Proof. It follows straightforwardly from the estimates (91) and (95), and the fact that

$$\|\mathbf{div}\,\boldsymbol{\tau}_{s,h}\|_{0,\Omega_s} + \|q_h\|_{1,\Omega_f} + \|\mathrm{div}\,\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f} + \|\mathbf{v}_h\|_{1,\Omega_s} \le \|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}$$

for each $\widehat{\boldsymbol{\tau}}_h := (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) := ((\boldsymbol{\tau}_{s,h}, q_h), (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h)) \in \mathbb{X}_h$. □

4.4. Well-posedness of the discrete formulation. We now aim to show the well-posedness of the augmented fully-mixed finite element scheme (61). For this purpose, as established by a classical result on projection methods for Fredholm operators of index zero (see, e.g. Theorem 13.7 in [26]), it suffices to prove that the Galerkin scheme associated to the isomorphism $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is well-posed. According to the above, in what follows we show that the bilinear forms $\mathbb{A}_0$ and $\mathbb{B}$ (cf. (45), (23)) satisfy the corresponding inf-sup conditions on the finite element subspace $\mathbb{X}_h \times \mathbb{Y}_h$.

We begin with the discrete analogue of Lemma 3.1, which was actually already proved in [20].

Lemma 4.6. *There exists $\beta > 0$, independent of h, such that*

$$\sup_{\substack{\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h \\ \widehat{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\,\mathbb{B}(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h)\,|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}} \,\geq\, \beta\,\|\boldsymbol{\eta}_h\|_{0,\Omega_s} \qquad \forall\,\boldsymbol{\eta}_h \,\in\, \mathbb{Y}_h\,.$$

Proof. See [20, Lemma 5.5]. □

We now let $\mathbb{V}_h$ be the discrete kernel of $\mathbb{B}$, that is

$$\begin{aligned} \mathbb{V}_h \,:=\, & \Big\{ \widehat{\boldsymbol{\tau}}_h \,:=\, (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) \,=\, ((\boldsymbol{\tau}_{s,h}, q_h), (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h)) \,\in\, \mathbb{X}_h \,: \\ & \qquad \mathbb{B}(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h) \,=\, 0 \quad \forall\,\boldsymbol{\eta}_h \,\in\, \mathbb{Y}_h \Big\} \\ =\, & \Big\{ \widehat{\boldsymbol{\tau}}_h \,:=\, (\widehat{\boldsymbol{\tau}}_{s,h}, \widehat{\boldsymbol{\tau}}_{f,h}) \,=\, ((\boldsymbol{\tau}_{s,h}, q_h), (\boldsymbol{\tau}_{f,h}, \mathbf{v}_h)) \,\in\, \mathbb{X}_h \,: \\ & \qquad \int_{\Omega_s} \boldsymbol{\tau}_{s,h} : \boldsymbol{\eta}_h \,=\, 0 \quad \forall\,\boldsymbol{\eta}_h \in \mathbb{Y}_h \Big\}\,. \end{aligned}$$

Then, the discrete analogue of Lemma 3.7 is established as follows.

Lemma 4.7. *There exist $C,\ h_0 \,>\, 0$, independent of h, but depending on* $\min\Big\{1, \dfrac{1}{\kappa_s^2}\Big\}$ *and* $\min\Big\{1, \dfrac{1}{\kappa_f^2}\Big\}$, *such that for each $h \,\leq\, h_0$ there holds*

$$\sup_{\substack{\widehat{\boldsymbol{\zeta}}_h \in \mathbb{V}_h \\ \widehat{\boldsymbol{\zeta}}_h \neq \mathbf{0}}} \frac{|\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\zeta}}_h)\,|}{\|\widehat{\boldsymbol{\zeta}}_h\|_{\mathbb{X}}} \,\geq\, C\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \qquad \forall\,\widehat{\boldsymbol{\tau}}_h \in \mathbb{V}_h\,. \tag{96}$$

Proof. We first define $\mathbf{T}_h \,:=\, (\mathbf{I} \,-\, 2\,\mathbf{P}_h) \,:\, \mathbb{X}_h \to \mathbb{X}_h$, which constitutes a discrete approximation of the operator $\mathbf{T} \,:=\, (\mathbf{I} \,-\, 2\,\mathbf{P}) \,:\, \mathbb{X} \to \mathbb{X}$ (cf. (52)), and observe, as a straightforward consequence of Lemma 4.5, that

$$\|\mathbf{T}(\widehat{\boldsymbol{\tau}}_h) \,-\, \mathbf{T}_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbb{X}} \,\leq\, C\,h^{\epsilon}\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \qquad \forall\,\widehat{\boldsymbol{\tau}}_h \,\in \mathbb{X}_h\,.$$

Then, adding and substracting $\mathbf{T}\big(\overline{\widehat{\boldsymbol{\tau}}}_h\big)$, using the boundedness of $\mathbb{A}_0$, and applying the inequality (54) (cf. Lemma 3.6), we find that for each $\widehat{\boldsymbol{\tau}}_h \,\in\, \mathbb{X}_h$ there holds

$$\begin{aligned} \Big|\,\mathrm{Re}\Big\{\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}}_h, \mathbf{T}_h(\overline{\widehat{\boldsymbol{\tau}}}_h))\,\Big\}\,\Big| \,&\geq\, \Big|\,\mathrm{Re}\Big\{\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}}_h, \mathbf{T}(\overline{\widehat{\boldsymbol{\tau}}}_h)\,\Big\}\,\Big| \,-\, \tilde{C}\,h^{\epsilon}\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}^2 \\ &\geq\, \Big\{C_1 \,-\, \tilde{C}\,h^{\epsilon}\Big\}\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}^2\,, \end{aligned}$$

with C_1 depending on $\min\Big\{1, \dfrac{1}{\kappa_s^2}\Big\}$ and $\min\Big\{1, \dfrac{1}{\kappa_f^2}\Big\}$, from which we deduce the existence of $h_0 > 0$ such that

$$\Big|\,\mathrm{Re}\Big\{\,\mathbb{A}_0(\widehat{\boldsymbol{\tau}}_h, \mathbf{T}_h(\overline{\widehat{\boldsymbol{\tau}}}_h))\,\Big\}\,\Big| \,\geq\, c\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}^2 \qquad \forall\,\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h\,, \quad \forall\,h \leq h_0\,. \tag{97}$$

It is clear from this inequality that $\mathbf{T}_h(\widehat{\boldsymbol{\tau}}_h) \,\neq\, \mathbf{0}$ for each $\widehat{\boldsymbol{\tau}}_h \,\neq\, \mathbf{0}$. In addition, it follows from the second equation of (90) and the above characterization of $\mathbb{V}_h$ that $\mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)$, and hence $\mathbf{T}_h(\widehat{\boldsymbol{\tau}}_h)$, belong to $\mathbb{V}_h$ for each $\widehat{\boldsymbol{\tau}}_h \,\in\, \mathbb{V}_h$. Finally, it is easy to see that (97) and the uniform boundedness of $\mathbf{T}_h$ imply the discrete inf-sup condition (96). □

We end this section with the well-posedness and convergence of our discrete scheme (61).

Theorem 4.1. *Assume that the homogeneous problem associated to* (21) *has only the trivial solution. Let* $h_0 > 0$ *be the constant provided by Lemma 4.7. Then, there exists* $h_1 \in]0, h_0]$ *such that for each* $h \le h_1$*, the mixed finite element scheme* (61) *has a unique solution* $(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\gamma}_h) \in \mathbb{X}_h \times \mathbb{Y}_h$*, where* $\widehat{\boldsymbol{\sigma}}_h := (\widehat{\boldsymbol{\sigma}}_{s,h}, \widehat{\boldsymbol{\sigma}}_{f,h}) := ((\boldsymbol{\sigma}_{s,h}, p_h), (\boldsymbol{\sigma}_{f,h}, \mathbf{u}_h))$*. In addition, there exist* $C_1, C_2 > 0$*, independent of* h*, but depending on* $\min\left\{1, \frac{1}{\kappa_s^2}\right\}$, $\min\left\{1, \frac{1}{\kappa_f^2}\right\}$, $\max\left\{1, \frac{1}{\kappa_s^2}\right\}$*, and* $\max\left\{1, \frac{1}{\kappa_f^2}\right\}$*, such that for each* $h \le h_1$ *there hold*

$$\|(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\gamma}_h)\|_{\mathbb{X}\times\mathbb{Y}} \;\le\; C_1 \sup_{\substack{\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h \\ \widehat{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbb{F}(\widehat{\boldsymbol{\tau}}_h)|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}} \;\le\; C_1 \left\{ \|\mathbf{f}\|_{0,\Omega_s} + \|p_i\|_{1/2,\Gamma} \right\}$$

and

$$\|(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\gamma}_h)\|_{\mathbb{X}\times\mathbb{Y}} \le C_2 \inf_{(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h) \in \mathbb{X}_h \times \mathbb{Y}_h} \|(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) - (\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h)\|_{\mathbb{X}\times\mathbb{Y}}\,, \tag{98}$$

where $\widehat{\boldsymbol{\sigma}} := (\widehat{\boldsymbol{\sigma}}_s, \widehat{\boldsymbol{\sigma}}_f) := ((\boldsymbol{\sigma}_s, p), (\boldsymbol{\sigma}_f, \mathbf{u})) \in \mathbb{X}$ *and* $\boldsymbol{\gamma} \in \mathbb{Y}$ *constitute the unique solution of* (21)*. Furthermore, if there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_s \in \mathbb{H}^\delta(\Omega_s)$, $\mathbf{div}\,\boldsymbol{\sigma}_s \in \mathbf{H}^\delta(\Omega_s)$, $p \in H^{1+\delta}(\Omega_f)$, $\boldsymbol{\sigma}_f \in \mathbf{H}^\delta(\Omega_f)$, $\operatorname{div} \boldsymbol{\sigma}_f \in H^\delta(\Omega_f)$, $\mathbf{u} \in \mathbf{H}^{1+\delta}(\Omega_s)$*, and* $\boldsymbol{\gamma} \in \mathbb{H}^\delta(\Omega_s)$*, then for each* $h \le h_1$ *there holds*

$$\begin{aligned} \|(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\gamma}_h)\|_{\mathbb{X}\times\mathbb{Y}} \le C_3\, h^\delta \Big\{ & \|\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\mathbf{div}\,\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|p\|_{1+\delta,\Omega_f} \\ + \;\; & \|\boldsymbol{\sigma}_f\|_{\delta,\Omega_f} + \|\operatorname{div} \boldsymbol{\sigma}_f\|_{\delta,\Omega_f} + \|\mathbf{u}\|_{1+\delta,\Omega_s} + \|\boldsymbol{\gamma}\|_{[H^\delta(\Omega_s)]^{3\times 3}} \Big\}, \end{aligned} \tag{99}$$

with a constant $C_3 > 0$*, independent of* h*.*

Proof. Because of Lemmas 4.6 and 4.7, the proof of the first part is a straightforward application of [26, Theorem 13.7]. In turn, the rate of convergence (99) follows directly from the Cea estimate (98) and the approximation properties of $\mathbb{X}_{1,h}$ (cf. Lemma 4.1), $\mathbb{X}_{2,h}$ (cf. Lemma 4.2), and $\mathbb{Y}_h$ (cf. $(\mathrm{AP}_h^{\boldsymbol{\gamma}})$). Note here that the corresponding inequalities (73) and (74) are bounded above thanks to $(\mathrm{AP}_h^{\boldsymbol{\sigma}_s})$, (AP_h^{p}), $(\mathrm{AP}_h^{\boldsymbol{\sigma}_f})$, and $(\mathrm{AP}_h^{\mathbf{u}})$. We omit additional details. □

We end this section by remarking that the fact that the stability constants C_1 and C_2 in Theorem 4.1 depend on $\min\left\{1, \frac{1}{\kappa_s^2}\right\}$, $\min\left\{1, \frac{1}{\kappa_f^2}\right\}$, $\max\left\{1, \frac{1}{\kappa_s^2}\right\}$, and $\max\left\{1, \frac{1}{\kappa_f^2}\right\}$, clearly indicates that they are quite sensitive to the choice of κ_s and κ_f. The specific limited ranges where these wavenumbers must lie will depend on how small and how large, respectively, the above minima and maxima are allowed to be.

5. Numerical results

In this section we present two examples illustrating the performance of the augmented fully-mixed finite element scheme (61) on a finite sequence of quasi-uniform triangulations of the domain. We begin by introducing additional notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathbb{X}_h$ and $\mathbb{Y}_h$, and the individual and global errors are by:

$$\mathsf{e}(\boldsymbol{\sigma}_s) := \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s}\,, \quad \mathsf{e}(p) := \|p - p_h\|_{1,\Omega_f}\,, \quad \mathsf{e}(\boldsymbol{\sigma}_f) := \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{\mathrm{div};\Omega_f}\,,$$

$$\mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_s}\,, \qquad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega_s}\,, \qquad \text{and}$$

$$\mathsf{e}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) := \Big\{ \big\{\mathsf{e}(\boldsymbol{\sigma}_s)\big\}^2 + \big\{\mathsf{e}(p)\big\}^2 + \big\{\mathsf{e}(\boldsymbol{\sigma}_f)\big\}^2 + \big\{\mathsf{e}(\mathbf{u})\big\}^2 + \big\{\mathsf{e}(\boldsymbol{\gamma})\big\}^2 \Big\}^{1/2}.$$

Also, we let $r(\boldsymbol{\sigma}_s)$, $r(p)$, $r(\boldsymbol{\sigma}_f)$, $r(\mathbf{u})$, $r(\boldsymbol{\gamma})$, and $r(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$ be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}_s) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma}_s)/\mathsf{e}'(\boldsymbol{\sigma}_s))}{\log(h/h')}\,, \quad r(p) := \frac{\log(\mathsf{e}(p)/\mathsf{e}'(p))}{\log(h/h')}\,,$$

$$r(\boldsymbol{\sigma}_f) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma}_f)/\mathsf{e}'(\boldsymbol{\sigma}_f))}{\log(h/h')}\,, \quad r(\mathbf{u}) := \frac{\log(\mathsf{e}(\mathbf{u})/\mathsf{e}'(\mathbf{u}))}{\log(h/h')}\,,$$

$$r(\boldsymbol{\gamma}) := \frac{\log(\mathsf{e}(\boldsymbol{\gamma})/\mathsf{e}'(\boldsymbol{\gamma}))}{\log(h/h')}\,, \quad \text{and} \quad r(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) := \frac{\log(\mathsf{e}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})/\mathsf{e}'(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}))}{\log(h/h')}\,,$$

where h and h' denote two consecutive meshsizes with corresponding errors e and e'.

In Example 1 we consider the domains $\Omega_s :=]0.25, 0.75[^3$ and $\Omega_f :=]0,1[^3 \setminus \Omega_s$, and take the parameters $\omega = v_0 = 2$, and $\rho_s = \rho_f = \lambda = \mu = 1$, whence $\kappa_f = 1$ and $\kappa_s = 2$. Then, we choose the data $\mathbf{f}$ and p_i so that, with the above constants, the exact solution of (21) is determined by

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi\, r} \left\{ a(r) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{b(r)}{r^2} \begin{pmatrix} (x_1 - 2)^2 \\ x_2(x_1 - 2) \\ x_3(x_1 - 2) \end{pmatrix} \right\} \qquad \forall\, \mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega_s\,,$$

and

$$p(\mathbf{x}) = \frac{1}{r} \exp(r\iota) \qquad \forall\, \mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega_f\,,$$

where

$$r = \sqrt{(x_1 - 2)^2 + x_2^2 + x_3^2}\,,$$

$$a(r) = \left\{ 1 - \frac{1}{4r^2} + \frac{\iota}{2r} \right\} \exp(2r\iota) - \frac{1}{3} \left\{ -\frac{3}{4r^2} + \frac{\sqrt{3}\,\iota}{2r} \right\} \exp\left(\frac{2r\iota}{\sqrt{3}} \right),$$

and

$$b(r) = \left\{ \frac{3}{4r^2} - 1 - \frac{3\iota}{2r} \right\} \exp(2r\iota) - \frac{1}{3} \left\{ \frac{9}{4r^2} - 1 - \frac{3\sqrt{3}\,\iota}{2r} \right\} \exp\left(\frac{2r\iota}{\sqrt{3}} \right).$$

Actually, $\mathbf{u}$ is the fundamental solution, centered at $(2,0,0)$, of the elastodynamic equation, which yields $\mathbf{f} = \mathbf{0}$ in Ω_s, and p is the fundamental solution, also centered at $(2,0,0)$, of the Helmholtz equation in Ω_f. Next, in Example 2 we assume the same geometry, parameters, and solution $\mathbf{u}$ from Example 1, but the exact pressure is replaced by a plane wave in the direction $(-1,0,0)^{\mathtt{t}}$, that is

$$p(\mathbf{x}) = \exp(-\, x_1\, \iota) \qquad \forall\, \mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega_f\,.$$

We remark that in these academic examples with known solutions, the transmission conditions defining the spaces $\mathbb{X}_1$ and $\mathbb{X}_2$ do not necessarily hold, and hence we simply replace them by the corresponding jump relations that arise. These non-homogeneous jumps are then handled at the discrete level by introducing suitable Lagrange multipliers in the computational implementation of (61). We remark that these multipliers are employed only at the algebraic level, not in the functional setting of the problem, which means that no hybrid additional variables are introduced on the interface.

In Tables 1 to 4 we present the convergence history of Examples 1 and 2 for a sequence of quasi-uniform triangulations of the computational domain $\bar{\Omega}_s \cup \bar{\Omega}_f$. We remark that the rate of convergence $O(h)$ predicted by Theorem 4.1 (when $\delta = 1$) is attained for all the unknowns and in all the cases presented. In particular, the use of different pairs of parameters satisfying the stabilization conditions $\kappa_1 \in]0, 2\mu[$ and $\kappa_2 \in]0, 1[$ confirms the robustness of the discrete scheme (61) with respect to them.

We end this paper by remarking that the devising of more efficient numerical methods for solving the Galerkin scheme (61) should consider a decoupled procedure combined with a preconditioning technique. We are currently working in this direction by following a similar approach to the one developed in [28].

TABLE 1. Convergence history of EXAMPLE 1 with $\kappa_1 = \mu$ and $\kappa_2 = 0.5$

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$
1/4	3229	8.237E−03	–	9.268E−02	–	4.447E−02	–
1/8	25845	3.953E−03	1.059	4.879E−02	0.926	2.196E−02	1.018
1/12	86213	2.622E−03	1.013	3.292E−02	0.970	1.461E−02	1.005
1/16	203977	1.964E−03	1.004	2.481E−02	0.983	1.095E−02	1.003
1/20	390525	1.570E−03	1.002	1.990E−02	0.989	8.756E−03	1.002
1/24	684053	1.309E−03	1.001	1.660E−02	0.992	7.295E−03	1.001

h	N	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathsf{e}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$	$r(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$
1/4	3229	1.329E−02	–	3.308E−03	–	1.040E−01	–
1/8	25845	4.929E−03	1.430	8.960E−04	1.884	5.388E−02	0.949
1/12	86213	2.937E−03	1.277	4.834E−04	1.522	3.624E−02	0.979
1/16	203977	2.068E−03	1.219	3.297E−04	1.331	2.727E−02	0.988
1/20	390525	1.597E−03	1.160	2.510E−04	1.223	2.185E−02	0.992
1/24	684053	1.306E−03	1.103	2.032E−04	1.158	1.823E−02	0.994

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TABLE 2. Convergence history of EXAMPLE 1 with $\kappa_1 = \mu/2$ and $\kappa_2 = 0.75$

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$
1/4	3229	8.219E−03	–	9.675E−02	–	7.747E−02	–
1/8	25845	3.946E−03	1.059	4.888E−02	0.985	2.257E−02	1.779
1/12	86213	2.620E−03	1.010	3.294E−02	0.973	1.473E−02	1.053
1/16	203977	1.963E−03	1.003	2.482E−02	0.984	1.099E−02	1.017
1/20	390525	1.570E−03	1.001	1.990E−02	0.990	8.775E−03	1.009
1/24	684053	1.308E−03	1.000	1.660E−02	0.993	7.305E−03	1.005

h	N	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathsf{e}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$	$r(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$
1/4	3229	9.089E−02	–	3.414E−03	–	1.540E−01	–
1/8	25845	9.760E−03	3.219	9.167E−04	1.897	5.486E−02	1.489
1/12	86213	4.417E−03	1.955	4.944E−04	1.523	3.645E−02	1.008
1/16	203977	2.722E−03	1.682	3.360E−04	1.343	2.735E−02	0.998
1/20	390525	1.940E−03	1.519	2.548E−04	1.239	2.189E−02	0.998
1/24	684053	1.506E−03	1.389	2.057E−04	1.175	1.825E−02	0.998

TABLE 3. Convergence history of EXAMPLE 2 with $\kappa_1 = \mu$ and $\kappa_2 = 0.5$

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$
1/4	3229	8.115E−03	–	6.535E−02	–	4.837E−02	–
1/8	25845	3.938E−03	1.043	3.338E−02	0.969	2.394E−02	1.015
1/12	86213	2.617E−03	1.008	2.238E−02	0.987	1.593E−02	1.004
1/16	203977	1.962E−03	1.002	1.682E−02	0.992	1.194E−02	1.002
1/20	390525	1.569E−03	1.000	1.347E−02	0.995	9.553E−03	1.001
1/24	684053	1.308E−03	0.999	1.123E−02	0.996	7.959E−03	1.001

h	N	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathsf{e}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$	$r(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma})$
1/4	3229	1.231E−02	–	2.975E−03	–	8.268E−02	–
1/8	25845	4.557E−03	1.433	8.248E−04	1.851	4.153E−02	0.994
1/12	86213	2.799E−03	1.202	4.589E−04	1.446	2.774E−02	0.995
1/16	203977	2.008E−03	1.155	3.186E−04	1.268	2.082E−02	0.997
1/20	390525	1.567E−03	1.111	2.451E−04	1.175	1.666E−02	0.998
1/24	684053	1.290E−03	1.067	1.997E−04	1.123	1.389E−02	0.999

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TABLE 4. Convergence history of EXAMPLE 2 with $\kappa_1 = \mu/2$ and $\kappa_2 = 0.75$

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$
1/4	3229	8.099E−03	–	7.129E−02	–	8.104E−02	–
1/8	25845	3.934E−03	1.042	3.352E−02	1.089	2.451E−02	1.725
1/12	86213	2.616E−03	1.006	2.241E−02	0.993	1.605E−02	1.045
1/16	203977	1.961E−03	1.001	1.683E−02	0.995	1.198E−02	1.015
1/20	390525	1.569E−03	1.000	1.348E−02	0.996	9.571E−03	1.007
1/24	684053	1.308E−03	0.999	1.123E−02	0.997	7.969E−03	1.004

h	N	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathsf{e}(\gamma)$	$r(\gamma)$	$\mathsf{e}(\widehat{\boldsymbol{\sigma}}, \gamma)$	$r(\widehat{\boldsymbol{\sigma}}, \gamma)$
1/4	3229	9.272E−02	–	2.911E−03	–	1.426E−01	–
1/8	25845	9.264E−03	3.323	8.359E−04	1.800	4.274E−02	1.738
1/12	86213	4.144E−03	1.984	4.645E−04	1.449	2.800E−02	1.043
1/16	203977	2.586E−03	1.639	3.214E−04	1.280	2.092E−02	1.013
1/20	390525	1.868E−03	1.457	2.466E−04	1.187	1.671E−02	1.006
1/24	684053	1.467E−03	1.328	2.006E−04	1.132	1.392E−02	1.004

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