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Jordan centers and Martindale-like covers

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Abstract

In this paper we show that the scalar center of a nondegenerate quadratic Jordan algebra is contained in the scalar center of any of its Martindale-like covers.

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Introduction

The notion of (weak) scalar center, introduced by Fulgham in [3], has revealed a central tool in the study of Martindale-like quotients [1,4] of linear Jordan algebras mainly due to two facts:

(i) any nonzero ideal of a nondegenerate PI Jordan algebra contains nonzero central elements [2, 3.6], and

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(ii) the scalar center of a nondegenerate linear Jordan algebra is contained in the scalar center of any of its Martindale-like algebras of quotients [1, 4.1].

Our aim in this paper is showing that the general (quadratic) version of (ii) holds. Indeed we will work at the slightly more general setting of what we call “Martindale-like covers,” defined in terms of natural “ideal absorption properties.” This result is basic in our forthcoming paper on polynomial identities and speciality of quadratic Martindale-like quotients, as well as we expect it to be useful in the description of Martindale-like quotients of strongly prime quadratic Jordan algebras satisfying a polynomial identity.

The proof of our main result is purely combinatorial, based on the fact that $2J + \text{Ker} \text{Id}_J$ is an essential ideal of any nondegenerate Jordan algebra $J$, which, with the use of annihilators, allows to split the problem into the 2-torsion free and the characteristic 2 cases.

The paper is divided into four sections. Section 0 is devoted to recalling basic facts and notions, including the essentialness of $2J + \text{Ker} \text{Id}_J$, mentioned above, and the definition of the scalar center. In Section 1 we study characteristic 2 phenomena needed in the sequel, and their natural extensions to arbitrary Jordan algebras in terms of the annihilator $\text{Ann}_J(\text{Ker} \text{Id}_J)$ of $2J$. In the next section we establish the fundamental properties of Martindale-like covers. Finally, in Section 3, we prove our main theorem asserting the inheritance of the scalar center by Martindale-like covers of nondegenerate Jordan algebras. It turns out that for a central element $z$ of $J$, and a cover $Q$ of $J$, $Vz$ is in the centroid of $Q$ as soon as $Q$ satisfies the natural outer ideal absorption properties, while for the fact that $z$ is indeed central in $Q$, the inner ideal absorption property must be assumed too.

0. Preliminaries

0.1. We will deal with Jordan algebras over a ring of scalars $\Phi$. The reader is referred to [5,7,11] for definitions and basic properties not explicitly mentioned or proved in this section. Given a Jordan algebra $J$, its products will be denoted $x^2, U_{xy}$, for $x, y \in J$. They are quadratic in $x$ and linear in $y$ and have linearizations denoted $V_{xy} = x \circ y, U_{x,zy} = \{x, y, z\} = V_{x,y}z$, respectively.

A Jordan algebra $J$ is said to be unital if there is an element $1 \in J$ satisfying $U_{1} = \text{Id}_J$ and $U_{x,1} = x^2$, for any $x \in J$ (such an element can be shown to be unique and it is called the unit of $J$).

Every Jordan algebra $J$ embeds in a unital Jordan algebra $\hat{J} = J \oplus \Phi 1$ called its (free) unitization [11, 0.6].

A Jordan algebra $J$ is said to be nondegenerate if zero is the only absolute zero divisor, i.e., zero is the only $x \in J$ such that $U_x = 0$.

0.2. We will need the following identities valid for arbitrary Jordan algebras.

(i) $(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$,
(ii) $z \circ U_{xy} = \{z, x, y\} \circ x - y \circ U_{xz}$,
(iii) $U_{U_{xy},z} = U_x U_y U_z, U_{x,z} = (U_x)^2$,
(iv) $\{x, U_{xy}, y\} = \{U_{x,z}, z, y\}$,
(v) $2U_{x,y} = x \circ (x \circ y) - x^2 \circ y$,
(vi) $\{z, x, U_{y_1}, y_2\} = \{z, \{x, y_1, y_2\}, y_1\} - \{z, y_2, U_{y_1}x\}$,
(vii) $U_x(y \circ z) = \{x \circ y, z, x\} - y \circ U_{xz}$,
(viii) $U_{z,xy} = U_z U_{xy} + U_{z,x} U_z y + z \circ U_x(y \circ z) - \{U_z x, y, x\}$,
(ix) \((Ux,y)^2 = UxUx\), 
(x) \((x \circ y)^2 = UxUx^2 + x \circ Ux\), 
(xi) \(U_{a,x,b}\)z = \(UaUxUbx + UaUxUbx + [a, x, Ua]z\) - \(UaUxUbx, z, b\), 
(xii) \(2UaUxUax = [a, x, UaUxUax] - [UaUxUax, z, b]\), 
(xiv) \(Ux\{x, b, c\} = [x, a, \{b, c, x\}] - \{UxUxb, c\}\).

Indeed, (vi) is [7, JP10], (xi) is [7, JP21], (xiv) is [7, JP12], and the rest of them follow from Macdonald’s theorem [6].

0.3. We recall that an ideal \(I\) of a Jordan algebra \(J\) is just a \(\Phi\)-submodule of \(J\) satisfying \(UIJ + I^2 + UJ I \subseteq I\), equivalently, \(UIJ + UJ I \subseteq I\), which implies \(I, J, J \subseteq I\) using (0.2)(i). An ideal \(I\) of \(J\) is said to be essential if it hits every nonzero ideal of \(J\), i.e., \(I \cap \not= 0\) for any nonzero ideal \(L\) of \(J\).

0.4. In a Jordan algebra \(J\), the annihilator \(Ann_J(I)\) of an ideal \(I\) of \(J\) is an ideal of \(J\) which, when \(J\) is nondegenerate, is given by

\[Ann_J(I) = \{x \in J \mid UxI = 0\} = \{x \in J \mid UxI = 0\}\]

[8, 1.3, 1.7], [12, 1.3]. An ideal \(I\) of \(J\) will be said sturdy if \(Ann_J(I) = 0\). It is easy to prove that essential ideals coincide with sturdy ideals in any semiprime Jordan algebra.

0.5. The centroid \(\Gamma(J)\) of a Jordan algebra \(J\) is the set of linear maps acting “scalarly” in Jordan products [10]:

\[\Gamma(J) = \{T \in \text{End}_{\Phi}(J) \mid TUx = UxT, TVx = VxT, T^2(x^2) = (T(x))^2, T^2Ux = U_{T(x)}, \text{for any } x \in J\}\].

It is immediate that \(TVx,y = Vx,y T, TUx,y = Ux,y T\) for any \(T \in \Gamma(J)\), and any \(x, y \in J\). Clearly, \(\Phi \text{Id}_J \subseteq \Gamma(J)\). By [10, 2.5], when \(J\) has no nonzero extreme elements (for example, when \(J\) is nondegenerate), \(\Gamma(J)\) is a unital associative commutative \(\Phi\)-algebra and \(J\) is a Jordan algebra over \(\Gamma(J)\).

0.6. Lemma. If \(J\) is a nondegenerate Jordan algebra and \(T \in \Gamma(J)\), then

(i) the sum \(T(J) + \text{Ker} T\) is direct and, indeed, \(\text{Ker} T = \text{Ker} T^n\) for any positive integer \(n\),
(ii) \(T(J)\) and \(\text{Ker} T\) are ideals of \(J\),
(iii) \(T(J) + \text{Ker} T\) is an essential ideal of \(J\).

Proof. (i) Let \(x \in J\) such that \(T^2(x) = 0\). Then, for any \(y \in J\), we have

\[U_{T(x)y} = UT(x)YxUT(x) = UT(x)YxT^2Ux = T^2UT(x)UxYx = UT^2(x)UxYx = 0,\]

hence \(U_{T(x)y} = 0\) by nondegeneracy. This shows \(U_{T(x)} = 0\), hence \(T(x) = 0\) again by nondegeneracy. We have proved \(\text{Ker} T = \text{Ker} T^2\), which readily implies our assertion.
(ii) By [10, 2.6] we already know that $T(J)$ is an ideal of $J$ and Ker $T$ is an outer ideal of $J$. But, under nondegeneracy, Ker $T$ is also an inner ideal of $J$: for any $x \in$ Ker $T$, $y \in \hat{J}$, $T^2(U_x y) = U_{T(x)} y = 0$, hence $U_x y \in$ Ker $T^2 =$ Ker $T$ by (i).

(iii) Given a nonzero ideal $L$ of $J$, if $T(L) = 0$, then $0 \neq L \subseteq L \cap \text{Ker} T \subseteq L \cap (T(J) + \text{Ker} T)$. Otherwise, there exists $x \in L$ such that $T(x) \neq 0$. By nondegeneracy, $0 \neq UT(J) = T^2 U_x J = U_x T^2(J) \subseteq U_L J \cap T(J) \subseteq L \cap (T(J) + \text{Ker} T)$. □

0.7. Following [3], the (weak) center of $J$ is the set $C(J)$ of all elements $z \in J$ such that $U_z$, $V_z \in \Gamma(J)$, which is a subalgebra of $J$ when $J$ is nondegenerate [3, Theorems 1, 2]. More explicitly, $z \in J$ lies in $C(J)$ if and only if

\[ c_i(z, J, J) = 0, \quad \text{for } i = 1, 2, 3, 5, 6, \quad \text{and} \quad c_i(z, J) = 0 \quad \text{for } i = 4, 7, \]

where

\[
\begin{align*}
c_1(z, x, y) &= V_z U_x y - U_x V_z y = z \circ U_x y - U_x (z \circ y), \\
c_2(z, x, y) &= V_z V_x y - V_x V_z y = z \circ (x \circ y) - x \circ (z \circ y), \\
c_3(z, x, y) &= U_{V_z x y} - V_z^2 U_x y = U_{z0x y} - z \circ (z \circ U_x y), \\
c_4(z, x) &= (V_z x)^2 - V_z^2 x^2 = (z \circ x)^2 - z \circ (z \circ x^2), \\
c_5(z, x, y) &= U_x U_z y - U_z U_x y, \\
c_6(z, x, y) &= U_z V_x y - V_x U_z y = U_z (x \circ y) - x \circ U_z y, \\
c_7(z, x) &= (U_z x)^2 - U_z^2 x^2,
\end{align*}
\]

since $c_5(z, J, J) = 0$ and (0.2)(iii) imply $U_{U_z x} = U_z U_x U_z = U_z^2 U_x$, for any $x \in J$.

We claim that $z \in C(J)$ also satisfies $c_9(z, J, J) = 0$, where $c_9(z, x, y) = \{ U_z x, y, x \} - 2 U_z U_x y$, which readily follows from the fact that $U_z \in \Gamma(J)$. If $J$ is nondegenerate then also $c_9(z, J) = 0$ for $c_9(z, x) = U_z x^2 - U_x z^2$, since $c_9(z, x) = c_5(z, x, 1)$ and $C(J) \subseteq C(\hat{J})$ by [3, Corollary 1].

1. Characteristic 2 phenomena

1.1. We remark that, by applying (0.6) to $T = 2 \text{Id}_J$ in a nondegenerate Jordan algebra $J$, $2x = 0$ if and only if $2^n x = 0$ for a positive integer $n$.

On the other hand, if $2x = 0$ and $x \in \text{Ann}(\text{Ker} 2 \text{Id}_J)$, then $x = 0$: $x \in \text{Ker} 2 \text{Id}_J \cap \text{Ann}(\text{Ker} 2 \text{Id}_J) = 0$ since $J$ is semiprime and Ker $2 \text{Id}_J$ is an ideal of $J$ by (0.6)(ii).

1.2. Remark. In a nondegenerate Jordan algebra $J$, $U_y x = U_y (-x)$, i.e., $U_y 2x = 0$, for any $x \in J$, $y \in \text{Ker} 2 \text{Id}_J$: $U_y 2x = 2 U_y x = 0$ since $U_y x \in \text{Ker} 2 \text{Id}_J$ by (0.6)(ii).

1.3. Lemma. Let $J$ be a nondegenerate Jordan algebra, and let $a, b \in J$. If $(U_a - U_b) J \subseteq \text{Ann}_J (2 \text{Id}_J)$, then $a - b \in \text{Ann}_J (2 \text{Id}_J)$.

Proof. (I) $U_{[a, J, b]} J \subseteq \text{Ann}_J (2 \text{Id}_J)$: for any $y \in 2 \text{Id}_J$, $x, z \in J$, using (0.2)(xi),
\[ U_y U_{\{a,x,b\} z} = U_y \left[ U_a U_x U_b z + U_b U_x U_a z + \{ a, x, U_b \{ x, a, z \} \} - \{ U_a U_b, z, b \} \right] \]
\[ = U_y \left[ U_a U_x U_a z + U_a U_x U_a z + \{ a, x, U_a \{ x, a, z \} \} - \{ U_b U_a, b, U_z \} \right] \]
\[ \text{(for } t \in J, U_y U_t - U_b t \in \text{Ann}_J (\text{Ker} \ 2 \text{Id}_J), \text{ which is an ideal of } J) \]
\[ = U_y \left[ 2U_a U_x U_a z + \{ a, x, U_a \{ x, a, z \} \} - \{ U_b U_a, x, U_z \} \right] \]
\[ \text{(by } (0.2)(\text{xii}) \) \]
\[ = U_y \left[ 2U_a U_x U_a z + \{ a, x, U_a \{ x, a, z \} \} - \{ U_b U_a, x, U_z \} \right] \]
\[ \text{(for } t \in J, U_y U_t - U_b t \in \text{Ann}_J (\text{Ker} \ 2 \text{Id}_J), \text{ which is an ideal of } J) \]
\[ = U_y \left[ 4U_a U_x U_a z \right] \]
\[ \text{(by } (0.2)(\text{xiii}) \) \]
\[ = 0 \]

by (1.2).

(II) \( \{ a, J, b \} \subseteq \text{Ann}_J (\text{Ker} \ 2 \text{Id}_J) \): using (0.2)(iii), for any \( y \in \text{Ker} \ 2 \text{Id}_J \), \( x \in J \), \( U_y \{ a, x, b \} = U_y U_{\{ a, x, b \} y} = 0 \) by (I), hence \( U_y \{ a, x, b \} = 0 \) by nondegeneracy of \( J \).

(III) \( U_{a-b} J \subseteq \text{Ann}_J (\text{Ker} \ 2 \text{Id}_J) \): for any \( y \in \text{Ker} \ 2 \text{Id}_J \), \( x \in J \),

\[ U_y U_{a-b} x = U_y \left[ U_a x + U_b x - \{ a, x, b \} \right] = U_y \left[ U_a x + U_a x - \{ a, x, b \} \right] \]
\[ = U_y \left[ 2U_a x - \{ a, x, b \} \right] = 0 \]

by (1.2) and (II).

Finally, for any \( y \in \text{Ker} \ 2 \text{Id}_J \), \( U_{a-b} y = U_{a-b} U_y U_{a-b} \) (by (0.2)(iii)) \( = 0 \), by (III), hence \( U_{a-b} y = 0 \) by nondegeneracy, and \( a - b \in \text{Ann}_J (\text{Ker} \ 2 \text{Id}_J) \) (0.4). \( \square \)

Under the assumption of characteristic 2, (1.3) turns into the following result of independent interest, though it is not explicitly needed in the sequel.

1.4. Corollary. Let \( J \) be a nondegenerate Jordan algebra of characteristic two \((2J = 0)\), \( a, b \in J \). If \( U_a = U_b \), then \( a = b \).

Proof. Use (1.3) and the fact that \( \text{Ann}_J (\text{Ker} \ 2 \text{Id}_J) = \text{Ann}_J (J) = 0 \) by nondegeneracy. \( \square \)

2. Martindale-like covers

2.1. When \( J \) and \( Q \) are Jordan algebras such that \( J \) is a subalgebra of \( Q \), we will say that \( Q \) is a cover of \( J \). We will consider the following ideal absorption properties for a cover \( Q \) of \( J \):

- the outer ideal absorption properties:
  (IA1) for any \( \neq q \in Q \) there exists an essential ideal \( I \) of \( J \) such that \( \neq U_I q \subseteq J \),
  (IA2) for any \( q \in Q \) there exists an essential ideal \( I \) of \( J \) such that \( I \circ q \subseteq J \),
  and the inner ideal absorption property:
  (IA3) for any \( q \in Q \) there exists an essential ideal \( I \) of \( J \) such that \( U_q I \subseteq J \).

A cover \( Q \) of \( J \) will be said a Martindale-like cover if it satisfies (IA1)–(IA3).

2.2. Remark. Assuming (IA1), condition (IA2) can be replaced by

(IA2') For any \( q \in Q \) there exists an essential ideal \( I \) of \( J \) such that \( \{ q, I, I \} \subseteq J \).
Indeed, (0.2)(i) implies that \(\{q, I, I\} \subseteq (q \circ I) \circ I + \{I, q, I\} \subseteq J\) when \(I\) is the intersection of the ideals in (IA1) and (IA2) for the element \(q\). Conversely, if \(I\) and \(L\) are essential ideals satisfying \(U_I q + \{q, L, L\} \subseteq J\), then \(K := U_I \cap L(I \cap L)\) is an essential ideal of \(J\) by [12, 1.2(a)], and (0.2)(ii) yields

\[
q \circ K = q \circ U_I \cap L(I \cap L) \subseteq \{q, I \cap L, I \cap L\} \circ (I \cap L) + (I \cap L) \circ U_I \cap L q
\]

\[
\subseteq \{q, L, L\} \circ J + J \circ U_I q \subseteq J.
\]

2.3. Remark. Notice that any cover \(Q\) of \(J\) satisfying (IA1) is tight over \(J\), i.e., any nonzero ideal of \(Q\) hits \(J\). As a consequence, if \(J\) is nondegenerate then \(Q\) is also nondegenerate (cf. [9, 2.9(iii)]). Similarly, \(J\) is free of 2-torsion if and only if \(Q\) is free of 2-torsion, using tightness, (0.6)(ii), and the obvious fact that \(\text{Ker} 2 \text{Id}_J = J \cap \text{Ker} 2 \text{Id}_Q\).

In the next result we go further in the tightness of Martindale-like covers, in fact of covers just satisfying (IA1).

2.4. Proposition. Let \(J\) be a nondegenerate Jordan algebra and \(Q\) be a cover of \(J\) satisfying (IA1). Then, for any \(0 \neq q \in Q\), and any essential ideal \(L\) of \(J\), \(U_L q \neq 0\) and \(U_q L \neq 0\). If \(J\) has not 2-torsion, then also \(L \circ q \neq 0\).

Proof. Given \(0 \neq q \in Q\), let \(I\) be an essential ideal of \(J\) such that \(0 \neq U_I q \subseteq J\), so that we can take \(x \in I\) such that \(0 \neq U_x q\). For any essential ideal \(L\) of \(J\), \(0 \neq U_{U_x q} L\) since \(\text{Ann}_J(L) = 0\). But \(U_{U_x q} L = U_x U_{U_x} U_x L\) by (0.2)(iii) \(\subseteq U_x U_q L\), which implies \(U_q L \neq 0\).

If \(U_L q = 0\), then \(U_{U_L q} q = 0\) in the algebra \(Q[t]\) of polynomials over \(Q\). Notice that \(Q\) is nondegenerate by (2.3), which readily implies that \(Q[t]\) is also nondegenerate. For any \(h \in L[t]\), let \(a := U_h U_q h \in Q[t]\). By (0.2)(iii),

\[
U_a Q[t] = U_h U_q U_h U_q U_h Q[t] = U_{U_h q} U_q U_h Q[t] = 0
\]

since \(U_h q = 0\), hence \(a = 0\) by nondegeneracy. For \(x, y \in L\), the coefficient of \(t\) in \(U_{x+y} U_q (x + ty)\) is \(U_x U_q y + U_{x+y} U_q x\), which is then zero. But, on the other hand, \(U_{x+y} U_q x = \{U_x q, q, y\}\) by (0.2)(iv) \(\neq 0\), hence we obtain \(U_{U_q} L = 0\). Fixing \(x \in L\) such that \(U_q x \neq 0\), we then have \(0 \neq U_{U_q} L = U_q U_x U_q L \subseteq U_q U_L U_q L\), which contradicts \(U_L U_q L = 0\). This shows \(U_L q \neq 0\).

Finally, in case \(J\) has not 2-torsion, \(0 \neq 2 U_L q \subseteq L \circ (L \circ q) + L^2 \circ q\) (by (0.2)(v)) \(\subseteq L \circ (L \circ q) + L \circ q\) implies \(L \circ q \neq 0\).

As a consequence, we can choose a single ideal to nontrivially absorb any given finite set of elements in the cover.

2.5. Corollary. Let \(J\) be a nondegenerate Jordan algebra and \(Q\) be a cover of \(J\) satisfying (IA1). Given a finite set \(q_1, \ldots, q_n\) of nonzero elements in \(Q\), there exists an essential ideal \(I\) of \(J\) such that \(0 \neq U_I q_i \subseteq J\), for all \(i = 1, \ldots, n\).

If \(Q\) also satisfies (IA2) and/or (IA3), then the ideal \(I\) above can also be assumed to satisfy \(I \circ q_i + \{q_i, I, I\} \subseteq J\) (with \(0 \neq I \circ q_i\) in case \(J\) has not 2-torsion), and/or \(0 \neq U_q I \subseteq J\), respectively, for all \(i = 1, \ldots, n\).
Proof. Apply (2.4) and (2.2) together with the fact that the finite intersection of essential ideals is also essential. □

2.6. If $J$ is a nondegenerate Jordan algebra without 2-torsion, a cover $Q$ of $J$ is a Martindale-like cover of $J$ if and only if for any $0 \neq q \in Q$ there exists an essential ideal $I$ of $J$ such that $0 \neq I \circ q \subseteq J$ (when $1/2 \in \Phi$, this just amounts to saying that $Q$ is a Jordan algebra of Martindale-like quotients of $J$ with respect to the filter of all essential ideals of $J$ in the sense of [4, 5.1]).

Indeed, a Martindale-like cover of $J$ satisfies (IA2) and, moreover, $I \circ q \neq 0$ for any $0 \neq q \in Q$ by (2.4) in the absence of 2-torsion. Conversely, assume that, for any $0 \neq q \in Q$, there exists an essential ideal $I$ of $J$ such that $0 \neq I \circ q \subseteq J$. Clearly, $M := 2I$ is an essential ideal of $J$ and

$$U_M q = 2(2U_I q) \subseteq 2(I \circ (I \circ q) + I^2 \circ q) \quad \text{(by (0.2)(v))}$$

$$\subseteq I \circ J + I \circ q \subseteq J.$$

Moreover, for $x \in I$ such that $x \circ q \neq 0$, we have, by sturdiness of $I$ (cf. (0.4)),

$$0 \neq U_I (x \circ q) \subseteq \{I \circ x, q, I\} + x \circ U_I q \quad \text{(by (0.2)(vii))}$$

$$\subseteq U_I q + x \circ U_I q,$$

which implies $U_I q \neq 0$, hence $0 \neq 4U_I q = U_M q$, and we have established (IA1).

Furthermore, $M \circ q \subseteq J$, and $\{q, M, M\} \subseteq J$ as in the proof of (2.2). We now just need to show (IA3). Let $L$ be an essential ideal of $J$ such that $q^2 \circ L \subseteq J$, and let $K := U_M M \cap L$, which is an essential ideal of $J$ by [12, 1.2(a)], and we will show $U_q 2K \subseteq J$. First, $q \circ U_M M \subseteq M$: for any $x, y \in M$,

$$q \circ U_q y = \{q, x, y\} \circ x - y \circ U_q q \quad \text{(by (0.2)(ii))}$$

$$\subseteq \{q, M, M\} \circ M + M \circ U_M q \subseteq J \circ M \subseteq M.$$

Thus, by (0.2)(v), $U_q 2K = 2U_q K \subseteq q \circ (q \circ K) + q^2 \circ K \subseteq q \circ (q \circ U_M M) + q^2 \circ L \subseteq q \circ M + q^2 \circ L \subseteq J$.

3. Center inheritance in Martindale-like covers

The proof of the next result is just the quadratic version of the proof of [1, 4.1]. In the generalization a factor 2 comes out.

3.1. Lemma. Let $J$ be a nondegenerate Jordan algebra, $Q$ be a cover of $J$ satisfying (IA1) and (IA2), and $z \in C(J)$. Then, $2z \circ (p \circ q) = 2(z \circ p) \circ q$, for any $p, q \in Q$, i.e., $2V_z V_q = 2V_q V_z$, for any $q \in Q$.

Proof. (I) For any $q \in Q$ and any $x \in J$ such that $x \circ q \in J$, $z \circ (x \circ q) = (z \circ x) \circ q$:

Use (2.5) to find an essential ideal $I$ of $J$ such that $U_I q + \{q, I, I\} \subseteq J$. For any $y_1, y_2 \in I$, and $t \in \hat{J}$,
\begin{align*}
\{z \circ t, q, U_{y_1} y_2\} &= \{z \circ t, \{q, y_1, y_2\}, y_1\} - \{z \circ t, y_2, U_{y_1} q\} \quad \text{(by (0.2)(vi))} \\
&= z \circ \{t, \{q, y_1, y_2\}, y_1\} - z \circ \{t, y_2, U_{y_1} q\} \\
&= z \circ \{t, q, U_{y_1} y_2\} \quad \text{(by (0.2)(vi))} \\
&= z \circ \{t, q, U_{y_1} y_2\} \\
\end{align*}

since \(\{q, y_1, y_2\}, U_{y_1} q \in J, z \in C(J),\) and \(C(J) \subseteq C(\hat{J})\) [3, Corollary 1]. Now, if \(K := U_I I\) and \(y \in K,\)

\begin{align*}
U_y((z \circ x) \circ q) &= \{y \circ (z \circ x), q, y\} - (z \circ x) \circ U_y q \quad \text{(by (0.2)(vii))} \\
&= z \circ \{y \circ x, q, y\} - z \circ (x \circ U_y q) \quad \text{(since \(x, y, U_y q \in J, z \in C(J)\))} \\
&= z \circ U_y(x \circ q) \quad \text{(by (0.2)(vii))} \\
&= U_y(z \circ (x \circ q))
\end{align*}

since \(y, x \circ q \in J\) and \(z \in C(J).\) We have shown that \(U_K ((z \circ x) \circ q - z \circ (x \circ q)) = 0,\) which implies \((z \circ x) \circ q - z \circ (x \circ q) = 0\) by (2.4) since \(K\) is an essential ideal of \(J\) by [12, 1.2(a)].

(II) Let \(q \in Q,\) and \(I\) be an essential ideal of \(J\) satisfying \(I \circ q + U_I q + \{q, I, I\} \subseteq J,\) that can be found by (2.5). Then \((z \circ q) \circ x = z \circ (q \circ x)\) for any \(x \in U_I I:\)

\begin{align*}
(z \circ q) \circ x &= 2[z, q, x] - z \circ (q \circ x) + (z \circ x) \circ q \quad \text{(by linearized (0.2)(v))} \\
&= 2[z, q, x] \quad \text{(by (I))} \\
&= \{z \circ 1, q, x\} \\
&= z \circ \{1, q, x\} \quad \text{(by (I))} \\
&= z \circ (q \circ x).
\end{align*}

(III) For any \(p, q \in Q,\) \(2(z \circ p) \circ q = 2z \circ (p \circ q)\):

By (2.5), we can find an essential ideal \(I\) of \(J\) such that \(I \circ p + U_I p + \{p, I, I\} + I \circ q + U_I q + \{q, I, I\} + I \circ (p \circ q) + U_I (p \circ q) + \{p \circ q, I, I\} \subseteq J.\) Let \(K := U_I I\) and \(L := U_K K.\) Notice that

\[U_L q \subseteq K.\] (2)

Indeed, \(U_L q\) is spanned by elements of the form \(U_{U_a b} q\) and \(U_{a'b'} q, U_{a'b} b\), where \(a, b, a', b' \in K,\) and

\[U_{U_a b} q = U_a U_{b} q \quad \text{(by (0.2)(iii))} \]
\[\subseteq U_K U_K U_I q \subseteq U_K U_K J \subseteq K,\]

whereas

\[\{U_{a'b'} q, U_{a'b} b\} \subseteq \{K, q, U_{a'b} b\} \subseteq \{K, \{q, a, b\}, a\} + \{K, b, U_{a'b} q\} \quad \text{(by (0.2)(vi))} \]
\[\subseteq \{K, \{q, I, I\}, K\} + \{K, K, U_I q\} \subseteq \{K, J, K\} + \{K, K, J\} \subseteq K.\]
Now, for any \( y \in L \),

\[
U_y (2(z \circ p) \circ q) = 2\left[ \{ y \circ (z \circ p), q, y \} - (z \circ p) \circ U_y q \right] \quad \text{(by (0.2)(vii))}
\]

\[
= 2\left[ \{ z \circ (y \circ p), q, y \} - z \circ (p \circ U_y q) \right] \quad \text{(by (II) since \( y, U_y q \in K \) by (2))}
\]

\[
= 2\left[ z \circ \{ y \circ (p \circ q), q, y \} - z \circ (p \circ U_y q) \right] \quad \text{(by (1) since \( y \circ p \in J \) and \( y \in K \))}
\]

\[
= 2z \circ U_y (p \circ q) \quad \text{(by (0.2)(viii))}
\]

\[
= z \circ \left[ (y \circ (p \circ q)) \circ y - y^2 \circ (p \circ q) \right] \quad \text{(by (0.2)(v))}
\]

\[
= \left[ (y \circ (z \circ (p \circ q))) \circ y - y^2 \circ (z \circ (p \circ q)) \right] \quad \text{(by (II))}
\]

\[
= 2U_y (z \circ (p \circ q)) \quad \text{(by (0.2)(v))}
\]

\[
= U_y (2z \circ (p \circ q)).
\]

We have shown \( U_L (2(z \circ p) \circ q - 2z \circ (p \circ q)) = 0 \), which implies \( 2(z \circ p) \circ q - 2z \circ (p \circ q) = 0 \) by (2.4), since \( L \) is an essential ideal of \( J \) by [12, 1.2(a)].

3.2. Theorem. Let \( J \) be a nondegenerate Jordan algebra, \( Q \) be a cover of \( J \) satisfying (IA1) and (IA2), and \( z \in C(J) \). Then,

\[
2c_i(z, Q, Q) = 0, \quad \text{for } i = 1, 2, 3, 5, 6, 8 \quad \text{and} \quad 2c_i(z, Q) = 0, \quad \text{for } i = 4, 7, 9.
\]

Proof. By (1.1), it is enough to prove \( 2^n c_i(z, Q, \ldots) = 0 \) for some positive integer \( n \). On the other hand, we claim that, for any \( c_i, i = 1, \ldots, 9 \), there exists a positive integer \( n \) such that \( 2^n c_i \) can be expressed in terms of 2 times “\( \circ \)-products.” As an example, for \( p, q \in Q \), using (0.2)(v) yields

\[
8c_3(z, p, q) = 8\left[ U_{zop} q - z \circ (z \circ U_p q) \right]
\]

\[
= 4\left[ (z \circ p) \circ ((z \circ p) \circ q) - (z \circ p)^2 \circ q - z \circ \left[ p \circ (p \circ q) - p^2 \circ q \right] \right]
\]

\[
= 2\left[ 2(z \circ p) \circ ((z \circ p) \circ q) - \left[ (z \circ p) \circ (z \circ p) \right] \circ q \right.
\]

\[
- z \circ \left[ 2p \circ (p \circ q) - (p \circ p) \circ q \right] \].
\]

Now, our result follows from (3.1).

The above result is enough to obtain a generalization of [1, 4.1] for 2-torsion free Jordan algebras.

3.3. Corollary. Let \( J \) be a nondegenerate Jordan algebra without 2-torsion, \( Q \) be a cover of \( J \) satisfying (IA1) and (IA2). Then, \( C(J) \subseteq C(Q) \).

Proof. Use (0.7), (3.2), and the fact that \( Q \) has not 2-torsion by (2.3).

3.4. Corollary. Let \( J \) be a nondegenerate Jordan algebra, \( Q \) be a cover of \( J \) satisfying (IA1) and (IA2). Then, \( 2C(J) \subseteq C(Q) \).
**Proof.** For any $z \in C(J)$, and any $i = 1, \ldots, 7$, $c_i(2z, Q, \ldots) = 2^k c_i(z, Q, \ldots)$ (for some positive integer $k$) $= 0$ by (3.2), hence $2z \in C(Q)$ by (0.7). □

In order to extend (3.3) to the general quadratic case we will proceed in two steps. In the first one we will study the centrality in $Q$ of the operator $V_z$ for a central element of $J$, and show that only conditions (IA1) and (IA2) are needed. Our first result is the natural generalization of [3, Corollary 2].

**3.5. Lemma.** In a nondegenerate Jordan algebra $J$, $C(J) \circ \ker 2 \Id_J = 0$.

**Proof.** Let $z \in C(J)$, $x \in \ker 2 \Id_J$, and $y \in J$. By (0.2)(viii),

$$U_{zox} y = U_z U_x y + U_x U_z y + z \circ U_x (y \circ z) - \{U_z x, y, x\}$$

$$= 2U_z U_x y + (z \circ z) \circ U_x y - 2U_x U_z y = 2z^2 \circ U_x y \in 2J.$$

But $4U_{zox} y = U_{zox} y = 0$, hence $U_{zox} y = 0$ by (1.1). We have shown $U_{zox} J = 0$, hence $z \circ x = 0$ by nondegeneracy. □

The next two results are meant to “lift” (3.5) to covers satisfying (IA1).

**3.6. Lemma.** If $J$ is a nondegenerate Jordan algebra and $Q$ is a cover of $J$ satisfying (IA1), then $C(J) \circ \ker 2 \Id_Q = 0$.

**Proof.** Let $z \in C(J)$, $q \in \ker 2 \Id_Q$, and let $I$ be an essential ideal of $J$ such that $U_I q \subseteq J$. Notice that $L := I \cap (2J + \ker 2 \Id_J)$ is an essential ideal of $J$ by (0.6)(iii). For any $y \in L$, using (0.2)(vii),

$$U_y (z \circ q) = \{y \circ z, q, y\} - z \circ U_y q.$$

But writing $y = 2a + b$ for $a \in J$, $b \in \ker 2 \Id_J$, $\{y \circ z, q, y\} = \{2a \circ z, q, y\} + \{b \circ z, q, y\} = \{a \circ z, 2q, y\}$ (since $b \circ z = 0$ by (3.5)) = 0 since $q \in \ker 2 \Id_Q$. On the other hand, $2U_y q = U_{2y} q = 0$, hence $U_y q \in J \cap \ker 2 \Id_Q = \ker 2 \Id_J$, so that $z \circ U_y q = 0$ by (3.5). We have shown $U_L (z \circ q) = 0$, which implies $z \circ q = 0$ by (2.4). □

**3.7. Lemma.** If $J$ is a nondegenerate Jordan algebra and $Q$ is a cover of $J$ satisfying (IA1), then $C(J) \circ Q \subseteq \Ann_Q(\ker 2 \Id_Q)$.

**Proof.** Let $z \in C(J)$, $p \in \ker 2 \Id_Q$, and $q \in Q$. By (0.2)(vii),

$$U_p (z \circ q) = \{p \circ z, q, p\} - z \circ U_p q = 0$$

by (3.6) since $p, U_p q \in \ker 2 \Id_Q$. This shows $z \circ q \in \Ann_Q(\ker 2 \Id_Q)$ (cf. (0.4) since $Q$ is nondegenerate by (2.3)). □

**3.8. Theorem.** If $J$ is a nondegenerate Jordan algebra and $Q$ is a cover of $J$ satisfying (IA1) and (IA2), then $V_z \in \Gamma(Q)$ for any $z \in C(J)$, equivalently,

$$c_1(z, Q, Q) = c_2(z, Q, Q) = c_3(z, Q, Q) = c_4(z, Q) = 0.$$
Proof. Notice that, $c_1(z, Q, Q)$, $c_2(z, Q, Q)$, $c_3(z, Q, Q)$, $c_4(z, Q)$ are contained in $\text{Ann}_Q(\text{Ker} 2 \text{Id}_Q)$ by (3.7), since they lie in the ideal of $Q$ generated by $z \circ Q$. Now, the result follows by using (3.2) and (1.1). □

3.9. Theorem. Let $J$ be a nondegenerate Jordan algebra, $Q$ be a cover of $J$ satisfying (IA1) and (IA2), and $z \in C(J)$. Then

(i) $\{z, p, q\} = \{z, q, p\} = \{p, z, q\}$, for any $p, q \in Q$,
(ii) $c_6(z, Q, Q) = c_7(z, Q) = c_9(z, Q) = 0$.

Proof. (i) By (0.2)(i) and (3.8),

$\{z, p, q\} = -\{p, z, q\} + (p \circ z) \circ q = -\{p, z, q\} + p \circ (z \circ q) = [z, q, p], \quad \text{and}$

$\{z, p, q\} = -\{z, q, p\} + (p \circ q) \circ z = -\{z, q, p\} + p \circ (q \circ z) = [p, z, q].$

(ii) If $c_9(z, Q) = 0$ then, for any $p \in Q$,

$c_7(z, p) = (U_z p)^2 - U_z p^2 = (U_z p)^2 - U_z U_p z^2 = 0$

by (0.2)(ix). Thus we will show $c_6(z, Q, Q) = c_9(z, Q) = 0$, and we just need to prove that $c_6(z, Q, Q), c_9(z, Q) \subseteq \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q)$ by (3.2) and (1.1). For any $p, q \in Q$, $y \in \text{Ker} 2 \text{Id}_Q$,

$U_y c_6(z, p, q) = U_y (U_z (p \circ q) - p \circ U_z q)$

$= U_y (U_z (p \circ q) + p \circ U_z q) \quad (\text{by (1.2) since } Q \text{ is nondegenerate by (2.3)})$

$= U_y ([z \circ p, q, z]) \quad (\text{by (0.2)(vii)})$

$= 0$

since $z \circ p \in \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q)$ (by (3.7)) implies $\{z \circ p, q, z\} \in \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q)$. Also

$U_y c_9(z, p) = U_y (U_z p^2 - U_p z^2)$

$= U_y (U_z p^2 + U_p z^2) \quad (\text{by (1.2) since } Q \text{ is nondegenerate by (2.3)})$

$= U_y ([z \circ p]^2 - z \circ U_p z) \quad (\text{by (0.2)(x)})$

$= 0$

since $z \circ Q \subseteq \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q)$ by (3.7). □

3.10. Lemma. Let $J$ be a nondegenerate Jordan algebra, $Q$ be a cover of $J$ satisfying (IA1) and (IA2), and $z \in C(J)$. Then

$U_z U_p q = -U_p U_z q + \{U_z p, q, p\}, \quad \text{for any } p, q \in Q.$
Proof. Notice that \( c_{10}(z, p, q) := U_z U_p q + U_p U_z q - \{U_z p, q, p\} = -c_5(z, p, q) - c_8(z, p, q) \), hence \( 2c_{10}(z, p, q) = 0 \) by (3.2). Using (1.1), we just need to prove \( c_{10}(z, Q, Q) \subseteq \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q) \). But using (0.2)(viii) yields \( c_{10}(z, p, q) = U_z \circ p - z \circ U_p (q \circ z) \in \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q) \) by (3.7).

Notice that, up to now, only the outer ideal absorption properties have been needed. The next results, aimed at studying the centrality of \( U_z \), will make explicit use of inner ideal absorption.

3.11. Lemma. Let \( J \) be a nondegenerate Jordan algebra, \( Q \) be a Martindale-like cover of \( J \), and \( z \in C(J) \). Then \( c_5(z, J, Q) = 0 \).

Proof. Let \( x \in J, q \in Q \), and \( I \) be an essential ideal of \( J \) such that \( U_q I \subseteq J \), \( U_I (U_x U_q) \subseteq J \), and \( U_I (U_x U_q) \subseteq J \), which exists by (2.5). For any \( y \in I, a \in J \),

\[
U_{U_y U_z U_x} a = U_y U_x U_q U_z a \quad \text{(by (0.2)(iii))}
\]
\[
= U_y U_x U_q U_z U_a \quad \text{(since \( U_q U_x U_z a \subseteq U_q I \subseteq J \), and \( U_z \in \Gamma(J) \))}
\]
\[
= U_y U_x U_q U_z U_a \quad \text{(since \( U_y a \in J \), and \( U_z \in \Gamma(J) \))}
\]
\[
= U_y U_x U_q U_z U_a \quad \text{(by (0.2)(iii))}.
\]

By (1.3), we have \( U_y c_5(z, x, q) = U_y U_z U_x q - U_y U_x U_z q = 0 \in \text{Ann}_J(\text{Ker} 2 \text{Id}_J) \). But \( 2U_y c_5(z, x, q) = U_y 2c_5(z, x, q) = 0 \) by (3.2), hence \( U_y c_5(z, x, q) = 0 \) by (1.1).

We have shown that \( U_I c_5(z, x, q) = 0 \), which implies \( c_5(z, x, q) = 0 \) by (2.4).

3.12. Lemma. Let \( J \) be a nondegenerate Jordan algebra, \( Q \) be a Martindale-like cover of \( J \), and \( z \in C(J) \) for any \( x, y \in J, q \in Q \),

\[
(\text{i)} \quad \{U_z x, q, x\} = 2U_z U_x q \in \mathbb{Q} \text{, so that } \{U_z x, q, x\} \in \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q),
\]
\[
(\text{ii)} \quad \{U_z x, q, y\} + \{x, q, U_z y\} \in \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q).
\]

Proof. By (3.10), \( U_z U_x q = -U_x U_z q + \{U_x, q, x\} \), which implies (i) using (3.11), (2.3), and (1.2), whereas (ii) follows by linearizing (i).

3.13. Lemma. Let \( J \) be a nondegenerate Jordan algebra, \( Q \) be a Martindale-like cover of \( J \), \( z \in C(J) \), \( q \in Q \), and \( I \) be an essential ideal of \( J \) such that \( U_q I + U_I q \subseteq J \). Then,

\[
\{U_z q, y, q\} = 2U_z U_q y \in \mathbb{Q}, \quad \text{so that } \{U_z q, y, q\} \in \text{Ann}_Q(\text{Ker} 2 \text{Id}_Q),
\]

for any \( y \in I \).

Proof. For any \( x \in I, u \in \text{Ker} 2 \text{Id}_Q \),

\[
U_u U_x \{U_z q, y, q\} = U_u \left( \{x, U_z q, \{y, q, x\} \} - \{U_x U_z q, q, y\} \right) \quad \text{(by (0.2)(xvi))}
\]
\[
= U_u \left( U_z \{x, q, \{y, q, x\} \} - \{U_x U_z q, q, y\} \right) \quad \text{(by applying (3.11) to both terms since } \{y, q, x\} \in U_I q \subseteq J)\]
= U_u[\{U_xq, q, U_z y\} - \{U_xq, q, U_z y\}] - \{U_xq, q, U_z y\}
(by (3.12)(ii) since \(U_xq \in U_I q \subseteq J\), and (1.2))
= U_u[\{x, U_q x, U_z y\}] (by (0.2)(iv))
= U_u[\{x, q, \{y, q, x\}\}] - \{x, U_q x, y\} - \{x, q, \{y, q, x\}\}
(since \(U_q x \in U_q I \subseteq J\) and \(U_z \in \Gamma(J)\))
= U_u[\{x, q, \{y, q, x\}\}] - \{x, U_q x, y\} (by (0.2)(xiv))
= U_u[\{y, q, x\}] - \{x, U_q x, y\}
(by (3.9)(i) and (3.14) since \(U_xq \in I\))
= U_uU_zU_xU_q y = 0

by (1.2). Also \(U_uU_x2U_zU_q y = 0\) by (1.2), hence we have shown

\[ U_I c_8(z, q, y) = U_I \left[ \{U_xq, q, y\} - 2U_zU_q y \right] \subseteq \text{Ann}_Q(\text{Ker } 2 \text{Id}_Q). \]

But \(2U_I c_8(z, q, y) = U_I 2c_8(z, q, y) = 0\) by (3.2), so that \(U_I c_8(z, q, y) = 0\) by (1.1). Therefore \(c_8(z, q, y) = 0\) by (2.4).

\[ \text{3.14. Lemma. Let } J \text{ be a nondegenerate Jordan algebra, } Q \text{ be a Martindale-like cover of } J, z \in C(J), q \in Q, \text{ and } I \text{ be an essential ideal of } J \text{ such that } U_q I + U_I q \subseteq J. \text{ Then, for any } y \in I, c_5(z, q, y) = 0. \]

\[ \text{Proof. By (3.10), } U_q U_z y = -U_z U_q y + \{U_zq, y, q\} = U_z U_q y \text{ using (3.13).} \]

\[ \text{3.15. Proposition. Let } J \text{ be a nondegenerate Jordan algebra, } Q \text{ be a Martindale-like cover of } J, \text{ and } z \in C(J). \text{ Then } c_5(z, Q, Q) = 0. \]

\[ \text{Proof. Let } p, q \in Q \text{ and } I \text{ be an essential ideal of } J \text{ such that } U_I q + \{q, I, I\} + U_I p + U_p I \subseteq J, \text{ which exists by (2.5). If we take } K := U_I I, \text{ we also have that } K \text{ is an essential ideal of } J \text{ [12, 1.2(a)], and } U_K q \subseteq I, \text{ as in (III)(2) of the proof of (3.1).} \]

\[ \text{For any } x \in K, \]

\[ U_x U_z U_p q = U_{\{p, z, x\} q} - U_p U_z U_x q - \{p, z, U_x z \{p, q\}\} + \{U_p U_z x, q, x\} \]

\[ = U_{\{z, p, x\} q} - U_z U_p U_x q - \{z, p, U_x \{p, z, q\}\} + \{U_z U_p x, q, x\} \]

\[ \text{(by (3.9)(i) and (3.14) since } U_x q \in I) \]

\[ = U_x U_p U_z q \]

using again (0.2)(xi). We have shown that \(U_K c_5(z, p, q) = 0\), which implies that \(c_5(z, p, q) = 0\) by (2.4).

\[ \text{3.16. Theorem. Let } J \text{ be a nondegenerate Jordan algebra, } Q \text{ be a Martindale-like cover of } J. \text{ Then } C(J) \subseteq C(Q). \]

\[ \text{Proof. Put together (3.8), (3.9)(ii), and (3.15), and use (0.7).} \]
References