

IRREDUCIBLE REPRESENTATIONS OF THE EXCEPTIONAL CHENG-KAC SUPERALGEBRA

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To the memory of our dear friend Hyo Chul Myung

Abstract

We classify all conformal irreducible modules of finite type over the Cheng Kac superalgebra CK_6 .

1 Introduction

The study of Lie conformal superalgebras and their representations was initiated by V. Kac ([K2]) in view of their connections to the free fields realizations in conformal field theory. A complete classification of simple Lie conformal superalgebras of finite type was achieved in [FK]. The list consists of current Lie superalgebras, $Cur(\mathcal{G})$, where \mathcal{G} is a simple finite dimensional

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Lie superalgebra; four series of Lie conformal superalgebras of Cartan type and the exceptional Lie conformal superalgebra CK_6 .

For classification of representations of finite type of current Lie superalgebras and Lie superalgebras of Cartan type see [BKLR], [BKL1], [BKL2], [CK1].

In this paper we classify all conformal irreducible modules of finite type over the superalgebra CK_6 . We use this classification and the results of [MZ4] to classify conformal irreducible Jordan bimodules of finite type over the Jordan superalgebra $JCK(6)$.

For a different approach to this classification see [BKL2].

2 Basic Definitions

Let A be an arbitrary (not necessarily associative) algebra over \mathbf{C} . By a formal distribution

$$a(z) = \sum_{i \in \mathbf{Z}} a(i)z^{-i-1} \in A[[z]]$$

we mean a power series over A , which is infinite in both directions.

Two formal distributions $a(z)$, $b(z)$ are said to be mutually local if there exists an integer $N = N(a, b) \geq 0$ such that $a(z)b(w)(z-w)^N = b(w)a(z)(z-w)^N = 0$.

We will consider a countable family of operations:

$$a(z) \circ_n b(z) = \text{Res}_w a(w)b(z)(w-z)^n, \quad n \geq 0, \quad n \in \mathbf{Z}.$$

Here Res_w means the coefficient at w^{-1} .

If $a(z)$, $b(z)$ are mutually local then only finitely many products $a \circ_n b$ may be different from zero.

Definition 2.1 *A vector space $C \subseteq A[[z^{-1}, z]]$ is called a conformal algebra of formal distributions if $\partial C \subseteq C$, $\partial = \frac{d}{dz}$, $C \circ_n C \subseteq C$ for an arbitrary $n \geq 0$ and every two elements from C are mutually local.*

By Dong Lemma (see [K2]) if A is an associative or Lie algebra then for an arbitrary collection C of pairwise mutually local distributions the closure of C with respect to the action of ∂ and to all operations \circ_n , $n \geq 0$, is a conformal algebra of formal distributions.

Examples 2.1 (1) Let \mathcal{G} be an arbitrary algebra and let $A = \mathcal{G}[t, t^{-1}]$ be the algebra of Laurent polynomials over \mathcal{G} . For an arbitrary element $a \in \mathcal{G}$ let $\tilde{a} = \sum_{i \in \mathbf{Z}} (at^i)z^{-i-1} \in A[[z^{-1}, z]]$.

Any two formal distributions \tilde{a}, \tilde{b} are mutually local.

(2) Let $\mathcal{V}ir = \text{Der} \mathbf{C}[\mathbf{t}^{-1}, \mathbf{t}]$ be the (centerless) Virasoro algebra. The formal distribution

$$L = \sum_{i \in \mathbf{Z}} t^{i+1} \frac{d}{dt} z^{-i-2} \in \mathcal{V}ir[[z^{-1}, z]]$$

is mutually local with itself.

(3) Let $W = \langle t^{-1}, t, \frac{d}{dt} \rangle$ be the (associative) Weyl algebra of differential operators on $C[t^{-1}, t]$. Let $J_k = \sum_{i \in \mathbf{Z}} t^i (\frac{d}{dt})^k z^{-i-1}$, $k \geq 0$. Any two formal distributions J_k, J_l are mutually local.

In all three cases (1), (2) and (3) we can talk about the conformal algebras $\text{Cur}(\mathcal{G})$, $\mathcal{V}ir$, W respectively, generated by them.

Now we are ready to introduce an abstract definition of a conformal algebra.

Let C be a module over a polynomial algebra $\mathbf{C}[\partial]$, which is equipped with countably many binary bilinear operations $C \circ_n C \rightarrow C$, $n \geq 0$.

Definition 2.2 We say that (C, ∂, \circ_n) is an abstract conformal algebra if for arbitrary elements $a, b \in C$ arbitrary $n \geq 0$, we have:

- 1) $\partial(a \circ_n b) = \partial a \circ_n b + a \circ_n \partial b$,
- 2) $\partial a \circ_n b = -na \circ_{n-1} b$; for $n = 0$ the condition turns into $\partial a \circ_0 b = 0$.
- 3) (Locality) There exists an integer $N = N(a, b) \geq 0$ such that for an arbitrary $n \geq N$ we have $a \circ_n b = 0$.

Every conformal algebra of formal distributions is an abstract conformal algebra. The converse is also true: every conformal algebra can be realized as an algebra of formal distributions over some algebra of coefficients. Moreover, among these algebras of coefficients there is a universal one $\text{Coeff}(C)$.

Definition 2.3 We say that a conformal algebra C is a Lie (resp. associative, Jordan) algebra iff $\text{Coeff}(C)$ is a Lie (associative, Jordan) algebra.

Now let C be a Lie conformal algebra and let M be another $\mathbf{C}[\partial]$ -module. Suppose that we have a family of bilinear maps $C \circ_n M \subseteq M$, $n \geq 0$.

Definition 2.4 *We say that M is a conformal C -module if the null split extension $C + M$ is a Lie conformal algebra.*

As above, M can be realized as a space of formal distributions over $\text{Coeff}(M)$, where $\text{Coeff}(M)$ is a universal (with this property) Lie module over $\text{Coeff}(C)$.

Important Remark If there is a natural (and standard) way to arrange elements of a (super)algebra L in formal distributions then we will talk about L and modules over L even if we have in mind their conformal counterparts.

3 The Cheng-Kac Superalgebra

The exceptional conformal superalgebra CK_6 was introduced in [CK2] and in [GLS]. In [MZ1] we constructed, for an arbitrary associative commutative superalgebra R with an even derivation $d : R \rightarrow R$, a superalgebra $CK(R, d)$ so that $CK_6 \simeq CK(\mathbf{C}[\mathbf{t}^{-1}, \mathbf{t}], \frac{d}{d\mathbf{t}})$.

Lets recall the construction of $CK(R, d)$ from [MZ1].

Consider the associative Weyl algebra $W = \sum_{i \geq 0} R d^i$, where the variable d does not commute with a coefficient $a \in R$, but $da = ad + d(a)$. We will realize the $CK(R, d)$ as a superalgebra of 8×8 matrices over W .

The simple finite dimensional Lie superalgebra $P(n-1)$ is the superalgebra of $2n \times 2n$ matrices of the type $\begin{pmatrix} a & k \\ h & -a^t \end{pmatrix}$, where a, h, k are $n \times n$ matrices over \mathbf{C} , $\text{tr}(a) = 0$, $k^t = -k$, $h^t = h$. The superalgebras $P(n)$, $n \neq 3$, are centrally closed. However, $P(3)$ has a nontrivial central cover $\hat{P}(3)$. Its existence comes from the fact that the Lie algebra $K_4(\mathbf{C})$ of skew-symmetric 4×4 matrices is a direct sum of two ideals $K_4(\mathbf{C}) = \mathfrak{sl}_2(\mathbf{C}) \oplus \mathfrak{sl}_2(\mathbf{C})$. For an arbitrary element $k \in K_4(\mathbf{C})$ we consider its decomposition $k = k' + k''$ and let $\varphi(k) = k' - k''$. The universal central cover $\hat{P}(3)$ of $P(3)$ can be realized as a superalgebra of 8×8 -matrices over the polynomial algebra $\mathbf{C}[d]$ of the type

$$\begin{pmatrix} a & k \\ \varphi(k)d + h & -a^t \end{pmatrix} + \alpha d I_8,$$

where a, k, h are 4×4 matrices over \mathbf{C} , $\text{tr}(a) = 0$, $k = -k^t$, $h = h^t$, $\alpha \in \mathbf{C}$ and I_8 is the identity matrix.

The superalgebra $CK(R, d)$ is a subsuperalgebra of 8×8 matrices over W generated by $P(\hat{3})$ and by all matrices $\begin{pmatrix} e_{ij}(a) & 0 \\ 0 & -e_{ji}(a) \end{pmatrix}$ where $a \in R$, $1 \leq i \neq j \leq 4$.

The Cartan subalgebra H of $CK(R, d)$ consists of diagonal matrices

$$H = \{h = \text{diag}(a_1, \dots, a_4, -a_1, \dots, -a_4), a_i \in \mathbf{C}, \sum_{i=1}^4 a_i = 0\},$$

the even and the odd roots of the $CK(R, d)$ with respect to the action of H are:

$$\Delta_{\bar{0}} = \{w_i - w_j \mid 1 \leq i \neq j \leq 4\},$$

$$\Delta_{\bar{1}} = \{w_i + w_j, 1 \leq i \neq j \leq 4, -w_i - w_j, 1 \leq i, j \leq 4\}.$$

Notice that $w_i(a) = a_i$, $1 \leq i \leq 4$.

Thus, the superalgebra $CK(R, d)$ is graded by the abelian group

$$\sum_{i=1}^4 \mathbf{Z}w_i / \mathbf{Z}(w_1 + w_2 + w_3 + w_4),$$

$$CK(R, d) = \sum_{\alpha \in \Delta \cup \{0\}} CK(R, d)_\alpha.$$

Let us fix the notation for the following weight elements:

$$e_{w_i - w_j} = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, \quad e_{w_i - w_j}(a) = \begin{pmatrix} e_{ij}(a) & 0 \\ 0 & -e_{ji}(a) \end{pmatrix},$$

$$h_{w_i - w_j}(a) = \begin{pmatrix} e_{ii}(a) - e_{jj}(a) & 0 \\ 0 & e_{jj}(a) - e_{ii}(a) \end{pmatrix}, \quad q_{-w_i - w_j} = \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix},$$

$$q_{-w_i - w_j}(a) = \begin{pmatrix} 0 & 0 \\ e_{ij}(a) + e_{ji}(a) & 0 \end{pmatrix}, \quad q_{w_i + w_j} = \begin{pmatrix} 0 & e_{ij} - e_{ji} \\ \varphi(e_{ij} - e_{ji})d & 0 \end{pmatrix},$$

$a \in R$.

In [MZ3] it was shown that $CK(R, d)_{w_i - w_j} = e_{w_i - w_j}(R)$, $1 \leq i \neq j \leq 4$; $CK(R, d)_{-2w_i} = q_{-2w_i}(R)$; $CK(R, d)_{w_i + w_j} = [q_{w_i + w_k}, e_{w_j - w_k}(R)] + q_{-w_k - w_l}(R)$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

For an arbitrary element $a \in R$ consider the element

$$\begin{aligned} & [[e_{w_4 - w_1}(a), q_{w_3 + w_1}], q_{w_2 + w_1}] = \\ & \begin{pmatrix} e_{11}(da) + e_{22}(ad) + e_{33}(ad) + e_{44}(ad) & 0 \\ 0 & e_{11}(ad) + e_{22}(da) + e_{33}(da) + e_{44}(da) \end{pmatrix} \\ & = I_8(ad) - \begin{pmatrix} e_{11}(a') & 0 \\ 0 & -e_{11}(a') + I_4(a') \end{pmatrix}, \quad a' = [a, d] = d(a) \end{aligned}$$

We will denote the element on the right hand side as $Vir(a)$. The mapping $ad \rightarrow Vir(a)$ from $Rd \rightarrow \mathcal{V}ir(R)$ is an isomorphism of Lie algebras.

It was shown in [MZ3] that $CK(R, d)_0 = H \otimes R + \mathcal{V}ir(R)$.

Consider the functional

$$f : \sum_{i=1}^4 \mathbf{Z}w_i / \mathbf{Z}(\sum_{i=1}^4 w_i) \rightarrow \mathbf{Z}$$

given by $f(w_1) = 5$, $f(w_2) = -3$, $f(w_3) = 2$, $f(w_4) = -4$.

Notice that $f(\pm w_i \pm w_j) \neq 0$, unless $\pm w_i \pm w_j = 0$.

From now on we will denote $L = CK_6 = CK(\mathbf{C}[t^{-1}, t], \frac{d}{dt})$.

Note that $L_0 = H \otimes \mathbf{C}[t^{-1}, t] \rtimes \mathcal{V}ir(R) \leq Cur(sl_4) \rtimes \mathcal{V}ir(R) \leq L$.

The algebra L has a triangular decomposition $L = L_- + L_0 + L_+$, $L_- = \sum_{f(\alpha) < 0} L_\alpha$, $L_+ = \sum_{f(\alpha) > 0} L_\alpha$.

Let M be a conformal module of finite type over the Lie conformal algebra CK_6 . Then the subalgebra $sl_4 \subseteq L$ acts on M and the action of sl_4 commutes with the action of the polynomial algebra $\mathbf{C}[\partial]$. Hence M decomposes into a finite direct sum of eigenspaces with respect to the action of H ,

$$M = \sum_{\gamma \in H^*} M_\gamma.$$

If M is irreducible, then there exists a unique highest weight $\lambda \in H^*$ such that $M_\lambda \neq (0)$ and $L_+ \circ_n M_\lambda = (0)$ for all $n \geq 0$; M_λ is an irreducible conformal module over L_0 .

We have mentioned above that $L_0 \subset Cur(sl_4) + Vir \subset L$.

Let M' be the $Cur(sl_4) + Vir$ -module generated by M_λ . Let M'' be the largest submodule of M' such that $M'' \cap M_\lambda = 0$. Then M'/M'' is an irreducible $Cur(sl_4) + Vir$ -module and $(M'/M'')_\lambda = M_\lambda$. Let $V = Coeff(M)$ be an L -module.

From the description of irreducible modules of finite type over $Cur(sl_4) \rtimes Vir(R)$ (see [CK1]) it follows that the module V_λ can be identified with $\mathbf{C}[t^{-1}, t]$, say $V_\lambda = \overline{\mathbf{C}[t^{-1}, t]}$. For arbitrary elements $a, b \in \mathbf{C}[t^{-1}, t]$, $h \in H$ we have $(h \otimes a)\bar{b} = \langle \lambda, h \rangle \overline{ab}$. Moreover, there exist scalars $\alpha, \beta \in \mathbf{C}$ such that for arbitrary $a, b \in \mathbf{C}[t^{-1}, t]$ we have

$$Vir(a)\bar{b} = \overline{-ab' + \beta a'b + \alpha ab}.$$

Denote this L_0 -module as $V(\lambda, \beta, \alpha)$. It is well known that, for an arbitrary $\lambda \in H^*$, given an irreducible L_0 -module W such that the elements $h \in H$ act on W as scalar multiplications $\langle \lambda, h \rangle$, there exists a unique L -module with the highest weight λ under the action of H , whose λ -space is isomorphic to W as L_0 -module. If we consider the irreducible L_0 -module $V(\lambda, \beta, \alpha)$, then the corresponding irreducible L -module will be denoted as $Irr(\lambda, \beta, \alpha)$.

It follows from the above that every irreducible conformal module over CK_6 is isomorphic to $Irr(\lambda, \beta, \alpha)$ for some $\lambda \in H^*$, $\beta, \alpha \in \mathbf{C}$. This gives rise to the question:

For which parameters $\lambda \in H^$, $\beta, \alpha \in \mathbf{C}$, the irreducible conformal module $Irr(\lambda, \Delta, \alpha)$ is of finite type?*

Let λ be an integral dominant weight, that is, $\langle \lambda, w_1 - w_3 \rangle, \langle \lambda, w_3 - w_2 \rangle, \langle \lambda, w_2 - w_4 \rangle$ all lie in $\mathbf{Z}_{\geq 0}$.

Theorem 3.1 *The conformal module $Irr(\lambda, \beta, \alpha)$ is of finite type if and only if*

- (1) $\langle \lambda, h_{w_1 - w_3} \rangle \geq 2$; $\beta, \alpha \in \mathbf{C}$, or
- (2) $\langle \lambda, h_{w_1 - w_3} \rangle = 1$; $\langle \lambda, h_{w_2 - w_3} \rangle = 0$, $\beta = -1$, $\alpha \in \mathbf{C}$.

These modules exhaust all conformal irreducible CK_6 -modules of finite type.

Since $V(\lambda, \beta, \alpha)$ are known to be conformal modules of finite type over L_0 (see [CK1]), we can easily conclude that $Irr(\lambda, \beta, \alpha)$ is of finite type if and only if it has finitely many weights with respect to the action of H . At this point we can forget about conformal modules and address the question:

For which $\lambda \in H^*$, $\beta, \alpha \in \mathbf{C}$, the L -module $Irr(\lambda, \beta, \alpha)$ has finitely many weights?

Lemma 3.1 *Let $\alpha = w_i - w_j$ or $-w_i - w_j$. For an element $a \in R$, let $X_\alpha(a) = e_{w_i - w_j}(a)$ or $q_{-w_i - w_j}(a)$ defined as above. Suppose that $\alpha < 0$ and for any decomposition $-\alpha = \alpha_1 + \dots + \alpha_r$ into a sum of positive roots, for any elements $x_i \in L_{\alpha_i}$, $1 \leq i \leq r$, there exist an element $h \in H$ and an element $b \in R$ such that $[x_1, [x_2, \dots [x_r, X_\alpha(a)] \dots]] = h \otimes ab$ for an arbitrary $a \in R$. Then for an arbitrary element $v_\lambda \in V_\lambda$ we have $X_\alpha(a)v_\lambda = X_\alpha(1)(av)_\lambda$.*

Proof: It is sufficient to show that

$$U(L_+)(X_\alpha(a)v_\lambda - X_\alpha(1)(av)_\lambda) \cap V_\lambda = (0).$$

Otherwise there exists a decomposition $-\alpha = \alpha_1 + \dots + \alpha_r$, $\alpha_i > 0$ and elements $x_i \in L_{\alpha_i}$ such that $x_1 \dots x_r(X_\alpha(a)v_\lambda - X_\alpha(1)(av)_\lambda) \neq 0$.

But

$$x_1 \dots x_r X_\alpha(a)v_\lambda = [x_1, [x_2, \dots, [x_r, X_\alpha(a)] \dots]]v_\lambda = (h \otimes ab)v_\lambda =$$

$$h(abv)_\lambda = [x_1, [x_2, \dots, [x_r, X_\alpha(1)] \dots]](av)_\lambda = x_1 \dots x_r X_\alpha(1)(av)_\lambda,$$

a contradiction. The lemma is proved.

Lemma 3.2 *The negative roots $w_2 - w_3$, $w_4 - w_3$, $-w_1 - w_2$, $-w_1 - w_3$, $-w_1 - w_4$, $w_4 - w_2$ satisfy the assumptions of Lemma 3.1*

Proof: We list all possible decompositions. The roots $w_3 - w_2$, $w_1 + w_4$ and $w_2 - w_4$ do not have nontrivial decompositions. Then, $w_1 + w_2 = (w_1 + w_4) + (w_2 - w_4)$, $w_1 + w_3 = (w_1 + w_2) + (w_3 - w_2) = (w_1 + w_4) + (w_2 - w_4) + (w_3 - w_2) = (w_3 - w_4) + (w_4 + w_1)$; $w_3 - w_4 = (w_2 - w_4) + (w_3 - w_2)$.

The condition of Lemma 3.1 is checked by a straightforward computation in the superalgebra L . The lemma is proved.

Lemma 3.3 For arbitrary elements $a, b \in R$ we have

$$[q_{w_1+w_4}, e_{w_2-w_4}(a)][q_{w_1+w_4}, e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = \gamma(abv)_\lambda,$$

where $\gamma = \langle \lambda, h_{w_1-w_2} \rangle (1 - \langle \lambda, h_{w_1-w_3} \rangle)$.

Proof: We have $[q_{w_1+w_4}, e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = (I) - (II)$, where

$$(I) = q_{w_1+w_4}e_{w_3-w_4}(b)q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda =$$

$$q_{w_1+w_4}q_{-w_1-w_2}e_{w_3-w_4}(b)q_{-w_1-w_3}v_\lambda =$$

$$q_{w_1+w_4}q_{-w_1-w_2}[e_{w_3-w_4}(b), q_{-w_1-w_3}]v_\lambda + q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_3}e_{w_3-w_4}(b)v_\lambda =$$

$$-q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_4}(b)v_\lambda = -q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_4}(bv)_\lambda,$$

by Lemma 3.1.

$$(II) = e_{w_3-w_4}(b)q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda =$$

$$-e_{w_3-w_4}(b)e_{w_4-w_2}q_{-w_1-w_3}v_\lambda - e_{w_3-w_4}(b)q_{-w_1-w_2}q_{w_1+w_4}q_{-w_1-w_3}v_\lambda =$$

$$-e_{w_3-w_4}(b)q_{-w_1-w_3}e_{w_4-w_2}v_\lambda - q_{-w_1-w_2}e_{w_3-w_4}(b)q_{w_1+w_4}q_{-w_1-w_3}v_\lambda.$$

Now

$$e_{w_3-w_4}(b)q_{-w_1-w_3}e_{w_4-w_2}v_\lambda =$$

$$[e_{w_3-w_4}(b), q_{-w_1-w_3}]e_{w_4-w_2}v_\lambda + q_{-w_1-w_3}e_{w_3-w_4}(b)e_{w_4-w_2}v_\lambda.$$

The second summand is 0 since $f(w_3 - w_4) = 6$ whereas $f(w_4 - w_2) = -1$.

The first summand is

$$-q_{-w_1-w_4}(b)e_{w_4-w_2}v_\lambda =$$

$$-[q_{-w_1-w_4}e_{w_4-w_2}]v_\lambda - e_{w_4-w_2}q_{-w_1-w_4}(b)v_\lambda =$$

$$-q_{-w_1-w_2}(b)v_\lambda - e_{w_4-w_2}q_{-w_1-w_4}(b)v_\lambda =$$

$$-q_{-w_1-w_2}(bv)_\lambda - e_{w_4-w_2}q_{-w_1-w_4}(bv)_\lambda,$$

by Lemma 3.2.

As for the other summand of (II),

$$q_{-w_1-w_2}e_{w_3-w_4}(b)q_{w_1+w_4}q_{-w_1-w_3}v_\lambda =$$

$$\begin{aligned} q_{-w_1-w_2}[e_{w_3-w_4}(b), [q_{w_1+w_4}, q_{-w_1-w_3}]]v_\lambda &= \\ q_{-w_1-w_2}h_{w_3-w_4}(b)v_\lambda &= q_{-w_1-w_2}h_{w_3-w_4}(bv)_\lambda. \end{aligned}$$

We proved that

$$[q_{w_1+w_4}, e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = P(bv)_\lambda,$$

where P is an operator that does not involve b . Choosing $b = 1$ we get

$$\begin{aligned} P &= ad([q_{w_1+w_4}, e_{w_3-w_4}])ad(q_{-w_1-w_2})ad(q_{-w_1-w_3}) = \\ &ad(q_{w_3+w_1})ad(q_{-w_1-w_2})ad(q_{-w_1-w_3}). \end{aligned}$$

Now we have to consider the element

$$[q_{w_1+w_4}, e_{w_2-w_4}(a)]q_{w_3+w_1}q_{-w_1-w_2}q_{-w_1-w_3}(bv)_\lambda.$$

Remark that $[[q_{w_1+w_4}, e_{w_2-w_4}(a)], q_{w_3+w_1}] \in e_{w_1-w_4}(R)$ and

$$e_{w_1-w_4}(R)q_{-w_1-w_2}q_{-w_1-w_3}(bv)_\lambda = [[e_{w_1-w_4}(R), q_{-w_1-w_2}], q_{-w_1-w_4}](bv)_\lambda = (0).$$

Hence our expression becomes

$$-q_{w_1+w_3}[q_{w_1+w_4}, e_{w_2-w_4}(a)]q_{-w_1-w_2}q_{-w_1-w_3}(bv)_\lambda.$$

Denote $X = [q_{w_1+w_4}, e_{w_2-w_4}(a)]$, $Y = q_{-w_1-w_2}$, $Z = q_{-w_1-w_3}$. Then $XYZ = [[X, Y], Z] - Y[X, Z] + YZX + Z[X, Y]$, $[X, Y] = h_{w_2-w_1}(a)$, $[X, Z] = e_{w_2-w_3}(a)$, $[[X, Y], Z] = q_{-w_1-w_3}(a)$. By Lemma 2,

$$h_{w_2-w_1}(a)(bv)_\lambda = h_{w_2-w_1}(abv)_\lambda, \quad e_{w_2-w_3}(a)(bv)_\lambda = e_{w_2-w_3}(abv)_\lambda$$

and

$$q_{-w_1-w_3}(a)(bv)_\lambda = q_{-w_2-w_3}(abv)_\lambda.$$

As we did above, we can conclude that

$$[q_{w_1+w_4}, e_{w_2-w_4}(a)][q_{w_1+w_4}, e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = \tilde{P}(abv)_\lambda,$$

where \tilde{P} is an operator that does not involve a or b . Choosing $a = b = 1$ we get

$$\tilde{P} = ad(q_{w_2+w_1})ad(q_{w_3+w_1})ad(q_{-w_1-w_2})ad(q_{-w_1-w_3}).$$

Now

$$\begin{aligned}
\tilde{P}(abv)_\lambda &= q_{w_2+w_1}[q_{w_3+w_1}, q_{-w_1-w_2}]q_{-w_1-w_3}(abv)_\lambda - \\
& q_{w_2+w_1}q_{-w_1-w_2}q_{w_3+w_1}q_{-w_1-w_3}(abv)_\lambda = \\
& [q_{w_2+w_1}, [[q_{w_3+w_1}, q_{-w_1-w_2}], q_{-w_1-w_3}]](abv)_\lambda - \\
& [q_{w_2+w_1}, q_{-w_1-w_2}], [q_{w_3+w_1}, q_{-w_1-w_3}]](abv)_\lambda = \\
& -h_{w_2-w_1}(abv)_\lambda - h_{w_2-w_1}h_{w_3-w_1}(abv)_\lambda = \gamma(abv)_\lambda,
\end{aligned}$$

$\gamma = \langle \lambda, h_{w_1-w_2} \rangle (1 - \langle \lambda, h_{w_1-w_3} \rangle)$. The lemma is proved.

Remark. In what follows $[x_1, \dots, x_n]$ denotes the left-normed commutator $[\dots[x_1, x_2], x_3], \dots, x_n]$.

Lemma 3.4 $[[[e_{w_4-w_1}(a), [q_{w_1+w_4}, e_{w_3-w_4}(c)], [q_{w_1+w_4}, e_{w_2-w_4}(b)]]] = h_{w_1-w_4}(ab'c) - Vir(abc)$.

Proof: Since $[e_{w_4-w_1}(a), q_{w_1+w_4}] = 0$ the left hand side is equal to

$$\begin{aligned}
& [e_{w_4-w_1}(a), e_{w_3-w_4}(c), q_{w_1+w_4}, e_{w_2-w_4}(b), q_{w_1+w_4}] = \\
& -[e_{w_3-w_1}(ac), q_{w_1+w_4}, e_{w_2-w_4}(b), q_{w_1+w_4}].
\end{aligned}$$

Furthermore, $e_{w_2-w_4}(b) = -[q_{w_1+w_2}, q_{-w_1-w_4}(b)]$. Substituting this expression we get:

$$\begin{aligned}
LHS &= [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{w_1+w_2}, q_{-w_1-w_4}(b), q_{w_1+w_4}] + \\
& [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{-w_1-w_4}(b), q_{w_1+w_2}, q_{w_1+w_4}] = (I) + (II).
\end{aligned}$$

Recall that we use the notation $[e_{w_4-w_1}(a), q_{w_3+w_1}, q_{w_2+w_1}] = Vir(a)$. Now,

$$\begin{aligned}
[e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{w_1+w_2}] &= -[e_{w_4-w_1}(ac), e_{w_3-w_4}, q_{w_1+w_4}, q_{w_1+w_2}] = \\
& -[e_{w_4-w_1}(ac), [e_{w_3-w_4}, q_{w_1+w_4}], q_{w_1+w_2}]
\end{aligned}$$

since $[e_{w_4-w_1}(ac), q_{w_1+w_4}] \subseteq L_{2w_4} = (0)$.

Using that $[e_{w_3-w_4}, q_{w_1+w_4}] = -q_{w_3+w_1}$, our expression becomes

$$[e_{w_4-w_1}(ac), q_{w_3+w_1}, q_{w_2+w_1}] = Vir(ac).$$

Hence,

$$(I) = [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{w_1+w_2}, [q_{-w_1-w_4}(b), q_{w_1+w_4}]] = \\ -[Vir(ac), h_{w_1-w_4}(b)] = -h_{w_1-w_4}(ab'c).$$

On the other hand,

$$(II) = [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{-w_1-w_4}(b), q_{w_1+w_2}, q_{w_1+w_4}] = \\ [e_{w_3-w_1}(ac), [q_{w_1+w_4}, q_{-w_1-w_4}(b)], q_{w_1+w_2}, q_{w_1+w_4}] = \\ [e_{w_3-w_1}(ac), h_{w_1-w_4}(b), q_{w_1+w_2}, q_{w_1+w_4}] = [e_{w_3-w_1}(abc), q_{w_1+w_2}, q_{w_1+w_4}] = \\ -[e_{w_3-w_1}(abc), q_{w_1+w_4}, q_{w_1+w_2}] = -Vir(abc),$$

as we have seen above. This proves the lemma.

Lemma 3.4 implies that

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]e_{w_4-w_1}(a)v_\lambda = \\ -[e_{w_4-w_1}(a), [q_{w_1+w_4}, e_{w_3-w_4}(c)], [q_{w_1+w_4}, e_{w_2-w_4}(b)]]v_\lambda = \\ (-h_{w_1-w_4}(ab'c) - Vir(abc))v_\lambda = \\ -\langle \lambda, h_{w_1-w_4} \rangle (abcv')_\lambda + ((\mu_0 abc + \mu_1 a'bc + \mu_2 ab'c + \mu_3 abc')v)_\lambda,$$

here v is viewed as an element from $R = F[t^{-1}, t]$; $\mu_0, \mu_1, \mu_2, \mu_3$ are scalars from F .

Choosing $a = 1$ we get:

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]e_{w_4-w_1}v_\lambda = \\ ((\mu_0 bc + \mu_2 b'c + \mu_3 bc')v)_\lambda - \langle \lambda, h_{w_1-w_4} \rangle (bcv')_\lambda.$$

Hence, $[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]e_{w_4-w_1}(av)_\lambda =$

$$((\mu_0 abc + \mu_2 ab'c + \mu_3 abc')v)_\lambda - \langle \lambda, h_{w_1-w_4} \rangle ((abcv')_\lambda + (a'bcv)_\lambda).$$

This implies

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)](e_{w_4-w_1}(a)(v)_\lambda - e_{w_4-w_1}(av)_\lambda) = \\ (\mu_1 + \langle \lambda, h_{w_1-w_4} \rangle)(a'bcv)_\lambda. \quad (*)$$

4 The case $\langle \lambda, h_{w_1-w_3} \rangle \geq 2$

In this section we will prove that if λ is an integral dominant functional and $\langle \lambda, h_{w_1-w_3} \rangle \geq 2$, then for arbitrary $\beta, \alpha \in F$ the irreducible module $V(\lambda, \beta, \alpha)$ has only finitely many weights with respect to the action of H .

Let $\gamma = \langle \lambda, h_{w_1-w_2} \rangle (1 - \langle \lambda, h_{w_1-w_3} \rangle)$ (see Lemma 3.3). Since $w_1 - w_2 = (w_1 - w_3) + (w_3 - w_2)$ and the root $w_3 - w_2$ is positive, we conclude that $\langle \lambda, h_{w_1-w_2} \rangle \geq 2$ and therefore $\gamma \neq 0$. Let $\xi = \frac{\mu_1 + \langle \lambda, h_{w_1-w_4} \rangle}{\gamma}$.

Lemma 4.1 $e_{w_4-w_1}(a)v_\lambda - e_{w_4-w_1}(av)_\lambda - \xi q_{-w_1-w_2} q_{-w_1-w_3}(a'v)_\lambda = 0$.

Proof. Denote the left hand side of the above equality as w . In order to prove that $w = 0$ we need only to check that $L_+w \cap V_\lambda = (0)$. From the equality (*) and Lemma 3.3 it follows that

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]w = 0.$$

Consider the element $q_{-w_3-w_4}(b)w$. We have $q_{-w_3-w_4}(b)e_{w_4-w_1}(a)v_\lambda = q_{-w_1-w_3}(ab)v_\lambda = q_{-w_1-w_3}(abv)_\lambda$ by Lemma 3.2. The last expression is equal also to $q_{-w_3-w_4}(b)e_{w_3-w_1}(av)_\lambda$. Furthermore,

$$q_{-w_3-w_4}(b)q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = q_{-w_1-w_2}q_{-w_1-w_3}q_{-w_3-w_4}(b)v_\lambda = 0,$$

since $-w_3 - w_4$ is positive.

We have shown that $q_{-w_3-w_4}(R)w = (0)$.

Similarly $q_{-w_2-w_4}(R)w = (0)$.

Let us show that $e_{w_1-w_4}(R)w = (0)$. Indeed, $f(w_1 - w_4) = 9$, $f(-w_1 - w_2) = -2$, $f(-w_1 - w_3) = -7$. Hence,

$$e_{w_1-w_4}(b)q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = [e_{w_1-w_4}(b), q_{-w_1-w_2}, q_{-w_1-w_3}]v_\lambda = 0.$$

Now

$$\begin{aligned} e_{w_1-w_4}(b)w &= e_{w_1-w_4}(b)e_{w_4-w_1}(a)v_\lambda - e_{w_1-w_4}(b)e_{w_4-w_1}(av)_\lambda = \\ &= h_{w_1-w_4}(ab)v_\lambda - h_{w_1-w_4}(b)(av)_\lambda = 0. \end{aligned}$$

Since $[L_{w_1+w_2}, L_{w_1+w_3}] \subseteq e_{w_1-w_4}(R)$ it follows that for arbitrary elements $x \in L_{w_1+w_2}$, $y \in L_{w_1+w_3}$, $xyw = -yxw$.

We have $L_{w_1+w_2} = [q_{w_1+w_4}, e_{w_2-w_4}(R)] + q_{-w_3-w_4}(R)$, $L_{w_1+w_3} = [q_{w_1+w_4}, e_{w_3-w_4}(R)] + q_{-w_2-w_4}(R)$.

From what we proved above, it follows that $L_{w_1+w_2}L_{w_1+w_3}w = (0)$. Together with $e_{w_1-w_4}(R)w = (0)$ it implies that $U(L_+)w \cap V_\lambda = (0)$ and therefore $w = 0$. Lemma is proved.

Lemma 4.2 $e_{w_3-w_1}(a)v_\lambda = e_{w_3-w_1}(av)_\lambda - \xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda$.

Proof. By Lemma 2.5,

$$e_{w_3-w_4}e_{w_4-w_1}(a)v_\lambda = e_{w_3-w_4}e_{w_4-w_1}(av)_\lambda + \xi e_{w_3-w_4}q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda.$$

Since $w_3 - w_4$ is positive, it implies

$$[e_{w_3-w_4}, e_{w_4-w_1}(a)]v_\lambda = [e_{w_3-w_4}, e_{w_4-w_1}](av)_\lambda +$$

$$\xi q_{-w_1-w_2}[e_{w_3-w_4}, q_{-w_1-w_3}](a'v)_\lambda = e_{w_3-w_1}(av)_\lambda - \xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda.$$

This implies the result and proves the lemma.

From now on in this section, unless otherwise stated, we will assume that $\langle \lambda, h_{w_1-w_3} \rangle = 2$. Our first aim is to show that $e_{w_3-w_1}^3 V_\lambda = (0)$.

Lemma 4.3 $e_{w_1-w_3}(a)e_{w_3-w_1}^3 v_\lambda = 6 \xi e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda$.

Proof. Taking into account that

$$[e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}, e_{w_3-w_1}] = e_{w_1-w_3}(a)e_{w_3-w_1}^3 -$$

$$3e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1}^2 + 3e_{w_3-w_1}^2e_{w_1-w_3}(a)e_{w_3-w_1} - e_{w_3-w_1}^3e_{w_1-w_3}(a) = 0$$

and

$$\begin{aligned} e_{w_1-w_3}(a)e_{w_3-w_1}^2 &= [e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}] + \\ &2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1} - e_{w_3-w_1}^2e_{w_1-w_3}(a) = \\ &-2e_{w_3-w_1}(a) + 2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1} - e_{w_3-w_1}^2e_{w_1-w_3}(a), \end{aligned}$$

we get

$$\begin{aligned} e_{w_1-w_3}(a)e_{w_3-w_1}^3 v_\lambda &= 3e_{w_3-w_1}(-2e_{w_3-w_1}(a) + 2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1})v_\lambda - \\ 3e_{w_3-w_1}^2e_{w_1-w_3}(a)e_{w_3-w_1}v_\lambda &= -6e_{w_3-w_1}e_{w_3-w_1}(a)v_\lambda + 3e_{w_3-w_1}^2h_{w_1-w_3}(a)v_\lambda = \\ -6e_{w_3-w_1}(e_{w_3-w_1}(av)_\lambda - \xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda) &+ 6e_{w_3-w_1}^2(av)_\lambda = \\ 6\xi e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda. \end{aligned}$$

The lemma is proved.

Lemma 4.4 $e_{w_1-w_3}(a)e_{w_1-w_3}(b)e_{w_3-w_1}^3v_\lambda = 0$.

Proof. By Lemma 4.3, the left hand side is equal to

$$6\xi e_{w_1-w_3}(a)e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda.$$

We notice that

$$e_{w_1-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda = [e_{w_1-w_3}(a), q_{-w_1-w_2}, q_{-w_1-w_4}](b'v)_\lambda = 0.$$

$$\text{Hence,} \quad e_{w_1-w_3}(a)e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda =$$

$$[e_{w_1-w_3}(a), e_{w_3-w_1}]q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda = h_{w_1-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda.$$

From Lemma 3.2 it follows that the last expression is equal to

$$\begin{aligned} & h_{w_1-w_3}(1)q_{-w_1-w_2}q_{-w_1-w_4}(ab'v)_\lambda = \\ & \langle \lambda + w_3 - w_1, h_{w_1-w_3}(1) \rangle q_{-w_1-w_2}q_{-w_1-w_4}(ab'v)_\lambda = 0. \end{aligned}$$

The lemma is proved.

Lemma 4.5 $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda = q_{w_4+w_1}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(av)_\lambda$.

Proof. We have $[q_{w_1+w_3}, e_{w_4-w_3}(a)] = q_{w_1+w_3}e_{w_4-w_3}(a) - e_{w_4-w_3}(a)q_{w_1+w_3}$. Now, since the total weight of the expression $q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}$ is $3w_3 - w_1$ that is positive, we only need to consider the expression

$$q_{w_1+w_3}e_{w_4-w_3}(a)e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda = (I) + (II)$$

where $(I) = q_{w_1+w_3}e_{w_4-w_1}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda$, and

$(II) = q_{w_1+w_3}e_{w_3-w_1}e_{w_4-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda$.

Let us consider these expressions separately.

$$(I) = q_{w_1+w_3}q_{-w_1-w_2}e_{w_4-w_1}(a)q_{-w_1-w_4}v_\lambda = \underbrace{q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_1}(a)v_\lambda}_{I.1} + \underbrace{q_{w_1+w_3}q_{-w_1-w_2}[e_{w_4-w_1}(a), q_{-w_1-w_4}]v_\lambda}_{I.2}.$$

$$(I.1) = q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_1}(a)v_\lambda =$$

$$q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}(e_{w_4-w_1}(av)_\lambda + \xi q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda) =$$

$$q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_1}(av)_\lambda,$$

because $q_{-w_1-w_2}q_{-w_1-w_4}q_{-w_1-w_2} = 0$;

$$(I.2) = -q_{w_1+w_3}q_{-w_1-w_2}q_{-2w_1}(a)v_\lambda =$$

$$q_{-w_1-w_2}q_{w_1+w_3}q_{-2w_1}(a)v_\lambda - [q_{w_1+w_3}, q_{-w_1-w_2}]q_{-2w_1}(a)v_\lambda = (I.2.1) + (I.2.2);$$

$$(I.2.1) = q_{-w_1-w_2}[q_{w_1+w_3}, q_{-2w_1}(a)]v_\lambda = -2q_{-w_1-w_2}e_{w_3-w_1}(a)v_\lambda =$$

$$-2q_{-w_1-w_2}(e_{w_3-w_1}(av)_\lambda - \xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda) = -2q_{-w_1-w_2}e_{w_3-w_1}(av)_\lambda;$$

$$(I.2.2) = e_{w_3-w_2}q_{-2w_1}(a)v_\lambda = q_{-2w_1}(a)e_{w_3-w_2}v_\lambda = 0$$

since $w_3 - w_2$ is positive.

$$(II) = q_{w_1+w_3}e_{w_3-w_1}e_{w_4-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda =$$

$$q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}e_{w_4-w_3}(a)q_{-w_1-w_4}v_\lambda =$$

$$\underbrace{q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_3}(a)v_\lambda}_{II.1} +$$

$$\underbrace{q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}[e_{w_4-w_3}(a), q_{-w_1-w_4}]v_\lambda}_{II.2}.$$

But

$$(II.1) = q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_3}(a)v_\lambda =$$

$$q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_3}(av)_\lambda$$

by Lemma 3.2;

$$(II.2) = -q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_3}(a)v_\lambda =$$

$$-q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_3}(av)_\lambda$$

again by Lemma 3.2.

To summarize, we have proved that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}q_{-w_1-w_3}q_{-w_1-w_4}v_\lambda = P(av)_\lambda,$$

where P is an operator that does not involve a . Choosing $a = 1$, we get $P = ad(q_{w_4+w_1})ad(e_{w_3-w_1})ad(q_{-w_1-w_2})ad(q_{-w_1-w_4})$. The lemma is proved.

Lemma 4.6 (i) $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^3 v_\lambda =$
 $-q_{w_1+w_4}e_{w_3-w_1}^3(av)_\lambda - 3\xi e_{w_3-w_1}^2 q_{-w_1-w_2}(a'v)_\lambda.$

(ii) $[q_{w_1+w_3}, e_{w_4-w_3}(a)][q_{w_1+w_3}, e_{w_4-w_3}(b)]e_{w_3-w_1}^3 v_\lambda =$
 $3\xi q_{w_1+w_4}e_{w_3-w_1}^2 q_{-w_1-w_2}((ab' - a'b)v)_\lambda.$

(iii) $[q_{w_1+w_3}, e_{w_4-w_3}(a)][q_{w_1+w_3}, e_{w_4-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}^3 v_\lambda = 0.$

Proof. (i) The element $q_{w_1+w_3}$ commutes with $e_{w_3-w_1}$ and w_1+w_3 is positive. Hence, $q_{w_1+w_3}e_{w_3-w_1}^3 v_\lambda = 0$. Furthermore, $[e_{w_4-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}] = 0$. Hence,

$$\begin{aligned} e_{w_4-w_3}(a)e_{w_3-w_1}^3 v_\lambda &= 3e_{w_3-w_1}^2 e_{w_4-w_3}(a)e_{w_3-w_1} v_\lambda - 2e_{w_3-w_1}^3 e_{w_4-w_3}(a)v_\lambda = \\ &3e_{w_3-w_1}^2 e_{w_4-w_1}(a)v_\lambda + e_{w_3-w_1}^3 e_{w_4-w_3}(a)v_\lambda = \\ &3e_{w_3-w_1}^2 (e_{w_4-w_1}(av)_\lambda + \xi q_{-w_1-w_2} q_{-w_1-w_3}(a'v)_\lambda) + e_{w_3-w_1}^3 e_{w_4-w_3}(av)_\lambda. \end{aligned}$$

We have proved that $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^3 v_\lambda =$

$$\begin{aligned} &[q_{w_1+w_3}, e_{w_4-w_3}]e_{w_3-w_1}^3 (av)_\lambda + 3\xi q_{w_1+w_3}e_{w_3-w_1}^2 q_{-w_1-w_2} q_{-w_1-w_3}(a'v)_\lambda = \\ &-q_{w_1+w_4}e_{w_3-w_1}^3 (av)_\lambda + 3\xi e_{w_3-w_1}^2 q_{w_1+w_3} q_{-w_1-w_2} q_{-w_1-w_3}(a'v)_\lambda = \\ &-q_{w_1+w_4}e_{w_3-w_1}^3 (av)_\lambda - 3\xi e_{w_3-w_1}^2 e_{w_3-w_2} q_{-w_1-w_3}(a'v)_\lambda - \\ &3\xi e_{w_3-w_1}^2 q_{-w_1-w_2} q_{w_1+w_3} q_{-w_1-w_3}(a'v)_\lambda = \\ &-q_{w_1+w_4}e_{w_3-w_1}^3 (av)_\lambda + 3\xi e_{w_3-w_1}^2 q_{-w_1-w_2}(a'v)_\lambda - 3\xi e_{w_3-w_1}^2 q_{-w_1-w_2} h_{w_1-w_3}(a'v)_\lambda = \\ &-q_{w_1+w_4}e_{w_3-w_1}^3 (av)_\lambda - 3\xi e_{w_3-w_1}^2 q_{-w_1-w_2}(a'v)_\lambda. \end{aligned}$$

The assertion (i) is proved.

(ii) Let us apply (i) to $[q_{w_1+w_3}, e_{w_4-w_3}(b)]e_{w_3-w_1}^3 v_\lambda$ and consider both summands of the right hand side of (i) separately.

We have, $[q_{w_1+w_3}, e_{w_4-w_3}(a)]q_{w_1+w_4}e_{w_3-w_1}^3 (bv)_\lambda =$
 $-q_{w_1+w_4}[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^3 (bv)_\lambda =$
 $q_{w_1+w_4}(q_{w_1+w_4}e_{w_3-w_1}^3 (abv)_\lambda + 3\xi e_{w_3-w_1}^2 q_{-w_1-w_2}(a'bv)_\lambda) =$

$$3\xi q_{w_1+w_4} e_{w_3-w_1}^2 q_{-w_1-w_2} (a'bv)_\lambda.$$

Acting on the second summand, we get

$$\begin{aligned} & [q_{w_1+w_3}, e_{w_4-w_3}(a)] e_{w_3-w_1}^2 q_{-w_1-w_2} (b'v)_\lambda = \\ & -e_{w_3-w_1}^2 [q_{w_1+w_3}, e_{w_4-w_3}(a)] q_{-w_1-w_2} (b'v)_\lambda + \\ & 2e_{w_3-w_1} [q_{w_1+w_3}, e_{w_4-w_3}(a)] e_{w_3-w_1} q_{-w_1-w_2} (b'v)_\lambda = \\ & e_{w_3-w_1}^2 [q_{w_1+w_3}, e_{w_4-w_3}(a)] q_{-w_1-w_2} (b'v)_\lambda + \\ & 2e_{w_3-w_1} [q_{w_1+w_3}, e_{w_4-w_3}(a), e_{w_3-w_1}] q_{-w_1-w_2} (b'v)_\lambda = \\ & e_{w_3-w_1}^2 [q_{w_1+w_3}, e_{w_4-w_3}(a), q_{-w_1-w_2}] (b'v)_\lambda + \\ & 2e_{w_3-w_1} [q_{w_1+w_3}, e_{w_4-w_1}(a)] q_{-w_1-w_2} (b'v)_\lambda = \\ & e_{w_3-w_1}^2 e_{w_4-w_2}(a) (b'v)_\lambda + 2e_{w_3-w_1} [q_{w_1+w_3}, e_{w_4-w_1}(a)] q_{-w_1-w_2} (b'v)_\lambda. \end{aligned}$$

The first summand of this sum is equal to $e_{w_3-w_1}^2 e_{w_4-w_2} (ab'v)_\lambda$ by Lemma 3.2. As for the second summand,

$$\begin{aligned} & e_{w_3-w_1} [q_{w_1+w_3}, e_{w_4-w_1}(a)] q_{-w_1-w_2} (b'v)_\lambda = \\ & e_{w_3-w_1} q_{w_1+w_3} e_{w_4-w_1}(a) q_{-w_1-w_2} (b'v)_\lambda = \\ & e_{w_3-w_1} q_{w_1+w_3} q_{-w_1-w_2} e_{w_4-w_1}(a) (b'v)_\lambda = \\ & e_{w_3-w_1} q_{w_1+w_3} q_{-w_1-w_2} e_{w_4-w_1} (ab'v)_\lambda + \\ & \xi e_{w_3-w_1} q_{w_1+w_3} \underbrace{q_{-w_1-w_2} q_{-w_1-w_2}}_0 q_{-w_1-w_3} (a'b'v)_\lambda = \\ & e_{w_3-w_1} q_{w_1+w_3} q_{-w_1-w_2} e_{w_4-w_1} (ab'v)_\lambda. \end{aligned}$$

We have shown that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)] e_{w_3-w_1}^2 q_{-w_1-w_2} (b'v)_\lambda = P(ab'v)_\lambda,$$

where P is an operator which does not involve a or b . Choosing $a = 1$, $b = t$, we get $P = -ad(q_{w_1+w_4})ad(e_{w_3-w_1})^2ad(q_{-w_1-w_2})$ which finishes the proof of (ii).

(iii) By using (ii) we need only to show that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)] q_{w_1+w_4} e_{w_3-w_1}^2 q_{-w_1-w_2} v_\lambda = 0.$$

Since $[q_{w_1+w_3}, e_{w_4-w_3}(a)]$ and $q_{w_1+w_4}$ commute, the expression above is $-q_{w_1+w_4}[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^2 q_{-w_1-w_2} v_\lambda$.

We proved above that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^2 q_{-w_1-w_2} v_\lambda = -q_{w_1+w_4}e_{w_3-w_1}^2 q_{-w_1-w_2}(av)_\lambda.$$

Now multiplying this expression on the left by $q_{w_1+w_4}$ we get 0. This concludes the proof of the lemma.

Lemma 4.7 $[q_{w_1+w_3}, e_{w_2-w_3}(R)]^3 e_{w_3-w_1}^3 v_\lambda = (0)$.

Proof. Apply $ad(e_{w_2-w_4})^3$ to the equality $[q_{w_1+w_3}, e_{w_4-w_3}(R)]^3 e_{w_3-w_1}^3 v_\lambda = (0)$ of Lemma 4.6(iii).

Since $[e_{w_2-w_4}, q_{w_1+w_3}] = [e_{w_2-w_4}, e_{w_3-w_1}] = [e_{w_4-w_3}(R), e_{w_2-w_4}, e_{w_2-w_4}] = (0)$, we will get

$$[q_{w_1+w_3}, [e_{w_2-w_4}, e_{w_4-w_3}(R)]]^3 e_{w_3-w_1}^3 v_\lambda = (0),$$

completing the proof of the lemma.

Lemma 4.8 $e_{w_3-w_1}^3 v_\lambda = 0$.

Proof. If $e_{w_3-w_1}^3 v_\lambda \neq 0$, then there exist positive roots $\alpha_1, \dots, \alpha_s$ such that $(0) \neq L_{\alpha_1} \cdots L_{\alpha_s} e_{w_3-w_1}^3 v_\lambda \subseteq V_\lambda$. Let s be the minimal number with this property. Since we can move each L_{α_i} to the right modulo shorter products, we can assume that for each i , $1 \leq i \leq s$, $\alpha_i + w_3 - w_1$ is a root or 0 and $f(\alpha_i) \leq f(w_1 - w_3) = 3$. Among all positive roots, only $w_1 - w_3$, $w_1 + w_2$, $w_1 + w_4$ have this properties. Suppose that

$$(0) \neq L_{w_1+w_2}^i L_{w_1+w_4}^j e_{w_1-w_3}(R)^k e_{w_3-w_1}^3 v_\lambda \subseteq V_\lambda.$$

Then $i(w_1 + w_2) + j(w_1 + w_4) + (3 - k)(w_3 - w_1) = m(w_1 + w_2 + w_3 + w_4)$; $0 \leq i, j, k \in \mathbf{Z}$, $m \in \mathbf{Z}$.

This implies $i = j = m$, $k = 3 - m \geq 0$. Hence, we have 3 options:

1) $k = 2$ or 3. This contradicts Lemma 4.4.

2) $k = 1$. By Lemma 4.3

$$L_{w_1+w_2}^2 L_{w_1+w_4}^2 e_{w_1-w_3}(R) e_{w_3-w_1}^3 v_\lambda \subseteq L_{w_1+w_2}^2 L_{w_1+w_4}^2 e_{w_3-w_1} q_{-w_1-w_2} q_{-w_1-w_4} v_\lambda.$$

The factors in $L_{w_1+w_2}^2 L_{w_1+w_4}^2$ on the right hand side anticommute, because of the minimality of s and the fact that $[L_{w_1+w_2}, L_{w_1+w_4}] \subseteq e_{w_1-w_3}(R)$, which leads to the case 2).

Suppose that at least one of the two $L_{w_1+w_4}$ factors lies in $q_{-w_2-w_3}(R)$. Then

$$\underbrace{q_{-w_2-w_3}(a)e_{w_3-w_1}} q_{-w_1-w_2} q_{-w_1-w_4} v_\lambda = e_{w_3-w_1} q_{-w_2-w_3}(a) q_{-w_1-w_2} q_{-w_1-w_4} v_\lambda + q_{-w_2-w_1}(a) q_{-w_1-w_2} q_{-w_1-w_4} v_\lambda.$$

The first summand is 0 because $-w_2 - w_3$ is positive. The second summand is equal to

$$q_{-w_1-w_2} q_{-w_1-w_4} q_{-w_2-w_1}(a) v_\lambda = q_{-w_1-w_2} q_{-w_1-w_4} q_{-w_2-w_1}(av)_\lambda$$

by Lemma 3.2. Now it remains to notice that $q_{-w_1-w_2} q_{-w_1-w_4} q_{-w_1-w_2} = 0$.

Thus, we can assume that both factors from $L_{w_1+w_4}$ are $[q_{w_1+w_3}, e_{w_4-w_3}(a_i)]$, $i = 1, 2$.

By Lemma 4.5 we have

$$[q_{w_1+w_3}, e_{w_4-w_3}(a_1)][q_{w_1+w_3}, e_{w_4-w_3}(a_2)] e_{w_3-w_1} q_{-w_1-w_2} q_{-w_1-w_4} v_\lambda = [q_{w_1+w_3}, e_{w_4-w_3}(a_1)] q_{w_4+w_1} e_{w_3-w_1} q_{-w_1-w_2} q_{-w_1-w_4} (a_2 v)_\lambda.$$

The element $[q_{w_1+w_3}, e_{w_4-w_3}(a_1)]$ anticommutes with $q_{w_4+w_1}$. Hence again by Lemma 4.5

$$[q_{w_1+w_3}, e_{w_4-w_3}(a_1)] q_{w_4+w_1} e_{w_3-w_1} q_{-w_1-w_2} q_{-w_1-w_4} (a_2 v)_\lambda = -q_{w_4+w_1} q_{w_4+w_1} e_{w_3-w_1} q_{-w_1-w_2} q_{-w_1-w_4} (a_1 a_2 v)_\lambda = 0.$$

3) $k = 0$. We have to examine $L_{w_1+w_2}^3 L_{w_1+w_4}^3 e_{w_3-w_1}^3 v_\lambda$. As above, we conclude that factors from $L_{w_1+w_2}$ and $L_{w_1+w_4}$ anticommute module the previous cases ($k \geq 1$).

From $q_{-w_3-w_4}(R)^2 e_{w_3-w_1}^3 v_\lambda = (0)$, it follows that no more than one factor from $L_{w_1+w_4}$ lies in $q_{-w_2-w_3}(R)$.

On the other hand, Lemma 4.6 (iii) implies that exactly one factor from $L_{w_1+w_4}$ lies in $q_{-w_2-w_3}(R)$. Similarly, $q_{-w_3-w_4}(R)^2 e_{w_3-w_1}^3 v_\lambda = (0)$ and Lemma 4.7 imply that exactly one factor from $L_{w_1+w_2}$ lies in $q_{-w_3-w_4}(R)$.

Now we need to show that

$$[q_{w_1+w_3}, e_{w_2-w_3}(a_1)][q_{w_1+w_3}, e_{w_2-w_3}(a_2)][q_{w_1+w_3}, e_{w_4-w_3}(b_1)][q_{w_1+w_3}, e_{w_4-w_3}(b_2)] \\ q_{-w_3-w_4}(c_1)q_{-w_2-w_3}(c_2)e_{w_3-w_1}^3 v_\lambda = 0.$$

First notice that $q_{-w_2-w_3}(c_2)e_{w_3-w_1}^3 v_\lambda =$

$$3e_{w_3-w_1}^2 [q_{-w_2-w_3}(c_2), e_{w_3-w_1}] v_\lambda = \\ 3e_{w_3-w_1}^2 q_{-w_1-w_2}(c_2) v_\lambda = 3e_{w_3-w_1}^2 q_{-w_1-w_2}(c_2 v)_\lambda$$

by Lemma 3.2. Hence, without loss of generality, we can assume that $c_2 = 1$ and similarly, $c_1 = 1$. Moreover,

$$q_{-w_3-w_4} q_{-w_2-w_3} e_{w_3-w_1}^3 v_\lambda = \\ 6e_{w_3-w_1} [q_{-w_3-w_4}, e_{w_3-w_1}] [q_{-w_2-w_3}, e_{w_3-w_1}] v_\lambda = 6e_{w_3-w_1} q_{-w_1-w_4} q_{-w_1-w_2} v_\lambda.$$

Now,

$$[q_{w_1+w_3}, e_{w_4-w_3}(b_1)][q_{w_1+w_3}, e_{w_4-w_3}(b_2)] e_{w_3-w_1} q_{-w_1-w_4} q_{-w_1-w_2} v_\lambda = 0$$

follows from Lemma 4.5. This concludes the proof of the lemma.

Lemma 4.9 1) $(e_{w_4-w_3})^{\langle \lambda, h_{w_3-w_4} \rangle + 1} v_\lambda = 0,$

2) $(e_{w_4-w_2})^{\langle \lambda, h_{w_2-w_4} \rangle + 1} v_\lambda = 0.$

Proof. The only positive roots α such that $\alpha + w_4 - w_3$ is a root and $f(\alpha) \leq f(w_3 - w_4) = 6$ are $w_3 - w_4, w_1 + w_2, w_3 - w_2, w_2 - w_4$. Suppose that

$$(0) \neq L_{w_1+w_2}^i L_{w_3-w_2}^j L_{w_2-w_4}^k L_{w_3-w_4}^l e_{w_4-w_3}^q v_\lambda \subseteq V_\lambda, \quad q = \langle \lambda, h_{w_3-w_4} \rangle + 1.$$

Then $i(w_1 + w_2) + j(w_3 - w_2) + k(w_2 - w_4) + (q - l)(w_4 - w_3) = m(w_1 + w_2 + w_3 + w_4)$, where $i, j, k, l, m \in \mathbf{Z}$.

This implies $i = m, i - j + k = m, j - (q - l) = m, -k + (q - l) = m$. Hence $i = m = 0, j = k = q - l \geq 0$.

Now we have to examine the expression $L_{w_3-w_2}^j L_{w_2-w_4}^j L_{w_3-w_4}^l e_{w_4-w_3}^q v_\lambda$, where $l = q - j$.

Suppose that $l \geq 1$. There exist rational numbers μ, ν such that for an arbitrary element $a \in R$,

$$\begin{aligned} e_{w_3-w_4}(a)e_{w_4-w_3}^q v_\lambda &= \mu e_{w_4-w_3}^{q-2} e_{w_4-w_3}(a)v_\lambda + \nu e_{w_4-w_3}^{q-1} h_{w_3-w_4} v_\lambda = \\ &= \mu e_{w_4-w_3}^{q-2} e_{w_4-w_3}(av)_\lambda + \nu e_{w_4-w_3}^{q-1} \langle \lambda, h_{w_3-w_4} \rangle (av)_\lambda \end{aligned}$$

by Lemma 3.2.

This implies that

$$e_{w_3-w_4}(a)e_{w_4-w_3}^q v_\lambda = e_{w_3-w_4} e_{w_4-w_3}^q (av)_\lambda = 0$$

since $q = \langle \lambda, h_{w_3-w_4} \rangle + 1$.

Now let $l = 0, j = q$. As above

$$e_{w_2-w_4}(a)e_{w_4-w_3}^q v_\lambda = q e_{w_4-w_3}^{q-1} e_{w_2-w_3}(a)v_\lambda = q e_{w_4-w_3}^{q-1} e_{w_2-w_3}(av)_\lambda.$$

This implies that

$$L_{w_3-w_2}^q L_{w_2-w_4}^q e_{w_4-w_3}^q V_\lambda = L_{w_3-w_2}^q e_{w_2-w_4}^q e_{w_4-w_3}^q V_\lambda$$

and, similarly, this expression is equal to $e_{w_3-w_2}^q e_{w_2-w_4}^q e_{w_4-w_3}^q V_\lambda$. We have shown above that

$$e_{w_2-w_4}^q e_{w_4-w_3}^q v_\lambda = q! e_{w_2-w_3}^q v_\lambda.$$

Now $e_{w_3-w_2}^q e_{w_2-w_3}^q v_\lambda = 0$ because $q = \langle \lambda, h_{w_3-w_4} \rangle + 1 \geq \langle \lambda, h_{w_3-w_2} \rangle + 1$. This proves 1). Let us prove now assertion 2). The only positive roots α such that $\alpha + w_4 - w_2$ is a root and $f(\alpha) \leq f(w_2 - w_4) = 1$ are $w_2 - w_4$ and $w_1 + w_4$. If

$$(0) \neq L_{w_1+w_4}^i L_{w_2-w_4}^j e_{w_4-w_2}^p v_\lambda \subseteq V_\lambda, \quad p = \langle \lambda, h_{w_2-w_4} \rangle + 1,$$

then $i(w_1 + w_4) + (p - j)(w_4 - w_2) = m(w_1 + w_2 + w_3 + w_4)$, which implies $i = m = 0, j = p$. Arguing as above, we see that $L_{w_2-w_4}^p e_{w_4-w_2}^p V_\lambda = e_{w_2-w_4}^p e_{w_4-w_2}^p V_\lambda = 0$. This completes the proof of the lemma.

Lemma 4.10 *Let $M \subseteq L$ a subspace such that $M^n v_\lambda = (0)$, where $v_\lambda \in V_\lambda$. Let $1 \leq i \neq j \leq 4$ and $e_{w_i-w_j}^m v_\lambda = 0$. Suppose further that $[e_{w_i-w_j}, M, M] = (0)$. Then $[M, e_{w_i-w_j}]^{m+n} v_\lambda = (0)$.*

Proof. From $[M, e_{w_i-w_j}, e_{w_i-w_j}] = (0)$ it follows that

$$[M, e_{w_i-w_j}]^{m+n} = \underbrace{[M \cdots M]_{m+n}}_{m+n}, \underbrace{[e_{w_i-w_j}, \dots, e_{w_i-w_j}]_{m+n}}_{m+n},$$

where products on the left hand side and in $M \cdots M$ are taken in the associative algebra $End_F(V)$. Hence,

$$[M, e_{w_i-w_j}]^{m+n} v_\lambda \subseteq \sum_{s+r=m+n} e_{w_i-w_j}^s \underbrace{M \cdots M}_{m+n} e_{w_i-w_j}^r v_\lambda.$$

In each nonzero summand on the right hand side $r \leq m-1$.

From $\underbrace{[M, [M, [M, \dots [M, e_{w_i-w_j}^r]] \cdots]}_{r+1} = (0)$ it follows that

$$\underbrace{M \cdots M}_{m+n} e_{w_i-w_j}^r \subseteq \sum_{p+q=m+n, p < m} M^p e_{w_i-w_j}^r M^q$$

which implies that $q \geq n$ and therefore $M^q v_\lambda = (0)$.

Lemma 4.11 *There exists $m \geq 1$ such that $e_{w_i-w_j}^m V_\lambda = (0)$ for any $1 \leq i \neq j \leq m$.*

Proof. By Lemmas 4.8 and 4.9 the elements $e_{\pm(w_1-w_3)}, e_{\pm(w_3-w_4)}, e_{\pm(w_2-w_4)}$ act nilpotently on V_λ . Now it remains to notice that those elements generate $sl(4)$ and to use Lemma 4.10.

Lemma 4.12 *For an arbitrary root α the subspace L_α acts nilpotently on V_λ .*

Proof. Let $\alpha = w_i - w_j$, $1 \leq i \neq j \leq 4$, $w_i - w_j$ negative. We have shown that $e_{w_i-w_j}^m V_\lambda = (0)$. Now, $L_{w_i-w_j} = [e_{w_j-w_i}(R), e_{w_i-w_j}, e_{w_i-w_j}]$ and $[e_{w_j-w_i}(R), e_{w_i-w_j}, e_{w_i-w_j}, e_{w_i-w_j}] = (0)$. This implies that

$$L_{w_i-w_j}^m = [L_{w_j-w_i}^m, \underbrace{e_{w_i-w_j}, \dots, e_{w_i-w_j}}_{2m}] \subseteq \sum_{p+q=2m} e_{w_i-w_j}^p L_{w_j-w_i}^m e_{w_i-w_j}^q.$$

If $q \geq m$ then $e_{w_i-w_j}^q V_\lambda = (0)$. If $q \leq m-1$, then $f(m(w_j - w_i) + q(w_i - w_j)) > 0$ and again $L_{w_j-w_i}^m e_{w_i-w_j}^q V_\lambda = (0)$. We have shown that $L_{w_j-w_i}^m V_\lambda = (0)$.

Let α be an odd root such that L_α acts on V_α nilpotently, α is not of the form $-2w_k$. Then for arbitrary $1 \leq i \neq j \leq 4$ the subspace $[L_\alpha, e_{w_i-w_j}]$ acts on V_λ nilpotently. Indeed, since $\alpha \neq -2w_i$, we have $[L_\alpha, e_{w_i-w_j}, e_{w_i-w_j}] = [e_{w_i-e_j}, L_\alpha, L_\alpha] = (0)$. Now the claim follows from Lemma 4.10.

Consider a root space $L_{w_i+w_j}$, $1 \leq i \neq j \leq 4$. If one of i, j is equal to 1, then $w_i + w_j > 0$. Let $i \neq 1, j \neq 1$. Then $[L_{w_i+w_j}, e_{w_i-w_1}] = (0)$, but $\langle w_i + w_j, h_{w_i-w_1} \rangle \neq 0$. Hence, $L_{w_i+w_j} = [[e_{w_i-w_1}, e_{w_1-w_i}], L_{w_i+w_j}] \subseteq [e_{w_i-w_1}, L_{w_1+w_j}]$.

From what we proved above it follows that $L_{w_i+w_j}$ acts on V_λ nilpotently.

Next, $L_{-2w_i} = [[e_{w_j-w_i}, e_{w_i-w_j}], L_{-2w_i}] \subseteq [e_{w_j-w_i}, L_{-w_i-w_j}]$, which implies that L_{-2w_i} acts on V_λ nilpotently. This completes the proof of the lemma.

5 Tensor product of modules $V(\lambda, \beta, \alpha)$

In this section we will discuss a realization of modules $V(\beta, \alpha)$ and define a tensor product in this class.

Let R be an arbitrary commutative F -algebra with a derivation $d : R \rightarrow R$. Recall that the Weyl algebra W is $W = \sum_{i=0}^{\infty} R d^i$, $da = ad + d(a)$. For an arbitrary scalar $\beta \in F$ consider the vector space $W_\beta(R, d) = \{a_0 d^\beta + a_1 d^{\beta-1} + a_2 d^{\beta-2} + \dots, a_i \in R\}$, the (infinite) sums are understood formally, $\tilde{W}(R, d) = \sum_{\beta \in F} W_\beta(R, d)$. The rule $d^\gamma a = \sum_{i=0}^{\infty} \binom{\gamma}{i} d^i(a) d^{\gamma-i}$, where $d^i(a)$ is the i -th derivative of the element a , makes $\tilde{W}(R, d)$ an associative algebra, $W \subseteq \tilde{W}(R, d)$. Moreover, for each $\beta \in F$ we have $[Rd, W_\beta(R, d)] \subseteq W_\beta(R, d)$. Hence $W_\beta(R, d)$ is a module over the Virasoro algebra Rd .

Now consider the associative commutative algebra $\tilde{R} = R + Rv$, $v^2 = 0$. Extend the derivation d via $d(v) = -\alpha v$, $\alpha \in F$. Then the subspace $W_\beta(R, v, d) = \sum_{i=0}^{\infty} R v d^{\beta-i} \subset W_\beta(\tilde{R}, d)$ is an Rd -submodule of $W_\beta(\tilde{R}, d)$.

The following proposition is straightforward.

Proposition 5.1 $W_\beta(R, v, d)/W_{\beta-1}(R, v, d) \simeq V(\beta, \alpha)$.

The tensor product $V(\beta_1, \alpha_1) \otimes_F V(\beta_2, \alpha_2)$ can be identified with $W_{\beta_1+\beta_2}(R, v_1 v_2, d)/W_{\beta_1+\beta_2-1}(R, v_1 v_2, d)$, where $R = F[t_1^{-1}, t_1, t_2^{-1} t_2]$, $d = -d/dt_1 - d/dt_2$.

Since $d(t_1 - t_2) = 0$ it follows that $(t_1 - t_2)(V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2))$ is a submodule of $V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2)$.

Clearly, $V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2)/(t_1 - t_2) \simeq V(\beta_1 + \beta_2, \alpha_1 + \alpha_2)$.

Proposition 5.2 *If $V(\lambda_i, \beta_i, \alpha_i)$, $i = 1, 2$ are conformal modules of finite type, then so is $V(\lambda_1 + \lambda_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2)$.*

Proof. The L -modules $V(\lambda_i, \beta_i, \alpha_i)$ have finitely many weight spaces with respect to the Cartan subalgebra H of L . The tensor product $V = V(\lambda_1, \beta_1, \alpha_1) \otimes V(\lambda_2, \beta_2, \alpha_2)$ also has finitely many weight spaces. The subspace of V of weight $\lambda_1 + \lambda_2$ can be identified with $V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2)$. Let M be the submodule of V generated by $(t_1 - t_2)(V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2))$. Then $(V/M)_{\lambda_1 + \lambda_2} \simeq V(\beta_1 + \beta_2, \alpha_1 + \alpha_2)$. The L -module $V(\lambda_1 + \lambda_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2)$ is a homomorphic image of the submodule of V/M generated by $(V/M)_{\lambda_1 + \lambda_2}$. Hence $V(\lambda_1 + \lambda_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2)$ has finitely many weight spaces with respect to H . This concludes the proof of the proposition.

Consider a copy of the algebra of Laurent polynomials $\overline{F[t, t^{-1}]}$ and make it a W -module via $a\bar{b} = \overline{ab}$, $d\bar{b} = -\bar{b}'$, $a, b \in F[t^{-1}, t]$. Then the space of 8-columns $\overline{F[t, t^{-1}]}^8$ becomes a left module over $M_8(W)$, hence a $CK(6)$ -module. It is easy to see that this $CK(6)$ -module is irreducible.

If we define the form $(w_i/w_j) = \delta_{ij}$ on $\sum_{i=1}^4 Fw_i$ and view functionals on H as elements of $\sum_{i=1}^4 Fw_i$, then the highest weight of the module $\overline{F[t, t^{-1}]}^8$ is w_1 , $(h_{w_i - w_j} \otimes a)(\bar{b}, 0, \dots, 0)^T = (w_i - w_j/w_1)(\bar{b}, 0, \dots, 0)^T$. Moreover $Vir(a)(\bar{b}, 0, \dots, 0)^T = (\overline{-ab' - a'b}, 0, \dots, 0)^T$. Hence $\overline{F[t, t^{-1}]}^8 \simeq V(w_1, -1, 0)$.

Proposition 5.3 *If λ is an integral dominant functional and $\langle \lambda, h_{w_1 - w_3} \rangle \geq 2$, then for arbitrary $\beta, \alpha \in F$ the irreducible module $V(\lambda, \beta, \alpha)$ has only finitely many weights with respect to the action of H .*

Proof. Let $\langle \lambda, h_{w_1 - w_3} \rangle = k \geq 2$. By Proposition 5.2 the module $V' = V(\lambda - (k-2)w_1, \beta + (k-2)\alpha)$ has finitely many H -weights. Tensoring V' with $\overline{F[t, t^{-1}]}^8 \simeq V(w_1, -1, 0)$ $k-2$ times and using Proposition 5.2 we get the result.

6 The case $\langle \lambda, h_{w_1-w_3} \rangle = 1$

The aim of this section is to prove the following

Proposition 6.1 *Let λ be an integral dominant weight, such that $\langle \lambda, h_{w_1-w_3} \rangle = 1$. Then $V(\lambda, \beta, \alpha)$ has finitely many weights with respect to H if and only if $\langle \lambda, h_{w_3-w_2} \rangle = 0$ and $\beta = -1$.*

Suppose at first that λ is an integral dominant weight such that $\langle \lambda, h_{w_1-w_3} \rangle = 1$ and $V(\lambda, \beta, \alpha)$ has finitely many H -weights.

Lemma 6.1 *For arbitrary elements $a \in R$, $v_\lambda \in V_\lambda$ we have $e_{w_3-w_1}(a)v_\lambda = e_{w_3-w_1}(av)_\lambda$.*

Proof. Since $V(\lambda, \beta, \alpha)$ is a finite sum of eigenspaces with respect to the H it follows that the element $e_{w_3-w_1}$ acts on V_λ nilpotently. The standard argument shows that $e_{w_3-w_1}^2 V_\lambda = (0)$. Now for an arbitrary $a \in R$ we have

$$\begin{aligned} 0 &= e_{w_1-w_3}(a)e_{w_3-w_1}^2 v_\lambda = \\ &= [e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}]v_\lambda + 2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1}v_\lambda = \\ &= -2e_{w_1-w_3}(a)v_\lambda + 2e_{w_3-w_1}h_{w_1-w_3}(a)v_\lambda = -2(e_{w_3-w_1}(a)v_\lambda - e_{w_1-w_3}(av)_\lambda). \end{aligned}$$

This concludes the proof of the lemma.

Lemma 6.2 $\beta = -1$.

Proof.

$$\begin{aligned} &[q_{w_1+w_3}, e_{w_2-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}(a)v_\lambda = \\ &[[q_{w_1+w_3}, e_{w_2-w_3}(b)], [q_{w_1+w_3}, e_{w_4-w_3}(c)], e_{w_3-w_1}(a)]v_\lambda = \\ &(-h_{w_1-w_3}(ab'c) + Vir(abc))v_\lambda \end{aligned}$$

as in Lemma 3.4. This element is equal to $(-ab'cv - abcv' + \beta(abc)'v + \alpha abc)v_\lambda$.

On the other hand, by Lemma 3.1 we have

$$\begin{aligned} &[q_{w_1+w_3}, e_{w_2-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}(a)v_\lambda = \\ &[q_{w_1+w_3}, e_{w_2-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}(av)_\lambda = \\ &(-h_{w_1-w_3}(b'c) + Vir(bc))(av)_\lambda = (-ab'cv - (bc)(av)' + \beta(bc)'(av) + \alpha abc)v_\lambda. \end{aligned}$$

Comparing these two expressions we see that $\beta a'bcv = -bca'v$, so $\beta = -1$. This finishes the proof of the lemma.

Lemma 6.3 $\langle \lambda, h_{w_3-w_2} \rangle = 0$.

Proof. We have $q_{w_1+w_2}q_{w_1+w_4}q_{-w_3-w_4}q_{-w_2-w_3}e_{w_3-w_1}^2v_\lambda = 0$. Now,

$$q_{-w_2-w_3}e_{w_3-w_1}^2v_\lambda = 2e_{w_3-w_1}[q_{-w_2-w_3}, e_{w_3-w_1}]v_\lambda = 2e_{w_3-w_1}q_{-w_1-w_2}v_\lambda.$$

$$\begin{aligned} \text{Hence, } 0 &= q_{w_1+w_2}q_{w_1+w_4}q_{-w_3-w_4}e_{w_3-w_1}q_{-w_1-w_2}v_\lambda = \\ &= q_{w_1+w_4}q_{-w_3-w_4}q_{w_1+w_2}e_{w_3-w_1}q_{-w_1-w_2}v_\lambda = \\ &= q_{w_1+w_4}q_{-w_3-w_4}[q_{w_1+w_2}, e_{w_3-w_1}]q_{-w_1-w_2}v_\lambda + \\ &= q_{w_1+w_4}q_{-w_3-w_4}e_{w_3-w_1}[q_{w_1+w_2}, q_{-w_1-w_2}]v_\lambda = \\ &= q_{w_1+w_4}q_{-w_3-w_4}q_{w_2+w_3}q_{-w_1-w_2}v_\lambda + \langle \lambda, h_{w_1-w_2} \rangle q_{w_1+w_4}q_{-w_3-w_4}e_{w_3-w_1}v_\lambda = \\ &= q_{w_1+w_4}[q_{-w_3-w_4}, q_{w_2+w_3}]q_{-w_1-w_2}v_\lambda - q_{w_1+w_4}q_{w_2+w_3}q_{-w_3-w_4}q_{-w_1-w_2}v_\lambda + \\ &= \langle \lambda, h_{w_1-w_2} \rangle [q_{w_1+w_4}, [q_{-w_3-w_4}, e_{w_3-w_1}]]v_\lambda = \\ &= -q_{w_1+w_4}q_{-w_1-w_4}v_\lambda + \langle \lambda, h_{w_1-w_2} \rangle \langle \lambda, h_{w_1-w_4} \rangle v_\lambda = \\ &= \langle \lambda, h_{w_1-w_4} \rangle (-1 + \langle \lambda, h_{w_1-w_2} \rangle)v_\lambda. \end{aligned}$$

Since $\langle \lambda, h_{w_1-w_4} \rangle \geq \langle \lambda, h_{w_1-w_3} \rangle = 1$ it follows that $\langle \lambda, h_{w_1-w_2} \rangle = 1$ and therefore $\langle \lambda, h_{w_3-w_2} \rangle = 0$. This concludes the proof of the lemma.

Now we will assume that λ is an integral dominant weight such that $\langle \lambda, h_{w_1-w_3} \rangle = 1$, $\langle \lambda, h_{w_3-w_2} \rangle = 0$. Let $\beta = -1$. We will prove that $V(\lambda, \beta, \alpha)$ is a finite sum of eigenspaces with respect to H .

Lemma 6.4 *Under the assumptions above, $e_{w_3-w_1}(a)v_\lambda = e_{w_3-w_1}(av)_\lambda$ for arbitrary $a \in R$, $v_\lambda \in V_\lambda$.*

Proof. The computations of Lemma 6.3 show that for $\langle \lambda, h_{w_1-w_3} \rangle = 1$, $\beta = -1$ we have

$$[q_{w_1+w_3}, e_{w_2-w_3}(R)][q_{w_1+w_3}, e_{w_4-w_3}(R)](e_{w_3-w_1}(a)v_\lambda - e_{w_3-w_1}(av))_\lambda = 0.$$

Also, $q_{-w_3-w_4}(R)(e_{w_3-w_1}(a)v_\lambda - e_{w_3-w_1}(av))_\lambda = q_{-w_2-w_3}(R)(e_{w_3-w_1}(a)v_\lambda - e_{w_3-w_1}(av))_\lambda = (0)$ by Lemma 3.2. This implies that $U(L_+)(e_{w_3-w_1}(a)v_\lambda - e_{w_3-w_1}(av))_\lambda = 0$. Lemma is proved.

Lemma 6.5 $e_{w_4-w_1}(a)v_\lambda = e_{w_4-w_1}(av)_\lambda$ for an arbitrary $a \in R$.

Proof. Denote $w = e_{w_4-w_1}(a)v_\lambda - e_{w_4-w_1}(av)_\lambda$. Clearly, $e_{w_1-w_4}(R)w = (0)$. Since $f(w_3 - w_4) > 0$, it follows that $e_{w_3-w_4}(b)w = e_{w_3-w_4}(ab)v_\lambda - e_{w_3-w_4}(b)(av)_\lambda = 0$ by Lemma 6.4.

From $[q_{w_1+w_4}, e_{w_4-w_1}(R)] = (0)$ we conclude that $q_{w_1+w_4}w = 0$. Hence, $[q_{w_1+w_4}, e_{w_3-w_4}(R)]w = (0)$. Also, $q_{-w_2-w_4}(R)w = (0)$ by Lemma 3.2 applied to the root $-w_1 - w_2$. We proved that $L_{w_1+w_3}w = (0)$.

If $w \neq 0$ then $U(L_+)w \cap V_\lambda \neq (0)$.

It means that there exist positive roots $\alpha_1, \dots, \alpha_s$ such that $\alpha_1 + \dots + \alpha_s + w_4 - w_1 \in \mathbf{Z}(w_1 + w_2 + w_3 + w_4)$ and, moreover, $\alpha_i + w_4 - w_1$ is a negative root or 0 for any i . If α_i is an even root and $\alpha_i + w_4 - w_1$ is one of the roots of Lemma 3.2 or 0 then $e_{\alpha_i}(b)w = 0$ again by Lemma 3.2. By Lemma 6.4 α_i is not supposed to be $w_3 - w_4$ as well. This rules out all even roots except $w_2 - w_4$.

Of odd roots, we have to examine $w_1 + w_2$ and $w_1 + w_3$, but the latter one has been ruled out above. Hence, $i(w_2 - w_4) + j(w_1 + w_2) + (w_4 - w_3) = k(w_1 + w_2 + w_3 + w_4)$; $i, j, k \in \mathbf{Z}$; $i, j \geq 0$. This equation does not have a solution. Hence $w = 0$. This finishes the proof of the lemma.

Lemma 6.6 $e_{w_3-w_1}^2 v_\lambda = 0$.

Proof. For an arbitrary element $a \in R$ we have $e_{w_1-w_3}(a)e_{w_3-w_1}^2 v_\lambda = [e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}]v_\lambda + 2e_{w_3-w_1}h_{w_1-w_3}(a)v_\lambda = -2e_{w_3-w_1}(a)v_\lambda + 2e_{w_3-w_1}h_{w_1-w_3}(a)v_\lambda = 0$ by Lemma 6.5.

Now, as in Section 4 we see that

$$U(L_+)e_{w_3-w_1}^2 v_\lambda \cap V_\lambda = L_{w_1+w_2}^2 L_{w_1+w_4}^2 e_{w_3-w_1}^2 v_\lambda.$$

We have $L_{w_1+w_4} = [q_{w_1+w_3}, e_{w_4-w_3}(R)] + q_{-w_2-w_3}(R)$.

Claim 1: $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^2 v_\lambda = [q_{w_1+w_3}, e_{w_4-w_3}]e_{w_3-w_1}^2 (av)_\lambda$ for an arbitrary $a \in R$.

Indeed, $[L_{w_1+w_4}, e_{w_3-w_1}, e_{w_3-w_1}] = (0)$ implies

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}^2 v_\lambda = 2e_{w_3-w_1}[[q_{w_1+w_3}, e_{w_4-w_3}(a)], e_{w_3-w_1}]v_\lambda =$$

$$2e_{w_3-w_1}q_{w_1+w_3}e_{w_4-w_1}(a)v_\lambda = 2e_{w_3-w_1}q_{w_1+w_3}e_{w_4-w_1}(av)_\lambda$$

by Lemma 6.5. This proves the claim.

Claim 2: $q_{-w_2-w_3}(a)e_{w_3-w_1}^2 v_\lambda = q_{-w_2-w_3}e_{w_3-w_1}^2 (av)_\lambda$.

Indeed, $q_{-w_2-w_3}(a)e_{w_2-w_1}^2 v_\lambda = 2e_{w_3-w_1}[q_{-w_2-w_3}(a), e_{w_3-w_1}]v_\lambda = 2e_{w_3-w_1}q_{-w_1-w_2}(a)v_\lambda = 2e_{w_3-w_1}q_{-w_1-w_2}(av)_\lambda$ by Lemma 3.2. This proves the claim.

These claims and the similar assertions for $L_{w_1+w_2}v_\lambda$ show that

$$\begin{aligned} & L_{w_1+w_2}^2 L_{w_1+w_4}^2 e_{w_3-w_1}^2 V_\lambda = \\ & (Fq_{w_1+w_2} + Fq_{-w_3-w_4})^2 (Fq_{w_1+w_4} + Fq_{-w_2-w_3})^2 e_{w_3-w_1}^2 V_\lambda = \\ & q_{w_1+w_2}q_{w_1+w_4}q_{-w_3-w_4}q_{-w_2-w_3}V_\lambda. \end{aligned}$$

The computations of Lemma 6.3 show that, under the assumption $\langle \lambda, h_{w_2-w_3} \rangle = 0$, this expression is equal to 0. This finishes the proof of the lemma.

Lemma 6.7 $e_{w_2-w_3}v_\lambda = 0$.

Proof. If $e_{w_2-w_3}v_\lambda \neq 0$ then there exist positive roots $\alpha_1, \dots, \alpha_s$ such that $\alpha_1 + \dots + \alpha_s = w_3 - w_2$, for each α_i the sum $\alpha_i + w_2 - w_3$ is a negative root or 0 and $L_{\alpha_1} \cdots L_{\alpha_s} e_{w_2-w_3}v_\lambda \neq (0)$.

The only positive roots with the properties above are $w_3 - w_2$ and $w_1 + w_4$. But

$$e_{w_3-w_2}(a)e_{w_2-w_3}v_\lambda = h_{w_3-w_2}(a)v_\lambda = \langle \lambda, w_3 - w_2 \rangle (av)_\lambda = 0.$$

Hence, all α_i have to be equal to $w_1 + w_4$, $L_{w_1+w_4}^s e_{w_2-w_3}v_\lambda \neq (0)$. But $[L_{w_1+w_4}, [L_{w_1+w_4}, e_{w_2-w_3}]] = (0)$, which leads to a contradiction and finishes the proof of the lemma.

Lemma 6.8 $e_{w_4-w_3}(a)v_\lambda = e_{w_4-w_3}(av)_\lambda$.

We have $e_{w_4-w_3}(a) = [e_{w_4-w_1}(a), e_{w_1-w_3}]$. Hence

$$\begin{aligned} e_{w_4-w_3}(a) &= -e_{w_1-w_3}e_{w_4-w_1}(a)v_\lambda = \\ & -e_{w_1-w_3}e_{w_4-w_1}(av)_\lambda = [e_{w_4-w_1}, e_{w_1-w_3}](av)_\lambda = e_{w_4-w_3}(av)_\lambda \end{aligned}$$

by Lemma 6.5.

Lemma 6.9 $e_{w_4-w_3}^{\langle \lambda, w_3-w_4 \rangle + 1} v_\lambda = 0$.

Proof. As in the proof of Lemma 6.7, if the assertion is not true, then there exist positive roots $\alpha_1, \dots, \alpha_s$, such that $\alpha_1 + \dots + \alpha_s + (\langle \lambda, w_3 - w_4 \rangle + 1)(w_4 - w_3) \in \mathbf{Z}(w_1 + \dots + w_4)$, $\alpha_i + w_4 - w_3$ is a negative root or 0. In fact, 0 is also excluded, because

$$e_{w_3-w_4}(a)e_{w_4-w_3}^{\langle \lambda, w_3-w_4 \rangle + 1}v_\lambda = e_{w_3-w_4}e_{w_4-w_3}^{\langle \lambda, w_3-w_4 \rangle + 1}(av)_\lambda = 0$$

by Lemma 6.8.

The only such positive roots are: $w_3 - w_2, w_2 - w_4, w_1 + w_2, -2w_4$.

Hence, there exist $i, j, k, l \in \mathbf{Z}_{\geq 0}, p \in \mathbf{Z}$, such that $i(w_3 - w_2) + j(w_2 - w_4) + k(w_1 + w_2) - 2lw_4 + m(w_4 - w_3) = p(w_1 + w_2 + w_3 + w_4)$, where $m = \langle \lambda, w_3 - w_4 \rangle + 1$. It means, that $k = p, -i + j + k = p, i - m = p, -j - 2l = p$. The first two equalities imply that $p = k \in \mathbf{Z}_{\geq 0}, i = j$. Now, adding the last two equalities we get $-m - 2l = 2p$, where the left hand side is negative, whereas the right hand side is positive. This concludes the proof of the lemma.

The element $e_{\pm(w_1-w_3)}, e_{\pm(w_2-w_3)}, e_{\pm(w_4-w_3)}$ generate $sl(4)$ and act on V_λ nilpotently. Arguing as in the proof of Lemmas 4.11 and 4.12 we get

Lemma 6.10 (1) *There exists $m \geq 1$ such that $e_{w_i-w_j}^m V_\lambda = (0)$ for any $1 \leq i \neq j \leq 4$.*

(2) *There exist $m \geq 1$ such that $L_\alpha^m V_\lambda = (0)$ for an arbitrary root α .*

This implies that for an integral dominant weight λ such that $1 = \langle \lambda, h_{w_1-w_3} \rangle, 0 = \langle \lambda, h_{w_3-w_2} \rangle$, the module $V(\lambda, -1, \alpha)$ is a finite sum of weight spaces with respect to the action of H .

7 The case $\langle \lambda, h_{w_1-w_3} \rangle = 0$

We have

$$[[e_{w_4-w_1}(a), q_{w_3+w_1}], q_{w_2+w_1}] = -[[e_{w_3-w_1}(a), q_{w_1+w_4}], q_{w_2+w_1}] = Vir(a).$$

If $\langle \lambda, h_{w_1-w_3} \rangle = 0$ and nevertheless $V(\lambda, \beta, \alpha)$ is of finite type, then $e_{w_3-w_1}V_\lambda = (0)$. This implies that

$$e_{w_3-w_1}(a)V_\lambda = -\frac{1}{2}[e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}]V_\lambda = (0).$$

Hence

$$q_{w_2+w_1}q_{w_1+w_4}e_{w_3-w_1}(a)V_\lambda = [q_{w_2+w_1}, [q_{w_1+w_4}, e_{w_3-w_1}(a)]]V_\lambda = Vir(a)V_\lambda = (0).$$

Since $H \subseteq [H \otimes R, Vir(R)]$ it follows that $HV_\lambda = (0)$, $\lambda = 0$. Then V is a 1-dimensional module with zero multiplication, which is not viewed as irreducible.

This concludes the proof of Theorem 3.1.

8 Jordan bimodules

Let V be a Jordan bimodule over a unital Jordan (super)algebra J . Then V can be represented as a direct sum $V = V_0 \oplus V_{1/2} \oplus V_1$, where $JV_0 = (0)$, $V_{1/2}$ is a one-sided Jordan bimodule (see [MZ2]), V_1 is a unital Jordan bimodule.

In [MZ2] it was shown that the universal associative enveloping algebra of $JCK(R, d)$ is $M_4(W(R, d))$. It means that one-sided Jordan $JCK(R, d)$ -bimodules are left modules over $M_4(W(R, d))$ or, equivalently, 4-tuples U^4 , where U is a left module over $W(R, d)$.

Now suppose that $R = F[t^{-1}, t]$, $d = \frac{d}{dt}$, and the one-sided Jordan bimodule over $JCK(6)$ is conformal. Then the left module M over $W(R, d)$ is a unital conformal module. Such modules correspond to left unital $\mathbf{C}[d]$ -modules. Irreducible $\mathbf{C}[d]$ -modules are one-dimensional and parametrized by scalars $\alpha \in F$. Indecomposable conformal modules of finite type correspond to Jordan blocks.

Let's be more precise. let N be a left $\mathbf{C}[d]$ -module. Then $N[t^{-1}, t] = \{\sum n_i t^i, n_i \in N, i \in \mathbf{Z}\}$ is a conformal left $W(F[t^{-1}, t], \frac{d}{dt})$ -module. It is of finite type if and only if $\dim_{\mathbf{C}} N < \infty$. The space of 4-tuples $N[t^{-1}, t]^4$ is a left associative conformal module over $M_4(W)$, hence a one-sided Jordan conformal module over $JCK(6)$.

Proposition 8.1 (1) *Every one-sided Jordan conformal bimodule over $JCK(6)$ is of the type $N[t^{-1}, t]^4$;*

(2) *The module $N[t^{-1}, t]^4$ is irreducible if and only if N is one-dimensional, $N = \mathbf{C}v$, $vd = \alpha v$, $\alpha \in F$;*

(3) *$N[t^{-1}, t]^4$ is an indecomposable module of finite type if and only if N is a Jordan block.*

Now let V be a unital irreducible conformal Jordan bimodule of finite type over $J = JCK(6)$. In [MZ4], [Z] it was shown that the Tits-Kantor-Koecher construction $K(V)$ is a Lie conformal module of finite type over the TKK-algebra $K(J) = CK(6)$. In [MZ4] we proved that: (1) the reduced module $\bar{K}(V)$ is irreducible over $K(J)$ and uniquely determines the J -bimodule V ; (2) the action of the Cartan subalgebra H on $\bar{K}(V)$ is diagonalizable, all weights of $\bar{K}(V)$ belong to the set $\{\pm w_i \pm w_j, 1 \leq i, j \leq 4\}$. Let λ be the highest weight of $\bar{K}(V)$. Since the Weyl group of sl_4 is the permutation group P_4 and $f(\lambda) \geq f(\sigma(\lambda))$ for all $\sigma \in P_4$, the only possibilities for λ are : $2w_1, w_1 - w_4, w_1 + w_3, -2w_4$. The last two cases are ruled out by Theorem 3.1. The modules $V(2w_1, \beta, \alpha), \beta, \alpha \in F$; and $V(w_1 - w_4, -1, \alpha), \alpha \in F$ indeed have Jordan structures.

Proposition 8.2 *Unital irreducible conformal Jordan $JCK(6)$ -bimodules of finite type form two parametric families, which correspond to $V(2w_1, \beta, \alpha), \beta, \alpha \in F$ and $V(w_1 - w_4, -1, \alpha), \alpha \in F$.*

Proof. We need to show that $V(2w_1, \beta, \alpha), V(w_1 - w_4, -1, \alpha), \alpha, \beta \in F$ are reduced Tits-Kantor-Koecher modules of the form $\bar{K}(V)$ for some unital Jordan bimodules V over $J = JCK(6)$.

As in section 5 consider the associative commutative algebra $\tilde{R} = R + Rv$, $R = \mathbf{C}[t^{-1}, t], v^2 = 0$ with the derivation $d, d(t) = -1, d(v) = \alpha v$. Consider the algebra $\tilde{W}(\tilde{R}, d) = \sum_{\beta \in F} W_\beta(\tilde{R}, d), W_\beta(R, v, d) = \sum_{i=0}^{\infty} Rvd^{\beta-i}$. The Lie superalgebra $L = CK(6)$ is embedded into $M_8(W)$, hence into $M_8(\tilde{W})$.

Consider the subspace $(W_\beta(R, v, d))_{1,5}$ of matrices having $W_\beta(R, v, d)$ at the intersection of the 1st row and 5th column and 0 elsewhere. It is easy to see that $[L_+, (W_\beta(R, v, d))_{1,5}] = (0)$ and for an arbitrary element $u \in (W_\beta(R, v, d))_{1,5}$, arbitrary $1 \leq i \neq j \leq 4$, we have $[h_{w_i - w_j}, u] = (2w_1/w_i - w_j)u$. Let $U_\beta = U(L)(W_\beta(R, v, d))_{1,5}$ be the L -submodule generated by $(W_\beta(R, v, d))_{1,5}$ in $M_8(\tilde{W})$. Clearly, $2w_1$ is the highest weight of this submodule and $(U_\beta)_{2w_1} = (W_\beta(R, v, d))_{1,5}$.

The L -submodule $U_{\beta-1}$ of U_β is generated by $(W_{\beta-1}(R, v, d))_{1,5}, U_\beta/U_{\beta-1} = U(L)(W_\beta(R, v, d)/W_{\beta-1}(R, v, d))$, hence $U_\beta/U_{\beta-1} = U(L)(U_\beta/U_{\beta-1})_{2w_1}$, and $(U_\beta/U_{\beta-1})_{2w_1} \simeq V(\beta, \alpha)$.

Hence $V(2w_1, \beta, \alpha)$ is a homomorphic image of the module $U_\beta/U_{\beta-1}$. Then all weights of $V(2w_1, \beta, \alpha)$ belong to the set $\{\pm w_i \pm w_j, 1 \leq i, j \leq 4\}$,

which implies that $V(2w_1, \beta, \alpha) \simeq K(\bar{V})$ for some irreducible unital Jordan J -bimodule V .

Now let's turn to bimodules $V(2w_1, \beta, \alpha)$. Consider the Cheng-Kac superalgebra $CK(\tilde{R}, d)$ and the subspace $e_{w_1-w_4}(Rv)$. This subspace generates the L -submodule which is isomorphic to $V(w_1 - w_4, -1, \alpha)$. This concludes the proof of the proposition.

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