Commuting U-operators in Jordan algebras[1]

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Abstract

For elements x, y in a non-degenerate non-unital Jordan algebra over a commutative ring, the relation $x \circ y = 0$ is shown to imply that the U-operators of x and y commute: $U_x U_y = U_y U_x$. The proof rests on the Ze′lmanov-McCrimmon classification [15] of strongly prime quadratic Jordan algebras.

1. Introduction.

In the present paper, we will be concerned with a problem that derives a considerable amount of its significance from the connection between Moufang sets and (quadratic) Jordan division rings established a few years ago by De Medts and Weiss [3]. More specifically, we will focus on a question, raised by K. Tent and communicated to the last-named author by Y. Segev, that may be phrased as follows: given a Jordan division ring with unit 1, U-operator U_x and circle product $x \circ y = U_{x,y}1$ (see 2.1, 2.4 below for precise definitions),

does the relation $x \circ y = 0$ imply that the linear maps U_x and U_y commute? (1)

Everyone expecting a short elementary answer to this simple-minded question is in for an unpleasant surprise: the answer we are going to provide in Theorem 4.7 below, though short, and an affirmative one at that, is by no means trivial, relying as it does on a substantial portion of the Ze′lmanov-McCrimmon classification [15] of arbitrary Jordan division rings. This is all the more regrettable since, from various points of view, it would be desirable to give a proof based exclusively on the manipulation of identities valid in arbitrary Jordan algebras. The hope would then be, for example, that these manipulations could somehow be mimicked in the setting of Moufang sets with abelian root groups, paving the way for new insights into this fascinating topic. On the other hand, they would also show that question (1) has an affirmative answer not just for division but, in fact, for arbitrary Jordan algebras.

Unfortunately, we have not been able to exhibit a proof of the desired kind. Instead, we must rely on the full arsenal of the Ze′lmanov-McCrimmon structure theory [15] combined with results of Thedy [20] in order to ensure (1) an affirmative answer for arbitrary

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non-degenerate, possibly non-unital, Jordan algebras without finiteness conditions (Theorem 9.5). The proof rests on the observation that the answer to (1) is trivially yes for special Jordan algebras (see 4.1), combined with the fact that the same conclusion holds for Albert algebras over arbitrary fields (Theorem 8.1); establishing the latter result turns out to be surprisingly delicate. We also obtain affirmative answers for classes of possibly degenerate Jordan algebras, e.g., in the presence of certain algebraicity conditions on the elements involved (Propositions 6.2−6.4), or for Jordan algebras of pointed quadratic forms without 2-torsion over commutative rings (Theorem 7.4).

For the convenience of the reader, we have worked hard to keep prerequisites from the theory of Jordan algebras at a minimum; in particular, the standard vocabulary of the theory will be recalled as we go along. Throughout we let k be an arbitrary commutative associative ring of scalars; occasionally it will be replaced by a field which we denote by F.

A (non-associative) k-algebra A is said to be *unital* if it contains an identity element $1 = 1_A$, in which case subalgebras of A are said to be *unital* if they contain the identity 1_A. For a quadratic map $Q\colon\ M \to N$ between k-modules, its polar map will be indicated by $Q(x, y) = Q(x + y) - Q(x) - Q(y)$.

2. Jordan algebras: generalities.

In this section, we recall some basic facts about arbitrary Jordan algebras that will be used frequently later on. Our main references are Jacobson [7, 8] and McCrimmon-Zel′manov [15].

2.1. The concept of a Jordan algebra. By a (unital) (quadratic) *Jordan algebra* over k we mean a k-module J together with a distinguished element $1 := 1_J \in J$ (the *unit*) and a quadratic map $U\colon\ J \to \mathrm{End}_k(J)$, $x \mapsto U_x$, (the *U-operator*) such that, setting

$$\{xyz\} := V_{x,y}z := U_{x,z}y := (U_{x+z} - U_x - U_z)y$$

(the *Jordan triple product*, which is obviously trilinear and symmetric in the outer variables), the following identities hold under all scalar extensions.

$$\begin{aligned} U_1 &= \mathrm{Id}_J, \\ U_x V_{y,x} &= V_{x,y} U_x, \\ U_{U_x y} &= U_x U_y U_x. \end{aligned}$$

By a Jordan *ring* we mean a Jordan algebra over $\mathbb{Z}$, the ring of rational integers.

In the remainder of this section, we fix a Jordan algebra J over k. A (Jordan) *subalgebra* of J is a k-submodule containing 1 and stable under the operation $U_x y$. We define the squaring and its bilinearization in J by

$$x^2 = U_x 1, \quad x \circ y := (x + y)^2 - x^2 - y^2. \tag{1}$$

Using this, we obtain linear maps $V_x\colon\ J \to J$ given by

$$V_x y := x \circ y$$

and recall the relations

$$x \circ y = V_{x,y}1 = V_{x,1}y = V_{1,x}y = U_{x,1}y = U_{x,y}1,$$

2.2. Linear Jordan algebras. If $2 \in k$ is a unit, with inverse $\frac{1}{2} \in k$, we introduce the (bilinear) *Jordan product*

$$xy := x \cdot y := \frac{1}{2} x \circ y,$$

making J a *linear Jordan algebra* in the sense that it is a unital commutative non-associative k-algebra with unit $1 = 1_J$ satisfying the *Jordan identity*

$$x(x^2y) = x^2(xy).$$

The U-operator of J can be recovered from the Jordan product via the formula $U_x y = 2x(xy) - x^2y$. Using this, one has no difficulty in setting up a categorical isomorphism between Jordan algebras and linear Jordan algebras over rings containing $\frac{1}{2}$.

2.3. Powers. Returning to the setting of an arbitrary base ring, powers of an element $x \in J$ with integer exponents ≥ 0 are defined inductively by

$$x^0 := 1, \quad x^1 := 1, \quad x^{n+2} = U_x x^n.$$

The submodule of J spanned by the powers of x will be written as

$$k[x] := \sum_{n \geq 0} kx^n.$$

It comes equipped with the surjective linear *evaluation map*

$$k[\mathbf{t}] \longrightarrow k[x], \quad f \longmapsto f(x),$$

where $\mathbf{t}$ is a variable. However, even though powers in Jordan algebras are quite well behaved, e.g., by satisfying the relations $U_{x^m} x^n = x^{2m+n}$, $\{x^m x^n x^p\} = 2x^{m+n+p}$, hence forcing $k[x] \subseteq J$ to be a Jordan subalgebra, this subalgebra will in general *not* be a linear associative k-algebra in its own right making the evaluation map an algebra homomorphism [7, 1.31-32]. On the other hand, by [7, 1.26, QJ37], we do have the formula $U_{(fg)(x)} = U_{f(x)} U_{g(x)}$ for all $f, g \in k[\mathbf{t}]$, which implies that

$$\text{the linear operators } U_{f(x)}, U_{g(x)},\ f, g \in k[\mathbf{t}], \text{ commute.} \tag{1}$$

Things become much simpler when dealing with linear Jordan algebras J (over rings containing $\frac{1}{2}$), because they are power associative, so $k[x] \subseteq J$ is always a unital commutative associative subalgebra.

2.4. Inverses and Jordan division algebras. An element $x \in J$ is said to be *invertible* (in J) if there exists an element $y \in J$, necessarily unique, such that $U_x y = x$ and $U_x y^2 = 1$. We call $x^{-1} := y$ the *inverse* of x (in J) and know from [8, Prop. 1.6.2] that x is invertible iff the linear map $U_x \colon J \to J$ is bijective iff 1 belongs to the range of U_x, in which case $x^{-1} = U_x^{-1} x$. We call J a *Jordan division algebra* if $J \neq \{0\}$ and all its non-zero elements are invertible.

2.5. Special versus exceptional Jordan algebras. Let A be a unital associative algebra over k. Then the k-module A together with the unit $1 := 1_A$ and the U-operator

$$U_x y := xyx \qquad (x, y \in A) \tag{1}$$

is a Jordan algebra, said to be *associated* with A and denoted by A^+. Its Jordan triple product and its circle product are respectively given by

$$\{xyz\} = xyz + zyx, \quad x \circ y = xy + yx \qquad (x, y, z \in A). \tag{2}$$

Powers in A and A^+ coincide, as do inverses, so A^+ is a Jordan division algebra iff A is an associative one. A Jordan algebra is said to be *special* if it is isomorphic to a subalgebra of A^+, for some unital associative algebra A. Typical examples of special Jordan algebras have the form

$$\mathrm{Her}(A,\tau) := \{x \in A \mid \tau(x) = x\},$$

A being as above and τ being an *involution* of A, i.e., an anti-automorphism of period 2. Jordan algebras which are not special are called *exceptional*. The most important examples of exceptional Jordan algebras are Albert algebras. They will be discussed in more detail in Section 4 below.

2.6. Identities in Jordan algebras. Jordan algebras satisfy a host of useful identities, some of them compiled in [7, 8]. In the present paper, the following ones will be needed.

(i) $V_a U_b = U_{a\circ b,b} - U_b V_a$,

(ii) $2U_a = V_a^2 - V_{a^2}$,

(iii) $a^2 \circ b^2 = \{a, a\circ b, b\} - a \circ U_b a$,

(iv) $a \circ U_b a = b \circ U_a b$,

(v) $U_a b^2 = -(a\circ b)^2 + U_b a^2 + \{a, b, a\circ b\}$,

(vi) $U_a U_b + U_b U_a - U_{a\circ b} = U_{U_a b, b} - V_a U_b V_a$,

(vii) $V_{a^2,b} = V_{a,a\circ b} - V_{U_a b}$, $V_{b,a^2} = V_{b\circ a,a} - V_{U_a b}$,

(viii) $V_a V_{b,a} = V_{U_a b} + V_b U_a$,

(ix) $V_a U_a = U_a V_a$,

(x) $V_{a^2} U_a = U_a V_{a^2}$.

Rather than pointing out a specific reference for the above identities (some of them are even part of the very definition of a Jordan algebra), we invoke Macdonald's Principle [9, p. 686] and simply note that they hold in special Jordan algebras, hence are valid in general since they either involve less than three variables or exactly three but are linear in one of them.

3. Motivation: the connection with Moufang sets.

In this section, we give a brief survey of how question (1.1) enters into the connection between Moufang sets and Jordan division rings. Referring to de Medts-Segev [2] and de Medts-Weiss [3] for more details, we recall that a *Moufang set* is a pair $\mathbb{M} = (X, \mathfrak{S})$ consisting of a set X with more than two elements and a family $\mathfrak{S} = (S_x)_{x\in X}$ of subgroups of $\mathrm{Sym}(X)$, the full permutation group of X, such that, writing $G = G_{\mathbb{M}}$ for the subgroup of $\mathrm{Sym}(X)$ generated by the S_x's ("the little projective group of $\mathbb{M}$"), the following conditions hold: (i) The S_x's make up a full conjugacy class of subgroups of G, (ii) for each $x \in X$, S_x is a normal subgroup of G_x (the stabilizer of x in G) and is simply transitive on $X \setminus \{x\}$.

For a Moufang set $\mathbb{M} = (X, \mathfrak{S})$ as above, the S_x's are called the *root groups* of $\mathbb{M}$.

Now suppose J is a Jordan division ring and, with a new symbol ∞, put $X := J \cup \{\infty\}$. We partially extend the algebraic operations of J to all of X via $a + \infty = \infty = \infty + a$ ($a \in J$), $-\infty = \infty$, $\infty^{-1} = 0$, $0^{-1} = \infty$ and use this to define permutations of X by means of $\alpha_a\colon X \to X$, $x \mapsto a + x$ ($a \in J$), $\tau\colon X \to X$, $x \mapsto -x^{-1}$. It then follows that $\mathbb{M}(J) := (X, \mathfrak{S})$, $\mathfrak{S} = (S_x)_{x\in X}$ with

$$S_\infty := \{\alpha_a \mid a \in J\}, \quad S_0 = \tau S_\infty \tau^{-1}, \quad S_a := \alpha_a S_0 \alpha_a^{-1} \qquad (a \in J^\times)$$

is a Moufang set with root groups all isomorphic to the additive group of J; in particular, they are abelian.

An important question in the theory of Moufang sets is the converse: does every Moufang set with abelian root groups come from a Jordan division ring? Since this question in its general form looks rather intractable at the moment, Segev has suggested to consider special cases, like the one defined by the Zassenhaus condition: a Moufang set $\mathbb{M} = (X, \mathfrak{S})$ is said to satisfy the *Zassenhaus condition* if $G := G_{\mathbb{M}}$ is not sharply 2-transitive on X and the pointwise stabilizer in G of three distinct points of X is trivial.

This concept gives rise to the following natural question: which Jordan division rings have the property that their associated Moufang sets satisfy the Zassenhaus condition? The following answer, due to Segev and Tent (unpublished), highlights the significance of question (1.1) in the present context: *If J is a Jordan division ring of characteristic not 2 such that $\mathbb{M}(J)$ satisfies the Zassenhaus condition, and (1.1) can be answered affirmatively for J, then J is a (commutative) field: more precisely, there exists a field K such that $J = K^+$.*

4. The original question.

In this section we will address question (1.1) in its original set-up of a Jordan division ring. We begin by disposing of a trivial but crucial side issue.

4.1. The case of a special Jordan algebra. Let J be a special Jordan algebra over k, so there is a unital associative k-algebra A such that J is a subalgebra of A^+. Hence the U-operator and the circle product of J are given by (2.5.1) and the second equation of (2.5.2) in terms of the associative product of A. Now suppose $x, y \in J$ satisfy the relation $x \circ y = 0$. Then $xy = -yx$ and the expression $U_xU_yz = xyzyx$ is symmetric in x and y. *Thus question* (1.1) *has an affirmative answer for special Jordan algebras.*

In order to proceed, we require a comparatively short digression into Albert algebras, which we will treat here in a slightly unusual way, by focussing exclusively on the objectives of the present paper. For simplicity, we therefore replace our commutative base ring k by an arbitrary field F. Free use will then be made of the differential calculus for polynomial maps as explained in Jacobson [6, Chap. VI], with a few notational adjustments taken from McCrimmon [10].

4.2. Cubic norm structures. Combining the conceptual framework of McCrimmon [10] with the terminology of Petersson-Racine [18]), we define a *cubic norm structure* over F as a quadruple $X = (X, 1, N, \sharp)$ consisting of a vector space X over F, a distinguished element $1 \in X$ (the *base point*), a cubic form $N\colon X \to F$ (the *norm*) and a quadratic map $X \to X$, $x \mapsto x^\sharp$ (the *adjoint*) such that $N(1) = 1$, $1^\sharp = 1$ and the following identities hold under all scalar extensions.

$$1 \times x = T(x)1 - x, \tag{1}$$

$$T(x^\sharp, y) = (\partial_y N)(x),$$

$$x^{\sharp\sharp} = N(x)x. \tag{2}$$

Here $x \times y = (x + y)^\sharp - x^\sharp - y^\sharp$ is the bilinearized adjoint, $T\colon X \times X \to F$ defined by

$$T(x, y) = (\partial_x N)(1)(\partial_y N)(1) - (\partial_x\partial_y N)(1) \tag{3}$$

is the *bilinear trace* of X, and $T(x) = T(x, 1)$. We also put $S(x) := T(x^\sharp)$, call the map $T\colon X \to k$ (resp. $S\colon X \to k$) the *linear* (resp. the *quadratic*) *trace* of X, and have

$$S(x, y) = T(x)T(y) - T(x, y). \tag{4}$$

The k-module X carries canonically a Jordan algebra structure $J = J(X)$ whose unit agrees with the base point of X and whose U-operator is given by

$$U_x y = T(x, y)x - x^\sharp \times y. \tag{5}$$

J is a *cubic Jordan algebra* over F in the sense that the relation

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0 \tag{6}$$

holds in all scalar extensions. Moreover, we have

$$x^\sharp = x^2 - T(x)x + T(x^\sharp)1, \tag{7}$$

$$x \times y = x \circ y - T(x)y - T(y)x + \big(T(x)T(y) - T(x, y)\big)1. \tag{8}$$

From (3) and (5) we deduce that any automorphism of X, i.e., any linear bijection $X \to X$ preserving unit, norm and adjoint, is an automorphism of J.

4.3. Alternative algebras of degree 3. Let A be a unital algebra over F which is *alternative* in the sense that the *associator* $[x, y, z] := (xy)z - x(yz)$ is alternating, equivalently, that any subalgebra on two generators is associative. Then the F-vector space A together with the unit $1 = 1_A$ and the U-operator $U_x y := (xy)x = x(yx) =: xyx$ is a Jordan algebra, actually a special one, said to be *associated* with A and denoted by A^+.

Now suppose A is a finite-dimensional alternative F-algebra of degree 3 and write $N := N_A$ for its (generic) norm. By [6, Theorems VI.1,VI.3] (see also Faulkner [5, Lemma]), the relations

$$N(1) = 1, \quad N(xy) = N(x)N(y), \quad N(\mathbf{t}1 - x) \text{ kills } x$$

hold in all scalar extensions. Defining

$$\begin{aligned} T(x, y) &= (\partial_x N)(1)(\partial_y N)(1) - (\partial_x \partial_y N)(1), \\ T(x) &= T(x, 1), \\ x^\sharp &= x^2 - T(x)x + (\partial_1 N)(x)1, \end{aligned}$$

as in 4.2, one checks that $X := X(A) := (A, 1, N, \sharp)$ is a cubic norm structure over F with bilinear trace $T(x, y) = T(xy)$ and associated cubic Jordan algebra $J(X) = A^+$. Moreover, T is also the generic trace of A and for $u, v \in A$, we have the additional relation

$$(uv)^\sharp = v^\sharp u^\sharp, \tag{1}$$

which (repeatedly) linearizes to

$$(u_1 v) \times (u_2 v) = v^\sharp (u_1 \times u_2), \tag{2}$$

$$(u v_1) \times (u v_2) = (v_1 \times v_2) u^\sharp, \tag{3}$$

$$(u_1 v_1) \times (u_2 v_2) + (u_1 v_2) \times (u_2 v_1) = (v_1 \times v_2)(u_1 \times u_2). \tag{4}$$

Most alternative algebras of degree 3 we encounter in the sequel will be *separable* in the sense that they stay semi-simple under all base field extensions, equivalently, that their bilinear trace is a non-degenerate symmetric bilinear form.

4.4. The first Tits construction. Let A be a finite-dimensional alternative F-algebra of degree 3, with generic norm N, generic trace T as in 4.3, and $\mu \in F$. Following McCrimmon [10, Theorem 6] if A is associative and Faulkner [5] or Petersson-Racine [18] in the general case, the direct sum of three copies of A as an F-vector space, written as

$$X := A \oplus Aj_1 \oplus Aj_2,$$

becomes a cubic norm structure $X(A, \mu)$ over F whose base point, norm, (bilinearized) adjoint, trace are extended from $X(A)$ by means of the formulas

$$\begin{aligned}
1 &= 1 + 0j_1 + 0j_2, \\
N(x) &= N(x_0) + \mu N(x_1) + \mu^2 N(x_2) - \mu T(x_0 x_1 x_2), \\
x^\sharp &= (x_0^\sharp - \mu x_1 x_2) + (\mu x_2^\sharp - x_0 x_1)j_1 + (x_1^\sharp - x_2 x_0)j_2, && (1)\\
x \times y &= (x_0 \times y_0 - \mu x_1 y_2 - \mu y_1 x_2) + (\mu x_2 \times y_2 - x_0 y_1 - y_0 x_1)j_1 + && (2)\\
&\quad (x_1 \times y_1 - x_2 y_0 - y_2 x_0)j_2, \\
T(x, y) &= T(x_0, y_0) + \mu T(x_1, y_2) + \mu T(x_2, y_1), && (3)\\
T(x) &= T(x_0) && (4)
\end{aligned}$$

for

$$x = x_0 + x_1 j_1 + x_2 j_2, \quad y = y_0 + y_1 j_1 + y_2 j_2 \qquad (x_i, y_i \in A,\ i = 0, 1, 2),$$

The cubic Jordan algebra corresponding to $X(A, \mu)$ will be denoted by $J(A, \mu)$. Both are said to arise from A, μ by means of the *first Tits construction*; there is also a second construction but we won't need it here. Note that $A^+ = J(X(A))$ embeds into $J(A, \mu)$ as a subalgebra through the initial summand.

4.5. The concept of an Albert algebra. The algebra $\mathrm{Mat}_3(F)$ of ordinary 3×3-matrices over F is central simple associative of degree 3, with generic norm (resp. generic trace) given by the ordinary determinant (resp. trace) of matrices, while their usual adjoint agrees with the adjoint of the corresponding cubic norm structure.

This being so, we call $J_{\mathrm{spl}} := J(\mathrm{Mat}_3(F), 1)$, i.e., the cubic Jordan algebra arising from $A = \mathrm{Mat}_3(F)$ and $\mu = 1$ by means of the first Tits construction, the *split Albert algebra* over F. By an *Albert algebra* over F, we mean a Jordan F-algebra which is an F-form of J_{spl}, i.e., which becomes isomorphic to J_{spl} after extending scalars to the separable closure of F. Albert algebras are central simple exceptional Jordan algebras of dimension 27 over F. By Galois descent, they inherit unit, norm, adjoint and trace from J_{spl}. Typical examples of Albert algebras are first Tits constructions $J(A, \mu)$, where A is any central simple associative algebra of degree 3 and $\mu \in F$ is a non-zero scalar. Conversely, suppose J is an Albert algebra, and A is a central simple associative algebra of degree 3. By [11, Theorem 8] and its proof, any embedding $A^+ \hookrightarrow J$ of Jordan algebras can be extended to an isomorphism $J(A, \mu) \xrightarrow{\sim} J$, for some non-zero scalar $\mu \in F$.

4.6. Albert division algebras. Albert division algebras, i.e., Albert algebras that are Jordan division algebras in the sense of 2.4, exist and are easy to construct. For example, given an associative F-algebra A of degree 3 and a non-zero scalar $\mu \in F$, the first Tits construction $J(A, \mu)$ is an Albert division algebra iff μ is not a generic norm of A, in which case A will be an associative division algebra. Moreover, an Albert algebra is division iff its norm is anisotropic. Hence, by a theorem of Springer [19, Lemma 4.2.11], the property of being an Albert division algebra is preserved under quadratic field extensions. Finally, subalgebras on two generators of an Albert division algebra exist only in dimensions $1, 3, 9$, by a result of Brühne [1, Prop. 3.2.2] combined with [21, p. 148].

We are now in a position to provide an affirmative answer to question (1.1) in its original form.

4.7. Theorem. *Suppose we are given elements x, y in a Jordan division ring satisfying $x \circ y = 0$. Then the U-operators U_x and U_y commute: $U_xU_y = U_yU_x$.*

Proof. Write J for the Jordan division ring in question. By the McCrimmon-Zel′manov structure theorem [15, 15.7], J is either special or an Albert division algebra over some field F. The former case having been settled in 4.1, we may assume the latter. Since J has degree 3 over F, it follows from [17, Cor. 3] that non-zero elements $x, y \in J$ having $x \circ y = 0$ do not exist unless F has characteristic 2. Thus, in view of (2.1.1), the proof of Theorem 4.7 will be complete once we have shown the following proposition.

4.8. Proposition. *Let J be an Albert division algebra over a field F of characteristic 2 and suppose $x, y \in J$ satisfy $x \circ y = 0$. Then $x \in F1$ or $y \in F[x]$.*

Proof. Assume the contrary, so $x \notin F1$ and $y \notin F[x]$. Then J', the subalgebra of J generated by x, y, has dimension 9 (4.6). Hence, by [21, p. 148], J' is either (i) a purely inseparable field extension of characteristic 3 and exponent at most 1 over F, (ii) of the form D^+ for some central associative division algebra D of degree 3 over F, or (iii) of the form $\mathrm{Her}(D, \tau)$, the Jordan algebra of τ-symmetric elements in a central associative division algebra (D, τ) of degree 3 with involution of the second kind over F. Here (i) is impossible since F has characteristic 2, and (iii) will be converted into (ii) after extending scalars to an appropriate separable quadratic field extension. We may therefore assume $J' = D^+$ as in (ii). Since F has characteristic 2, the relation $x \circ y = 0$ implies that x and y commute in D. But $k[x]$, being a separable cubic subfield of D, agrees with its own centralizer, which implies $y \in k[x]$, a contradiction. □

Remark. Roughly speaking, the reasons for the validity of Theorem 4.7 are (i) the result is trivial for special Jordan algebras, (ii) elements x, y in Albert *division* algebras satisfying $x \circ y = 0$ are extremely rare. The ubiquity of such elements in Jordan algebras where the hypothesis of being division has been dropped is well documented in the special cases below and seems to be responsible for the intractability of the problem in its most general form.

5. A first approach to the general case.

In this section, we fix an arbitrary Jordan algebra J over k and derive a number of consequences of the relation $x \circ y = 0$ that turn out to be useful in studying question (1.1) in its most general form.

5.1. Proposition. *Assume that $x, y \in J$ satisfy $x \circ y = 0$. Then:*

(i) $x^n \circ y = 0$ *for all odd integers* $n > 0$,

(ii) $V_xU_y = -U_yV_x$,

(iii) $x^2 \circ y = -2U_xy$,

(iv) $U_yx^2 = U_xy^2$,

(v) $x^2 \circ y^2 = -x \circ U_yx = 2U_yx^2$,

(vi) $U_xU_y - U_yU_x = U_{U_xy,y} + U_yV_{x^2} = -U_{U_xy,y} - V_{x^2}U_y$,

(vii) $V_{x^2,y} = -V_{U_xy} = V_{y,x^2}$.

The identities obtained from (i)−(vii) *by interchanging x and y also hold. For further reference we will indicate them with an asterisk as a superscript.*

Proof. (i) We proceed by induction on n. The assertion is true for $n = 1$. Given any odd integer $n \geq 3$, let us assume that $x^{n-2} \circ y = 0$. Then $y \circ x^n = V_yU_xx^{n-2} = U_{y \circ x,x}x^{n-2} - U_x(y \circ x^{n-2})$ (by 2.6(i)) $= 0$ by assumption and the induction hypothesis.

(ii) Apply 2.6(i) with $a = x$ and $b = y$.

(iii) Apply 2.6(ii) with $a = x$ acting on y.

(iv) Apply 2.6(v) with $a = x$ and $b = y$.

(v) Apply 2.6(iii) with $a = x$ and $b = y$ to obtain the first equality, and, for the second one, use (ii).

(vi) $U_xU_y + U_yU_x = U_{U_xy,y} - V_xU_yV_x$ (by 2.6(vi)) $= U_{U_xy,y} + U_yV_x^2$ (by (ii)) $= U_{U_xy,y} + 2U_yU_x + U_yV_{x^2}$ (by 2.6(ii)), which readily implies the first equality.
Again $U_xU_y + U_yU_x = U_{U_xy,y} - V_xU_yV_x$ (by 2.6(vi)) $= U_{U_xy,y} + V_x^2U_y$ (by (ii)) $= U_{U_xy,y} + 2U_xU_y + V_{x^2}U_y$ (by 2.6(ii)), which implies $U_xU_y - U_yU_x = -U_{U_xy,y} - V_{x^2}U_y$.

(vii) Apply 2.6(vii) with $a = x$ and $b = y$.

The final assertion is obvious. □

5.2. Proposition. *Assume that $x, y \in J$ satisfy $x \circ y = U_xy = 0$. Then:*

(i) $x^n \circ y = 0$ *for all positive integers n,*

(ii) $x^2 \circ y^2 = 2U_xy^2 = 2U_yx^2 = x \circ U_yx = 0$,

(iii) $V_{x^2,y} = V_{y,x^2} = 0$,

(iv) $2V_{x^2}U_y = 2U_yV_{x^2} = 0$,

(v) $2U_xU_y = 2U_yU_x$.

Proof. (i) We have $x^2 \circ y = 0$ as a consequence of 5.1(iii). Now, for any $n \geq 3$, $x^n = U_xx^{n-2}$ and we can prove the assertion by induction on n as in 5.1(i).
(ii) follows directly from 5.1(v) and 5.1(v)*.
(iii) follows directly from 5.1(vii).
(iv) Using (iii),

$$0 = 2V_yV_{x^2,y} = 2V_{U_yx^2} + 2V_{x^2}U_y \text{ (by 2.6(viii) with } a = y \text{ and } b = x^2) = V_{2U_yx^2} + 2V_{x^2}U_y = 2V_{x^2}U_y$$

by (ii). Now $2U_yV_{x^2} = -2V_{x^2}U_y = 0$ by 5.1(ii) applied to x^2 and y since $x^2 \circ y = 0$ by (i).
(v) Just notice that $2(U_xU_y - U_yU_x) = 0$ by using (iv) and 5.1(vi). □

5.3. Proposition. *Assume that $x, y \in J$ satisfy $x \circ y = 0$. If x is invertible then $x^{-1} \circ y = 0$.*

Proof. $U_x(x^{-1} \circ y) = U_xV_y(x^{-1}) = -V_yU_x(x^{-1})$ (by 5.1(ii)*) $= -V_yx = -y \circ x = 0$, hence $x^{-1} \circ y = 0$ since U_x is invertible. □

6. Algebraic elements of low degree in linear Jordan algebras.

In this section, we fix elements x, y in a linear Jordan algebra J over a field F of characteristic not 2 (cf. 2.2). Our aim is to answer question (1.1) affirmatively in the presence of certain algebraicity conditions. Referring to [6, VI.3] for details, let us begin by recalling some basic concepts.

6.1. Algebraic elements. x is said to be *algebraic* if the unital commutative associative subalgebra $F[x]$ of J is finite-dimensional over F. In this case, the *minimal polynomial* of x, denoted by $\mu_x(\mathbf{t}) \in F[\mathbf{t}]$, can be formed with respect to this subalgebra and has the usual properties. For example, x is invertible in J iff it is so in $F[x]$ iff $\mu_x(0) \neq 0$, in which case $x^{-1} \in F[x]$. At the other extreme, x is nilpotent iff it is algebraic with $\mu_x(\mathbf{t}) = \mathbf{t}^n$ for some positive integer n. The *degree* of an algebraic element in J is defined as the degree of its minimal polynomial.

6.2. Proposition. *If x is algebraic of degree at most 2 and $x \circ y = 0$, then $U_x U_y = U_y U_x$.*

Proof. If x is algebraic of degree 1, then it is a scalar multiple of 1, and $U_x U_y = U_y U_x$, for any $y \in J$. Hence we may assume that x is algebraic of degree 2, forcing $\mu_x(\mathbf{t}) = \mathbf{t}^2 + \alpha \mathbf{t} + \beta$ for some $\alpha, \beta \in F$ and, in particular, $x^2 + \alpha x + \beta 1 = 0$. If $\beta \neq 0$, then x is invertible by 6.1 with inverse $x^{-1} = -\beta^{-1}(x + \alpha 1)$. Thus, by 5.3,

$$0 = x^{-1} \circ y = -\beta^{-1}(x + \alpha 1) \circ y = -2\beta^{-1}\alpha y,$$

so either $y = 0$, which obviously implies the assertion, or $\alpha = 0$. In the latter case, $x^2 = -\beta 1$, $V_{x^2} = -2\beta \mathrm{Id}_J$, thus

$$V_{x^2} U_y = U_y V_{x^2}, \tag{1}$$

and

$$2U_x U_y = (V_x^2 - V_{x^2})U_y \text{ (by 2.6(ii))} = \\ U_y(V_x^2 - V_{x^2}) \text{ (by 5.1(ii) and (1))} = 2U_y U_x.$$

If $\beta = 0$, then $x^2 = -\alpha x$ and $x^2 \circ y = -\alpha x \circ y = 0$. Hence $2U_x y = 0$ by 5.1(iii), which yields $U_x y = 0$ since the characteristic is not two. Thus $2U_x U_y = 2U_y U_x$ by 5.2(v), and $U_x U_y = U_y U_x$, again using that the characteristic is not two. □

6.3. Proposition. *If x is algebraic of degree 3 and x, y are both invertible, then $x \circ y \neq 0$.*

Proof. Arguing indirectly, let us assume $x \circ y = 0$. By 6.1 we have $\mu_x(\mathbf{t}) = \mathbf{t}^3 + \alpha \mathbf{t}^2 + \beta \mathbf{t} + \gamma$ for some $\alpha, \beta, \gamma \in F$, $\gamma \neq 0$. Thus $x^{-1} = -\gamma^{-1}(x^2 + \alpha x + \beta 1)$, which implies that x^2 is a linear combination of x^{-1}, x, and 1. Therefore, using 5.3, $x^2 \circ y$ is a scalar multiple of y and the same holds true for $U_x y$ by 5.1(iii). Let us say

$$U_x y = \delta y \tag{1}$$

for some $\delta \in F$. We have

$$U_x y^2 = \frac{1}{2} U_x V_y y = -\frac{1}{2} V_y U_x y \text{ (by 5.1(ii)*)} = -\frac{1}{2} y \circ (\delta y) \text{ (by (1))} = -\delta y^2. \tag{2}$$

Thus

$$U_y x^2 = U_x y^2 \text{ (by 5.1(iv))} = -\delta y^2 \text{ (by (2))} = U_y(-\delta 1),$$

i.e., $U_y(x^2 + \delta 1) = 0$. But y is invertible by hypothesis, forcing $x^2 + \delta 1 = 0$, so x is algebraic of degree at most two, a contradiction. □

Remark. The preceding result generalizes [17, Cor. 3], which says that in cubic Jordan division algebras of characteristic not 2 non-zero elements x, y with $x \circ y = 0$ do not exist.

6.4. Proposition. *If x is algebraic of degree 3 and neither invertible nor nilpotent, then $x \circ y = 0$ implies $U_x U_y = U_y U_x$.*

Proof. This time 6.1 yields $\mu_x(\mathbf{t}) = \mathbf{t}^3 + \alpha \mathbf{t}^2 + \beta \mathbf{t}$ for some $\alpha, \beta \in F$ not both zero. If $\alpha \neq 0$, then x^2 is a linear combination of x^3 and x, hence $x^2 \circ y = 0$ by 5.1(i), $U_x y = 0$ by 5.1(iii), and $U_x U_y = U_y U_x$ by 5.2(v).

Hence we may assume $\alpha = 0$, forcing $x^3 = -\beta x$, $\beta \neq 0$. After extending scalars to the algebraic closure of F, we may replace x by $(-\beta)^{-\frac{1}{2}} x$ in order to ensure $x^3 = x$.

Now $U_x^3 = U_{x^3} = U_x$ implies that the minimal polynomial of the endomorphism $U_x : J \longrightarrow J$ divides $X^3 - X = (X - 1)(X + 1)X$. Hence U_x is diagonalizable and $J = J_0 \oplus J_1 \oplus J_{-1}$, where $J_\varepsilon = \{z \in J \mid U_x z = \varepsilon z\}$, $\varepsilon \in \{0, 1, -1\}$. The fact that U_x and V_x commute by 2.6(ix) implies

$$V_x(J_\varepsilon) \subseteq J_\varepsilon, \tag{1}$$

$\varepsilon \in \{0, 1, -1\}$ (if $z \in J_\varepsilon$, $U_x V_x z = V_x U_x z = V_x(\varepsilon z) = \varepsilon V_x z$). Similarly, using 2.6(x), we obtain

$$V_{x^2}(J_\varepsilon) \subseteq J_\varepsilon. \tag{2}$$

We can write $y = y_0 + y_1 + y_{-1}$, with $y_\varepsilon \in J_\varepsilon$, $\varepsilon \in \{0, 1, -1\}$, and $V_x(y) = 0$ together with (1), yields

$$V_x(y_\varepsilon) = x \circ y_\varepsilon = 0, \tag{3}$$

$\varepsilon \in \{0, 1, -1\}$. On the other hand, $e := x^2$ is an idempotent ($e^2 = x^4 = \frac{1}{2}x^3 \circ x = \frac{1}{2}x \circ x = x^2 = e$), and $e \circ y_1 = x^2 \circ y_1 = -2U_x y_1$ (by (3) and 5.1(iii)) $= -2y_1$, which implies

$$y_1 = 0, \qquad y = y_0 + y_{-1}$$

since the only possible eigenvalues of V_e are $0, 1, 2$ [14, II.8.1.4] because e is an idempotent.

Let $J = J_0(e) \oplus J_1(e) \oplus J_2(e)$ be the Peirce decomposition of J with respect to e [14, II.8.1.2(1)]. In what follows, free use will be made of the rules governing multiplication of the Peirce components [14, II, 8.2.1].

Since $U_e J_0 = U_{x^2} J_0 = U_x^2 J_0 = 0$, we have $J_0 \subseteq J_0(e) \oplus J_1(e)$, while for any $z \in J_i$, $i = \pm 1$, $U_e z = U_{x^2} z = U_x^2 z = i^2 z = z$, which implies $J_1 \oplus J_{-1} \subseteq J_2(e)$. Thus, we have

$$J_0 = J_0(e) \oplus J_1(e), \qquad J_1 \oplus J_{-1} = J_2(e). \tag{4}$$

For y_0 we can be more precise since $e \circ y_0 = x^2 \circ y_0 = -2U_x y_0$ (by 5.1(iii) using (3)) $= 0$, so that

$$y_0 \in J_0(e). \tag{5}$$

We recall a fact that will be need later,

$$U_x U_{y_0} = U_{y_0} U_x, \tag{6}$$

which follows directly from (3), the fact that $U_x y_0 = 0$, and 5.2(v).

Now we show $U_x U_y = U_y U_x$ by checking that both sides coincide when restricted to J_0 and $J_2(e)$.
(I) If $z \in J_0$, then

$$U_y z = U_{y_0 + y_{-1}} z = U_{y_0} z + U_{y_{-1}} z + \{y_0 z y_{-1}\} = U_{y_0} z + \{y_0 z y_{-1}\}$$

since $U_{y_{-1}} z \in U_{J_2(e)}(J_0(e) + J_1(e))$ (by (4)) $= 0$. Thus

$$U_x U_y z = U_x U_{y_0} z + U_x \{y_0 z y_{-1}\} = U_x U_{y_0} z$$

since $\{y_0, z, y_{-1}\} \in \{J_0(e), J_0(e) + J_1(e), J_2(e)\}$ (by (4) and (5)) $\subseteq J_1(e) \subseteq J_0$. But by (6), $U_x U_{y_0} z = U_{y_0} U_x z = 0$, and we have shown $U_x U_y z = 0 = U_y U_x z$.
(II) If $z \in J_2(e)$, then

$$U_y z = U_{y_0 + y_{-1}} z = U_{y_0} z + U_{y_{-1}} z + \{y_0 z y_{-1}\} = U_{y_{-1}} z \tag{7}$$

since $U_{y_0} z \in U_{J_0(e)} J_2(e)$ (by (5)) $= 0$, and $\{y_0 z y_{-1}\} \subseteq \{J_0(e) J_2(e) J\}$ (by (5)) $= 0$. Now $U_x z \in J_2(e)$ since $x = x^3 = U_x x \in J_1 \subseteq J_2(e)$ by (4), hence (7) implies $U_y U_x z = U_{y_{-1}} U_x z$, so that $U_x U_y z = U_y U_x z$ reduces to $U_x U_{y_{-1}} z = U_{y_{-1}} U_x z$ which holds by Prop. 6.2: $x, z, y_{-1} \in J_2(e)$ by (4), $J_2(e)$ is a Jordan subalgebra of J, $x \circ y_{-1} = 0$ by (3), and x is algebraic of degree two or less in $J_2(e)$ since $0 = x^2 - e = x^2 - 1_{J_2(e)}$. □

7. Pointed quadratic forms.

Since question (1.1) in its general form has an affirmative answer for special Jordan algebras (see 4.1), we have to focus attention on exceptional ones. While Jordan algebras of pointed quadratic forms are special under mild regularity conditions (see [8, Theorem 2.2.14] for a more precise statement), they are not so in general [8, p. 2.6]. Hence it makes sense to discuss them in the present context.

7.1. Trace and conjugation of a pointed quadratic form. In what follows we fix a *pointed quadratic form* $(M, q, 1)$ over k, so M is a k-module, $1 \in M$ is a distinguished element (the *base point*) and $q\colon\ M \to k$ is a quadratic forms (the *norm*) satisfying $q(1) = 1 \in k$. We call $t\colon\ M \to k$, $x \mapsto q(1, x)$, the *trace* and $\iota\colon\ M \to M$, $\quad x \mapsto \bar{x} := t(x)1 - x$, the *conjugation* of $(M, q, 1)$. By definition we have

$$q(1) = 1, \quad t(1) = 2, \tag{1}$$

which implies that the conjugation of $(M, q, 1)$ is a linear map of period 2 preserving base point, norm and trace:

$$\bar{1} = 1, \quad q(\bar{x}) = q(x), \quad t(\bar{x}) = t(x).$$

We also have

$$q(\bar{x}, y) = q(x, \bar{y}) = t(x)t(y) - q(x, y). \tag{2}$$

7.2. The Jordan algebra of a pointed quadratic form. We now consider the Jordan algebra $J = J(M, q, 1)$ associated with $(M, q, 1)$ [8, 2.1]. Recall, in particular, that $J = M$ as k-modules, $1_J = 1 \in M$ is the unit of J, and its U-operator acts on J via

$$U_x y = q(x, \bar{y})x - q(x)\bar{y}.$$

Linearizing gives $x \circ y = t(x)y + t(y)x - q(x, y)1$. In particular, the condition $x \circ y = 0$ is equivalent to

$$t(x)y + t(y)x = q(x, y)1. \tag{1}$$

Our aim is to derive a formula for $U_x U_y z$ (in terms of the norm and its polarization) that is symmetric in x and y. We will be able to do so but, unfortunately, only in the absence of 2-torsion. We begin by applying (7.1.2) to obtain

$$\begin{aligned} q(x)q(\bar{y}, z)\bar{y} &= q(x)\big(t(y)t(z) - q(y, z)\big)\big(t(y)1 - y\big) \\ &= q(x)t(y)^2 t(z)1 - q(x)t(y)t(z)y - q(x)t(y)q(y, z)1 + q(x)q(y, z)y, \end{aligned}$$

after which a straightforward verification yields

$$\begin{aligned} U_x U_y z = {}& q(x)q(y)z - q(x)q(y, z)y - q(y)q(x, z)x + \\ & q(x, \bar{y})q(\bar{y}, z)x - q(x)t(y)^2 t(z)1 + q(x)t(y)q(y, z)1 + q(x)t(y)t(z)y. \end{aligned} \tag{2}$$

Since the first three terms on the right-hand side of (2) form an expression that is symmetric in x and y, it will be enough to show that the remaining ones all belong to the 2-torsion part of J provided $x \circ y = 0$. This will be accomplished by the following lemma.

7.3. Lemma. *If $x, y \in J$ satisfy the relation $x \circ y = 0$, then*

$$2q(x)t(y) = 0 = 2t(x)q(y), \tag{1}$$

$$2q(x, y) = 2t(x)t(y), \tag{2}$$

$$2q(x, \bar{y}) = 0. \tag{3}$$

Proof. Given $z \in J$, we invoke (7.2.1) and obtain $q(x, y)t(z) = q(q(x, y)1, z) = q\big(t(x)y + t(y)x, z\big)$, hence

$$q(x, y)t(z) = t(x)q(y, z) + t(y)q(x, z). \tag{4}$$

Setting $z = x$ in (4) yields $q(x, y)t(x) = t(x)q(x, y) + 2q(x)t(y)$, hence the first relation of (1), which by symmetry implies the second. Setting $z = 1$ in (4) and applying (7.1.1) yields (2), which combines with (7.1.2) to yield (3). □

7.4. Theorem. *If $x, y \in J$ satisfy $x \circ y = 0$, then $2U_xU_y = 2U_yU_x$; in particular, if there is no 2-torsion, the operators U_x and U_y commute.*

Proof. Combining Lemma 7.3 with (7.2.2), we conclude that

$$2U_xU_yz = 2\big(q(x)q(y)z - q(x)q(y, z)y - q(y)q(x, z)x\big) \tag{1}$$

is symmetric in x and y, whence the assertion follows.

Remark. In the absence of 2-torsion, (7.4.1) yields the formula

$$U_xU_yz = q(x)q(y)z - q(x)q(y, z)y - q(y)q(x, z)x, \tag{2}$$

which continues to be symmetric in x and y. We do not know whether a similar formula holds in general.

8. Albert algebras.

In view of the results derived so far, particularly (7.4.2), one is tempted to conjecture that, given elements x, y, z satisfying $x \circ y = 0$ in a cubic Jordan algebra, there exists a formula, in terms of norm, trace and adjoint, for the expression U_xU_yz that is symmetric in x and y. Unfortunately, we have not been able to confirm this, even if the base ring is a field and low (positive) characteristics are excluded. Instead, we have to settle with the following theorem, giving an affirmative answer to question (1.1) for arbitrary Albert algebras.

8.1. Theorem. *Let J be an Albert algebra over a field F. If $x, y \in J$ satisfy the relation $x \circ y = 0$, then the operators U_x and U_y commute.*

The *proof* of this theorem requires a few preparations that will be developed as we go along.

8.2. Initiating the proof of Theorem 8.1. Changing scalars to the algebraic closure of F, we may assume that J is split. By the Jacobson embedding theorem [6, Theorem IX.11], which is valid in all characteristics [16], x is contained in a (unital) subalgebra of J isomorphic to A^+, $A := \mathrm{Mat}_3(F)$. We may therefore realize J as a first Tits construction via

$$J = J(A, 1) = A \oplus Aj_1 \oplus Aj_2 \tag{1}$$

as in 4.4 with $\mu = 1$ such that

$$x = x_0, \quad y = y_0 + y_1 j_1 + y_2 j_2 \qquad\qquad (x_0, y_0, y_1, y_2 \in A). \tag{2}$$

We wish to study the implication

$$x \circ y = 0 \Longrightarrow \forall z \in J : \; U_x U_y z = U_y U_x z, \tag{3}$$

and we will do so not only for $A = \mathrm{Mat}_3(F)$ as above. *Instead, we will assume from now on that A be any separable alternative algebra of degree 3 over F,* with generic norm N, generic trace T and adjoint $x \mapsto x^\sharp$. The reason for working in this more general context will become apparent in Corollary 8.12 below.

By linearity and (1), we may assume $z = z_0 \in A$ or $z = z_i j_i$, $z_i \in A$, $i = 1, 2$ in (3). If $\tau \colon A \to A$ is an anti-automorphism (e.g., $z \mapsto z^t$ in the special case $A = \mathrm{Mat}_3(F)$ considered before), it is readily checked that the linear bijection

$$J \xrightarrow{\sim} J, \quad z_0 + z_1 j_1 + z_2 j_2 \longmapsto \tau(z_0) + \tau(z_2) j_1 + \tau(z_1) j_2,$$

preserves base points, norms and adjoints, hence is an isomorphism. *For the purpose of proving Theorem 8.1 it will therefore be enough to consider the cases $z = z_0$ and $z = z_1 j_1$, $z_i \in A$, $i = 0, 1$ in (3).*

8.3. Lemma. *The following conditions are equivalent.*

(i) $x \circ y = 0$.

(ii) $x \circ y_0 = x \circ (y_1 j_1) = x \circ (y_2 j_2) = 0$.

(iii) $x_0 \circ y_0 = 0$, $x_0 y_1 = T(x_0) y_1$, $y_2 x_0 = T(x_0) y_2$.

Proof. From (4.4.2),(8.2.2) we conclude $x \times y = x_0 \times y_0 - (x_0 y_1) j_1 - (y_2 x_0) j_2$, and (4.2.8), (4.4.3), (4.4.4) imply

$$\begin{aligned} x \circ y = x_0 \times y_0 - (x_0 y_1) j_1 - (y_2 x_0) j_2 + T(x_0) y_0 + T(x_0)(y_1 j_1) + \\ T(x_0)(y_2 j_2) + T(y_0) x_0 - \big(T(x_0) T(y_0) - T(x_0, y_0)\big) 1 \\ = x_0 \circ y_0 + \big(T(x_0) y_1 - x_0 y_1\big) j_1 + \big(T(x_0) y_2 - y_2 x_0\big) j_2. \end{aligned}$$

The assertion follows. □

8.4. Iterated U-operators. Applying (4.2.5), (4.4.3),(4.4.1),(4.4.2), a straightforward computation shows, for all $z_0, z_1, z_2 \in A$.

$$U_x z_0 = U_{x_0} z_0 = x_0 z_0 x_0, \tag{1}$$

$$U_x(z_1 j_1) = (x_0^\sharp z_1) j_1, \tag{2}$$

$$U_x(z_2 j_2) = (z_2 x_0^\sharp) j_2, \tag{3}$$

$$\begin{aligned} U_y z_0 = {} & \big(U_{y_0} z_0 + (y_1 y_2) \times z_0\big) + \big(T(y_0 z_0) y_1 + z_0 y_2^\sharp - z_0 (y_0 y_1)\big) j_1 + \\ & \big(T(y_0 z_0) y_2 + y_1^\sharp z_0 - (y_2 y_0) z_0\big) j_2, \end{aligned} \tag{4}$$

$$\begin{aligned} U_y(z_1 j_1) = {} & \big(T(y_2 z_1) y_0 + z_1 y_1^\sharp - z_1 (y_2 y_0)\big) + \big(T(y_2 z_1) y_1 + y_0^\sharp z_1 - (y_1 y_2) z_1\big) j_1 + \\ & \big(U_{y_2} z_1 + (y_0 y_1) \times z_1\big) j_2, \end{aligned} \tag{5}$$

Making repeated use of (1)−(5), we now obtain

$$\begin{aligned} U_xU_yz_0 = \Big(&U_{x_0}U_{y_0}z_0 + x_0\big((y_1y_2)\times z_0\big)x_0\Big)+ \\ &\Big(T(y_0z_0)x_0^\sharp y_1 + x_0^\sharp(z_0y_2^\sharp) - x_0^\sharp\big(z_0(y_0y_1)\big)\Big)j_1+ \\ &\Big(T(y_0z_0)y_2x_0^\sharp + (y_1^\sharp z_0)x_0^\sharp - \big((y_2y_0)z_0\big)x_0^\sharp\Big)j_2, \end{aligned} \tag{6}$$

$$\begin{aligned} U_yU_xz_0 = \Big(&U_{y_0}U_{x_0}z_0 + (y_1y_2)\times(x_0z_0x_0)\Big)+ \\ &\Big(T\big((x_0y_0x_0)z_0\big)y_1 + (x_0z_0x_0)y_2^\sharp - (x_0z_0x_0)(y_0y_1)\Big)j_1+ \\ &\Big(T\big((x_0y_0x_0)z_0\big)y_2 + y_1^\sharp(x_0z_0x_0) - (y_2y_0)(x_0z_0x_0)\Big)j_2, \end{aligned} \tag{7}$$

$$\begin{aligned} U_xU_y(z_1j_1) = \Big(&T(y_2z_1)x_0y_0x_0 + x_0(z_1y_1^\sharp)x_0 - x_0\big(z_1(y_2y_0)\big)x_0\Big)+ \\ &\Big(T(y_2z_1)x_0^\sharp y_1 + x_0^\sharp(y_0^\sharp z_1) - x_0^\sharp\big((y_1y_2)z_1\big)\Big)j_1+ \\ &\Big((y_2z_1y_2)x_0^\sharp + \big((y_0y_1)\times z_1\big)x_0^\sharp\Big)j_2, \end{aligned} \tag{8}$$

$$\begin{aligned} U_yU_x(z_1j_1) = \Big(&T(y_2x_0^\sharp z_1)y_0 + (x_0^\sharp z_1)y_1^\sharp - (x_0^\sharp z_1)(y_2y_0)\Big)+ \\ &\Big(T(y_2x_0^\sharp z_1)y_1 + y_0^\sharp(x_0^\sharp z_1) - (y_1y_2)(x_0^\sharp z_1)\Big)j_1+ \\ &\Big(y_2(x_0^\sharp z_1)y_2 + (y_0y_1)\times(x_0^\sharp z_1)\Big)j_2. \end{aligned} \tag{9}$$

8.5. Lemma. *The following conditions are equivalent.*

(i) *Whenever $x \in A \subseteq J$ and $y \in J$ satisfy $x \circ y = 0$, then the linear operators U_xU_y and U_yU_x agree on $A \oplus Aj_1$.*

(ii) *Whenever $x_0, y_0, y_1, y_2 \in A$ satisfy the relations*

$$x_0 \circ y_0 = 0, \quad x_0y_1 = T(x_0)y_1, \quad y_2x_0 = T(x_0)y_2, \tag{1}$$

then

$$x_0\big((y_1y_2)\times z\big)x_0 = (y_1y_2)\times(x_0zx_0), \tag{2}$$
$$T(y_0z)x_0^\sharp y_1 - x_0^\sharp\big(z(y_0y_1)\big) = T\big((x_0y_0x_0)z\big)y_1 - (x_0zx_0)(y_0y_1), \tag{3}$$
$$x_0^\sharp(zy_2^\sharp) = (x_0zx_0)y_2^\sharp, \tag{4}$$
$$T(y_0z)y_2x_0^\sharp - \big((y_2y_0)z\big)x_0^\sharp = T\big((x_0y_0x_0)z\big)y_2 - (y_2y_0)(x_0zx_0), \tag{5}$$
$$(y_1^\sharp z)x_0^\sharp = y_1^\sharp(x_0zx_0), \tag{6}$$
$$T(y_2z)x_0y_0x_0 - x_0\big(z(y_2y_0)\big)x_0 = T(y_2x_0^\sharp z)y_0 - (x_0^\sharp z)(y_2y_0), \tag{7}$$
$$x_0(zy_1^\sharp)x_0 = (x_0^\sharp z)y_1^\sharp, \tag{8}$$
$$T(y_2z)x_0^\sharp y_1 - x_0^\sharp\big((y_1y_2)z\big) = T(y_2x_0^\sharp z)y_1 - (y_1y_2)(x_0^\sharp z), \tag{9}$$
$$x_0^\sharp(y_0^\sharp z) = y_0^\sharp(x_0^\sharp z), \tag{10}$$
$$(y_2zy_2)x_0^\sharp = y_2(x_0^\sharp z)y_2, \tag{11}$$
$$\big((y_0y_1)\times z\big)x_0^\sharp = (y_0y_1)\times(x_0^\sharp z) \tag{12}$$

for all $z \in A$.

Proof. By Lemma 8.3, elements $x_0, y_0, y_1, y_2 \in A$ satisfy (1) if and only if $x = x_0 \in A \subseteq J$ and $y = y_0 + y_1j_1 + y_2j_2 \in J$ satisfy $x \circ y = 0$. In this case, since A^+ is special, 4.1 implies $U_{x_0}U_{y_0} = U_{y_0}U_{x_0}$ on A. Hence the assertion follows from inspecting the equations (8.4.6)−(8.4.8). □

8.6. Continuing the proof of Theorem 8.1 *Until the end of the proof we assume that A is associative.* Combining the reductions carried out in 8.2 with Lemma 8.5, the implication (8.2.3) will follow once we have shown that (8.5.1) implies (8.5.2)−(8.5.12), so let us suppose from now on that (8.5.1) holds. By passing to A^{op} if necessary, (8.5.5) (resp. (8.5.4)) follows from (8.5.3) (resp. (8.5.6)). We are thus reduced to showing

$$(8.5.2),(8.5.3),(8.5.6)-(8.5.12). \tag{1}$$

Proof of (8.5.2). Applying (4.2.1),(8.5.1) and (4.3.4) first for $v_2 = 1$, then for $u_2 = 1$, we obtain

$$\begin{aligned} x_0\big((y_1y_2)\times z\big)x_0 &= T(x_0)\big((y_1y_2)\times z\big)x_0 - (x_0\times 1)\big((y_1y_2)\times z\big)x_0 \\ &= T(x_0)\big((y_1y_2)\times z\big)x_0 - \big((y_1y_2x_0)\times z\big)x_0 - \big((y_1y_2)\times (zx_0)\big)x_0 \\ &= -\big((y_1y_2)\times (zx_0)\big)x_0 \\ &= -T(x_0)\big(y_1y_2)\times (zx_0)\big) + \big((y_1y_2)\times (zx_0)\big)(x_0\times 1) \\ &= -T(x_0)\big((y_1y_2)\times (zx_0)\big) + (x_0y_1y_2)\times (zx_0) + (x_0zx_0)\times (y_1y_2) \\ &= (y_1y_2)\times (x_0zx_0). \end{aligned}$$

Proof of (8.5.3). This time we combine (4.2.1) with (4.3.2) and conclude

$$T(y_0z)x_0^\sharp y_1 - x_0^\sharp zy_0y_1 = x_0^\sharp\big(T(zy_0)1 - zy_0\big)y_1 = x_0^\sharp\big((zy_0)\times 1\big)y_1 = \big((zy_0x_0)\times x_0\big)y_1,$$

so by (4.2.8) and (8.5.1) we have

$$\begin{aligned} T(y_0z)x_0^\sharp y_1 - x_0^\sharp zy_0y_1 &= zy_0x_0^2y_1 + x_0zy_0x_0y_1 - T(zy_0x_0)x_0y_1 - \\ &\quad T(x_0)zy_0x_0y_1 + T(zy_0x_0)T(x_0)y_1 - T(zy_0x_0^2)y_1 \\ &= T(x_0)^2zy_0y_1 - x_0zx_0y_0y_1 - T(x_0)T(zy_0x_0)y_1 - \\ &\quad T(x_0)^2zy_0y_1 + T(zy_0x_0)T(x_0)y_1 + T(x_0y_0x_0z)y_1 \\ &= T(x_0y_0x_0z)y_1 - x_0zx_0y_0y_1. \end{aligned}$$

Proof of (8.5.10). Using (4.3.1) and (8.5.1), we obtain $x_0^\sharp y_0^\sharp z = (y_0x_0)^\sharp z = (-x_0y_0)^\sharp z = (x_0y_0)^\sharp z = y_0^\sharp x_0^\sharp z$, hence (8.5.10).
In view of (1), it remains to verify

$$(8.5.6)-(8.5.9),\ (8.5.11),(8.5.12). \tag{2}$$

In order to do so, we require a few further preparations.

8.7. Lemma. *Suppose $u, v \in A$ satisfy $uv = 0$. Then*

$$u^\sharp v = T(u^\sharp)v, \quad uv^\sharp = T(v^\sharp)u. \tag{1}$$

Moreover, $u^\sharp = 0$ or $v^\sharp = 0$.

Proof. By (4.2.7), $u^\sharp v = u^2v - T(u)uv + T(u^\sharp)1v = T(u^\sharp)v$, giving the first equation of (1). The second follows from the first by passing to A^{op}. Now suppose $v^\sharp \neq 0$. Then u cannot be invertible, forcing $N(u) = 0$. Taking adjoints in the first equation of (1) and applying (4.2.2),(4.3.1), we therefore obtain $0 = N(u)v^\sharp u = v^\sharp u^{\sharp\sharp} = (u^\sharp v)^\sharp = T(u^\sharp)^2v^\sharp$, hence $T(u^\sharp) = 0$. For any $a \in A$ we have $auv = 0$, so the preceding considerations apply to au in place of u and yield $T(u^\sharp a^\sharp) = 0$. Linearizing and applying (4.2.1), we conclude $0 = T(u^\sharp(1\times a)) = T(u^\sharp)T(a) - T(u^\sharp a) = -T(u^\sharp a)$, hence $u^\sharp = 0$ since A was assumed to be separable (cf. 4.3, 8.2). ☐

8.8. Lemma. *With the notations and assumptions of* 8.6 *we have*

$$x_0^\sharp y_1 = T(x_0^\sharp)y_1, \quad y_2 x_0^\sharp = T(x_0^\sharp)y_2, \tag{1}$$

$$y_1^\sharp x_0 = -T(x_0)y_1^\sharp, \quad x_0 y_2^\sharp = -T(x_0)y_2^\sharp. \tag{2}$$

If $y_1^\sharp \neq 0$ *or* $y_2^\sharp \neq 0$*, then*

$$x_0^\sharp = -T(x_0)x_0. \tag{3}$$

Proof. By passing to A^{op} if necessary, it suffices to establish the first equations of (1),(2) and to derive (3) under the assumption $y_1^\sharp \neq 0$.

Combining (4.2.7) with (8.5.1), we obtain $x_0^\sharp y_1 = x_0^2 y_1 - T(x_0)x_0 y_1 + T(x_0^\sharp)1y_1 = T(x_0)^2 y_1 - T(x_0)^2 y_1 + T(x_0^\sharp)y_1$, hence the first equation of (1). Moreover, combining (4.2.1) with (4.3.2) yields

$$\begin{aligned} y_1^\sharp x_0 &= T(x_0)y_1^\sharp - y_1^\sharp(x_0 \times 1) = T(x_0)y_1^\sharp - (x_0 y_1) \times y_1 \\ &= T(x_0)y_1^\sharp - T(x_0)y_1 \times y_1 = T(x_0)y_1^\sharp - 2T(x_0)y_1^\sharp, \end{aligned}$$

and this is the first equation of (2). It remains to prove (3) under the assumption $y_1^\sharp \neq 0$. Setting $\bar{x}_0 := T(x_0)1 - x_0$, we deduce $\bar{x}_0 y_1 = 0$ from (8.5.1), and Lemma 8.7 implies $0 = \bar{x}_0^\sharp = (T(x_0)1 - x_0)^\sharp = T(x_0)^2 1 - T(x_0)(1 \times x_0) + x_0^\sharp = T(x_0)^2 1 - T(x_0)^2 1 + T(x_0)x_0 + x_0^\sharp$, and (3) follows. ☐

8.9. Returning to the proof of Theorem 8.1. We now proceed to establish the equations of (8.6.2).

Proof of (8.5.6). The equation is obvious for $y_1^\sharp = 0$, allowing us to assume $y_1^\sharp \neq 0$. Then (8.8.3) holds, and if $T(x_0) \neq 0$, we conclude $y_1^\sharp x_0 z x_0 = -T(x_0)^{-1} y_1^\sharp x_0^\sharp z x_0 = -T(x_0)^{-1}(x_0 y_1)^\sharp z x_0 = -T(x_0)^{-1} T(x_0)^2 y_1^\sharp z x_0 = y_1^\sharp z x_0^\sharp$. Thus we are reduced to the case $T(x_0) = 0$, which implies $x_0^\sharp = 0$ by (8.8.3), hence $y_1^\sharp z x_0^\sharp = 0$, and (4.2.5),(8.8.2) yield $y_1^\sharp x_0 z x_0 = y_1^\sharp(U_{x_0} z) = y_1^\sharp T(x_0 z)x_0 = -T(x_0)T(x_0 z)y_1^\sharp = 0$.

Proof of (8.5.7). Consulting (4.2.1),(4.3.2),(8.5.1) and (4.2.8), we obtain

$$\begin{aligned} x_0^\sharp z y_2 &= T(zy_2)x_0^\sharp - x_0^\sharp\big((zy_2) \times 1\big) = T(zy_2)x_0^\sharp - (zy_2 x_0) \times x_0 \\ &= T(zy_2)x_0^\sharp - T(x_0)(zy_2) \times x_0 \\ &= T(zy_2)x_0^\sharp - T(x_0)zy_2 x_0 - T(x_0)x_0 z y_2 + T(x_0)T(zy_2)x_0 + \\ &\quad T(x_0)^2 z y_2 - T(x_0)^2 T(zy_2)1 + T(x_0)T(zy_2 x_0)1 \\ &= T(zy_2)x_0^\sharp - T(x_0)^2 z y_2 - T(x_0)x_0 z y_2 + T(x_0)T(zy_2)x_0 + \\ &\quad T(x_0)^2 z y_2 - T(x_0)^2 T(zy_2)1 + T(x_0)^2 T(zy_2)1 \\ &= T(zy_2)\big(x_0^\sharp + T(x_0)x_0\big) - T(x_0)x_0 z y_2. \end{aligned}$$

Combining this with (4.2.7) we deduce

$$x_0^\sharp z y_2 = T(zy_2)x_0^2 + T(x_0^\sharp)T(zy_2)1 - T(x_0)x_0 z y_2,$$

and (8.8.1) yields

$$\begin{aligned} T(y_2 x_0^\sharp z)y_0 - x_0^\sharp z y_2 y_0 &= T(x_0^\sharp)T(y_2 z)y_0 - T(zy_2)x_0^2 y_0 - \\ &\quad T(x_0^\sharp)T(zy_2)y_0 + T(x_0)x_0 z y_2 y_0 \\ &= T(zy_2)x_0 y_0 x_0 + x_0 z y_2 x_0 y_0 \\ &= T(y_2 z)x_0 y_0 x_0 - x_0 z y_2 y_0 x_0, \end{aligned}$$

as claimed.

Proof of (8.5.8). Again we may assume $y_1^\sharp \neq 0$, so (8.8.3) holds. Hence, by (8.8.2), $x_0 z y_1^\sharp x_0 = -T(x_0) x_0 z y_1^\sharp = x_0^\sharp z y_1^\sharp$.

Proof of (8.5.9),(8.5.11). We begin by applying (8.8.1) and obtain

$$T(y_2 z) x_0^\sharp y_1 - x_0^\sharp y_1 y_2 z = T(x_0^\sharp) T(y_2 z) y_1 - T(x_0^\sharp) y_1 y_2 z = T(y_2 x_0^\sharp z) y_1 - y_1 y_2 x_0^\sharp z,$$

giving (8.5.9). Similarly, $y_2 z y_2 x_0^\sharp = T(x_0^\sharp) y_2 z y_2 = y_2 x_0^\sharp z y_2$, giving (8.5.11).

We have thus verified all equations of (8.6.2) with the exception of (8.5.12), which turns out to be the most difficult. We begin with yet another technical result.

8.10. Lemma. *Let $c \in A$ be an element satisfying $T(c) = 1$ and $c^\sharp = 0$.*

(a) *c is an idempotent of A with the Peirce decomposition*

$$A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}, \quad A_{11} = Fc. \tag{1}$$

(b) *A_{22}^+ is the Jordan algebra of a pointed quadratic form. More precisely,*

$$A_{22}^+ = J\big(A_{22}, d, z_{22} \mapsto T(z_{22}^\sharp)\big), \quad d := 1 - c, \tag{2}$$

and the trace of A restricts to the trace of A_{22}^+. Furthermore

$$z_{22} \longmapsto \bar{z}_{22} = T(z_{22}) d - z_{22} = c \times z_{22} \tag{3}$$

is the conjugation of A_{22}^+ and an algebra involution of the associative algebra A_{22}. Finally,

$$c \times z_{12} = c \times z_{21} = z_{12}^\sharp = z_{21}^\sharp = 0, \tag{4}$$

$$z_{22}^\sharp = T(z_{22}^\sharp) c, \tag{5}$$

$$T(z_{12}) = T(z_{21}) = 0, \tag{6}$$

$$z_{12} \times z_{22} = -z_{12} \bar{z}_{22}, \tag{7}$$

$$z_{21} \times z_{22} = -\bar{z}_{22} z_{21}, \tag{8}$$

$$z_{12} \times z_{21} = -\overline{z_{21} z_{12}} \tag{9}$$

for all $z_{ij} \in A_{ij}$, $i, j = 1, 2$, $(i, j) \neq (1, 1)$.

Proof. (a) By (4.2.7), $c \in A$ is an idempotent, and if $x \in A$ satisfies $cx = x = xc$, then (4.2.5) yields $x = cxc = U_c x = T(cx) c$, proving (1).

(b) The first part is basically just Faulkner's lemma [4, Lemma 1.5]. Since A_{22} is an associative F-algebra of degree 2 in the sense of McCrimmon [13], its conjugation is an algebra involution by (4.2.4) and [13, Theorem 1.1]. The rest follows from a number of easy computations, using (2),(4.2.7),(4.2.8),(4.3.3),(4.3.2),(4.3.1): $c \times z_{22} = c z_{22} + z_{22} c - T(c) z_{22} - T(z_{22}) c + (T(c) T(z_{22}) - T(c z_{22})) 1 = -z_{22} - T(z_{22}) c + T(z_{22}) 1 = T(z_{22}) d - z_{22} = \bar{z}_{22}$, giving (3); $c \times z_{12} = c \times c z_{12} = (1 \times z_{12}) c^\sharp = 0$, $c \times z_{21} = c \times (z_{21} c) = c^\sharp (1 \times z_{21}) = 0$, $z_{12}^\sharp = (c z_{12})^\sharp = z_{12}^\sharp c^\sharp = 0$, $z_{21}^\sharp = (z_{21} c)^\sharp = c^\sharp z_{21}^\sharp = 0$, giving (4); $z_{22}^\sharp = z_{22}^2 - T(z_{22}) z_{22} + T(z_{22}^\sharp) 1 = -T(z_{22}^\sharp) d + T(z_{22}^\sharp) 1 = T(z_{22}^\sharp) c$, giving (5); $T(z_{12}) = T(c z_{12}) = T(z_{12} c) = 0$, $T(z_{21}) = T(z_{21} c) = T(c z_{21}) = 0$, giving (6); $z_{12} \times z_{22} = z_{12} z_{22} + z_{22} z_{12} - T(z_{12}) z_{22} - T(z_{22}) z_{12} + (T(z_{12}) T(z_{22}) - T(z_{12} z_{22})) 1 = z_{12} z_{22} - T(z_{22}) z_{12} d = -z_{12} \bar{z}_{22}$, giving (7); $z_{21} \times z_{22} = z_{21} z_{22} + z_{22} z_{21} - T(z_{21}) z_{22} - T(z_{22}) z_{21} + (T(z_{21}) T(z_{22}) - T(z_{21} z_{22})) 1 = z_{22} z_{21} - T(z_{22}) d z_{21} = -\bar{z}_{22} z_{21}$, giving (8); and finally, by (1),

$$\begin{aligned} z_{12} \times z_{21} &= z_{12} z_{21} + z_{21} z_{12} - T(z_{12}) z_{21} - T(z_{21}) z_{12} + \\ &\quad \big(T(z_{12}) T(z_{21}) - T(z_{12} z_{21})\big) 1 \\ &= T(z_{12} z_{21}) c + z_{21} z_{12} - T(z_{12} z_{21}) 1 \\ &= z_{21} z_{12} - T(z_{21} z_{12}) d = -\overline{z_{21} z_{12}}, \end{aligned}$$

giving (9). □

8.11. Finishing the proof of Theorem 8.1. We are now prepared to prove (8.5.12). To this end, we proceed in several steps.

1^0. We reduce to the case $y_1^\sharp = 0$. Indeed, if $y_1^\sharp \neq 0$, then (8.8.3) holds and (4.3.3) yields

$$\begin{aligned}\big((y_0y_1)\times z\big)x_0^\sharp &= (x_0y_0y_1)\times(x_0z) = -(y_0x_0y_1)\times(x_0z)\\ &= -(y_0y_1)\times\big(T(x_0)x_0z\big) = (y_0y_1)\times(x_0^\sharp z),\end{aligned}$$

as claimed.

2^0. Being allowed to assume $y_1^\sharp = 0$ by 1^0, we next reduce to the case $T(y_1) = 1$, forcing $y_1 \in A$ to be an idempotent as in Lemma 8.10. To see this, we assume (8.5.1) implies (8.5.12) if $y_1^\sharp = 0$ and $T(y_1) = 1$. Then suppose y_1 with (8.5.1) and $y_1^\sharp = 0$ is arbitrary, consider the subspace

$$I := \{v \in A \mid (v\times z)x_0^\sharp = v\times(x_0^\sharp z) \text{ for all } z \in A\}$$

of A and let $v \in I$, $w \in A$. Then (4.3.4) gives, for all $z \in A$,

$$\begin{aligned}\big((vw)\times z\big)x_0^\sharp + \big(v\times(zw)\big)x_0^\sharp &= (w\times 1)(v\times z)x_0^\sharp = (w\times 1)\big((v\times(x_0^\sharp z)\big)\\ &= (vw)\times(x_0^\sharp z) + \big(v\times(x_0^\sharp zw)\big)\\ &= (vw)\times(x_0^\sharp z) + \big(v\times(zw)\big)x_0^\sharp\end{aligned}$$

hence $vw \in I$, so $I \subseteq A$ is a right ideal. Now, by non-degeneracy of T, some $w \in A$ has $T(c) = 1$ with $c := y_1w$. Since, in addition, $c^\sharp = w^\sharp y_1^\sharp = 0$,

$$cy_1 = y_1wy_1 = T(c)y_1 - y_1^\sharp\times w = y_1, \quad x_0c = x_0y_1w = T(x_0)y_1w = T(x_0)c,$$

we conclude $y_0c \in I$ (by the special case whose validity we have assumed), forcing $y_0y_1 = y_0cy_1 \in I$ since $I \subseteq A$ is a right ideal. Hence (8.5.12) holds.

3^0. For the rest of the proof, we may and always will assume that the element $y_1 = c \in A$ satisfies $T(c) = 1$, $c^\sharp = 0$, allowing us to adopt the notation of Lemma 8.10. Since $x_0c = T(x_0)c$ by (8.5.1), we have, by (8.10.1),(8.10.6),

$$x_0 = T(x_0)c + x_{12} + x_{22}, \quad x_{12} \in A_{12}, \quad x_{22} \in A_{22}, \quad T(x_{22}) = 0. \tag{1}$$

This and (8.10.2),(8.10.3) yield

$$x_{22}^2 = -T(x_{22}^\sharp)d, \quad \bar{x}_{22} = -x_{22}. \tag{2}$$

Write

$$y_0 = \beta c + y_{12} + y_{21} + y_{22}, \quad \beta \in F, \quad y_{ij} \in A_{ij} \qquad (i,j = 1,2,\ (i,j) \neq (1,1)). \tag{3}$$

Comparing the Peirce components of $x_0y_0 = -y_0x_0$ (cf. (8.5.1)), we conclude

$$T(x_{12}y_{21}) = -2\beta T(x_0), \tag{4}$$

$$T(x_0)y_{12} + y_{12}x_{22} = -\beta x_{12} - x_{12}y_{22}, \tag{5}$$

$$x_{22}y_{21} = -T(x_0)y_{21}, \tag{6}$$

$$x_{22}y_{22} + y_{22}x_{22} = -y_{21}x_{12}. \tag{7}$$

Next we observe

$$y_0y_1 = y_0c = \beta c + y_{21} \tag{8}$$

by (3). Finally, we compute the Peirce components of $x_0^\sharp$ with the aid of (1), (2), (8.10.4), (8.10.5), (8.10.3), (8.10.7) to obtain

$$x_0^\sharp = T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22}. \tag{9}$$

In order to verify (8.5.12), it will be enough by linearity in z to consider the following cases.

4^0. $z = c$. Then by (8) and (8.10.4), $(y_0y_1) \times z = (\beta c + y_{21}) \times c = 2\beta c^\sharp + y_{21} \times c = 0$, so the left-hand side of (8.5.12) is zero as well, while (9) yields $x_0^\sharp z = (T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22})c = T(x_{22}^\sharp)c$, hence $(y_0y_1) \times (x_0^\sharp z) = T(x_{22}^\sharp)(\beta c + y_{21}) \times c = 0$, and the assertion is proved.

5^0. $z = z_{12} \in A_{12}$. Then by (8) and (8.10.4),(8.10.9), $(y_0y_1) \times z = (\beta c + y_{21}) \times z_{12} = -\overline{y_{21}z_{12}}$, forcing

$$\big((y_0y_1) \times z\big)x_0^\sharp = -\overline{y_{21}z_{12}}\big(T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22}\big) = T(x_0)\overline{y_{21}z_{12}}x_{22}. \tag{10}$$

On the other hand, $x_0^\sharp z = (T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22})z_{12} = T(x_{22}^\sharp)z_{12}$, which implies

$$(y_0y_1) \times (x_0^\sharp z) = T(x_{22}^\sharp)(\beta c + y_{21}) \times z_{12} = -T(x_{22}^\sharp)\overline{y_{21}z_{12}}. \tag{11}$$

Taking conjugates in (10),(11) and invoking (2), we therefore have to show

$$T(x_0)x_{22}y_{21} = T(x_{22}^\sharp)y_{21}. \tag{12}$$

But from (2),(6) we conclude that $T(x_0)x_{22}y_{21} = -x_{22}^2y_{21} = T(x_{22}^\sharp)dy_{21} = T(x_{22}^\sharp)y_{21}$, and (12) holds.

6^0. $z = z_{21} \in A_{21}$. Then by (8) and (8.10.4) linearized, $(y_0y_1) \times z = (\beta c + y_{21}) \times z_{21} = 0$, forcing the left-hand side of (8.5.12) to be zero as well. On the other hand, by (9),

$$\begin{aligned} x_0^\sharp z &= (T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22})z_{21} \\ &= x_{12}x_{22}z_{21} - T(x_0)x_{22}z_{21} = T(x_{12}x_{22}z_{21})c - T(x_0)x_{22}z_{21}, \end{aligned}$$

hence, again by by (8) and (8.10.4) linearized, $(y_0y_1) \times (x_0^\sharp z) = (\beta c + y_{21}) \times (T(x_{12}x_{22}z_{21})c - T(x_0)x_{22}z_{21}) = 0$, and the proof of (8.5.12) is complete.

7^0. $z = z_{22} \in A_{22}$. Then by (8) and (8.10.3),(8.10.8),

$$(y_0y_1) \times z = (\beta c + y_{21}) \times z_{22} = -\bar z_{22}y_{21} + \beta\bar z_{22},$$

which implies, using (9),

$$\big((y_0y_1) \times z\big)x_0^\sharp = \big(-\bar z_{22}y_{21} + \beta\bar z_{22}\big)\big(T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22}\big),$$

hence

$$\big((y_0y_1) \times z\big)x_0^\sharp = -T(x_{22}^\sharp)\bar z_{22}y_{21} - \bar z_{22}y_{21}x_{12}x_{22} - \beta T(x_0)\bar z_{22}x_{22}. \tag{13}$$

On the other hand, again by (9),

$$x_0^\sharp z = \big(T(x_{22}^\sharp)c + x_{12}x_{22} - T(x_0)x_{22}\big)z_{22} = x_{12}x_{22}z_{22} - T(x_0)x_{22}z_{22},$$

which by Lemma 8.10, particularly (8.10.3), (8.10.8), (8.10.9) and (2), implies

$$\begin{aligned} (y_0y_1) \times (x_0^\sharp z) &= \big(\beta c + y_{21}\big) \times \big(x_{12}x_{22}z_{22} - T(x_0)x_{22}z_{22}\big) \\ &= -\beta T(x_0)\overline{x_{22}z_{22}} - \overline{y_{21}x_{12}x_{22}z_{22}} + T(x_0)\overline{x_{22}z_{22}}y_{21} \\ &= -T(x_0)\bar z_{22}x_{22}y_{21} + \beta T(x_0)\bar z_{22}x_{22} + \bar z_{22}x_{22}\overline{y_{21}x_{12}}. \end{aligned}$$

Comparing this with (13), we see that it suffices to show

$$T(x_{22}^\sharp)y_{21} = T(x_0)x_{22}y_{21}, \quad y_{21}x_{12}x_{22} + 2\beta T(x_0)x_{22} + x_{22}\overline{y_{21}x_{12}} = 0.$$

Here the first equation agrees with (12), while the second one follows from (7),(4),(2) and

$$\begin{aligned}
y_{21}x_{12}x_{22} + {} & 2\beta T(x_0)x_{22} + x_{22}\overline{y_{21}x_{12}} \\
&= -x_{22}y_{22}x_{22} - y_{22}x_{22}^2 - T(y_{21}x_{12})x_{22} + x_{22}\overline{y_{21}x_{12}} \\
&= -x_{22}y_{22}x_{22} + T(x_{22}^\sharp)y_{22} - x_{22}\big(T(y_{21}x_{12})d - \overline{y_{21}x_{12}}\big) \\
&= -x_{22}y_{22}x_{22} + T(x_{22}^\sharp)y_{22} - x_{22}y_{21}x_{12} \\
&= -x_{22}y_{22}x_{22} + T(x_{22}^\sharp)y_{22} + x_{22}^2y_{22} + x_{22}y_{22}x_{22} \\
&= T(x_{22}^\sharp)y_{22} - T(x_{22}^\sharp)y_{22} = 0.
\end{aligned}$$

This concludes the proof of (8.5.12), hence of Theorem 8.1. □

8.12. Corollary. *Let A be a separable alternative algebra of degree 3 over F, with generic norm N, generic trace T, and adjoint $x \mapsto x^\sharp$. If $x_0, y_0, y_1, y_2 \in A$ satisfy* (8.5.1), *then the equations* (8.5.2)−(8.5.12) *hold.*

Proof. By hypothesis, the cubic Jordan algebra $J(A,1)$ over F is either special or Albert. In any event, we may conclude from 4.1 and Theorem 8.1 that Lemma 8.5 (i) holds. Hence so does Lemma 8.5 (ii). □

Remark. The only alternative algebras to which Corollary 8.12 applies but which are not associative have the form $A = F \oplus C$, with C an octonion algebra over F. It would be interesting to give a direct proof of the corollary in this special case.

9. Non-unital Jordan algebras.

After the preceding preparations, we are finally ready to tackle the main result of the paper. We begin by recalling the basic definitions.

9.1. The concept of a non-unital Jordan algebra. Following McCrimmon [12], we define a *non-unital (quadratic) Jordan algebra* over k as a k-module J together with two quadratic maps $J \to J$, $x \mapsto x^2$, (the *squaring*), and $U\colon J \to \mathrm{End}_k(J)$, $x \mapsto U_x$, (the *U-operator*) such that, setting

$$x \circ y := V_x y := (x+y)^2 - x^2 - y^2$$

(the *Jordan circle product*) and

$$\{xyz\} := V_{x,y}z := U_{x,z}y := (U_{x+z} - U_x - U_z)y$$

(the *Jordan triple product*), the following identities hold in all scalar extensions.

$$\begin{aligned}
V_{x,x}y &= x^2 \circ y, \\
U_x(x \circ y) &= x \circ U_x y \\
U_x x^2 &= (x^2)^2 \\
U_x U_y x^2 &= (U_x y)^2 \\
U_{x^2} &= {U_x}^2 \\
U_{U_x y} &= U_x U_y U_x.
\end{aligned}$$

Given a non-unital Jordan algebra J, a (Jordan) *subalgebra* of J is a k-submodule stable under the operations x^2 and $U_x y$. An *ideal* of J is k-submodule $I \subseteq J$ such that $I^2 + I \circ J + U_I J + U_J I + \{JJI\} \subseteq J$. An ideal $I \subseteq J$ is always a subalgebra, and the quotient J/I is a non-unital Jordan algebra in a natural way. If I_1, I_2 are ideals in J, so is $U_{I_1} I_2$; if even $U_{I_1} I_2 = \{0\}$, then the ideals I_1, I_2 are said to be *orthogonal*.

Every unital Jordan algebra can be viewed as a non-unital Jordan algebra with squaring defined by (2.1.1). Obviously, unital subalgebras then become subalgebras also in the non-unital sense.

9.2. Unitizations. Given a non-unital Jordan algebra J over k, let $k1$ be a free k-module of rank 1. Then the direct sum $\hat{J} := J \oplus k1$ becomes a unital Jordan algebra, the *(free) unital hull* of J, under the U-operator

$$U_{x+\lambda 1}(y + \mu 1) = (U_x y + \mu x^2 + 2\lambda\mu x + \lambda(x \circ y) + \lambda^2 y) + \lambda^2 \mu 1,$$

making $J \subseteq \hat{J}$ an ideal [12]. By factoring out a maximal ideal I of $\hat{J}$ not hitting J (which exists by Zorn's Lemma), we obtain the unital Jordan algebra $J' := \hat{J}/I$ (a *tight* unital hull of J) such that J embeds as an ideal in J' in a tight way, meaning that any nonzero ideal of J' hits J.

9.3. Extending results to the non-unital case. Since non-unital Jordan algebras can always be viewed as subalgebras of unital ones (9.2), many useful properties of the former retain their validity in the broader setting. In particular, 2.6 holds for arbitrary non-unital Jordan algebras, as do Propositions 5.1 and 5.2.

9.4. Non-degeneracy and primeness. Let J be a non-unital Jordan algebra over k. An element $z \in J$ is an *absolute zero divisor* if $U_z = 0$. We say J is *non-degenerate* if it does not contain absolute zero divisors other than zero. There is a unique smallest ideal in J, called its *McCrimmon radical* and denoted by $\mathrm{Mc}(J)$, making the quotient $J/\mathrm{Mc}(J)$ non-degenerate. J is said to *prime* if it does not contain non-zero orthogonal ideals. Non-unital Jordan algebras that are both prime and non-degenerate are called *strongly prime.*

After these preparations, we will now be able to establish the main result of the paper.

9.5. Theorem. *Let J be a non-degenerate non-unital Jordan algebra over k. If $x, y \in J$ satisfy $x \circ y = 0$, then U_x and U_y commute.*

Proof. We carry out a number of reductions that will eventually allow us to make use of our preceding answers to question (1.1). First of all, by [20, Corollary 4], J is a subalgebra of a direct product of strongly prime non-unital Jordan algebras. Hence we may assume that J itself is strongly prime. Now let J' be a tight unital hull of J. By tightness, J' is also prime (nonzero orthogonal ideals of J' would give rise to nonzero orthogonal ideals of J) and non-degenerate (we have $\mathrm{Mc}(J) = 0$, but also $\mathrm{Mc}(J) = \mathrm{Mc}(J') \cap J$ by [20, Corollary to Theorem 5], hence $\mathrm{Mc}(J') = 0$ by tightness). Thus we may assume that J is unital. Now the Zel′manov-McCrimmon structure theory [15, 15.1,15.4] implies that J is either special or an Albert form. Hence 4.1 and Theorem 8.1 yield the desired conclusion. □

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References

[1] P. Bruehne, *Ordnungen und die Tits-Konstruktionen von Albert-Algebren*, Ph.D. thesis, Fernuniversität in Hagen, 2000.

[2] T. De Medts and Y. Segev, *A course on Moufang sets*, Innov. Incidence Geom. **9** (2009), 79–122. MR 2658895 (2011h:20058)

[3] T. De Medts and R.M. Weiss, *Moufang sets and Jordan division algebras*, Math. Ann. **335** (2006), no. 2, 415–433. MR 2221120 (2007e:17027)

[4] J.R. Faulkner, *Octonion planes defined by quadratic Jordan algebras*, Memoirs of the American Mathematical Society, No. 104, American Mathematical Society, Providence, R.I., 1970. MR 0271180 (42 #6063)

[5] ______, *Finding octonion algebras in associative algebras*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1027–1030. MR 931729 (89g:17003)

[6] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, Vol. XXXIX, American Mathematical Society, Providence, R.I., 1968. MR 0251099 (40 #4330)

[7] ______, *Lectures on quadratic Jordan algebras*, Tata Institute of Fundamental Research, Bombay, 1969, Tata Institute of Fundamental Research Lectures on Mathematics, No. 45. MR 0325715 (48 #4062)

[8] ______, *Structure theory of Jordan algebras*, University of Arkansas Lecture Notes in Mathematics, vol. 5, University of Arkansas, Fayetteville, Ark., 1981. MR 634508 (83b:17015)

[9] R.E. Lewand and K. McCrimmon, *Macdonald's theorem for quadratic Jordan algebras*, Pacific J. Math. **35** (1970), 681–706. MR 0299648 (45 #8696)

[10] K. McCrimmon, *The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras*, Trans. Amer. Math. Soc. **139** (1969), 495–510. MR MR0238916 (39 #276)

[11] ______, *The Freudenthal-Springer-Tits constructions revisited*, Trans. Amer. Math. Soc. **148** (1970), 293–314. MR MR0271181 (42 #6064)

[12] ______, *Quadratic Jordan algebras and cubing operations*, Trans. Amer. Math. Soc. **153** (1971), 265–278. MR 0268239 (42 #3138)

[13] ______, *Nonassociative algebras with scalar involution*, Pacific J. Math. **116** (1985), no. 1, 85–109. MR MR769825 (86d:17003)

[14] ______, *A taste of Jordan algebras*, Universitext, Springer-Verlag, New York, 2004. MR 2014924 (2004i:17001)

[15] K. McCrimmon and E. Zel′manov, *The structure of strongly prime quadratic Jordan algebras*, Adv. in Math. **69** (1988), no. 2, 133–222. MR 946263 (89k:17052)

[16] H.P. Petersson, *The Jacobson embedding theorem for the split Albert algebra over arbitrary fields*, In preparation.

[17] ______, *On linear and quadratic Jordan division algebras*, Math. Z. **177** (1981), no. 4, 541–548. MR 82h:17013

[18] H.P. Petersson and M.L. Racine, *Jordan algebras of degree* 3 *and the Tits process*, J. Algebra **98** (1986), no. 1, 211–243. MR 87h:17038a

[19] T.A. Springer and F.D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. MR MR1763974 (2001f:17006)

[20] A. Thedy, *z-closed ideals of quadratic Jordan algebras*, Comm. Algebra **13** (1985), no. 12, 2537–2565. MR 811523 (87a:17026)

[21] J. Tits and R.M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. MR 1938841 (2003m:51008)