FINITE ELEMENT SPECTRAL ANALYSIS FOR THE MIXED FORMULATION OF THE ELASTICITY EQUATIONS

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Abstract. The aim of this paper is to analyze the linear elasticity eigenvalue problem formulated in terms of the stress tensor and the rotation. This is achieved by considering a mixed variational formulation in which the symmetry of the stress tensor is imposed weakly. We show that a discretization of the mixed eigenvalue elasticity problem with reduced symmetry based on the lowest order Arnold-Falk-Winther element provides a correct approximation of the spectrum and prove quasi-optimal error estimates. Finally, we report some numerical experiments.

Key words. Mixed elasticity equations, eigenvalue problem, finite elements, error estimates.

AMS subject classifications. 65N12, 65N15, 65N25, 65N30, 74B05

1. Introduction. We analyze in this paper a mixed finite element approximation of an eigenvalue problem arising in linear elasticity. The use of mixed methods for the numerical solution of elasticity problems may be motivated by the need of obtaining direct finite element approximations of stresses ensuring the equilibrium condition. It is also well-known that mixed methods are suitable to deal safely with nearly incompressible materials since they are free from the locking phenomenon.

The preservation of the stress tensor symmetry represents the more complicated issue in the construction of mixed finite elements in continuum mechanics. During the last decade, stable mixed finite element methods for linear elasticity, including strong and weakly imposed symmetry for the stresses, have been derived using mathematical tools based on the finite element exterior calculus (cf. [3, 4, 5, 6]). The first mixed finite elements known to be stable for the symmetric stress-displacement two-dimensional formulation is provided in [6]. A three-dimensional analogue of this element was proposed in [1]. We are interested here in mixed methods in which the symmetry of the stress tensor is imposed weakly by means of a suitable Lagrange multiplier. In spite of the introduction of an additional variable, these methods produce mixed finite elements with less degrees of freedom. One of the oldest methods in this category was introduced in [2]; it is based on the so called PEERS element. Recently, further stable elements with a weak symmetry condition for the stresses have been constructed in [3] and [5]. Proofs employing more classical techniques are given in [11] for some of the main results obtained in [3] and [5]. We illustrate here our spectral approximation theory for the mixed formulation of the elasticity problem by employing the lowest-order Arnold-Falk-Winther (AFW) element. It consists of piecewise linear approximations for the stress and piecewise constant functions for the rotation (as well as for the displacement, which will not appear as an unknown in our problem). We point out that we could as well have chosen other finite element

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methods, as the PEERS element, for instance, to obtain the same stability properties and error estimates.

The so called Babuška-Osborn abstract spectral approximation theory (cf. [7]) is a powerful tool to deal with compact operators. However, it is well know that, generally, the solution operator corresponding to mixed formulations fails to be compact. Actually, in our case, the mixed formulation of the problem admits an essential spectrum: the solution operator has $\mu = 1$ as an eigenvalue (that does not correspond to a physical vibration mode) with an infinite dimensional eigenspace. The approximation of eigenvalue problems in mixed form has been the object of several papers; among them we refer to [20]. An alternative analysis, covering the case of Stokes problem and the mixed form of a second order elliptic problem, is given in [12, 13] (see also Part 3 of [10] and the references therein). In particular, this analysis reveals that the natural conditions for the well-posedness and stability of source mixed problems are not sufficient to ensure a correct spectral approximation. Unfortunately, we can not take advantage here of the tools provided in [10, Part 3] since our mixed formulation can be casted neither to class $(f, 0)$ nor to class $(0, g)$ of the abstract framework considered there. Instead, we directly adapt results from [15] to prove that our mixed approximation is spectrally correct (safe from spurious modes) and to provide asymptotic error estimates.

The paper is organized as follows. In Section 2 we introduce a mixed formulation with reduced symmetry of the eigenvalue elasticity problem and define the solution operator. We point out that, in contrast to the usual dual-mixed formulation, the elastodynamic equation is used here to eliminate the displacement field. This leads to a method that is equivalent (both at the continuous and the discrete level) to the standard dual-mixed formulation (see [2]), which has the advantage of reducing the number of unknowns. Moreover, the discrete displacement field may be recovered by a local post-process procedure from the elastodynamic equation. Section 3 is devoted to the characterization of the spectrum of the solution operator. In Section 4 we introduce the discrete eigenvalue problem, describe the spectrum of the discrete solution operator, and provide the essential tools that allow us to show in Section 5 that the numerical scheme provides a correct spectral approximation. We also establish asymptotic error estimates for the eigenvalues and eigenfunctions. Finally, we present in Section 6 a set of numerical experiments to confirm that the method is not polluted with spurious modes and to show that the experimental rates of convergence are in accordance with the theoretical ones. We report numerical results obtained with the AFW and the PEERS elements, including nearly incompressible and perfectly incompressible materials, which demonstrate that both methods preserve its locking-free character when applied to the elasticity vibration problem. We end the paper with an appendix where we prove a couple of basic properties for spectral problems posed in terms of symmetric (although not positive definite) bilinear forms.

We end this section with some notation which will be used below. Given any Hilbert space V, let V^n and $V^{n\times n}$ denote, respectively, the space of vectors and tensors of order n ($n = 2$ or 3) with entries in V. In particular, $\boldsymbol{I}$ is the identity matrix of $\mathbb{R}^{n\times n}$. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{n\times n}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{n} \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^{n} \tau_{ij}\sigma_{ij}$.

Let Ω be a polyhedral Lipschitz bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$. For $s \geq 0$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathrm{H}^s(\Omega)^n$ or $\mathrm{H}^s(\Omega)^{n\times n}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define for $s \geq 0$

the Hilbert space $\mathrm{H}^s(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^{n\times n} : \ \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^n\}$, whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}^s(\mathbf{div};\Omega)} := \|\boldsymbol{\tau}\|^2_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{s,\Omega}$ and denote $\mathrm{H}(\mathbf{div};\Omega) := \mathrm{H}^0(\mathbf{div};\Omega)$.

Finally, we employ $\mathbf{0}$ to denote a generic null vector or tensor and use C to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The spectral problem. We assume that an isotropic and linearly elastic solid occupies a bounded and connected Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3). We denote by $\boldsymbol{\nu}$ the outward unit normal vector to the boundary $\partial\Omega$. We assume that $\partial\Omega$ admits a disjoint partition $\partial\Omega = \Gamma \cup \Sigma$, the structure being fixed on Γ and free of stress on Σ. For the sake of simplicity, we also assume that both Γ and Σ has positive measure.

The constitutive equation relating the displacement field $\boldsymbol{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega,$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ is the linearized strain tensor, ∇ being the gradient tensor, and $\mathcal{C}$ is the elasticity operator, which we assume given by Hooke's law, i.e.,

$$\mathcal{C}\boldsymbol{\tau} := \lambda_{\mathrm{S}}\,(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu_{\mathrm{S}}\boldsymbol{\tau},$$

where λ_{S} and μ_{S} are the Lamé coefficients, which we assume constant. It is easy to check that the inverse of the elasticity operator $\mathcal{C}$ is given by

$$\mathcal{C}^{-1}\boldsymbol{\tau} := \frac{1}{2\mu_{\mathrm{S}}}\boldsymbol{\tau} - \frac{\lambda_{\mathrm{S}}}{2\mu_{\mathrm{S}}(n\lambda_{\mathrm{S}} + 2\mu_{\mathrm{S}})}\,(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}. \tag{2.1}$$

For nearly incompressible materials λ_{S} is too large in comparison with μ_{S}. However, notice that the coefficients in (2.1) do not blow up as $\lambda_{\mathrm{S}} \to \infty$.

Under the hypothesis of small oscillations, the classical approximation yields the following eigenvalue problem for the free vibration modes of the system and the corresponding natural frequencies $\omega \geq 0$:

$$\begin{aligned} \boldsymbol{\sigma} &= \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{in } \Omega,\\ \mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho_{\mathrm{S}}\boldsymbol{u} &= \mathbf{0} && \text{in } \Omega,\\ \boldsymbol{\sigma}\boldsymbol{\nu} &= \mathbf{0} && \text{on } \Sigma,\\ \boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma, \end{aligned}$$

where ρ_{S} is the density of the material. We assume ρ_{S} is a strictly positive constant.

We aim to employ a dual-mixed approach to derive a variational formulation of this problem. The main unknown will be the stress tensor $\boldsymbol{\sigma}$, which will be sought in the following subspace of $\mathrm{H}(\mathbf{div};\Omega)$:

$$\boldsymbol{\mathcal{W}} := \{\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div};\Omega) : \ \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\}.$$

Thanks to the boundedness of the normal trace operator $\boldsymbol{\tau} \mapsto \boldsymbol{\tau}\boldsymbol{\nu}$ from $\mathrm{H}(\mathbf{div};\Omega)$ onto $\mathrm{H}^{-1/2}(\partial\Omega)$ and to the continuity of the restriction operator from $\mathrm{H}^{-1/2}(\partial\Omega)$ to $\mathrm{H}^{1/2}_{00}(\Sigma)'$, we conclude that $\boldsymbol{\mathcal{W}}$ is a closed subspace of $\mathrm{H}(\mathbf{div};\Omega)$. (We recall that $\mathrm{H}^{1/2}_{00}(\Sigma)'$ is the dual of the space of functions from $\mathrm{H}^{1/2}(\Sigma)$ whose extension by zero to the whole boundary $\partial\Omega$ belongs to $\mathrm{H}^{1/2}(\partial\Omega)$.)

As will be shown below, the displacement field $\boldsymbol{u}$ disappears from the formulation, while the rotation

$$\boldsymbol{r} := \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathrm{t}}\right]$$

is introduced as a further unknown. Notice that this new variable, $\boldsymbol{r}$, belongs to the space

$$\boldsymbol{\mathcal{Q}} := \left\{\boldsymbol{s} \in \mathrm{L}^2(\Omega)^{n\times n} : \ \boldsymbol{s}^{\mathrm{t}} = -\boldsymbol{s}\right\}$$

of skew-symmetric tensors.

We endow $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ with the $\mathrm{H}(\mathbf{div};\Omega) \times \mathrm{L}^2(\Omega)^{n\times n}$ norm, which we will simply denote $\|\cdot\|$, as well as the corresponding induced norm of operators acting from $\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ into the same space.

By using the new variable $\boldsymbol{r}$ and (2.1), the constitutive equation can be rewritten

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla\boldsymbol{u} - \boldsymbol{r}.$$

Testing this equation with $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$ and integrating by parts yield

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} = -\int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{\tau} : \boldsymbol{r}.$$

Hence, from the elastodynamic equation, $\mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho_{\mathrm{S}}\boldsymbol{u} = \mathbf{0}$, we obtain

$$\omega^2\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} - \int_\Omega \frac{1}{\rho_{\mathrm{S}}}\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \omega^2\int_\Omega \boldsymbol{\tau} : \boldsymbol{r} = 0 \qquad \forall\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

Finally, the symmetry of $\boldsymbol{\sigma}$ is imposed weakly through the following equation:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{s} = 0 \qquad \forall\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{2.2}$$

Combining the last two equations, we arrive at the following variational eigenvalue problem in which $\lambda := \omega^2$:

Problem 2.1. *Find $\lambda \in \mathbb{R}$, $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$, and $\boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$, such that $(\boldsymbol{\sigma}, \boldsymbol{r}) \neq \mathbf{0}$ and*

$$\int_\Omega \frac{1}{\rho_{\mathrm{S}}}\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} = \lambda\left(\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{r}\right) \qquad \forall\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

$$\lambda\int_\Omega \boldsymbol{\sigma} : \boldsymbol{s} = 0 \qquad \forall\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}.$$

Notice that to obtain a symmetric variational eigenvalue problem, we have multiplied the constraint (2.2) by the eigenvalue λ. Therefore, the symmetry of the stress $\boldsymbol{\sigma}$ imposed by this constraint is lost for $\lambda = 0$, which actually is an eigenvalue of Problem 2.1. However, this is not relevant in practice, because $\lambda = 0$ does not correspond to a physical vibration mode of the structure. In fact, it is a spurious eigenvalue of this mixed formulation for the elasticity equation, which would be present even though the last equation were not multiplied by λ.

We introduce the bilinear forms

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \frac{1}{\rho_{\mathrm{S}}}\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau}, \qquad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

$$b(\boldsymbol{\tau}, \boldsymbol{s}) := \int_\Omega \boldsymbol{\tau} : \boldsymbol{s}, \qquad \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \quad \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}.$$

Then, by a shift argument, the eigenvalue problem above can be rewritten as follows:

PROBLEM 2.2. *Find* $\lambda \in \mathbb{R}$, $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$, *and* $\boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$, *such that* $(\boldsymbol{\sigma}, \boldsymbol{r}) \neq \boldsymbol{0}$ *and*

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{r}) = (\lambda + 1) \left[\int_\Omega \mathcal{C}^{-1} \boldsymbol{\sigma} : \boldsymbol{\tau} + b(\boldsymbol{\tau}, \boldsymbol{r}) \right] \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$
$$b(\boldsymbol{\sigma}, \boldsymbol{s}) = (\lambda + 1)\, b(\boldsymbol{\sigma}, \boldsymbol{s}) \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}.$$

Next, we define the corresponding solution operator:

$$\begin{aligned} \boldsymbol{T} : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} &\longrightarrow \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}, \\ (\boldsymbol{f}, \boldsymbol{g}) &\longmapsto (\boldsymbol{\sigma}^*, \boldsymbol{r}^*), \end{aligned}$$

where $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)$ is the solution of the following source problem:

$$a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{r}^*) = \int_\Omega \mathcal{C}^{-1} \boldsymbol{f} : \boldsymbol{\tau} + b(\boldsymbol{\tau}, \boldsymbol{g}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{2.3}$$

$$b(\boldsymbol{\sigma}^*, \boldsymbol{s}) = b(\boldsymbol{f}, \boldsymbol{s}) \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{2.4}$$

The Babuška-Brezzi theory shows that this problem is well posed. Indeed, the inf-sup condition for the bilinear form b, namely

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{b(\boldsymbol{\tau}, \boldsymbol{s})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)}} \geq \beta \, \|\boldsymbol{s}\|_{0,\Omega} \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}},$$

is an immediate consequence of the following global condition (see, for instance, [11]):

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\displaystyle\int_\Omega \mathbf{div}\, \boldsymbol{\tau} \cdot \boldsymbol{v} + b(\boldsymbol{\tau}, \boldsymbol{s})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)}} \geq \beta \left(\|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega} \right) \qquad \forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}. \tag{2.5}$$

On the other hand, the identity

$$\mathcal{C}^{-1} \boldsymbol{\tau} : \boldsymbol{\tau} = \frac{1}{n\lambda_{\mathrm{S}} + 2\mu_{\mathrm{S}}} (\mathrm{tr}\, \boldsymbol{\tau})^2 + \frac{1}{2\mu_{\mathrm{S}}} \boldsymbol{\tau}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} \tag{2.6}$$

yields

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \geq \frac{1}{2\mu_S} \left\| \boldsymbol{\tau}^{\mathtt{D}} \right\|_{0,\Omega}^2 + \frac{1}{\rho_S} \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega}^2 \qquad \forall \boldsymbol{\tau} \in \mathrm{H}(\mathbf{div}; \Omega), \tag{2.7}$$

which allows us to prove the following lemma regarding the ellipticity of $a(\cdot, \cdot)$.

LEMMA 2.1. *There exists a constant* $\alpha > 0$, *depending on* μ_{S}, ρ_{S}, *and* Ω *(but not on* λ_{S}*), such that*

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \geq \alpha \, \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)}^2 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

Proof. For each $\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div}; \Omega)$, let $\boldsymbol{\tau}_0 := \boldsymbol{\tau} - \frac{1}{n|\Omega|} \left(\int_\Omega \mathrm{tr}\, \boldsymbol{\tau} \right) \boldsymbol{I}$. For $n = 2$, it is proved in [14, Proposition IV.3.1] that there exists $C > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau}_0\|_{0,\Omega}^2 \leq C \left(\left\| \boldsymbol{\tau}^{\mathtt{D}} \right\|_{0,\Omega}^2 + \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega}^2 \right) \qquad \forall \boldsymbol{\tau} \in \mathrm{H}(\mathbf{div}; \Omega).$$

The same proof runs for $n = 3$, too. On the other hand, the proof from [18, Lemma 2.2] can be easily adapted to our case to show that there exists $C > 0$, also depending only on Ω, such that

$$\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div};\Omega)} \leq C \|\boldsymbol{\tau}_0\|^2_{\mathrm{H}(\mathbf{div};\Omega)} \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

The result follows now directly from the last two inequalities, the fact that $\mathbf{div}\,\boldsymbol{\tau}_0 = \mathbf{div}\,\boldsymbol{\tau}$ in Ω, and (2.7). Thus we conclude the proof. □

The linear operator $\boldsymbol{T}$ is then well defined and bounded. The norm of this operator remains bounded in the nearly incompressible case (i.e., when $\lambda_S \to \infty$). Notice that $(\lambda, \boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbb{R} \times \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ solves Problem 2.1 if and only if $(\mu, (\boldsymbol{\sigma}, \boldsymbol{r}))$, with $\mu = 1/(1+\lambda)$, is an eigenpair of $\boldsymbol{T}$, i.e., if and only if

$$\boldsymbol{T}(\boldsymbol{\sigma}, \boldsymbol{r}) = \frac{1}{1+\lambda}\,(\boldsymbol{\sigma}, \boldsymbol{r})\,.$$

Because of this, next step is to obtain a spectral characterization of this operator.

3. Spectral characterization. We need to describe the spectrum of the solution operator, $\mathrm{sp}(\boldsymbol{T})$, to obtain complete information about the solutions of our original problem. To accomplish this task we will decompose the space $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ into a convenient direct sum. Let

$$\boldsymbol{\mathcal{K}} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} : \ \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \ \text{ in } \Omega\}\,.$$

From the definition of $\boldsymbol{T}$, it is clear that $\boldsymbol{T}|_{\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}} : \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ reduces to the identity. Thus, $\mu = 1$ is an eigenvalue of $\boldsymbol{T}$. Moreover, if $(\boldsymbol{\sigma}, \boldsymbol{r})$ is an associated eigenfunction, then, from the definition of $\boldsymbol{T}$ again, $\int_\Omega \frac{1}{\rho_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} = 0$ for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$. Hence, $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0}$ in Ω, so that $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$. Therefore, we have proved the following result:

LEMMA 3.1. *$\mu = 1$ is an eigenvalue of $\boldsymbol{T}$, with associated eigenspace $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$.*

Next, we define the auxiliary operator

$$\begin{aligned} \boldsymbol{P} : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} &\longrightarrow \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}, \\ (\boldsymbol{\sigma}, \boldsymbol{r}) &\longmapsto (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{r}}), \end{aligned}$$

where $(\widetilde{\boldsymbol{\sigma}}, (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \boldsymbol{\mathcal{W}} \times \left[\mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}\right]$ is the solution of the following problem:

$$\int_\Omega \mathcal{C}^{-1}\widetilde{\boldsymbol{\sigma}} : \boldsymbol{\tau} + \int_\Omega \widetilde{\boldsymbol{u}} \cdot \mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \widetilde{\boldsymbol{r}} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{3.1}$$

$$\int_\Omega \boldsymbol{v} \cdot \mathbf{div}\,\widetilde{\boldsymbol{\sigma}} + \int_\Omega \widetilde{\boldsymbol{\sigma}} : \boldsymbol{s} = \int_\Omega \boldsymbol{v} \cdot \mathbf{div}\,\boldsymbol{\sigma} \qquad \forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}. \tag{3.2}$$

The latter is a well posed mixed problem. In fact, the ellipticity in the kernel property is an immediate consequence of Lemma 2.1, whereas the corresponding inf-sup condition is (2.5). Therefore, $\boldsymbol{P}$ is a well posed linear operator.

Problem (3.1)–(3.2) is the well-known dual-mixed formulation with weakly imposed symmetry of the following classical elasticity problem with volumetric force density $-\,\mathbf{div}\,\boldsymbol{\sigma}$:

$$-\,\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} = -\,\mathbf{div}\,\boldsymbol{\sigma} \qquad \text{in } \Omega, \tag{3.3}$$

$$\widetilde{\boldsymbol{\sigma}} = \mathcal{C}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) \qquad \text{in } \Omega, \tag{3.4}$$

$$\widetilde{\boldsymbol{\sigma}}\boldsymbol{\nu} = \mathbf{0} \qquad \text{on } \Sigma, \tag{3.5}$$

$$\widetilde{\boldsymbol{u}} = \mathbf{0} \qquad \text{on } \Gamma. \tag{3.6}$$

In fact, it is straightforward to check that $(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}) \in \mathrm{H}(\mathbf{div};\Omega) \times \mathrm{H}^1(\Omega)^n$ satisfies these equations if and only if $(\widetilde{\boldsymbol{\sigma}}, (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \boldsymbol{\mathcal{W}} \times \left[\mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}\right]$ is the solution to (3.1)–(3.2), with $\widetilde{\boldsymbol{r}} = \frac{1}{2}\left[\nabla\widetilde{\boldsymbol{u}} - (\nabla\widetilde{\boldsymbol{u}})^{\mathrm{t}}\right]$.

Owing to the regularity result for the classical elasticity problem (see, for instance, [17]), we know that the solution $\widetilde{\boldsymbol{u}}$ to (3.3)–(3.6) belongs to $\mathrm{H}^{1+s}(\Omega)^n$ for some $s \in (0,1]$ depending on the geometry of Ω and the Lamé coefficients and

$$\|\widetilde{\boldsymbol{u}}\|_{1+s,\Omega} \leq C \,\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}\,, \tag{3.7}$$

with $C > 0$ independent of $\boldsymbol{\sigma}$. From now on, $s \in (0,1]$ denotes a constant such that this inequality holds true. The following lemma summarizes these additional regularity results.

LEMMA 3.2. *There exists $C > 0$ such that for all $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, if $(\widetilde{\boldsymbol{\sigma}}, (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \boldsymbol{\mathcal{W}} \times \left[\mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}\right]$ is the solution to equations* (3.1)–(3.2)*, then*

$$\|\widetilde{\boldsymbol{\sigma}}\|_{s,\Omega} + \|\widetilde{\boldsymbol{u}}\|_{1+s,\Omega} + \|\widetilde{\boldsymbol{r}}\|_{s,\Omega} \leq C \,\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}\,.$$

Consequently, $\boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \subset \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n}$.

Since (3.2) implies that $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} = \mathbf{div}\,\boldsymbol{\sigma}$ in Ω, it is easy to check that the operator $\boldsymbol{P}$ is idempotent and that its kernel is given by $\mathrm{Ker}(\boldsymbol{P}) = \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$. Therefore, being $\boldsymbol{P}$ a projector, $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} = (\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}) \oplus \boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$. In what follows we will obtain an alternative characterization of $\boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$, which will be useful to show that $\boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ is an invariant subspace of $\boldsymbol{T}$ (as is the case with $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$, too).

With this end, let us us rewrite the equations of Problem 2.2 as follows:

$$A((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) = (\lambda + 1)\, B((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) \qquad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}},$$

where, A and B are the bounded bilinear forms in $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ defined by

$$\begin{aligned} A((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) :&= a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{r}) + b(\boldsymbol{\sigma}, \boldsymbol{s}) \\ &= \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \frac{1}{\rho_{\mathrm{S}}}\,\mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{r} + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{s}, \\ B((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) :&= \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + b(\boldsymbol{\tau}, \boldsymbol{r}) + b(\boldsymbol{\sigma}, \boldsymbol{s}) \\ &= \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{r} + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{s}. \end{aligned}$$

Let

$$\boldsymbol{\mathcal{G}} := \left\{(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} : \ B((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) = 0 \ \ \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}\right\}.$$

In spite of the fact that B is not an inner product in $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, we prove in the appendix (cf. Proposition A.1) that $\boldsymbol{\mathcal{G}}$ is an invariant subspace of $\boldsymbol{T}$. In what follows, we will show that $\boldsymbol{\mathcal{G}} = \boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$. The first step is the following result.

LEMMA 3.3. $(\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}) \cap \boldsymbol{\mathcal{G}} = \{\mathbf{0}\}$.

Proof. Let $(\boldsymbol{\sigma}, \boldsymbol{r}) \in (\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}) \cap \boldsymbol{\mathcal{G}}$. Then, $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ and

$$\begin{aligned} &\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{r} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{K}}, \\ &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{s} = 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \end{aligned}$$

As an immediate consequence of (2.5) we have that the following inf-sup condition also holds true:

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}} \frac{\displaystyle\int_\Omega \operatorname{\mathbf{div}}\boldsymbol{\tau}\cdot\boldsymbol{v}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\operatorname{\mathbf{div}};\Omega)}} \geq \beta\,\|\boldsymbol{v}\|_{0,\Omega} \qquad \forall \boldsymbol{v}\in\mathrm{L}^2(\Omega)^n. \tag{3.8}$$

Hence, $(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$ is a solution to the problem above if and only if there exists $\boldsymbol{u}\in\mathrm{L}^2(\Omega)^n$ such that $(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{r}))\in\boldsymbol{\mathcal{W}}\times\left[\mathrm{L}^2(\Omega)^n\times\boldsymbol{\mathcal{Q}}\right]$ is a solution to the following problem:

$$\begin{aligned}
&\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma}:\boldsymbol{\tau}+\int_\Omega \boldsymbol{\tau}:\boldsymbol{r}+\int_\Omega \boldsymbol{u}\cdot\operatorname{\mathbf{div}}\boldsymbol{\tau}=0 \qquad \forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}},\\
&\int_\Omega \boldsymbol{\sigma}:\boldsymbol{s}+\int_\Omega \boldsymbol{v}\cdot\operatorname{\mathbf{div}}\boldsymbol{\sigma}=0 \qquad \forall(\boldsymbol{s},\boldsymbol{v})\in\boldsymbol{\mathcal{Q}}\times\mathrm{L}^2(\Omega)^n.
\end{aligned}$$

As stated above, this is a well posed problem, so that $(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{r}))=\mathbf{0}$ and we conclude the proof. ☐

Now we are in a position to prove the following result.

LEMMA 3.4. $\boldsymbol{P}(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})=\boldsymbol{\mathcal{G}}$.

Proof. Let $(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{r}})\in\boldsymbol{P}(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$. Then, from the definition of $\boldsymbol{P}$ and the fact that $\operatorname{\mathbf{div}}\boldsymbol{\tau}=\mathbf{0}$ for all $\boldsymbol{\tau}\in\boldsymbol{\mathcal{K}}$,

$$\begin{aligned}
&\int_\Omega \mathcal{C}^{-1}\widetilde{\boldsymbol{\sigma}}:\boldsymbol{\tau}+\int_\Omega \boldsymbol{\tau}:\widetilde{\boldsymbol{r}}=0 \qquad \forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{K}},\\
&\int_\Omega \widetilde{\boldsymbol{\sigma}}:\boldsymbol{s}=0 \qquad \forall\boldsymbol{s}\in\boldsymbol{\mathcal{Q}}.
\end{aligned}$$

Hence, from the definition of $\boldsymbol{\mathcal{G}}$, $(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{r}})\in\boldsymbol{\mathcal{G}}$.

Conversely, let $(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{G}}$ and let $(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{r}})=\boldsymbol{P}(\boldsymbol{\sigma},\boldsymbol{r})$. We have just proved that $(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{r}})\in\boldsymbol{\mathcal{G}}$, so that $(\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma},\widetilde{\boldsymbol{r}}-\boldsymbol{r})\in\boldsymbol{\mathcal{G}}$, too. Moreover, from the definition of $\boldsymbol{P}$, $\operatorname{\mathbf{div}}(\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma})=\mathbf{0}$ in Ω, so that $(\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma},\widetilde{\boldsymbol{r}}-\boldsymbol{r})\in\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$. Hence, according to Lemma 3.3, $(\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma},\widetilde{\boldsymbol{r}}-\boldsymbol{r})=\mathbf{0}$, so that $(\boldsymbol{\sigma},\boldsymbol{r})=(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{r}})=\boldsymbol{P}(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{P}(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$ and we conclude the proof. ☐

The following is the key point for the spectral characterization of $\boldsymbol{T}$.

PROPOSITION 3.5. *Subspace $\boldsymbol{\mathcal{G}}$ is invariant for $\boldsymbol{T}$,*

$$T(\boldsymbol{\mathcal{G}})\subset\left\{(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)\in\mathrm{H}^s(\Omega)^{n\times n}\times\mathrm{H}^s(\Omega)^{n\times n}:\ \operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega)^n\right\}, \tag{3.9}$$

and there exists $C>0$ such that for all $(\boldsymbol{f},\boldsymbol{g})\in\boldsymbol{\mathcal{G}}$, if $(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)=T(\boldsymbol{f},\boldsymbol{g})$, then

$$\|\boldsymbol{\sigma}^*\|_{s,\Omega}+\|\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\|_{1,\Omega}+\|\boldsymbol{r}^*\|_{s,\Omega}\leq C\,\|(\boldsymbol{f},\boldsymbol{g})\|. \tag{3.10}$$

Consequently, the operator $\boldsymbol{T}|_{\boldsymbol{\mathcal{G}}}:\boldsymbol{\mathcal{G}}\longrightarrow\boldsymbol{\mathcal{G}}$ is compact.

Proof. According to Proposition A.1 from the appendix, $\boldsymbol{T}(\boldsymbol{\mathcal{G}})\subset\boldsymbol{\mathcal{G}}$. Therefore, we have in particular that $\boldsymbol{T}|_{\boldsymbol{\mathcal{G}}}:\boldsymbol{\mathcal{G}}\longrightarrow\boldsymbol{\mathcal{G}}$ is correctly defined.

Let $(\boldsymbol{f},\boldsymbol{g})\in\boldsymbol{\mathcal{G}}$ and $(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)=\boldsymbol{T}(\boldsymbol{f},\boldsymbol{g})$. By testing (2.3) with $\boldsymbol{\tau}\in\mathcal{D}(\Omega)^{n\times n}\subset\boldsymbol{\mathcal{W}}$, we have that

$$\mathcal{C}^{-1}\boldsymbol{\sigma}^*-\nabla\left(\frac{1}{\rho_{\mathrm{S}}}\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\right)+\boldsymbol{r}^*=\mathcal{C}^{-1}\boldsymbol{f}+\boldsymbol{g}.$$

Then, since ρ_{S} is constant, we have that $\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega)^n$.

On the other hand, from Lemmas 3.4 and 3.2, we have that $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) \in \boldsymbol{T}(\boldsymbol{\mathcal{G}}) \subset \boldsymbol{\mathcal{G}} = \boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \subset \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n}$, so that (3.9) holds true. Moreover, by using Lemma 3.2 again, it is easy to check that (3.10) also hold true.

Finally, the compactness of $\boldsymbol{T}|_{\boldsymbol{\mathcal{G}}}$ follows from the fact that

$$\left\{(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) \in \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n} : \ \mathbf{div}\,\boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega)^n\right\} \cap (\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$$

is compactly included in $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$. Thus, we conclude the proof. ◻

The following result will be used combined with Proposition A.2 from the appendix to conclude that the eigenvalues of $\boldsymbol{T}$ are non-defective. Another immediate consequence of this result is that $\mu = 0$ is not an eigenvalue of $\boldsymbol{T}$.

Lemma 3.6. *For all non-vanishing* $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{G}}$,

$$A((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\sigma}, \boldsymbol{r})) \geq B((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\sigma}, \boldsymbol{r})) > 0.$$

Proof. The first inequality follows from the definition of A and B. To prove the second one, notice that by virtue (2.6) we have that

$$B((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\sigma}, \boldsymbol{r})) = \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \min\left\{\frac{n}{n\lambda_{\mathrm{S}} + 2\mu_{\mathrm{S}}}, \frac{1}{2\mu_{\mathrm{S}}}\right\} \|\boldsymbol{\sigma}\|_{0,\Omega}^2 \geq 0.$$

Moreover the expression above cannot vanish; otherwise $\boldsymbol{\sigma} = \mathbf{0}$ and, hence, $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \cap \boldsymbol{\mathcal{G}} = \{\mathbf{0}\}$ (cf. Lemma 3.3). Thus, we conclude the proof. ◻

We end this section with the spectral characterization of $\boldsymbol{T}$.

Theorem 3.7. *The spectrum of $\boldsymbol{T}$ decomposes as follows:* $\mathrm{sp}(\boldsymbol{T}) = \{0, 1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, *where:*

i) $\mu = 1$ *is an infinite-multiplicity eigenvalue of* $\boldsymbol{T}$ *and its associated eigenspace is* $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$;

ii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0, 1)$ *is a sequence of finite-multiplicity eigenvalues of* $\boldsymbol{T}$ *which converge to* 0 *and the corresponding eigenspaces lie on* $\boldsymbol{\mathcal{G}}$; *moreover the ascent of each of these eigenvalues is* 1;

iii) $\mu = 0$ *is not an eigenvalue of* $\boldsymbol{T}$.

Proof. Since $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} = (\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}) \oplus \boldsymbol{\mathcal{G}}$ (cf. Lemmas 3.3 and 3.4), $\boldsymbol{T}|_{\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}} : \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ is the identity and $\boldsymbol{T}|_{\boldsymbol{\mathcal{G}}} : \boldsymbol{\mathcal{G}} \longrightarrow \boldsymbol{\mathcal{G}}$ is compact (cf. Proposition 3.5), the decomposition of $\mathrm{sp}(\boldsymbol{T})$ follows from the spectral characterization of compact operators. Property (i) was established in Lemma 3.1. Property (ii) follows from Lemma 3.6 and Proposition A.2 from the appendix. Finally, property (iii) is an immediate consequence of Lemma 3.6, too. ◻

As an immediate consequence of Proposition 3.5 (cf. (3.10)) we have the following additional regularity result for the eigenfunctions of $\boldsymbol{T}$ lying on $\boldsymbol{\mathcal{G}}$.

Corollary 3.8. *Let* $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ *be an eigenfunction of* $\boldsymbol{T}$ *associated to an eigenvalue* $\mu \in (0, 1)$. *Then,* $\boldsymbol{\sigma}, \boldsymbol{r} \in \mathrm{H}^s(\Omega)^{n\times n}$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathrm{H}^1(\Omega)^n$, *and*

$$\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{1,\Omega} + \|\boldsymbol{r}\|_{s,\Omega} \leq C\,\|(\boldsymbol{\sigma}, \boldsymbol{r})\|,$$

with $C > 0$ *depending on the eigenvalue.*

4. The discrete problem. Let $\{\mathcal{T}_h(\Omega)\}_{h>0}$ be a shape-regular family of triangulations of the polyhedral (polygonal) region Ω by tetrahedrons (triangles) T with mesh size h. In what follows, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^n$, $\mathcal{P}_k(S)$ denotes the space of polynomials defined in S of total degree less or equal than k.

We define

$$\boldsymbol{\mathcal{W}}_h := \left\{ \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}} :\ \boldsymbol{\tau}_h|_T \in \mathcal{P}_1(T)^{n\times n}\ \forall T \in \mathcal{T}_h(\Omega) \right\}$$

and introduce the finite element subspace of $\boldsymbol{\mathcal{Q}}$ given by

$$\boldsymbol{\mathcal{Q}}_h := \left\{ \boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}} :\ \boldsymbol{s}_h|_T \in \mathcal{P}_0(T)^{n\times n}\ \forall T \in \mathcal{T}_h(\Omega) \right\}.$$

In addition, for the analysis below, we will also use the space

$$\boldsymbol{\mathcal{U}}_h := \left\{ \boldsymbol{v}_h \in \mathrm{L}^2(\Omega)^n :\ \boldsymbol{v}_h|_T \in \mathcal{P}_0(T)^n\ \forall T \in \mathcal{T}_h(\Omega) \right\}.$$

Notice that $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{U}}_h \times \boldsymbol{\mathcal{Q}}_h$ is the lowest-order mixed finite element of the family introduced for linear elasticity by Arnold, Falk and Winther (see [4, 5]).

Let us now recall some well-known approximation properties of the finite element spaces introduced above. Given $s \in (0,1]$, let $\boldsymbol{\Pi}_h : \mathrm{H}^s(\Omega)^{n\times n} \cap \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h$ be the usual BDM interpolation operator (see [14]), which is characterized by the identities

$$\int_F (\boldsymbol{\Pi}_h \boldsymbol{\tau})\, \boldsymbol{\nu}_F \cdot \boldsymbol{p} = \int_F \boldsymbol{\tau} \boldsymbol{\nu}_F \cdot \boldsymbol{p} \qquad \forall \boldsymbol{p} \in \mathcal{P}_1(F)^n$$

for all face (edge) F of $T \in \mathcal{T}_h(\Omega)$, with $\boldsymbol{\nu}_F$ being a unit normal vector to the face (edge) F. The following commuting diagram property holds true (cf. [14]):

$$\mathbf{div}(\boldsymbol{\Pi}_h \boldsymbol{\tau}) = \boldsymbol{L}_h(\mathbf{div}\, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^{n\times n} \cap \mathrm{H}(\mathbf{div};\Omega), \tag{4.1}$$

where $\boldsymbol{L}_h : \mathrm{L}^2(\Omega)^n \longrightarrow \boldsymbol{\mathcal{U}}_h$ is the $\mathrm{L}^2(\Omega)^n$-orthogonal projector. In addition, it is well known (see, e.g., [19, Theorem 3.16]) that there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^{n\times n} \cap \mathrm{H}(\mathbf{div};\Omega)$ there holds

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{0,\Omega} \leq C h^s \left(\|\boldsymbol{\tau}\|_{s,\Omega} + \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} \right). \tag{4.2}$$

Finally, we denote by $\boldsymbol{R}_h : \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $\mathrm{L}^2(\Omega)^{n\times n}$-norm. Then, for any $s \in (0,1]$, we have:

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)} \leq C h^s \|\boldsymbol{\tau}\|_{\mathrm{H}^s(\mathbf{div};\Omega)} \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\mathbf{div};\Omega) \cap \boldsymbol{\mathcal{W}}, \tag{4.3}$$

$$\|\boldsymbol{r} - \boldsymbol{R}_h \boldsymbol{r}\|_{0,\Omega} \leq C h^s \|\boldsymbol{r}\|_{s,\Omega} \qquad \forall \boldsymbol{r} \in \mathrm{H}^s(\Omega)^{n\times n} \cap \boldsymbol{\mathcal{Q}}, \tag{4.4}$$

$$\|\boldsymbol{v} - \boldsymbol{L}_h \boldsymbol{v}\|_{0,\Omega} \leq C h^s \|\boldsymbol{v}\|_{s,\Omega} \qquad \forall \boldsymbol{v} \in \mathrm{H}^s(\Omega)^n. \tag{4.5}$$

Notice that (4.3) is actually a straightforward consequence of (4.1), (4.2), and (4.5).

Let us now introduce the discrete counterpart of Problem 2.1:

Problem 4.1. *Find $\lambda_h \in \mathbb{R}$, $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$, and $\boldsymbol{r}_h \in \boldsymbol{\mathcal{Q}}_h$, such that $(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \neq \mathbf{0}$ and*

$$\int_\Omega \frac{1}{\rho_{\mathrm{S}}} \mathbf{div}\, \boldsymbol{\sigma}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h = \lambda_h \left(\int_\Omega \mathcal{C}^{-1} \boldsymbol{\sigma}_h : \boldsymbol{\tau}_h + \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{r}_h \right) \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h,$$

$$\lambda_h \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{s}_h = 0 \qquad \forall \boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h.$$

The discrete version of the operator $\boldsymbol{T}$ is then given by

$$\begin{aligned} \widetilde{\boldsymbol{T}}_h : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} &\longrightarrow \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}, \\ (\boldsymbol{f}, \boldsymbol{g}) &\longmapsto (\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*), \end{aligned}$$

where $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ is the solution of the following discrete source problem, in which the bilinear forms a and b are as in the previous section:

$$a(\boldsymbol{\sigma}_h^*, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{r}_h^*) = \int_\Omega \mathcal{C}^{-1}\boldsymbol{f} : \boldsymbol{\tau}_h + b(\boldsymbol{\tau}_h, \boldsymbol{g}) \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h,$$
$$b(\boldsymbol{\sigma}_h^*, \boldsymbol{s}_h) = b(\boldsymbol{f}, \boldsymbol{s}_h) \qquad \forall \boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h.$$

We can use the the classical Babuška-Brezzi theory to prove that $\widetilde{\boldsymbol{T}}_h$ is well defined and bounded uniformly with respect to h. Indeed, we already know from Lemma 2.1 that $a(\cdot,\cdot)$ is elliptic on the whole $\boldsymbol{\mathcal{W}}$ and the following discrete inf-sup condition is proved in [4, Theorem 11.9]: There exists $\beta^* > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \frac{\displaystyle\int_\Omega \operatorname{\mathbf{div}} \boldsymbol{\tau}_h \cdot \boldsymbol{v}_h + b(\boldsymbol{\tau}_h, \boldsymbol{s}_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega)}} \geq \beta^* \left(\|\boldsymbol{s}_h\|_{0,\Omega} + \|\boldsymbol{v}_h\|_{0,\Omega} \right) \tag{4.6}$$

for all $(\boldsymbol{v}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{U}}_h \times \boldsymbol{\mathcal{Q}}_h$. Moreover, the following Cea-like estimate holds true: There exists $C > 0$, independent of h, such that for all $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$,

$$\|\boldsymbol{T}(\boldsymbol{\sigma}, \boldsymbol{r}) - \widetilde{\boldsymbol{T}}_h(\boldsymbol{\sigma}, \boldsymbol{r})\| \leq C \inf_{(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{T}(\boldsymbol{\sigma}, \boldsymbol{r}) - (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|. \tag{4.7}$$

The reason why we have called this operator $\widetilde{\boldsymbol{T}}_h$, instead of just $\boldsymbol{T}_h$, is that we preserve this notation for its restriction onto the finite element space. In fact, since $\widetilde{\boldsymbol{T}}_h(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \subset \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$, we are allowed to define

$$\boldsymbol{T}_h := \widetilde{\boldsymbol{T}}_h|_{\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h} : \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \longrightarrow \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h.$$

It is well-known that $\operatorname{sp}(\widetilde{\boldsymbol{T}}_h) = \operatorname{sp}(\boldsymbol{T}_h) \cup \{0\}$ (see, for instance, [8, Lemma 4.1]).

Once more as in the continuous case, $(\lambda_h, \boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \mathbb{R} \times \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ solves Problem 4.1 if and only if $(\mu_h, (\boldsymbol{\sigma}_h, \boldsymbol{r}_h))$, with $\mu_h = 1/(1+\lambda_h)$, is an eigenpair of $\boldsymbol{T}_h$, i.e., if and only if

$$\boldsymbol{T}_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) = \frac{1}{1+\lambda_h} (\boldsymbol{\sigma}_h, \boldsymbol{r}_h).$$

To describe the spectrum of this operator, we will proceed as in the continuous case and decompose $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ in a convenient direct sum. With this end, we define

$$\boldsymbol{\mathcal{K}}_h := \boldsymbol{\mathcal{K}} \cap \boldsymbol{\mathcal{W}}_h = \{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h : \ \operatorname{\mathbf{div}} \boldsymbol{\tau}_h = \mathbf{0} \ \text{ in } \Omega\}.$$

Clearly $\boldsymbol{T}_h|_{\boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h} : \boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h \longrightarrow \boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h$ reduces to the identity. Thus, $\mu_h = 1$ is an eigenvalue of $\boldsymbol{T}_h$ and, from the definition of $\widetilde{\boldsymbol{T}}_h$, $(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)$ is an associated eigenfunction if and only if $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{K}}_h$. Therefore, we have the following discrete analogue to Lemma 3.1:

LEMMA 4.1. *$\mu_h = 1$ is an eigenvalue of $\boldsymbol{T}_h$, with associated eigenspace $\boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h$.*

Next step is to define the discrete analogue to the operator $\boldsymbol{P}$. Let

$$\boldsymbol{P}_h : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h,$$
$$(\boldsymbol{\sigma}, \boldsymbol{r}) \longmapsto (\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{r}}_h),$$

where $(\widetilde{\boldsymbol{\sigma}}_h, (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{r}}_h)) \in \boldsymbol{\mathcal{W}}_h \times (\boldsymbol{\mathcal{U}}_h \times \boldsymbol{\mathcal{Q}}_h)$ is the solution of the following problem:

$$\int_\Omega \mathcal{C}^{-1}\widetilde{\boldsymbol{\sigma}}_h : \boldsymbol{\tau}_h + \int_\Omega \widetilde{\boldsymbol{u}}_h \cdot \mathbf{div}\,\boldsymbol{\tau}_h + \int_\Omega \boldsymbol{\tau}_h : \widetilde{\boldsymbol{r}}_h = 0 \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h, \tag{4.8}$$

$$\int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_h + \int_\Omega \widetilde{\boldsymbol{\sigma}}_h : \boldsymbol{s}_h = \int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\boldsymbol{\sigma} \qquad \forall (\boldsymbol{v}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{U}}_h \times \boldsymbol{\mathcal{Q}}_h. \tag{4.9}$$

These equations are a finite element discretization of the mixed problem (3.1)–(3.2) used to define $\boldsymbol{P}$. The ellipticity in the kernel for the discrete problem follows easily from Lemma 2.1 and the fact that $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) \subset \boldsymbol{\mathcal{U}}_h$, whereas (4.6) is the corresponding discrete inf-sup condition. Hence, as a consequence of the Babuška-Brezzi theory, problem (4.8)–(4.9) is well posed, the operators $\boldsymbol{P}_h$ are bounded uniformly with respect to h, and the following Cea-like estimate also holds true:

$$\begin{aligned} &\|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{H}(\mathbf{div};\Omega)} + \|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{r}}_h\|_{0,\Omega} \\ &\quad \le C \left[\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega)} + \inf_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h} \|\widetilde{\boldsymbol{u}} - \boldsymbol{v}_h\|_{0,\Omega} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\widetilde{\boldsymbol{r}} - \boldsymbol{s}_h\|_{0,\Omega} \right], \end{aligned} \tag{4.10}$$

where $(\widetilde{\boldsymbol{\sigma}}, (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{r}}))$ and $(\widetilde{\boldsymbol{\sigma}}_h, (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{r}}_h))$ are the solutions to (3.1)–(3.2) and (4.8)–(4.9), respectively.

This estimate, combined with the approximation properties (4.3)–(4.5), lead to $\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}, \boldsymbol{r})\| \le Ch^s \left[\|\widetilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^s(\mathbf{div};\Omega)} + \|\widetilde{\boldsymbol{u}}\|_{s,\Omega} + \|\widetilde{\boldsymbol{r}}\|_{s,\Omega} \right]$. However, for this inequality to be meaningful, we need that $\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{r}} \in \mathrm{H}^s(\Omega)^{n\times n}$, $\widetilde{\boldsymbol{u}} \in \mathrm{H}^s(\Omega)^n$, and $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} \in \mathrm{H}^s(\Omega)^n$. According to Lemma 3.2 the former hold true. Instead, the latter cannot hold for an arbitrary $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$. In fact, from (3.2), $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} = \mathbf{div}\,\boldsymbol{\sigma}$ in Ω, so that $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}$ cannot be smoother than $\mathbf{div}\,\boldsymbol{\sigma}$. In spite of this fact, there are two cases in which an $\mathcal{O}(h^s)$ convergence for $\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}, \boldsymbol{r})\|$ can be proved; these cases are all what we will need for the spectral approximation theory in the following section.

LEMMA 4.2. *There exists $C > 0$ such that:*

i) if $(\boldsymbol{\sigma}, \boldsymbol{r})$ is an eigenfunction of $\boldsymbol{T}$ associated to an eigenvalue $\mu \in (0,1)$, then

$$\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}, \boldsymbol{r})\| \le Ch^s \|(\boldsymbol{\sigma}, \boldsymbol{r})\|;$$

ii) if $(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$, then

$$\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \le Ch^s \|\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega}.$$

Proof. Case (i). The estimate follows from (4.10), (4.3)–(4.5), Lemma 3.2, and Corollary 3.8.

Case (ii). For $(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$, let $(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{r}}) = \boldsymbol{P}(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)$ and $(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{r}}_h) = \boldsymbol{P}_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)$. By virtue of (4.10), (4.4), (4.5), and Lemma 3.2, we obtain

$$\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \le C \left[\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega)} + h^s \|\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} \right].$$

Since $\widetilde{\boldsymbol{\sigma}} \in \boldsymbol{\mathcal{W}} \cap \mathrm{H}^s(\Omega)^{n\times n}$ (cf. Lemma 3.2), we have that $\boldsymbol{\tau}_h := \boldsymbol{\Pi}_h \widetilde{\boldsymbol{\sigma}} \in \boldsymbol{\mathcal{W}}_h$ is well defined and, according to (4.2),

$$\|\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_h \widetilde{\boldsymbol{\sigma}}\|_{0,\Omega} \le Ch^s \left(\|\widetilde{\boldsymbol{\sigma}}\|_{s,\Omega} + \|\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}\|_{0,\Omega} \right).$$

On the other hand, from (3.2), $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} = \mathbf{div}\,\boldsymbol{\sigma}_h$ in Ω. Therefore, because of (4.1),

$$\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}}) = \boldsymbol{L}_h(\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}) = \boldsymbol{L}_h(\mathbf{div}\,\boldsymbol{\sigma}_h) = \mathbf{div}\,\boldsymbol{\sigma}_h = \mathbf{div}\,\widetilde{\boldsymbol{\sigma}},$$

which, with the last two inequalities and Lemma 3.2, allow us to end the proof. ∎

For $(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{r}}_h) = \boldsymbol{P}_h(\boldsymbol{\sigma}, \boldsymbol{r})$, (4.9) implies that $\int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_h = \int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\boldsymbol{\sigma}$ for all $\boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h$. Hence, it is easy to check that the operator $\boldsymbol{P}_h$ is idempotent and, then, so is $\boldsymbol{P}_h|_{\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h}$ too, because $\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}) \subset \boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h$. Moreover, it is easy to check that $\operatorname{Ker}(\boldsymbol{P}_h|_{\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h}) = \boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h$. Therefore, being $\boldsymbol{P}_h|_{\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h}$ a projector, we have that $\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h = (\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h)\oplus\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h)$.

Our next goal is to show that $\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h) = \boldsymbol{\mathcal{G}}_h$, where

$$\boldsymbol{\mathcal{G}}_h := \left\{(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h :\ B((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)) = 0 \ \ \forall(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h\right\},$$

with the bilinear form B being as in the previous section. Notice that as a consequence of Proposition A.1 from the appendix, $T_h(\boldsymbol{\mathcal{G}}_h) \subset \boldsymbol{\mathcal{G}}_h$. Moreover, we have the following discrete analogue to Lemma 3.3, too.

Lemma 4.3. $(\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h)\cap\boldsymbol{\mathcal{G}}_h = \{\mathbf{0}\}$.

Proof. Since the discrete inf-sup condition analogous to (3.8),

$$\sup_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h} \frac{\displaystyle\int_\Omega \mathbf{div}\,\boldsymbol{\tau}_h\cdot\boldsymbol{v}_h}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega)}} \geq \beta^*\,\|\boldsymbol{v}_h\|_{0,\Omega} \qquad \forall\boldsymbol{v}_h\in\boldsymbol{\mathcal{U}}_h,$$

follows from (4.6), the proof runs almost identically to that of Lemma 3.3. ∎

We skip the proofs of the following two lemmas, since they run almost identically to those of Lemmas 3.4 and 3.6, respectively.

Lemma 4.4. $\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h) = \boldsymbol{\mathcal{G}}_h$.

Lemma 4.5. *For all* $(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{G}}_h$,

$$A((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)) \geq B((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)) > 0.$$

Now, we are in a position to write down a characterization of the spectrum of the operator $\boldsymbol{T}_h$ and, hence, of the solutions to Problem 4.1.

Theorem 4.6. *The spectrum of* $\boldsymbol{T}_h$ *consists of* $M := \dim(\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h)$ *eigenvalues, repeated accordingly to their respective multiplicities. The spectrum decomposes as follows:* $\operatorname{sp}(\boldsymbol{T}_h) = \{1\}\cup\{\mu_{hk}\}_{k=1}^K$. *Moreover,*

i) the eigenspace associated to $\mu_h = 1$ *is* $\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h$*;*

ii) $\mu_{hk} \in (0,1)$, $k = 1, \ldots, K := M - \dim(\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h)$, *are non-defective eigenvalues with eigenspaces lying on* $\boldsymbol{\mathcal{G}}_h$*;*

iii) $\mu_h = 0$ *is not an eigenvalue of* $\boldsymbol{T}_h$.

Proof. Since $\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h = (\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h)\oplus\boldsymbol{\mathcal{G}}_h$ (cf. Lemmas 4.3 and 4.4), $\boldsymbol{T}_h|_{\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h} : \boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h \longrightarrow \boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h$ is the identity and $\boldsymbol{T}_h(\boldsymbol{\mathcal{G}}_h)\subset\boldsymbol{\mathcal{G}}_h$ (cf. Proposition A.1), the theorem follows from Lemmas 4.1 and 4.5 and Proposition A.2. ∎

5. Spectral approximation. To prove that $\boldsymbol{T}_h$ provides a correct spectral approximation of $\boldsymbol{T}$, we will resort to the corresponding theory for non-compact operators from [15]. With this end, for the sake of brevity, we will denote throughout this section $\boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{X}}_h := \boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h$. Moreover, when no confusion can arise, we will use indistinctly $\boldsymbol{x}$, $\boldsymbol{y}$, etc. to denote pairs of elements in $\boldsymbol{\mathcal{X}}$ and, analogously, $\boldsymbol{x}_h$, $\boldsymbol{y}_h$, etc. for those in $\boldsymbol{\mathcal{X}}_h$. Recall that $\|\cdot\|$ denotes the norm in $\boldsymbol{\mathcal{X}}$ as well as the corresponding induced norm on operators acting from $\boldsymbol{\mathcal{X}}$ into the same space. Finally, as in [15], we will use $\|\cdot\|_h$ to denote the norm of an operator restricted to the discrete subspace $\boldsymbol{\mathcal{X}}_h$; namely, if $\boldsymbol{S}:\boldsymbol{\mathcal{X}}\longrightarrow\boldsymbol{\mathcal{X}}$, then

$$\|\boldsymbol{S}\|_h := \sup_{\boldsymbol{x}_h\in\boldsymbol{\mathcal{X}}_h} \frac{\|\boldsymbol{S}\boldsymbol{x}_h\|}{\|\boldsymbol{x}_h\|}.$$

We recall some classical notation for spectral approximation. For $\boldsymbol{x}\in\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Z}}$ closed subspaces of $\boldsymbol{\mathcal{X}}$, we set

$$\delta(\boldsymbol{x},\boldsymbol{\mathcal{Y}}) := \inf_{\boldsymbol{y}\in\boldsymbol{\mathcal{Y}}} \|\boldsymbol{x}-\boldsymbol{y}\|, \qquad \delta(\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}) := \sup_{\boldsymbol{y}\in\boldsymbol{\mathcal{Y}}:\,\|\boldsymbol{y}\|=1} \delta(\boldsymbol{y},\boldsymbol{\mathcal{Z}}),$$

and

$$\widehat{\delta}(\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}) := \max\left\{\delta(\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}),\delta(\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Y}})\right\},$$

the latter being the so called *gap* between subspaces $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Z}}$.

The first step to apply [15] is to establish the following two properties:

- P1: $\|\boldsymbol{T}-\boldsymbol{T}_h\|_h\to 0,\quad$ as $h\to 0$.
- P2: $\forall\boldsymbol{x}\in\boldsymbol{\mathcal{X}},\quad \lim_{h\to 0}\delta(\boldsymbol{x},\boldsymbol{\mathcal{X}}_h)=0$.

The latter, P2, follows immediately from the approximation properties of the finite element spaces (4.3) and (4.5) and the density of smooth functions in $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{Q}}$. The former, P1, is a consequence of the following lemma.

LEMMA 5.1. *There exists $C>0$, independent of h, such that*

$$\|\boldsymbol{T}-\boldsymbol{T}_h\|_h \le Ch^s.$$

Proof. For $(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\in\boldsymbol{\mathcal{X}}_h=\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h$, we write

$$\begin{aligned}(\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{\sigma}_h,\boldsymbol{r}_h) &= (\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{P}_h(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)) + (\boldsymbol{T}-\boldsymbol{T}_h)((\boldsymbol{I}-\boldsymbol{P}_h)(\boldsymbol{\sigma}_h,\boldsymbol{r}_h))\\ &= (\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{P}_h(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)),\end{aligned}$$

the last equality because $(\boldsymbol{I}-\boldsymbol{P}_h)$ is a projector onto $\boldsymbol{\mathcal{K}}_h\times\boldsymbol{\mathcal{Q}}_h\subset\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$ and, on this subspace, $\boldsymbol{T}$ and $\boldsymbol{T}_h$ are both the identity. Now,

$$(\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{P}_h(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)) = \underbrace{(\boldsymbol{T}-\widetilde{\boldsymbol{T}}_h)((\boldsymbol{P}_h-\boldsymbol{P})(\boldsymbol{\sigma}_h,\boldsymbol{r}_h))}_{E_1} + \underbrace{(\boldsymbol{T}-\widetilde{\boldsymbol{T}}_h)(\boldsymbol{P}(\boldsymbol{\sigma}_h,\boldsymbol{r}_h))}_{E_2}.$$

For the first term we use Lemma 4.2 (ii) to write

$$\|E_1\| \le \left(\|\boldsymbol{T}\|+\|\widetilde{\boldsymbol{T}}_h\|\right)\|(\boldsymbol{P}_h-\boldsymbol{P})(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\| \le Ch^s\|\boldsymbol{\sigma}_h\|_{\mathrm{H}(\mathbf{div};\Omega)}.$$

For the second one, by virtue of the Cea-like estimate (4.7), we have that

$$\|E_2\| \le C \inf_{(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{T}(\boldsymbol{P}(\boldsymbol{\sigma}_h,\boldsymbol{r}_h))-(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\|.$$

Now, since $\boldsymbol{P}(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\in\boldsymbol{\mathcal{G}}$ (cf. Lemma 3.4), according to Proposition 3.5, if we denote $(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)=\boldsymbol{T}(\boldsymbol{P}(\boldsymbol{\sigma}_h,\boldsymbol{r}_h))$, then $\boldsymbol{\sigma}^*,\boldsymbol{r}^*\in\mathrm{H}^s(\Omega)^{n\times n}$, $\mathbf{div}\,\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega)^n$ and

$$\|\boldsymbol{\sigma}^*\|_{s,\Omega}+\|\mathbf{div}\,\boldsymbol{\sigma}^*\|_{1,\Omega}+\|\boldsymbol{r}^*\|_{s,\Omega} \le C\|\boldsymbol{P}(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\| \le C\|(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\|.$$

Then, from the last two inequalities and the approximation properties (4.3) and (4.5), we write

$$\|E_2\| \leq C \inf_{(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h} \|(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)-(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\| \leq Ch^s\,\|(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\|\,,$$

which together with the estimate of E_1 and the first two equalities of the proof allow us to conclude the lemma. ∎

Once properties P1 and P2 have been established, we are in a position to apply the results about spectral convergence from [15]. With this aim, we recall first the definition of the *resolvent operators* of, respectively, $\boldsymbol{T}$ and $\boldsymbol{T}_h$:

$$\begin{aligned}(z\boldsymbol{I}-\boldsymbol{T})^{-1} &: \boldsymbol{\mathcal{X}} \longrightarrow \boldsymbol{\mathcal{X}}, \qquad z\in\mathbb{C}\setminus\{\mathrm{sp}(\boldsymbol{T})\}\,,\\ (z\boldsymbol{I}-\boldsymbol{T}_h)^{-1} &: \boldsymbol{\mathcal{X}}_h \longrightarrow \boldsymbol{\mathcal{X}}_h, \qquad z\in\mathbb{C}\setminus\{\mathrm{sp}(\boldsymbol{T}_h)\}\,.\end{aligned}$$

The mapping $z\mapsto \left\|(z\boldsymbol{I}-\boldsymbol{T})^{-1}\right\|$ is continuous for all $z\notin \mathrm{sp}(\boldsymbol{T})$ and goes to zero as $|z|\to\infty$. Consequently, it is bounded on any closed subset of the complex plane not intersecting $\mathrm{sp}(\boldsymbol{T})$. The following theorem shows that the same happens uniformly for $\boldsymbol{T}_h$, provided h is small enough.

THEOREM 5.2. *Let $F\subset\mathbb{C}$ be a closed set such that $F\cap\mathrm{sp}(\boldsymbol{T})=\emptyset$. Then, there exist $h_0>0$ and $C>0$ such that, for all $h<h_0$, there hold $F\cap\mathrm{sp}(\boldsymbol{T}_h)=\emptyset$ and*

$$\left\|(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\right\|_h \leq C \qquad \forall z\in F.$$

Proof. It is proved in [15, Lemma 1] that this result follows from property P1. ∎

An equivalent form of the first assertion of this theorem is that any open set of the complex plane containing $\mathrm{sp}(\boldsymbol{T})$, also contains $\mathrm{sp}(\boldsymbol{T}_h)$ for h small enough. Thus, as a consequence of this theorem, we conclude that the proposed finite element method (i.e., Problem 4.1) does not introduce spurious modes with eigenvalues interspersed among the positive eigenvalues of Problem 2.1. Let us remark that such a spectral pollution could be in principle expected from the fact that the corresponding solution operator $\boldsymbol{T}$ has an infinite dimensional eigenvalue ($\mu=1$).

By applying the results from [15, Section 2] to our problem, we conclude the spectral convergence of $\boldsymbol{T}_h$ to $\boldsymbol{T}$ as $h\to 0$. More precisely, for all isolated eigenvalue μ of $\boldsymbol{T}$ with finite multiplicity m (and, hence, $\mu\in(0,1)$), for h small enough, there exist m eigenvalues $\mu_{h1},\dots,\mu_{hm}$ of $\boldsymbol{T}_h$ (repeated accordingly to their respective multiplicities) which converge to μ as $h\to 0$. Moreover, if $\boldsymbol{\mathcal{E}}$ is the eigenspace of $\boldsymbol{T}$ corresponding to μ and $\boldsymbol{\mathcal{E}}_h$ is the invariant subspace of $\boldsymbol{T}_h$ spanned by the eigenspaces of $\boldsymbol{T}_h$ corresponding to $\mu_{h1},\dots,\mu_{hm}$, then $\widehat{\delta}(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}}_h)\to 0$ as $h\to 0$, too.

Next step is to obtain error estimates for the spectral approximation. The classical reference for this issue on non-compact operators is [16]. However, we cannot apply the results from this reference directly to our problem, since the bilinear form A used to define $\boldsymbol{T}$ is not coercive. Instead of extending the results from this reference to our case, we will adapt the proofs from [15] to obtain error estimates.

With this end, first we recall the definition of spectral projectors. Let $\mu\neq 1$ be an isolated eigenvalue of $\boldsymbol{T}$. Let D be an open disk in the complex plane with boundary γ, such that μ is the only eigenvalue of $\boldsymbol{T}$ lying in D and $\gamma\cap\mathrm{sp}(\boldsymbol{T})=\emptyset$. The spectral projectors $\boldsymbol{E}:\boldsymbol{\mathcal{X}}\longrightarrow\boldsymbol{\mathcal{X}}$ and $\boldsymbol{E}_h:\boldsymbol{\mathcal{X}}_h\longrightarrow\boldsymbol{\mathcal{X}}_h$ are defined as follows:

$$\boldsymbol{E} := \frac{1}{2\pi i}\int_\gamma (z\boldsymbol{I}-\boldsymbol{T})^{-1}\,dz \qquad \text{and} \qquad \boldsymbol{E}_h := \frac{1}{2\pi i}\int_\gamma (z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\,dz.$$

Notice that the latter is well defined only if $\gamma \cap \mathrm{sp}(\boldsymbol{T}_h) = \emptyset$; however, according to Theorem 5.2, this always happens for h sufficiently small. The former is a projector onto the eigenspace $\boldsymbol{\mathcal{E}}$ of $\boldsymbol{T}$ associated to μ. The latter is a projector onto the invariant subspace $\boldsymbol{\mathcal{E}}_h$ of $\boldsymbol{T}_h$ associated to the eigenvalues of $\boldsymbol{T}_h$ lying in D.

The proofs of the following results are essentially identical to those of Lemma 2 and Theorem 2 and 3 from [15], but they use the estimates from Lemmas 5.1 and 4.2 (i) instead of properties P1 and P2, respectively. For the sake of completeness, we include brief proofs of these results.

LEMMA 5.3. *There exist constants $C > 0$ and $h_0 > 0$ such that, for all $h < h_0$,*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_h \leq Ch^s.$$

Proof. Let $h_0 > 0$ be such that, for all $h < h_0$, $\gamma \cap \mathrm{sp}(\boldsymbol{T}_h) = \emptyset$ (cf. Theorem 5.2). From the definition of the spectral projectors, we have

$$\begin{aligned}
\|\boldsymbol{E} - \boldsymbol{E}_h\|_h &\leq \frac{1}{2\pi} \int_\gamma \left\|(z\boldsymbol{I} - \boldsymbol{T})^{-1} - (z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}\right\|_h dz \\
&= \frac{1}{2\pi} \int_\gamma \left\|(z\boldsymbol{I} - \boldsymbol{T})^{-1} (\boldsymbol{T} - \boldsymbol{T}_h) (z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}\right\|_h dz \\
&\leq \frac{1}{2\pi} \int_\gamma \left\|(z\boldsymbol{I} - \boldsymbol{T})^{-1}\right\| \|\boldsymbol{T} - \boldsymbol{T}_h\|_h \left\|(z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}\right\|_h dz \leq Ch^s,
\end{aligned}$$

where, for the last inequality, we have used Lemma 5.1 and Theorem 5.2. □

THEOREM 5.4. *There exist constants $C > 0$ and $h_0 > 0$ such that, for all $h < h_0$,*

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \leq Ch^s.$$

Proof. Let h_0 be as in Lemma 5.3 and $h < h_0$. For all $\boldsymbol{x}_h \in \boldsymbol{\mathcal{E}}_h$, we have $\boldsymbol{E}_h\boldsymbol{x}_h = \boldsymbol{x}_h$, whereas $\boldsymbol{E}\boldsymbol{x}_h \in \boldsymbol{\mathcal{E}}$. Hence

$$\delta(\boldsymbol{x}_h, \boldsymbol{\mathcal{E}}) \leq \|\boldsymbol{E}_h\boldsymbol{x}_h - \boldsymbol{E}\boldsymbol{x}_h\| \leq \|\boldsymbol{E}_h - \boldsymbol{E}\|_h \|\boldsymbol{x}_h\| \leq Ch^s \|\boldsymbol{x}_h\|,$$

the last inequality because of Lemma 5.3. Then $\delta(\boldsymbol{\mathcal{E}}_h, \boldsymbol{\mathcal{E}}) \leq Ch^s$.

Now, for all $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}$, $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{x}$ and, since $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{G}}$, $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}$, too. Then,

$$\begin{aligned}
\|\boldsymbol{x} - \boldsymbol{E}_h\boldsymbol{P}_h\boldsymbol{x}\| &\leq \|\boldsymbol{E}(\boldsymbol{P}\boldsymbol{x} - \boldsymbol{P}_h\boldsymbol{x})\| + \|(\boldsymbol{E} - \boldsymbol{E}_h)\boldsymbol{P}_h\boldsymbol{x}\| \\
&\leq \|\boldsymbol{E}\| \|\boldsymbol{P}\boldsymbol{x} - \boldsymbol{P}_h\boldsymbol{x}\| + \|\boldsymbol{E} - \boldsymbol{E}_h\|_h \|\boldsymbol{P}_h\boldsymbol{x}\| \leq Ch^s \|\boldsymbol{x}\|,
\end{aligned}$$

the last inequality because of Lemmas 4.2 (i) and 5.3 and the fact that the operators $\boldsymbol{P}_h$ are bounded uniformly in h. Then, $\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \leq Ch^s$, too, and we conclude the proof. □

We recall that $\mu \in (0,1)$ is an eigenvalue of $\boldsymbol{T}$ with multiplicity m if and only $\lambda := (1/\mu) - 1$ is an eigenvalue of Problem 2.1 with the same multiplicity and the corresponding eigenfunctions coincide. Analogously, μ_{hi}, $i = 1, \ldots, m$, are the eigenvalues of $\boldsymbol{T}_h$ (repeated accordingly to their respective multiplicities) which converge to μ if and only if $\lambda_{hi} := (1/\mu_{hi}) - 1$ are the eigenvalues of Problem 4.1 converging to λ and the corresponding eigenfunctions also coincide.

Thus, the theorem above provides an error estimate for the approximation of the eigenfunctions of Problem 2.1 by means of those of Problem 4.1, which is the discrete problem implemented in practice. The last step is the following theorem, in which we establish a double order of convergence for the corresponding eigenvalues.

THEOREM 5.5. *There exist constants $C > 0$ and $h_1 > 0$ such that, for all $h < h_1$,*

$$|\lambda - \lambda_{hi}| \leq Ch^{2s}, \qquad i = 1, \ldots, m.$$

Proof. Let h_0 be as in Lemma 5.3 and $h < h_0$. Let $\boldsymbol{x}_{hi} = (\boldsymbol{\sigma}_{hi}, \boldsymbol{r}_{hi})$ be an eigenfunction of Problem 4.1 corresponding to λ_{hi}, normalized so that $\|\boldsymbol{x}_{hi}\| = 1$. According to Theorem 5.4, $\delta(\boldsymbol{x}_{hi}, \boldsymbol{\mathcal{E}}) \leq Ch^s$, so that there exists $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}$ (i.e., $\boldsymbol{x} = (\boldsymbol{\sigma}, \boldsymbol{r})$ an eigenfunction of Problem 2.1 corresponding to λ) satisfying

$$\|\boldsymbol{x}_{hi} - \boldsymbol{x}\| \leq Ch^s. \tag{5.1}$$

Notice that, in spite of the notation, $\boldsymbol{x}$ actually depends on h.

By writing Problems 2.1 and 4.1 in terms of the bilinear forms A and B, we have

$$A(\boldsymbol{x}, \boldsymbol{y}) = (\lambda + 1)\, B(\boldsymbol{x}, \boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}},$$
$$A(\boldsymbol{x}_{hi}, \boldsymbol{y}_h) = (\lambda_{hi} + 1)\, B(\boldsymbol{x}_{hi}, \boldsymbol{y}_h) \qquad \forall \boldsymbol{y}_h \in \boldsymbol{\mathcal{X}}_h,$$

Then, it is easy to check the following equality, which is a variation of a well-known equation (see, for instance, [7, Lemma 9.1]):

$$A(\boldsymbol{x} - \boldsymbol{x}_{hi}, \boldsymbol{x} - \boldsymbol{x}_{hi}) - (\lambda + 1)\, B(\boldsymbol{x} - \boldsymbol{x}_{hi}, \boldsymbol{x} - \boldsymbol{x}_{hi}) = (\lambda_{hi} - \lambda)\, B(\boldsymbol{x}_{hi}, \boldsymbol{x}_{hi}).$$

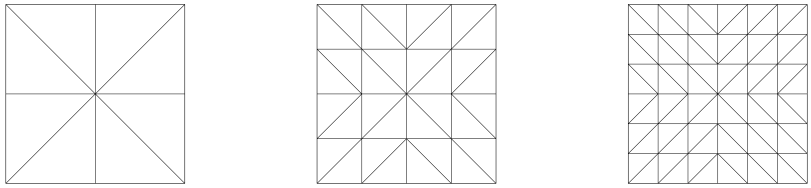
Now, according to (5.1), there exists $h_0' > 0$ ($h_0' < h_0$) such that, for all $h < h_0'$, we have $\|\boldsymbol{x}_{hi} - \boldsymbol{x}\| \leq \frac{1}{2}$. Therefore, since $\|\boldsymbol{x}_{hi}\| = 1$, we have that $\frac{1}{2} \leq \|\boldsymbol{x}\| \leq \frac{3}{2}$. Hence, since B continuous and $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{G}}$ is finite-dimensional, by virtue of Lemma 3.6, there exists $c > 0$, independent of h, such that $B(\boldsymbol{x}, \boldsymbol{x}) \geq c$. Thus, because of (5.1) and the continuity of B again, we know that there exists $h_1 > 0$ ($h_1 < h_0'$) such that, for all $h < h_1$, $B(\boldsymbol{x}_{hi}, \boldsymbol{x}_{hi}) \geq \frac{c}{2}$. Therefore, the theorem follows from (5.1) and the fact that A and B are continuous bilinear forms in $\boldsymbol{\mathcal{X}}$. Thus we conclude the proof. ☐

6. Numerical Results. We report in this section the results of some numerical tests carried out with the method based on AFW elements proposed in Section 4 and with the analogous one based on PEERS elements (cf. [2]), which confirm the theoretical results proved above. The numerical methods have been implemented in a MATLAB code.

We recall that the Lamé coefficients of a material are defined in terms of the Young modulus E and the Poisson ratio ν as follows:

$$\lambda_{\mathrm{S}} := \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu_{\mathrm{S}} := \frac{E}{2(1+\nu)}. \tag{6.1}$$

6.1. Test 1: Approximation of the elasticity vibration problem with AFW finite elements. We have considered an elastic body occupying the two-dimensional domain $\Omega := (0,1) \times (0,1)$, fixed at its bottom ($\Gamma$) and free at the rest of the boundary (Σ). We have used uniform meshes as shown in Figure 6.1. The refinement parameter N used to label each mesh is the number of elements on each edge.



$N = 2$ $N = 4$ $N = 6$

Fig. 6.1. *Test 1 (AFW). Uniform meshes.*

We have used the following values for the material coefficients, which correspond to steel: density $\rho_S = 7.7 \times 10^3\,\mathrm{Kg/m^3}$, Young modulus $E = 1.44 \times 10^{11}\,\mathrm{Pa}$, and Poisson ratio $\nu = 0.35$

We report in Table 6.1 the lowest computed vibration frequencies $\omega_{hi} := \sqrt{\lambda_{hi}}$ obtained with this method. We have used four different meshes with increasing levels of refinement. The table also includes the estimated orders of convergence, as well as more accurate values of the vibration frequencies extrapolated from the computed ones by means of a least-squares fitting.

Table 6.1
Test 1 (AFW). Computed lowest vibration frequencies ω_{hi}, $i = 1, \ldots, 6$, for $\nu = 0.35$.

	$N = 10$	$N = 20$	$N = 30$	$N = 40$	Order	Extrapolated
ω_{h1}	2949.897	2945.996	2945.141	2944.811	1.72	2944.295
ω_{h2}	7360.490	7352.006	7350.318	7349.698	1.88	7348.840
ω_{h3}	7956.297	7899.968	7889.117	7885.257	1.94	7880.084
ω_{h4}	13019.896	12816.938	12778.482	12764.909	1.96	12746.802
ω_{h5}	13169.008	13082.780	13065.967	13059.934	1.92	13051.758
ω_{h6}	15060.813	14935.298	14910.771	14902.027	1.92	14890.114

To test the locking-free character of the method, we have solved similar problems with Poisson ratios $\nu = 0.49$ (nearly incompressible) and $\nu = 0.5$ (perfectly incompressible material). In the last case $\lambda_S = \infty$ (cf. (6.1)), but we have used the expression $\mathcal{C}^{-1}\boldsymbol{\tau} := 1/(2\mu_S)\left[\boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}\right]$, which follows from (2.1) by taking limit as $\lambda_S \to \infty$.

We report in Tables 6.2 and 6.3 the same results as in Tables 6.1 for $\nu = 0.49$ and $\nu = 0.5$, respectively.

Table 6.2
Test 1 (AFW). Computed lowest vibration frequencies ω_{hi}, $i = 1, \ldots, 6$, for $\nu = 0.49$.

	$N = 10$	$N = 20$	$N = 30$	$N = 40$	Order	Extrapolated
ω_{h1}	3030.973	3027.216	3026.270	3025.874	1.48	3025.120
ω_{h2}	7970.766	7952.238	7948.513	7947.128	1.86	7945.193
ω_{h3}	8127.648	8067.882	8056.421	8052.362	1.95	8046.967
ω_{h4}	12768.147	12688.783	12673.338	12667.772	1.92	12660.250
ω_{h5}	13430.465	13229.412	13191.638	13178.340	1.98	13161.057
ω_{h6}	15855.380	15643.181	15601.973	15587.265	1.92	15567.043

TABLE 6.3
Test 1 (AFW). Computed lowest vibration frequencies ω_{hi}, $i = 1, \dots, 6$, for $\nu = 0.5$.

	$N = 10$	$N = 20$	$N = 30$	$N = 40$	Order	Extrapolated
ω_{h1}	3039.966	3036.179	3035.213	3034.806	1.46	3034.018
ω_{h2}	8021.350	8001.786	7997.854	7996.391	1.86	7994.348
ω_{h3}	8149.142	8088.826	8077.261	8073.166	1.95	8067.720
ω_{h4}	12746.042	12666.976	12651.586	12646.036	1.92	12638.546
ω_{h5}	13464.906	13263.891	13226.137	13212.848	1.98	13195.563
ω_{h6}	15888.149	15671.954	15630.001	15615.029	1.93	15594.866

It can be seen from these tables that the method is thoroughly locking-free. Moreover, evidence of a double order of convergence for the vibration frequencies can be also clearly observed in all cases.

Let us remark that the eigenfunctions of this problem may present singularities at the points where the boundary condition change from Dirichlet (fixed) to Neumann (free). According to [17], estimate (3.7) holds in this case for all $s < s_0$, where s_0 is the smallest positive root of the following characteristic equation:

$$\sin^2 s_0\theta = \frac{(\lambda_S + 2\mu_S)^2 - (\lambda_S + \mu_S)^2 s_0^2 \sin^2\theta}{(\lambda_S + \mu_S)(\lambda_S + 3\mu_S)},$$

with θ being the size of the inner angle of the domain at the point where the boundary conditions change (in this test, $\theta = \frac{\pi}{2}$). Solving this equation for the used values of ν, we obtained the results reported in Table 6.4.

TABLE 6.4
Test 1. Sobolev exponents.

ν	s_0
0.35	0.6797
0.49	0.5999
0.5	0.5946

According to this table, the computed vibration frequencies reported in Tables 6.1 to 6.3 all actually converge with (at least) a double order.

Figures 6.2 and 6.3 show the vibration modes of the four lowest vibration frequencies for the first case ($\nu = 0.35$).

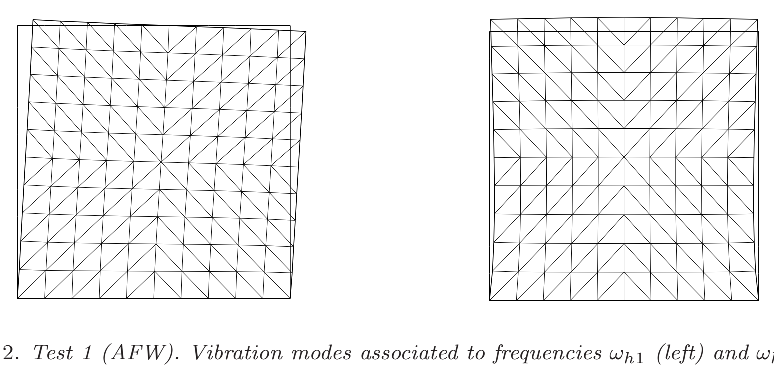


FIG. 6.2. *Test 1 (AFW). Vibration modes associated to frequencies ω_{h1} (left) and ω_{h2} (right).*

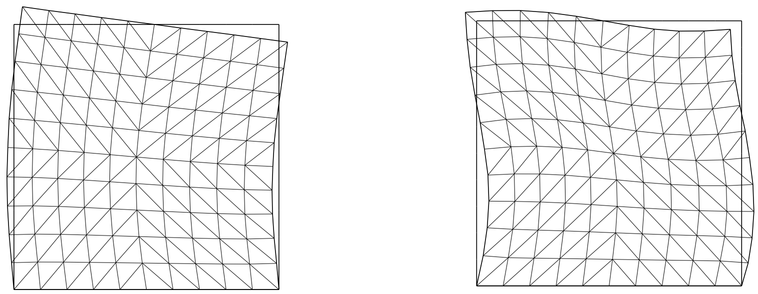
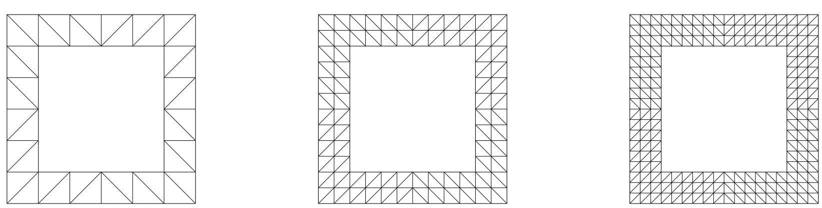


Fig. 6.3. *Test 1 (AFW). Vibration modes associated to frequencies ω_{h3} (left) and ω_{h4} (right).*

6.2. Test 2: Approximation of the elasticity vibration problem with PEERS finite elements. As stated in the introduction, although the analysis in Sections 4 and 5 was presented for the lowest-order AFW elements, the same could be done for other methods satisfying properties (4.1)–(4.6). As an example of this, we use in this test the PEERS finite elements introduced by Arnold, Brezzi and Douglas in [2] to discretize Problem 2.1. Spurious-free approximation results and error estimates similar to those from Theorems 5.2, 5.4, and 5.5 can be proved for these elements by repeating the arguments from the previous sections.

To demonstrate the performance of these elements we have repeated the previous tests, changing only the domain. We have chosen now $\Omega := \left(-\frac{1}{4}, \frac{5}{4}\right)^2 \setminus [0,1]^2$, which corresponds to a two-dimensional closed vessel with vacuum inside. The vessel has been taken fixed at its bottom (Γ) and free at the rest of the boundary (Σ). We have used the same material parameters as in the previous test and uniform meshes as shown in Figure 6.4. The refinement parameter N used to label each mesh is now the number of element layers across the thickness of the solid.



$N = 1$ $N = 2$ $N = 3$

Fig. 6.4. *Test 2 (PEERS). Uniform meshes.*

Tables 6.5 to 6.7 are the analogues to Tables 6.1 to 6.3 for this test. In Tables 6.5 ($\nu = 0.35$) we also include an additional column with the results reported in [9] for the same problem solved with standard piecewise linear continuous elements applied to the direct displacement formulation.

TABLE 6.5
Test 2 (PEERS). Computed lowest vibration frecuencies $\omega_{hi}, i = 1, \ldots, 6$ for $\nu = 0.35$.

	$N = 16$	$N = 24$	$N = 32$	$N = 40$	Order	Extrapolated	[9]
ω_{h1}	658.143	660.766	661.959	662.634	1.26	664.699	665.918
ω_{h2}	2274.912	2278.765	2280.357	2281.201	1.52	2283.277	2284.617
ω_{h3}	3769.740	3781.765	3787.064	3790.002	1.34	3798.392	3803.266
ω_{h4}	3839.291	3853.616	3860.221	3863.989	1.22	3875.980	3879.595
ω_{h5}	4496.992	4500.655	4502.364	4503.342	1.19	4506.556	4507.626
ω_{h6}	5450.111	5457.685	5460.962	5462.753	1.40	5467.594	5470.751

TABLE 6.6
Test 2 (PEERS). Computed lowest vibration frecuencies $\omega_{hi}, i = 1, \ldots, 6$ for $\nu = 0.49$.

	$N = 16$	$N = 24$	$N = 32$	$N = 40$	Order	Extrapolated
ω_{h1}	705.913	709.167	710.655	711.499	1.24	714.140
ω_{h2}	2414.882	2419.620	2421.609	2422.675	1.48	2425.376
ω_{h3}	3968.908	3983.087	3989.368	3992.861	1.33	4002.904
ω_{h4}	4145.538	4164.380	4173.086	4178.057	1.22	4193.836
ω_{h5}	4893.125	4898.831	4901.495	4903.022	1.19	4908.030
ω_{h6}	5803.292	5812.016	5815.820	5817.908	1.38	5823.650

TABLE 6.7
Test'2 (PEERS). Computed lowest vibration frecuencies $\omega_{hi}, i = 1, \ldots, 6$ for $\nu = 0.50$.

	$N = 16$	$N = 24$	$N = 32$	$N = 40$	Order	Extrapolated
ω_{h1}	710.507	713.824	715.341	716.201	1.24	718.893
ω_{h2}	2428.223	2433.053	2435.082	2436.172	1.48	2438.925
ω_{h3}	3987.814	4002.202	4008.579	4012.129	1.33	4022.318
ω_{h4}	4175.243	4194.541	4203.458	4208.551	1.21	4224.955
ω_{h5}	4932.287	4938.209	4940.974	4942.558	1.19	4947.756
ω_{h6}	5837.940	5846.777	5850.633	5852.752	1.37	5858.653

The eigenfunctions of this problem may present singularities similar to those of the previous test and with the same Sobolev exponents reported in Table 6.4. Additional singularities can arise at the reentrant angles of the domain. According to [17], since at both sides of each reentrant angle the eigenfunction satisfies Neumann boundary conditions, estimate (3.7) holds in this case for all $s < s_1$, where s_1 is the smallest positive root of the following characteristic equation:

$$\sin^2 s_1\theta = s_1^2 \sin^2 \theta,$$

with θ being the size of the reentrant angle of the domain (in this test, $\theta = \frac{3\pi}{2}$). Notice that, in this case, the characteristic equation does not depend on the Lamé coefficients and, then, are independent of the values of ν. Solving this equation, we obtain $s_1 = 0.5445$. Comparing this value with those of Table 6.4, we observe that, in this test, for all values of ν the strongest singularities arise at the reentrant angles.

Essentially the same conclusions as in the previous test can be obtained for PEERS elements from Tables 6.5 to 6.7.

Finally, Figures 6.5 and 6.6 show the vibration modes corresponding to the four lowest vibration frequencies.

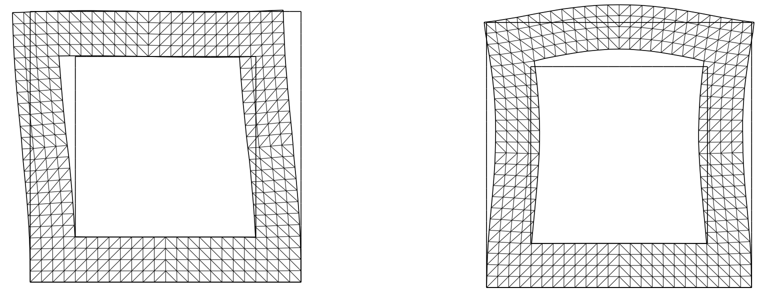


Fig. 6.5. *Test 2 (PEERS). Vibration modes associated to frequencies ω_{h1} (left) and ω_{h2} (right).*

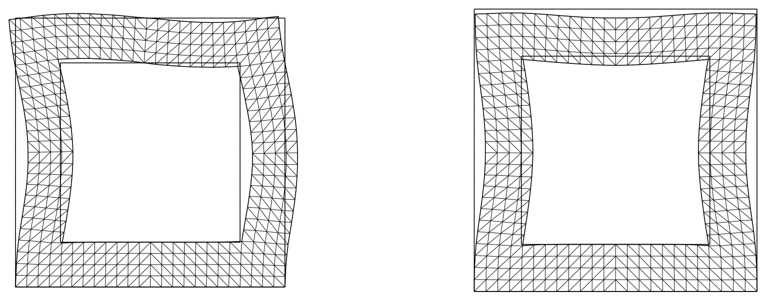


Fig. 6.6. *Test 2 (PEERS). Vibration modes associated to frequencies ω_{h3} (left) and ω_{h4} (right).*

REFERENCES

[1] S. Adams and B. Cockburn, *A mixed finite element for elasticity in three dimensions*, J. Sci. Comput., 25 (2005), pp. 515-521.

[2] D. N. Arnold, F. Brezzi, and J. Douglas, *PEERS: A new mixed finite element method for plane elasticity*, Japan J. Appl. Math., 1 (1984), pp. 347–367.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Differential complexes and stability of finite element methods. II: The elasticity complex*, in: Compatible Spatial Discretizations, D. N. Arnold, et al., eds., IMA Volumes in Mathematics and its Applications, Vol. 142, Springer Verlag 2005, pp. 47–67.

[4] ———, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica, 15 (2006), pp. 1–155.

[5] ———, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*, Math. Comp., 76 (2007), pp. 1699–1723.

[6] D. N. Arnold and R. Winther, *Mixed finite elements for elasticity*, Numer. Math., 92 (2002), pp. 401–419.

[7] I. Babuška and J. Osborn, *Eigenvalue problems*, in Handbook of Numerical Analysis, Vol. II, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 641–787.

[8] A. Bermúdez, R. Durán, M. A. Muschietti, R. Rodríguez, and J. Solomin, *Finite element vibration analysis of fluid-solid systems without spurious modes*, SIAM J. Numer. Anal., 32 (1995), pp. 1280–1295.

[9] A. Bermúdez and R. Rodríguez, *Finite element computation of the vibration modes of a fluid-solid system*, Comput. Methods Appl. Mech. Engrg., 119 (1994), pp. 355–370.

[10] D. Boffi *Finite element approximation of eigenvalue problems*, Acta Numerica, 19 (2010), pp. 1–120.

[11] D. Boffi, F. Brezzi, and M. Fortin, *Reduced symmetry elements in linear elasticity*, Comm. Pure Appl. Anal., 8 (2009), pp. 1–28.

[12] D. Boffi, F. Brezzi, and L. Gastaldi, *On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form*, Math. Comp. 69 (2000), pp. 121–140.

[13] ———, *On the convergence of eigenvalues for mixed formulations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 25 (1998), pp. 131–154.
[14] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Verlag, New York, 1991.
[15] J. Descloux, N. Nassif, and J. Rappaz, *On spectral approximation. Part 1: The problem of convergence*, RAIRO Anal. Numér., 12 (1978), pp. 97–112.
[16] ———, *On spectral approximation. Part 2: Error estimates for the Galerkin method*, RAIRO Anal. Numér., 12 (1978), pp. 113–119.
[17] P. Grisvard, *Probléms aux limites dans les polygones. Mode démploi*, EDF, Bulletin de la Direction des Etudes et Recherches (Serie C), 1 (1986), pp. 21–59.
[18] G. N. Gatica, *Analysis of a new augmented mixed finite element method for linear elasticity allowing* $\mathbb{RT}_0 - \mathbb{P}_1 - \mathbb{P}_0$ *approximations*, M2AN Math. Model. Numer. Anal., 40 (2006), pp. 1–28.
[19] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica, 11 (2002), pp. 237–339.
[20] B. Mercier, J. Osborn, J. Rappaz, and P.A. Raviart, *Eigenvalue approximation by mixed and hybrid methods*, Math. Comp. 36 (1981), pp. 427–453.

Appendix. Two auxiliary results on variational spectral problems.

Let V be a real (or complex) Hilbert space. Let $A, B : V \times V \longrightarrow \mathbb{R}$ (resp. $\mathbb{C}$) be two symmetric (resp. Hermitian) bounded bilinear forms. We assume that A is such that for all $f \in V'$, there exists a unique $u \in V$ satisfying

$$A(u, v) = f(v) \qquad \forall v \in V$$

and that there exists a constant C, independent of f, such that $\|u\|_V \leq C \|f\|_{V'}$. Let $T : V \longrightarrow V$ be the bounded linear operator defined for all $u \in V$ as follows:

$$Tu \in V : \qquad A(Tu, v) = B(u, v) \qquad \forall v \in V.$$

When B is an inner product in V, T is a self-adjoint operator (with respect to B). However, in general it is not. In spite of this, we prove in what follows a couple of properties of T which are standard for self-adjoint operators.

Proposition A.1. *Let $E \subset V$ be an invariant subspace of T (i.e., $T(E) \subset E$). Let $F := \{u \in V : \; B(u, v) = 0 \;\; \forall v \in E\}$. Then, F is also an invariant subspace of T.*

Proof. Let $u \in F$. From the symmetric (resp. Hermitian) character of the bilinear forms and the definition of T, we have

$$B(Tu, v) = \overline{B(v, Tu)} = \overline{A(Tv, Tu)} = A(Tu, Tv) = B(u, Tv) = 0 \qquad \forall v \in E,$$

the latter because $Tv \in E$ and $u \in F$. Hence $Tu \in F$ and we end the proof. □

Proposition A.2. *Let $\mu \in \mathbb{C}$ be an eigenvalue of T and u a corresponding eigenfunction; namely, $u \in V$, $u \neq 0$, and $Tu = \mu u$. If $A(u, u) \neq 0$, then:*
i) $\mu \in \mathbb{R}$;
ii) μ is non defective (i.e., its ascent is 1).

Proof. From the definition of T and the symmetric (resp. Hermitian) character of the bilinear forms, we have

$$\mu\, A(u, u) = B(u, u) = \overline{B(u, u)} = \overline{\mu\, A(u, u)} = \overline{\mu}\, A(u, u).$$

Hence $(\mu - \overline{\mu})\, A(u, u) = 0$. Thus, for $A(u, u) \neq 0$, $\mu = \overline{\mu}$ and we conclude (i).

The proof of (ii) is by contradiction. Let us assume that μ is defective; namely, there exists $\widehat{u} \in V$ such that $T\widehat{u} = \mu\widehat{u} + u$. Then, we have that

$$Tu = \mu u \implies \mu\, A(u, v) = B(u, v) \quad \forall v \in V;$$
$$T\widehat{u} = \mu\widehat{u} + u \implies \mu\, A(\widehat{u}, v) + A(u, v) = B(\widehat{u}, v) \quad \forall v \in V.$$

We take $v = \widehat{u}$ in the first equation above and $v = u$ in the second one, to write

$$\mu\, A(u, \widehat{u}) = B(u, \widehat{u}),$$
$$\mu\, A(\widehat{u}, u) + A(u, u) = B(\widehat{u}, u).$$

Then, by subtracting the first equation (resp. its conjugate) from the second one and using that μ is real and the bilinear forms are symmetric (resp. Hermitian), we obtain $A(u, u) = 0$, which contradicts the assumption of the proposition. Thus, we end the proof. □