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Supermembrane origin of type II gauged supergravities in 9D

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Abstract: The M-theory origin of the IIB gauged supergravities in nine dimensions, classified according to the inequivalent classes of monodromy, is shown to exactly corresponds to the global description of the supermembrane with central charges. The global description is a realization of the sculpting mechanism of gauging (arXiv:1107.3255) and it is associated to particular deformation of fibrations. The supermembrane with central charges may be formulated in terms of sections on symplectic torus bundles with $\mathrm{SL}(2,\mathbb{Z})$ monodromy. This global formulation corresponds to the gauging of the abelian subgroups of $\mathrm{SL}(2,Z)$ associated to monodromies acting on the target torus. We show the existence of the trombone symmetry in the supermembrane formulated as a non-linear realization of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry and construct its gauging in terms of the supermembrane formulated on an inequivalent class of symplectic torus fibration. The supermembrane also exhibits invariance under T-duality and we find the explicit T-duality transformation. It has a natural interpretation in terms of the cohomology of the base manifold and the homology of the target torus. We conjecture that this construction also holds for the IIA origin of gauged supergravities in 9D such that the supermembrane becomes the origin of all type II supergravities in 9D. The geometric structure of the symplectic torus bundle goes beyond the classification on conjugated classes of $\mathrm{SL}(2,\mathbb{Z})$. It depends on the elements of the coinvariant group associated to the monodromy group. The possible values of the (p,q) charges on a given symplectic torus bundle are restricted to the corresponding equivalence class defining the element of the coinvariant group.

Keywords: Supergravity Models, M-Theory, p-branes, Discrete and Finite Symmetries

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Contents

1 Introduction

The M-theory origin of gauged supergravities is a interesting open problem. The aim of this paper is to show that the 11D supermembrane compactified on a torus is the M-theory origin of **all** supergravities in 9D: not only the maximal supergravity [1] but also the gauged sector [2]–[9]. In the picture we propose, there are two well-differentiated sectors: The first one is associated to trivial compactifications of the supermembrane on a 2-torus, its low energy limit corresponds to the $N=2$ maximal supergravity in 9D, and globally it corresponds to a trivial symplectic torus bundle. The second sector corresponds to a formulation on a nontrivial symplectic torus bundle. It may occur because a nontrivial monodromy or even in the case of trivial monodromy (the identity) because of a nontrivial cohomology class of the base manifold. The central charge condition is exactly the condition of non-trivial cohomology. The supermembrane with nontrivial central charges corresponds to this sector ([10–12]). In particular we will analyze the formulation on a symplectic torus bundle with nontrivial monodromy. From the physical point of view, the consequence of being a nontrivial cohomology, is very relevant. The spectrum of the hamiltonian becomes discrete with finite multiplicity. By this we refer to the spectrum of the exact hamiltonian, not only to its semiclassical approximation.

It is well-established that the 11D supergravity equations of motion appear as a consequence of imposing kappa symmetry to the supermembrane action formulated on a general background. This supports the conjecture that the low energy description of the supermembrane is the 11D supergravity.[1] The maximal dimension for gauged supergravities is 9D. There are four different classes of gauging appearing in type IIB gauged supergravities in 9D as was initially established by [2, 3]. If we include also the deformations coming from the type IIA sector, there are four more, but only seven of them are independent deformations and they constitute the type II 9D gauged supergravity [4], where it is also included the gauging of scaling symmetries [5, 6]. Very recently the most general gaugings in 9D (expressed in the tensor embedding formalism [7, 8]), have been found in [9].

Nowadays, the double field theory has become a interesting arena to try to realize in a bottom-up approach, some of the properties of string theory. It is a global approach that describe sigma models with double coordinates on a T^{2d} torus fibrations such that the transition functions will be evaluated in the T-duality group $O(d, d, \mathbb{Z})$. The type II realization has been done recently in [13, 14]. The proposed action is such that it is invariant under duality transformations. In 9D the duality transformations correspond to $\mathrm{SL}(2, \mathbb{Z}) \times Z_2$ [15].

There is evidence that string theory can be consistently defined in non-geometric backgrounds in which the transition functions between coordinate patches involve not only diffeomorphisms and gauge transformations but also duality transformations [16, 17]. Some global aspects of T-duality in String theory were formerly analyzed in [18], and more recently by [19]. Such backgrounds can arise from compactifications with duality twists [20] or from acting on geometric backgrounds with fluxes with T-duality [16, 17, 21]. In special cases, the compactifications with duality twists are equivalent to asymmetric orbifolds which can give consistent string backgrounds [22–25]. In this type of compactifications, T-folds are constructed by using strings formulated on a doubled torus T^{2n} with n-coordinates conjugate to the momenta and the other n-coordinates conjugate to the winding modes [17], plus a constraint to guarantee the correct number of propagating degrees of freedom.

T-duality transformation at the worldsheet level were studied in [41]. The relation of duality and M-theory was also analyzed in [42]. In [16, 17] it was argued that a fundamental formulation of string/M-theory should exist in which the T- and U-duality symmetries are

[1]Indeed this conjecture means that the groundstate of the 11D supermembrane corresponds to the supermultiplet associated to the 11D supergravity, though, a rigorous proof of this difficult open problem is still lacking.

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manifest from the start. In particular, it was argued that many massive, gauged supergravities cannot be naturally embedded in string theory without such a framework [21, 26–28]. Examples of generalized T-folds can be obtained by constructing torus fibrations over base manifolds with non-contractible cycles. However, in spite of these important advances, up to our knowledge, a full-fledged realization of these ideas in terms of worldvolume theories in M-theory is still lacking.

The aim of this paper is to prove that the action of the Supermembrane with nontrivial central charges, whose local structure was given in [10–12], may be globally defined in terms of sections of a symplectic torus bundle with nontrivial monodromy characterizing at low energies the gaugings of the type II supergravities. This global description was derived following the sculpting mechanism in [61]. Earlier attempts to establish the connection between the gauging of the supermembrane and that of 9D gauged supergravities can be found in [29, 30].

We prove it in the context of IIB monodromies. The supermembrane formulation on a symplectic torus bundle with monodromy has all the geometrical structure required to derive at low energies the IIB gauged supergravities in 9D. At the level of the supermembrane we are gauging the abelian subgroups of $\mathrm{SL}(2,\mathbb{Z})$, the group of isotopy classes of symplectomorphisms or equivalently area preserving diffeomorphims. It is then natural to think that type IIB gauge supergravities can only interact with the corresponding class of gauged supermembranes in this work. According to the inequivalent classes of monodromies, more precisely, to the elements of the coinvariant group of the given monodromy, there is a classification of the corresponding symplectic torus bundles that describe globally the supermembrane. The monodromy is given as a representation of the fundamental group $\Pi_1(\Sigma)$ (where Σ is the base manifold of the supermembrane) into $\mathrm{SL}(2,\mathbb{Z})$, the isotopy group of homotopic classes of symplectomorphisms (symplectomorphism group on 2-dimensions or equivalently area preserving diffeomorphisms is the local symmetry of the supermembrane in the Light Cone Gauge). The $\mathrm{SL}(2,\mathbb{Z})$ group acts naturally on the first homology group of the fiber, which in our case corresponds to the target torus. The monodromy defines an automorphism on the fibers providing the global structure of the geometrical setting. We also show the existence of a new Z_2 symmetry that plays the role of T-duality in the supermembrane interchanging the winding and KK charges but leaving the Hamiltonian invariant, so that the complete symmetry group in the ungauged supermembrane corresponds to: $(\mathrm{SL}(2,\mathbb{Z})_\Sigma \times \mathrm{SL}(2,\mathbb{Z})_{T^2})/Z_2$. T-duality becomes an exact symmetry of the symplectic torus bundle description of the supermembrane by fixing its energy tension.

In type IIB nine dimensional supergravities, there are four inequivalent gaugings of $GL(2,\mathbb{R})$ global symmetry: three of them are associated to the gauging of the $\mathrm{SL}(2,\mathbb{R})$ global symmetry: the parabolic, elliptic and hyperbolic inequivalent classes and we find their respective symplectic torus bundles. The fourth gauging corresponds to the gauging of the trombone symmetry associated to the $\mathbb{R}^+$ scalings. At quantum level the realization of this last gauging is more involved since the scaling is not included in the arithmetic subgroup $GL(2,\mathbb{Z})$. In [31] they provided a way to realize this symmetry as a rigid symmetry, by studying a nonlinear realization of this symmetry that was called active $\mathrm{SL}(2,\mathbb{Z})$

symmetry. A way to realize this scaling is by a nonlinear representation of $\mathrm{SL}(2,\mathbb{Z})$. We show that this 'symmetry' is present in the ungauged supermembrane with central charges theory. The symplectic torus bundle associated to the gauging of this scaling symmetry is constructed and it corresponds from the point of view of fibration to a inequivalent class of symplectic torus bundles. This proves the supermembrane origin of the type IIB gauged supergravities. The monodromies with type IIA origin are infered from the fact that T-duality invariance of the mass operator of the supermembrane with central charges.

The paper is structured in the following way: In section 2 we made a summary of the results of inequivalent classes of type IIB gauged supergravities in 9D and its relation with the different monodromies. In section 3 we summarize the construction of the supermembrane with central charges, the two $\mathrm{SL}(2,\mathbb{Z})_\Sigma \times \mathrm{SL}(2,\mathbb{Z})_{T^2}$ discrete global symmetries. In section 4 we explain the sculpting mechanism in which principle torus fibration is deformed to acquire a monodromy of the fiber bundle. The corresponding action is gauged with respect to the one already published in several works, see for example [11, 12]. The new results are presented in sections 5,6,7, and 8. In section 5 we show the explicit global construction of the gauged supermembrane with central charges, and the inequivalent classes of symplectic torus bundles associated to the the inequivalent classes of monodromies. It is important to remark that for monodromies which include, elliptic, parabolic and hyperbolic classes there are torsion elements in the second cohomology group of the base manifold with coefficients in the module associated to the monodromy and this provides an extra restriction on the possible values of the charges of the theory. In section 6 we present the classification of the supermembrane theory formulated on the symplectic torus fibrations, and its relation to the different gaugings. We also discuss the residual symmetries of the theory after the gauge fixing. In section 7 we discuss the fiber bundle construction for the supermembrane with the gauging of the trombone symmetry. The effect of the nonlinear representation of the monodromy induces changes in the homology coefficients of the torus of the fiber leading to inequivalent fibrations. In section 8 we show the existence of a new Z_2 symmetry that plays the role of T-duality in the supermembrane. For other approaches to the supermembrane T-duality see [32–34]. In section 9 we present our discussion and conclusions.

2 Preliminars

The gauged supergravities were firstly discovered by [35] by compactifying the 11D supergravity on a S^7 a compact manifold with nontrivial holonomy, soon after this result, the gauging mechanism was also applied to theories with noncompact symmetry groups in [36]. The first paper of supergravity in nine dimensions containg a gauged sector was studied long time ago by [37]. Since then, the field has been very active and it has been found a number of ways to obtain a consistent deformation of a given maximal supergravity formulated in a target space with $d < 11$: by means of twisting in a Scherk-Schwarz compactification (SS), through compactification on manifolds with fluxes, noncommutative geometries etc.. For very nice reviews see for example: [28, 38].

In this section we will only review aspects -all of them previously found in the literature-, that are relevant for our constructions: those in which monodromy plays a fundamental role. SS-compactifications appeared as a generalization of Kaluza-Klein (KK)-reductions in which the fields are allowed to have a nontrivial dependence on the compactified variables, but in such a way that the truncation of the Langrangian in lower dimensions is still consistent. SS-compactifications of supergravity may be expressed the D-dimensional backgrounds in terms of principal fiber bundles over circles with a twisting given by the monodromy [39, 40, 43]. The background possesses a group of global isometries G associated to the compactification manifold over which it is fibered. The principal fiber bundles of fiber G have a monodromy $\mathcal{M}(g)$ valued in the Lie algebra g of the symmetry group G. The invariant functional of the actions are expressed in terms of the local sections of this bundle. The monodromy $\mathcal{M}(g)$ can be expressed in terms of a mass matrix M, as $\mathcal{M}(g) = \exp M$. The maps in terms of the compactified variables $g(y)$ are not periodic, but have a monodromy $g(y) = \exp(My)$ [43].

As explained in [20] twisted compactification induces a SS-potential in the moduli space. For certain values of the moduli space it is equivalent to introduce fluxes along the internal coordinates of the compactified torus. In [15] it is conjectured that at quantum level the global symmetry of the supergravity action breaks to its arithmetic subgroup also called the U-duality group $G(\mathbb{Z})$. The quantization condition is imposed to preserve the quantization of lattice of charges of the p-brane considered. At quantum level all twisting must then belong to the $G(\mathbb{Z})$ duality group what implies also the restriction to quantized parameters of mass matrix M. Indeed this condition was explored in further detail in [45] for the case of gauged supergravities in 9D, where in addition to impose the elements of the mass matrix to be integer, they have to satisfy in many cases, the diophantine equation to guarantee that the monodromy lies in the inequivalent classes of $\mathrm{SL}(2,\mathbb{Z})$.

For the case of interest here, the type II gauged supergravities in 9D, the monodromies are associated to the $GL(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$ global symmetry group. In the $\mathrm{SL}(2,\mathbb{R})$ sector, there are three inequivalent classes of theories, corresponding to the hyperbolic, elliptic and parabolic $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes and represented by the monodromy matrices of the form [43]

$$\mathcal{M}_p = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{M}_h = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}, \qquad \mathcal{M}_e = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.1}$$

where each class is specified by the coupling constant (k, γ or θ). In 9D the theory can also be described in terms of the mass matrix M with three parameters [2]

$$M = \frac{1}{2}\begin{pmatrix} m^1 & m^2 + m^3 \\ m^2 - m^3 & -m^1 \end{pmatrix}. \tag{2.2}$$

This mass matrix, as already explained in [2], belongs to the Lie algebra $sl(2,\mathbb{R})$ and transforms in the adjoint irreducible representation. It is characterized by the vector of mass $\vec{m} = (m_1, m_2, m_3)$. At low energies the gauged supergravity is determined by the mass matrix M for a given monodromy $\mathcal{M}$.

The field content of 9D II supergravity following the notation of [2, 4] is composed of a supervielbein ${e_\mu}^a$, three scalars ϕ, φ, χ, three gauge fields $(A_\mu, \{A_\mu^{(1)}, A_\mu^{(2)}\} \equiv \vec{A})$ two antisymmetric 2-forms $\{B^{(1)}_{\mu\nu}, B^{(2)}_{\mu\nu}\} \equiv \vec{B}$, a three form $C_{\mu\nu\rho}$ for the bosonic sector and in the fermionic side the contribution is a spinor ψ_μ and two dilatinos $\lambda, \tilde{\lambda}$ where the D=9 global $\Lambda = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ symmetry acts in the ungauged theory in the following way:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \vec{A} \to \Lambda\vec{A}, \qquad \vec{B} \to \Lambda\vec{B}, \tag{2.3}$$

plus the fermionic transformations. One of the scalars φ and the three form C remain invariant. As explained in [2, 4] the gauge transformations correspond to

$$A \to A - d\lambda \qquad\qquad \vec{B} \to \vec{B} - \vec{A}d\lambda. \tag{2.4}$$

The massive deformations from the type IIB sector are labeled by four parameters $m = (m_i, m_4)$ $\quad i = 1, \ldots, 3$. Three of them $\vec{m} = (m_1, m_2, m_3)$ belong to the $\mathrm{SL}(2,\mathbb{R})$ deformations and the last m_4 has its origin in the gauging of the scaling symmetry $\mathbb{R}^+$. The parameters of m gauge a subgroup of the global symmetry $\mathrm{SL}(2,\mathbb{R})$ and $\mathbb{R}^+$ respectively, with parameter $\Lambda = e^{\widetilde{M}\lambda}$ and gauge field transformations become modified as follows:

$$A \to A - d\lambda \qquad \vec{B} \to \Lambda(\vec{B} - \vec{A}d\lambda). \tag{2.5}$$

where we define $\widetilde{M} = (M, m_4)$, to group both type of deformations. Following [2, 4], consider in first place the massive deformations associated to $\Lambda_{\mathrm{SL}(2,\mathbb{R})}$ to the gauging of the subgroup of $\mathrm{SL}(2,\mathbb{R})$ with generator the mass matrix M employed in the reduction. There are three distinct cases depending on the value of $\vec{m}^2 = \frac{1}{4}({m_1}^2 + {m_2}^2 - {m_3}^2)$ [43, 44] characterizing the a set of three conjugacy classes already shown in (2.1): $\mathbb{R}, \mathrm{SO}(1,1)^+, \mathrm{SO}(2)$. Since we will make use of them we will describe them shortly.[2] Each of the subgroups is generated by a $\mathrm{SL}(2,\mathbb{R})$ group element Λ with $\det\Lambda = 1$. They are classified according to their trace:

- The parabolic gauged supergravity is associated to the gauging of the subgroup $\mathbb{R}$ with parameter ζ generated by

$$\Lambda_p = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$

  The conjugacy class corresponds to matrices with $|Tr\Lambda_p| = 2$.

- The hyperbolic gauged supergravity is associated to the gauging of the subgroup $\mathrm{SO}(1,1)^+$ with parameter γ

$$\Lambda_h = \begin{pmatrix} e^\gamma & 0 \\ 0 & e^{-\gamma} \end{pmatrix}. \tag{2.7}$$

  The conjugacy class is formed with matrices whose $|Tr\Lambda_h| > 2$

[2]To simplify the notation we keep the one used in [4] and summarize their results focusing only in the monodromy analysis.

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- The elliptic gauged supergravity is associated to the gauging of the subgroup SO(2) generated by elements Λ_e of $\mathrm{SL}(2,\mathbb{R})$ with parameter θ,

$$\Lambda_e = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{2.8}$$

  The elliptic conjugacy class correspond to matrices with $|Tr\Lambda_e| < 2$.

The group $\mathbb{R}^+$ is a one-parameter conjugacy class. It corresponds to the scalings that leave invariant the field equations but scale globally the lagrangian. These symmetries where called trombone by [31]. Its gauging was studied for example in [5, 6]. It corresponds to the reduction with $m_4 \neq 0; m_1 = m_2 = m_3 = 0$. Following [4] the $\mathbb{R}^+$-symmetry has been gauged with parameter $\Lambda_{\mathbb{R}^+} = e^{m_4\lambda}$

As explained, in [4] the complete set of deformations $\{m_i, m_4\}$ for the IIB reductions corresponds to

$$\Lambda_{GL(2,R)} = \Lambda_{\mathrm{SL}(2,R)}\Lambda_{\mathbb{R}^+}. \tag{2.9}$$

At quantum level the realization of these symmetries G is proposed to be associated to their arithmetic subgroups $G(Z)$ [15]. The quantum realization of the trombone symmetry is more involved. The problem at quantum level is the following: The group $GL(2,\mathbb{R})$ should break to its arithmetic subgroup to guarantee the quantization of the BPS charge lattice, however the set of matrices $Mat(2,\mathbb{Z})$ whose determinant is an integer does not form a group since the inverse of an integer is not necessarily an integer. (The arithmetic subgroup of $GL(2,\mathbb{R})$ is the group $GL(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})\times Z_2$, but it does fail in incorporating the scalings). In [31] they found a proper way to model out the scalings at quantum level by introducing nonlinear representations of $\mathrm{SL}(2,\mathbb{Z})$ that they called active, to distinguish from those associated to the U-duality. This symmetry is characterised by the fact that it acts on the lattice charge transforming integer charges into integer charges by $\mathrm{SL}(2,\mathbb{Z})$ transformation but leaving the moduli fixed. This is achieved by the use of a compensation transformation, that it is applied once the U-duality transforms charges and moduli by the linear $\mathrm{SL}(2,\mathbb{Z})$, acting on the transformed moduli to get it back to its original value.

3 The supermembrane with a topological condition

In this section we will make a self-contained summary of the construction of supermembrane with central charges due to a topological condition. The hamiltonian of the $D = 11$ Supermembrane [46] may be defined in terms of maps X^M, $M = 0, \dots, 10$, from a base manifold $R \times \Sigma$, where Σ is a Riemann surface of genus g onto a target manifold which we will assume to be 11D Minkowski. Following [47, 48] one may now fix the Light Cone Gauge, (LCG),

$$X^+ = T^{-2/3}P^{0+}\tau = -T^{-2/3}P^0_-\tau, \quad P_- = P^0_-\sqrt{W}, \quad \Gamma^+\Psi = 0 \tag{3.1}$$

where $\sqrt{W}$ is a time independent density introduced in order to preserve the density behavior of P_-. X^-, P_+ are eliminated from the constraints and solve the fermionic second class constraints in the usual way [48].

The canonical reduced hamiltonian to the light-cone gauge has the expression [48]

$$\mathcal{H} = \int_\Sigma d\sigma^2 \sqrt{W} \left(\frac{1}{2} \left(\frac{P_M}{\sqrt{W}} \right)^2 + \frac{1}{4} \{X^M, X^N\}^2 - \overline{\Psi}\Gamma_- \Gamma_M \{X^M, \Psi\} \right) \tag{3.2}$$

subject to the constraints

$$\phi_1 := d\left(\frac{P_M}{\sqrt{W}} dX^M - \overline{\Psi}\Gamma_- d\Psi \right) = 0 \tag{3.3}$$

and

$$\phi_2 := \oint_{C_s} \left(\frac{P_M}{\sqrt{W}} dX^M - \overline{\Psi}\Gamma_- d\Psi \right) = 0, \tag{3.4}$$

where the range of M is now $M = 1, \ldots, 9$ corresponding to the transverse coordinates in the light-cone gauge, C_s, $s = 1, 2$ is a basis of 1-dimensional homology on Σ,

$$\{X^M, X^N\} = \frac{\epsilon^{ab}}{\sqrt{W(\sigma)}} \partial_a X^M \partial_b X^N. \tag{3.5}$$

$a, b = 1, 2$ and σ^a are local coordinates over Σ. ϕ_1 and ϕ_2 are generators of area preserving diffeomorphisms, see [49]. That is

$$\sigma \to \sigma' \quad \to \quad W'(\sigma) = W(\sigma).$$

When the target manifold is simply connected dX^M are exact one-forms.

We consider now the compactified Supermembrane embedded on a target space $M_9 \times T^2$ where T^2 is a flat torus defined in terms of a lattice $\mathcal{L}$ on the complex plane C:

$$\mathcal{L} : z \to z + 2\pi R(l + m\tau), \tag{3.6}$$

where m, l are integers, R is real and represent the radius, $R > 0$, and τ a complex moduli $\tau = Re\tau + iIm\tau$, $Im\tau > 0$, T^2 is defined by $C/\mathcal{L}$. τ is the complex coordinate of the Teichmuller space for $g = 1$, that is the upper half plane. The Teichmuller space is a covering of the moduli space of Riemann surfaces, it is a $2g - 1$ complex analytic simply connected manifold for genus g Riemann surfaces.

The conformally equivalent tori are identified by the parameter τ modulo the Teichmuller modular group, which in the case $g = 1$ is $SL(2, \mathbb{Z})$. It acts on the Teichmuller space through a Mobius transformation and it has a natural action on the homology group $H_1(T^2)$.

We consider maps X^m, X^r from $M_9 \times T^2$ to the target space , with $r = 1, 2; m = 3, \ldots, 9$ where X^m are single valued maps onto the Minkowski sector of the target space while X^r maps onto the T^2 compact sector of the target. The winding condition corresponds to

$$\begin{aligned} &\oint_{\mathcal{C}_s} dX = 2\pi R(l_s + m_s\tau) \\ &\oint_{\mathcal{C}_s} dX^m = 0 \end{aligned} \tag{3.7}$$

where $dX = dX^1 + idX^2$ and $l_s, m_s, s = 1, 2$, are integers. We denote $d\widehat{X}^r$, $r = 1, 2$, the normalized harmonic one-forms with respect to $\mathcal{C}_s$, $s = 1, 2$, a canonical basis of homology on Σ:

$$\oint_{\mathcal{C}_s} d\widehat{X}^r = \delta^r_s. \tag{3.8}$$

We now impose a topological restriction on the winding maps [10]: the irreducible winding constraint,

$$\int_\Sigma dX^r \wedge dX^s = n\epsilon^{rs}\mathrm{Area}(T^2) \quad r, s = 1, 2 \tag{3.9}$$

Using $\mathrm{Area}(T^2) = (2\pi R)^2 Im\tau$, condition (3.9) implies that the winding matrix $\mathbb{W} = \left(\begin{smallmatrix} l_1 & l_2 \\ m_1 & m_2 \end{smallmatrix}\right)$ has $det\mathbb{W} = n \neq 0$. That is, all integers $l_s, m_s, s = 1, 2$ are admissible provided $det\mathbb{W} = n$ when n is assumed to be different from zero. ϵ^{rs} is the symplectic antisymmetric tensor associated to the symplectic 2-form on the flat torus T^2. In the case under consideration ϵ^{rs} is the Levi Civita antisymmetric symbol.

We may decompose the closed one-forms dX^r into

$$dX^r = M^r_s d\widehat{X}^s + dA^r \quad r = 1, 2 \tag{3.10}$$

where $d\widehat{X}^s, s = 1, 2$ is the basis of harmonic one-forms we have already introduced, dA^r are exact one-forms and M^r_s are constant coefficients. This condition is satisfied provided

$$M^1_s + iM^2_s = 2\pi R(l_s + m_s\tau) \tag{3.11}$$

Consequently, the most general expression for the maps X^r, $r = 1, 2$, is

$$dX = 2\pi R(l_s + m_s\tau)d\widehat{X}^s + dA, \tag{3.12}$$

$l_s, m_s,\ s = 1, 2$, arbitrary integers.

An important point implied by the assumption $n \neq 0$ is that the cohomology class in $H^2(\Sigma, Z)$ is non-trivial. It also implies that at global level the theory is described by an action formulated on a principal torus bundle over Σ. There exists a infinite set of possible gauge connections associated to it.

The topological condition (3.9) does not change the field equations of the hamiltonian (3.2). In fact, any variation of I^{rs} under a change δX^r, single valued over Σ, is identically zero. In addition to the field equations obtained from (3.2), the classical configurations must satisfy the condition (3.9). It is only a topological restriction on the original set of classical solutions of the field equations. In the quantum theory the space of physical configurations is also restricted by the condition (3.9). There is a compatible election for W on the geometrical picture we have defined. We define

$$\sqrt{W} = \frac{1}{2}\epsilon_{rs}\partial_a\widehat{X}^r\partial_b\widehat{X}^s\epsilon^{ab}, \tag{3.13}$$

it is a regular density globally defined over Σ. It is invariant under a change of the canonical basis of homology.

The physical hamiltonian in the LCG is given by

$$\begin{aligned}\mathcal{H} = &\int_\Sigma T^{-2/3}\sqrt{W}\left[\frac{1}{2}\Big(\frac{P_m}{\sqrt{W}}\Big)^2+\frac{1}{2}\Big(\frac{P_r}{\sqrt{W}}\Big)^2+\frac{T^2}{2}\{X^r,X^m\}^2+\frac{T^2}{4}\{X^r,X^s\}^2\right]\\ &+\int_\Sigma T^{-2/3}\sqrt{W}\left[\frac{T^2}{4}\{X^m,X^n\}^2-\overline{\Psi}\Gamma_-\Gamma_m\{X^m,\Psi\}-\overline{\Psi}\Gamma_-\Gamma_r\{X^r,\Psi\}\right]\end{aligned} \tag{3.14}$$

subject to the constraints

$$d(P_r dX^r + P_m dX^m - \overline{\Psi}\Gamma_- d\Psi) = 0 \tag{3.15}$$

$$\oint_{\mathcal{C}_s}(P_r dX^r + P_m dX^m - \overline{\Psi}\Gamma_- d\Psi) = 0 \tag{3.16}$$

and the global restriction (3.9). $\mathcal{C}_s$ is a canonical basis of homology on Σ. This is the case with trivial monodromy and hence without the gauging of the SL$(2,\mathbb{Z})$ symmetries described below. It is a symplectic gauge theory on a given isotopy class of symplectomorphisms.

The Mass operator of the supermembrane with central charges and KK modes found in [50] is

$$Mass^2 = T^2((2\pi R)^2 nIm\tau)^2+\frac{1}{R^2}\left(m_1^2+\left(\frac{m|q\tau-p|}{Im\tau}\right)^2\right)+T^{2/3}H \tag{3.17}$$

where the H is defined in terms of the above hamiltonian $\mathcal{H}$ once the winding contribution has been extracted $H=\mathcal{H}-T^{-2/3}\int_\Sigma\sqrt{W}\frac{T^2}{4}\{X_h^r,X_h^s\}^2$.

3.1 The SL$(2,\mathbb{Z})$ symmetries of the supermembrane with central charges

The supermembrane is invariant under are preserving diffeomorphisms on the base manifold. This symmetry is realized by the first class constraints on the theory. This is a gauge symmetry associated to a trivial principle bundle with structure group the symplectomorphisms homotopic to the identity. Besides this standard symmetry of the supermembrane, when the theory is restricted by the central charge condition (the irreducible winding condition), the theory is invariant under two SL$(2,\mathbb{Z})$ symmetries. One of them acting on the homology basis of the base manifold Σ, a two-torus. This SL$(2,\mathbb{Z})$ realizes the modular transformations[3] on the upper-half plane. The other SL$(2,\mathbb{Z})$ acts on the target space, on the moduli of the target torus: the complex τ and R parameters of the target torus. On τ acts as a Moebius transformations, however since the transformation of R is nontrivial, the equivalence classes of tori under this transformation are not conformally equivalent. Using these two SL$(2,\mathbb{Z})$ symmetries, it can be seen [50] that the mass contribution of the stringy states in the supermembrane with central charges exactly agree with the perturbative mass spectrum of (p,q) IIB and IIA superstring. Let us discuss it in more detail.

[3]In particular the supermembrane with central charges is invariant under the conformal maps homotopic to the identity.

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3.1.1 SL(2, $\mathbb{Z}$) of the Riemann surface

The supermembrane with central charges is invariant under area preserving diffeomorphisms homotopic to the identity. Those are diffeomorphisms which preserve $d\widehat{X}^r, r = 1, 2,$ the harmonic basis of one-forms. W is then invariant:

$$W^{'}(\sigma) = W(\sigma). \tag{3.18}$$

Moreover the supermembrane with central charges is invariant under diffeomorphisms changing the homology basis, and consequently the normalized harmonic one-forms, by a modular transformation on the Teichmüller space of the base torus Σ. In fact, if

$$d\widehat{X}^{'r}(\sigma) = S^r_s d\widehat{X}^s(\sigma) \tag{3.19}$$

provided

$$\epsilon_{rs} S^r_t S^s_u = \epsilon_{tu} \tag{3.20}$$

that is $S \in \mathrm{Sp}(2, Z) \equiv \mathrm{SL}(2, \mathbb{Z})$. We then conclude that the supermembrane with central charge, has an additional symmetry with respect to the compactified $D = 11$ Supermembrane without the topological irreducibility condition. All conformal transformations on Σ are symmetries of the supermembrane with central charges [51]–[57]. We notice that under (3.19)

$$dX \to 2\pi R(l^{'}_s + m^{'}_s \tau) d\widehat{X}^{'s} + dA^{'} \tag{3.21}$$

where $A^{'}(\sigma^{'}) = A(\sigma)$ is the transformation law of a scalar. Defining the winding matrix as $\mathbb{W} = \left(\begin{smallmatrix} l_1 & l_2 \\ m_1 & m_2 \end{smallmatrix}\right)$, then

$$\mathbb{W} \to \mathbb{W} S^{-1} \tag{3.22}$$

3.1.2 The U-duality invariance

The supermembrane with central charges is also invariant under the following transformation on the target torus T^2:

$$\begin{aligned} \tau &\to \frac{a\tau + b}{c\tau + d} \\ R &\to R|c\tau + d| \\ A &\to A e^{i\varphi} \\ \mathbb{W} &\to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \mathbb{W} \end{aligned} \tag{3.23}$$

where $c\tau + d = |c\tau + d| e^{-i\varphi}$ and $\Lambda = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(2, Z)$. As shown in [50] the hamiltonian density of the supermembrane with central charges is then invariant under (3.23). The SL(2, $\mathbb{Z}$) matrix now acts from the left of the matrix $\mathbb{W}$.

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The two actions from the left and from the right by $\mathrm{SL}(2,\mathbb{Z})$ matrices are not equivalent, they are complementary. The following remarks are valid. The general expression for the dX maps is then

$$dX = dX_h + dA \tag{3.24}$$

The harmonic part of dX,

$$dX_h = 2\pi R[(m_1\tau + l_1)d\widehat{X}^1 + (m_2\tau + l_2)d\widehat{X}^2]. \tag{3.25}$$

X_h is a minimal immersion from Σ to T^2 on the target, moreover it is directly related to a holomorphic immersion of Σ onto T^2. The extension of the theory of supermembranes restricted by the topological constraint to more general compact sectors in the target space is directly related to the existence of those holomorphic immersions.

4 The sculpting mechanism for gauging theories

In this section we summarize the results of the paper [61]. The mechanism of gauging proposed there consists in a specific change in the global description of a theory in terms of fibration, it is called *sculpting mechanism*. It consists in a deformation of the homotopy-type of the complete fibration preserving the homotopy-type of the base and the fiber. We will restrict here to the application of this mechanism to the supermembrane. Taking as the un-gauged theory the compactified supermembrane on a 2-torus. It corresponds to a invariant functional (action) over a Riemann base manifold whose fiber is the tangent space: $T^2 \times M_9$ for simplicity. The topologically nontrivial part of the fiber corresponds to the torus manifold associated to the tangent space. The global formulation of the un-gauged theory is a trivial torus bundle over a base manifold that for simplicity we also choose to be homotopically a torus.

The change of the total fiber bundle can be viewed in terms of two main steps: the first one is due to the introduction of a topological condition that we will explain below (the central charge) by which the trivial torus bundle is deformed into a principal bundle. On the physical side, it can be seen a restrictions on the maps allowed in the compactified target space. Secondly, the process of extracting the gauge field from the closed form in a consistent way implies the modification of the principal torus bundle in a symplectic torus bundle with monodromy. The total fiber bundle may or may not be symplectic according to the fact that the monodromy is given by the torsion class associated to the MCG of the $\Pi_1(\Sigma)$ base manifold. The resulting supermembrane is therefore, gauged in this new *sculpting* sense and it corresponds geometrically to a supermembrane minimally immersed in the target space. As a result of this procedure, the global symmetry of the un-gauged theory is partially broken to a subgroup $H \in G$. A new gauge symmetry A appears due to the global as a restriction of the diffeomorphism invariance gauge symmetry of the compact base manifold by the discrete symmetry subgroup Γ associated to the monodromy representation ρ of the harmonic forms. This symmetry gets promoted to a connection by the action of the principal fiber bundle to which the symplectic torus bundle is associated to.

The change in the homotopy-type of the complete manifold is produced by extracting properly the gauge connection from the closed 1-forms. The supermembrane with central charges as a global manifold corresponds to a symplectic torus bundle with nontrivial monodromy ρ. The Cohomology of the torus bundle change in this case in the following way

$$\mathbb{I} \rightsquigarrow H^2(\Sigma,\mathbb{Z}) \rightsquigarrow H^2(\Sigma,\mathbb{Z}_\rho) \tag{4.1}$$

being ρ a representation of the large diffeomorphims group of the base manifold. Notice that arrows do not imply a spectral sequence. Closely following [61] we just emphasize the three main steps needed to produce the *sculpting* deformation of the fiber bundle: I The first step is to impose the central charge condition which represents a obstruction to the triviality called that produces a twist in the fibration generating a principal fiber bundle whose cohomology is $H^2(\Sigma,\mathbb{Z})$. The lagrangian of the undeformed fiber has the following symmetries: a gauge symmetry $DPA_0(\Sigma_1^2)$, target space susy $N=2$, a discrete global symmetry $G\equiv \mathrm{Sp}(2,Z)$ associated to the wrapping condition of the embedded maps $\Sigma_1^2\to T^2$: There exists a infinite set of connections that can be attached to the principal bundle . The winding condition defines closed 1-forms dX_r that admit a Hodge decomposition in terms of harmonic one-forms $d\widehat{X}_r$ and a exact one-form dA_r:

$$dX_r = P_r^s d\widehat{X}_s + dA_r \tag{4.2}$$

the matrix P_r^s is associated to the 4 global degrees of freedom associated to the winding condition, whose coefficients depend on time. In presence of the central charge condition, the matrix $\mathcal{P}_r^s$ becomes constant and non-degenerate, (we are freezing the wrapping).

The harmonic one-forms due to the wrapping condition have a global Sp(2,Z) symmetry of the mapping class group. As a consequence of the nontrivial fibration now

$$P_r^s = M_r^s = 2\pi R^r S_r^s \quad \textit{with} \quad S_r^s\in \mathrm{SL}(2,\mathbb{Z}) \tag{4.3}$$

Once a fixed basis $\{d\hat{X}^s\}$, is chosen , the decomposition is unique, and P_r^s is fixed (for example to δ_s^r) there is a partial fixing of the symmetry that breaks the original global symmetry to a residual one, that leaves a global invariance under the subgroup that will be related to the monodromies associated to the gauging.

The next step is to *extract* a one-form connection to the nontrivial fiber bundle. We define a symplectic connection A preserving the structure of the fiber under holonomies. To this end, first we define a rotated derivative associated to the Weyl bundle [63]:

$$D_r\bullet = (2\pi R^r l^r)\theta_l^r \frac{\epsilon^{ab}}{\sqrt{W(\sigma)}}\partial_a\widehat{X}^l(\sigma)\partial_b\bullet \tag{4.4}$$

with $\theta\in \mathrm{SL}(2,\mathbb{Z})$ which depends on the monodromy ρ.

In 2-dim the area preserving diffeomorphisms are the same as the symplectomorphisms. The third relevant choice is the election for W on the geometrical picture we have defined. We define

$$\sqrt{W} = \frac{1}{2}\epsilon_{rs}\partial_a\widehat{X}^r\partial_b\widehat{X}^s\epsilon^{ab}, \tag{4.5}$$

it is a regular density globally defined over Σ. It is invariant under a change of the canonical basis of homology.

The matrix θ carries the information of the discrete global symmetry residual associated to transition functions of the patching of the different charts in the compact base manifold for a fixed base of harmonic forms. It plays a analogous role to the embedding tensor in the Noether gauging of supergravities theories. Let us signal that here the place where the discrete global symmetries appear together with the derivative operator instead of appearing besides the gauge field since its origin its topological associated to the p-brane base manifold compact surface.

The definition of this rotated derivative, we are performing an *extension of the covariant derivative definition*, in which the associated bundle has a nontrivial monodromy from the $\pi_1(\Sigma)$ on the homology of the fiber $H_1(T^2)$. The related derivative fixes a scale in the theory and breaks the former $H = \mathrm{Sp}(2,\mathbb{Z})$ theory to a subgroup $\Gamma \in \mathrm{Sp}(2,\mathbb{Z})$ by specifying the integers of S_r^s. Fixing R^r also fixes the Kahler and complex structure geometrical moduli.

The symplectic covariant derivative [63], is then:

$$\mathcal{D}_r \bullet = D_r \bullet + \{A_r, \bullet\} \tag{4.6}$$

and then the connection transform with the symplectomorphism like:

$$\delta_\epsilon A = \mathcal{D}_r \epsilon \tag{4.7}$$

The *sculpted* fiber bundle is a symplectic torus bundle with cohomology $H^2(\Sigma, \mathbb{Z}_\rho)$.

This symplectic form is one in particular different to the canonical one associated to the flat torus t^2 taken as a starting point in the compactified supermembrane case associated to the trivial torus bundle. This means that the nontrivial fibration implies a deformation in the base manifold, indeed the isometry group closely related to the harmonic group of symmetry is not the associated to a flat torus.Since a Riemann manifold has three compatible structures g_{ab}, J, Λ_{ab} the metric is associated to the harmonic one-forms that preserve the fiber associated to the MR-monopoles [62], the induced symplectomorphism do not lie in the same conformal class of the flat torus. There is a compatible election for W on the geometrical picture we have defined. We consider the $2g$ dimensional space of harmonic one-forms on Σ. We denote dX^r, $r = 1, 2$, the normalized harmonic one-forms with respect to $\mathcal{C}_s$, $s = 1, 2$, a canonical basis of homology on Σ:

$$\oint_{\mathcal{C}_s} d\widehat{X}^r = \delta^r_s. \tag{4.8}$$

We define

$$\sqrt{W} = \frac{1}{2} \epsilon_{rs} \partial_a \widehat{X}^r \partial_b \widehat{X}^s \epsilon^{ab}, \tag{4.9}$$

it is a regular density globally defined over Σ. It is invariant under a change of the canonical basis of homology.

It also implies that there is an U(1) nontrivial principle bundle over Σ and a connection on it whose curvature is given by $d\widehat{X}^r \wedge d\widehat{X}^s$. This U(1) nontrivial principal fiber bundle are associated to the presence of monopoles on the worldvolume of the supermembrane explicitly discussed in [62].

After replacing this expression in the hamiltonian (3.2) one obtain the gauged supermembrane in this new *sculpting* sense gauging the $SL(2,\mathbb{Z})$ that is the hamiltonian of the supermembrane with central charges [11, 12]:

$$H = \int_\Sigma \sqrt{W} d\sigma^1 \wedge d\sigma^2 \Big[\frac{1}{2}\Big(\frac{P_m}{\sqrt{W}}\Big)^2 + \frac{1}{2}\Big(\frac{P^r}{\sqrt{W}}\Big)^2 + \frac{1}{4}\{X^m, X^n\}^2 + \frac{1}{2}(\mathcal{D}_r X^m)^2 + \frac{1}{4}(\mathcal{F}_{rs})^2 \\ + (n^2 \mathrm{Area}^2_{T^2}) + \int_\Sigma \sqrt{W} \Lambda \Big(\mathcal{D}_r\Big(\frac{P_r}{\sqrt{W}}\Big) + \Big\{X^m, \frac{P_m}{\sqrt{W}}\Big\}\Big)\Big] \\ + \int_\Sigma \sqrt{W}[-\overline{\Psi}\Gamma_- \Gamma_r \mathcal{D}_r \Psi - \overline{\Psi}\Gamma_- \Gamma_m \{X^m, \Psi\} - \Lambda\{\overline{\Psi}\Gamma_-, \Psi\}]$$

where $\mathcal{D}_r X^m = D_r X^m + \{A_r, X^m\}$, $\mathcal{F}_{rs} = D_r A_s - D_s A_r + \{A_r, A_s\}$,
$D_r = 2\pi l_r \theta_r^l R_r \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a \widehat{X}^l \partial_b$ and P_m and P_r are the conjugate momenta to X^m and A_r respectively. $\mathcal{D}_r$ and $\mathcal{F}_{rs}$ are the covariant derivative and curvature of a symplectic noncommutative theory [63], constructed from the symplectic structure $\frac{\epsilon^{ab}}{\sqrt{W}}$ introduced by the central charge. The last term represents its supersymmetric extension in terms of Majorana spinors. Λ are the lagrange multiplier associated to the constrains. The physical degrees of the theory are the X^m, A_r, Ψ_α they are single valued fields on Σ.

5 The supermembrane as a symplectic torus bundle with monodromy in $SL(2, \mathbb{Z})$

In this section we develop the global construction found in [64], characterizing in deeper detail its connection with the $SL(2,\mathbb{Z})$ gaugings in supergravity in 9D.

We consider in this section the global structure of the supermembrane in the Light Cone Gauge when the fields X, Ψ are sections and A is a symplectic connection on a nontrivial symplectic torus bundle. A symplectic torus bundle ξ is a smooth fiber bundle $F \to E \xrightarrow{\pi} \Sigma$ whose structure group G is the group of symplectomorphisms preserving a symplectic two-form on the fiber F. Σ is the base manifold which we consider to be a closed, compact Riemann surface modeling the spacial piece of the foliation of the supermembrane worldvolume, and E is the total space. We will take the fiber as the target-space manifold $M_9 \times T^2$ consider in section 3, as in [61, 64]. The only topologically nontrivial part corresponds to the T^2, so from now on, we will only refer to this part that is the one that characterizes the fiber bundle. We consider in particular Σ, as already explained, a genus $g = 1$ surface with a non-flat induced metric. We remark that when $g > 1$, the first homotopy group $\Pi_1(\Sigma)$ is non-abelian allowing the construction of symplectic torus bundles with non-abelian monodromies. In this paper we will restrict to the abelian case only.

On T^2, a flat torus, we consider the canonical symplectic 2-form. Its pullback, using the harmonic maps from the base manifold to T^2, defines the symplectic 2-form ω on Σ. In terms of a harmonic basis of one-forms $d\widehat{X}^r$, $r = 1, 2$ in the notation of section 3:

$\omega = [(2\pi R)^2 n Im\tau]\epsilon_{rs} d\widehat{X}^r \wedge d\widehat{X}^s$. The symplectomorphisms[4] on Σ homotopic to the identity are generated by the first class constraints (3.3), (3.4). Moreover, the symplectomorphisms preserving ω define isotopic classes. These classes form a group $\Pi_0(G)$ where G is the group of all symplectomorphisms. In the case we are considering, where the fiber is T^2, $\Pi_0(G)$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. The action of G on the fiber T^2 produces an action on the homology and cohomology of T^2. This action reduces to an action of $\Pi_0(G)$, since on a given isotopy class two symplectomorphisms are connected by a continuous path within the class, and hence one cannot change the element of the homology or cohomology group. The action of G on the fiber over a point $x \in \Sigma$ when one goes around an element of $\Pi_1(\Sigma)$ defines a homomorphism

$$\Pi_1(\Sigma) \to \Pi_0(G) \approx \mathrm{SL}(2,\mathbb{Z}) \tag{5.1}$$

which may be called the monodromy of the symplectic torus bundle.[5] The monodromy may be trivial or not, but even when it is trivial, the symplectic torus bundle can be nontrivial. In fact, one could have a nontrivial transition within the symplectomorphisms on a isotopy class. If the monodromy is trivial, the symplectic torus bundle is trivial if and only if there exists a global section. When Σ is a 2-torus, as we are considering, $\Pi_1(\Sigma)$ is abelian and the homomorphism defines a representation $\rho : \Pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{Z})$, realized in terms of an abelian subgroup of $\mathrm{SL}(2,\mathbb{Z})$. It naturally acts on $H_1(T^2)$ the first homology group on T^2. This provides to $H_1(T^2)$ the structure of a $Z[\pi_1(\Sigma)]$-module which may be denoted Z^2_ρ. Given ρ there is a bijective correspondence between the equivalence classes of symplectic torus bundles with base Σ and $Z^2_\rho-$module, and the elements of $H^2(\Sigma, Z^2_\rho)$, the second cohomology group of Σ with coefficients Z^2_ρ [66]. Following [66]: the element of $H^2(\Sigma, Z^2_\rho)$ is called the cohomology class of the symplectic torus bundle and it is denoted $C(E)$. $C(E) = 0$ if and only if there exists a global section on E. If ρ is trivial, $C(E) = 0$ if and only if E is trivial.

The supermembrane theory with nontrivial central charge has $C(E) \neq 0$ and hence E is always nontrivial. The supermembrane on a eleven dimensional Minkowski target space [48] was formulated on a trivial symplectic bundle, as well as the supermembrane on a compactified space in [67]. The $C(E) \neq 0$ condition is the relevant condition which ensures a discrete spectrum of supermembrane with nontrivial central charges [51]–[57]. In the case of a trivial symplectic torus bundle the spectrum spectrum of the supermembranes was proven to be continuous from $[0, \infty)$ [68]. There is a third case , which has not been discussed in the literature: $C(E) = 0$ but a nontrivial monodromy. The analysis of the spectrum of a supermembrane on such a symplectic torus bundle could render a supermembrane theory with discrete spectrum on the $C(E) = 0$ sector, which is excluded by the supermembrane with the nontrivial central charges. This important point will be analyzed elsewhere.

[4] On a 2-dimensional surface symplectomorphisms and area preserving diffeomorphisms define the same group.

[5] It would be interesting to see if there is a relation (if any) with a construction on torus bundles with monodromy that has recently appeared [65].

The second cohomology $H^2(\Sigma, Z^2_\rho)$ may be equal to Z, as in the case of the representation

$$\rho(\alpha,\beta)=\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}, \quad or \quad \rho(\alpha,\beta)=\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \tag{5.2}$$

where (α,β) denotes the element of $\Pi_1(\Sigma)$. But it may also have a finite number of elements as in the case of [66],

$$\rho(\alpha,\beta)=\begin{pmatrix}-2mn+1 & 2mn^2+n\\ -m & mn+1\end{pmatrix}^{(\alpha+\beta)} \tag{5.3}$$

where the integers $m,n>0$. In this case $H^2(\Sigma, Z^2_\rho)=Z_m\oplus Z_n$. The number of inequivalent symplectic torus bundles is, in this case, mn. Hence given ρ the number of inequivalent symplectic torus bundles is in general not in the correspondence with Z as one could in principle think. This remark has relevant consequences in the analysis of the symmetry groups associated to the theory at quantum level. From a geometrical point there is a qualitative difference between the symplectic torus bundle associated with the representations (5.2) and (5.3). A theorem in [66] ensures the existence of symplectic 2-form on E which reduces to the the symplectic 2-form on each fiber if and only if the element $H^2(\Sigma, Z^2_\rho)$ associated to E is a torsion element. In case (5.3) all elements are torsion while in case (5.2) only $C(E)=0$, which is excluded if we consider a supermembrane with nontrivial central charge. Let us now consider the transformation law of the fields describing the supermembrane with nontrivial central charge. In section 3, we showed the transformation law under a rigid $\mathrm{SL}(2,\mathbb{Z})$ transformations. There are two $\mathrm{SL}(2,\mathbb{Z})$ invariances, one associated to the basis Σ and one to the moduli on the target space. We now consider a supermembrane on a symplectic torus bundle with monodromy $\rho(\alpha,\beta)$. Under a rigid $\mathrm{SL}(2,\mathbb{Z})$ on the target the symplectic connection $A(x)$ transforms with a global factor $e^{i\varphi}$ where $e^{-i\varphi}=\frac{c\tau+d}{|c\tau+d|}$ and $\Lambda=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ acts on the moduli and winding matrix as already stated. On the symplectic torus bundle with monodromy $\rho(\alpha,\beta)$, $A(x)$ transforms with a phase factor $e^{i\varphi_\rho}$ with $\varphi_\rho\equiv\varphi(\rho(\alpha,\beta))$ but now $\Lambda\equiv\rho(\alpha,\beta)$. That is a,b,c,d are integers which depend on (α,β). For example, if we consider $\alpha=\beta=0$ corresponding to a trivial element of $\Pi_1(\Sigma)$ then $\varphi=0$, while if $(\alpha,\beta)\neq(0,0)$ then φ can be different from zero , for example in case (5.3). If we write $A(x)=|A(x)|e^{i\lambda(x)}$ then associated to $(\alpha,\beta)\in\Pi_1(\Sigma)$ we have $A(x)=|A(x)|e^{i\lambda(x)+\varphi_\rho}$. We then have,

$$d\left(|A(x)|e^{i\lambda(x)+e^{i\varphi_\rho}}\right)=dA(x)e^{i\varphi_\rho}. \tag{5.4}$$

In order to take into account the phase factor $e^{i\varphi_\rho}$ we may multiply the symplectic covariant derivative in the formulation by this phase factor and leave $A(x)$ as a single-valued one-form connection. In the hamiltonian of section 3, the phase factor $e^{i\varphi_\rho}$ is canceled by its complex conjugate contribution consequently, the hamiltonian is well-defined on a symplectic torus bundle with nontrivial monodromy. Another important aspect of the supermembrane formulated on a symplectic torus bundle with monodromy is that the (p,q)

Kaluza-Klein charges in the mass squared formula take value on the Z^2_ρ-module. In fact, the (p,q) charges are naturally associated to the element of $H_1(T^2)$. We then have a nice geometrical interpretation: The KK charges are associated to the homology of T^2 on the target, while the winding is associated to the cohomology on the base Σ. In [64] we proved that the hamiltonian together with the constrains are invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ on the homology group $H_1(T^2)$ of the fibre 2-torus T^2. So that the supermembrane with central charges may be formulated in terms of sections of symplectic torus bundles with a representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{Z})$ inducing a $Z[\pi_1(\Sigma)]$-module in terms of the $H_1(T^2)$ homology group of the fiber. Locally the target is a product of $M_9 \times T^2$ but globally we cannot split the target from the base Σ since T^2 is the fiber of the non trivial symplectic torus bundle $T^2 \to \Sigma$. The formulation of the supermembrane in terms of sections of the symplectic torus bundle with a monodromy is a nice geometrical structure to analyze global aspects of gauging procedures on effective theories arising from M-theory. The allowed classes of monodromy are those subgroups corresponding to the elliptic, parabolic and hyperbolic inequivalent classes of $\mathrm{SL}(2,\mathbb{Z})$ showed in the section 2. But as already explained, the global classification depends on the cohomology class of the fibration, so it is more refined at global level, i.e. there are more inequivalent classes of symplectic torus bundles which may be related to different domain-wall solutions of supergravity.

6 Classification of symplectic torus bundles

Two conjugate representations ρ and $U\rho U^{-1}$, with $U \in \mathrm{SL}(2,\mathbb{Z})$, define Z^2_ρ and $Z^2_{U\rho U^{-1}}$ modules with isomorphic cohomology groups $H^2(\Sigma, Z^2_\rho) \sim H^2(\Sigma, Z^2_{U\rho U^{-1}})$. They define equivalent symplectic torus bundles. An equivalent way to see it is to consider the group of coinvariants associated to ρ and $U\rho U^{-1}$. There is an isomorphism between the group of coinvariants associated to ρ and to $U\rho U^{-1}$, they define equivalent symplectic torus bundles. In order to classify them, we must determine first the conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$ and then then the associated coinvariants. Once this has been done the correspondence with the nine-dimensional gauged supergravities follows directly. $\mathrm{SL}(2,\mathbb{Z})$ may be generated by S and ST^{-1} where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{6.1}$$

Every conjugacy class of $\mathrm{SL}(2,\mathbb{Z})$ can be represented by one of the following [71]

$$\begin{array}{ll} \pm S & \text{with Trace=0.} \\ \pm T^{-1}S, \quad \pm(T^{-1}S)^2, & \text{with} \quad |\text{Trace}| = 1. \\ \pm T^n, n \in \mathbb{Z} & \text{with} \quad |\text{Trace}| = 2. \\ \pm T^{r_0}ST^{r_1}S\dots T^{r_k}S & r_i \le -2, r_0 < -2, i = 1,\dots,k, \quad \text{and} \quad |\text{Trace}| > 2. \end{array} \tag{6.2}$$

The representations:

$$\begin{aligned} \rho(\alpha,\beta) &= (\pm S)^{\alpha+\beta} \\ \rho(\alpha,\beta) &= (+T^{-1}S)^{\alpha+\beta} \end{aligned} \tag{6.3}$$

$$\rho(\alpha, \beta) = (+(T^{-1}S)^2)^{\alpha+\beta}$$
$$\rho(\alpha, \beta) = (-\mathbb{I})^{\alpha+\beta}$$

define finite subgroups isomorphic to Z_4, Z_6, Z_3, Z_2 respectively, associated to the monodromies $\mathcal{M}_4, \mathcal{M}_6, \mathcal{M}_3, \mathcal{M}_2$ in [20]. The representations

$$\rho(\alpha, \beta) = (-T^{-1}S)^{\alpha+\beta} \quad \text{and} \quad \rho(\alpha, \beta) = (-(T^{-1}S)^2)^{\alpha+\beta} \tag{6.4}$$

define subgroups isomorphic to Z_3 and Z_6 respectively. The associated coinvariant groups are the trivial one and Z_6 respectively. In terms of the representation

$$\rho_{mn}(\alpha, \beta) = \begin{pmatrix} -2mn+1 & 2mn^2+n \\ -m & 1+mn \end{pmatrix}^{\alpha+\beta} \tag{6.5}$$

with $m, n > 0$ [66], $[(T^{-1}S)^2]^{\alpha+\beta}$ is conjugate to $\rho_{31}(\alpha, \beta)$, $S^{\alpha+\beta}$ is conjugate to $\rho_{21}(\alpha, \beta)$ and $[T^{-1}S]^{\alpha+\beta}$ to $\rho_{11}(\alpha, \beta)$. The inequivalent symplectic torus bundles associated to $\rho_{mn}(\alpha, \beta)$ are mn and all of them correspond to the torsion classes in $H^2(B, Z^2_\rho) \equiv \mathbb{Z}_n \oplus \mathbb{Z}_m$ equivalently to the coinvariant group $Z_n \oplus Z_m$. It is interesting that beyond the finite group cases $(\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6)$ associated to the elliptic case, there are monodromies defining non-finite subgroups associated to a finite number of symplectic torus bundle. For example $\rho_{41}(\alpha, \beta)$ is conjugate to $(-T^{-1})^{\alpha+\beta} \equiv \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{\alpha+\beta}$, which generates a non-finite subgroup, the associated number of symplectic torus bundles is finite, four in this case. The group of coinvariants is isomorphic to Z_4. For the parabolic conjugacy class $|\text{Trace}| = 2$, there are two cases, the first one is associated to monodromies with a positive trace, they generate infinite symplectic torus bundles in correspondence to $\mathbb{Z}$, while the second case, with negative trace, generates a finite number of inequivalent symplectic torus bundles. The group of coinvariants is always Z_4. In both cases the subgroups generated by the monodromy representation are not finite. If $mn > 4$, $\text{Trace}\rho_{mn}(\alpha, \beta) < -2$. These are hyperbolic representations of $\text{SL}(2, \mathbb{Z})$. In this case there is a finite number of inequivalent symplectic torus bundles generated by non-finite subgroups.

In this case, $mn > 4$, the matrix $M \equiv \rho_{mn}(\alpha, \beta)$ (6.5) with $\alpha + \beta = 1$ is conjugate, according to (6.3) to $\pm T^{r_0} S T^{r_1} S \ldots T^{r_k} S$, $r_i \leq -2, r_0 < 2$, and $i = 1, \ldots, k$. In particular, we obtain for $n = 1, m \geq 5$ that M is conjugate to $-T^{-3} S (T^{-2} S)^{m-5}$. See appendix B. The group of coinvariants associated to the corresponding monodromy is Z_m, $m \geq 5$. There are m inequivalent symplectic torus bundles corresponding to this monodromy. The sign is very relevant. For example, for $m = 5$ $\rho_{51}(\alpha, \beta) = (-T^3 S)^{\alpha+\beta}$ has a coinvariant group Z_5 while $(+T^3 S)^{\alpha+\beta}$ has a trivial coinvariant group, with only the identity element. The latter case is not contained in (6.5), since it corresponds to positive trace.

6.1 Gauge fixing and residual symmetries

We may now consider the gauge freedom associated to the gauging of the abelian subgroups of $\text{SL}(2, Z)$. It corresponds to equivalent symplectic torus bundles arising in particular from conjugate representations $U\rho(\alpha, \beta)U^{-1}$, $U \in \text{SL}(2, Z)$. Two conjugate representations ρ and $U\rho U^{-1}$, with $U \in \text{SL}(2, \mathbb{Z})$, define Z^2_ρ and $Z^2_{U\rho U^{-1}}$ modules with isomorphic cohomology groups $H^2(\Sigma, Z^2_\rho) \sim H^2(\Sigma, Z^2_{U\rho U^{-1}})$. They define equivalent symplectic torus bundles. An equivalent way to see it is to consider the group of coinvariants associated to ρ and $U\rho U^{-1}$. In fact, the group $H^2(\Sigma, Z^2_\rho)$ is isomorphic, via Poincare duality, to the coinvariants group associated to ρ. There is then an isomorphism between the group of coinvariants associated to ρ and to $U\rho U^{-1}$, they define equivalent symplectic torus bundles. Given $\mathcal{Q} \equiv \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in H_1(T^2)$, the group of coinvariants of monodromy ρ is the abelian group of equivalence classes

$$\{\mathcal{Q} - \Lambda\widehat{\mathcal{Q}} - \widehat{\mathcal{Q}}\} \tag{6.6}$$

for any $\Lambda \in \rho$ and any $\widehat{\mathcal{Q}} = \begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} \in H_1(T^2)$. It follows that this class is mapped to the class associated to $U\mathcal{Q}$ under the representation $U\rho U^{-1}$:

$$\{U\mathcal{Q} - U\Lambda U^{-1}\widetilde{\mathcal{Q}} - \widetilde{\mathcal{Q}}\} \tag{6.7}$$

where $\widetilde{\mathcal{Q}} = U\widehat{\mathcal{Q}}$, but any $\widetilde{\mathcal{Q}} \in H_1(T^2)$ may always be expressed as $U\widehat{\mathcal{Q}}$ for some other $\widehat{\mathcal{Q}} \in H_1(T^2)$, since U is invertible. There is then an isomorphism between the group of coinvariants associated to ρ and to $U\rho U^{-1}$, they define equivalent symplectic torus bundles.

We may choose U in order to leave freezed the winding matrix under the action of the monodromy transformation. The gauge fixing procedure goes as follows. We re-arrange the winding matrix as $M = \left(\begin{smallmatrix} m_1 & l_1 \\ m_2 & l_2 \end{smallmatrix}\right)$, with $detM = n$. Under the symmetry of section 3 it transforms as

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} m_1 & l_1 \\ m_2 & l_2 \end{pmatrix} \Lambda^{-1} \tag{6.8}$$

The SL$(2, Z)$ symmetry associated to the base manifold may be interpreted as having independence on the basis of homology on the base manifold. In fact, the winding matrix is associated to a particular basis of homology. Hence, since the change of homology basis corresponds to a SL$(2,\mathbb{Z})$ transformations, the theory should only depend on the equivalence classes constructed from the application from the left by a SL$(2,\mathbb{Z})$ matrix:

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} m_1 & l_1 \\ m_2 & l_2 \end{pmatrix}. \tag{6.9}$$

Under this transformation the winding matrix may always be reduced to the canonical form

$$\begin{pmatrix} \lambda_1 & 0 \\ \beta & \lambda_2 \end{pmatrix} \tag{6.10}$$

with $\lambda_1\lambda_2 = n$ the central charge defined in section 2, and $|\beta| \leq \lambda_1/2$. In particular, if $\lambda_1 = n$, $\lambda_2 = 1$ then $|\beta| \leq \frac{n}{2}$. we notice that in addition to the central charge integer n there are additional degrees of freedom represented by the integer β. We may now consider the supermembrane formulated as a symplectic torus bundle with monodromy $U\rho(\alpha,\beta)U^{-1}$. The action on the winding matrix is given by

$$\begin{pmatrix} \lambda_1 & 0 \\ \beta & \lambda_2 \end{pmatrix} U\rho^{-1}U^{-1}. \tag{6.11}$$

We may also act from the left by a $\mathrm{SL}(2,\mathbb{Z})$ matrix which we take of the form $V^{-1}\rho^* V$. We can take U and V both $\mathrm{SL}(2,\mathbb{Z})$ matrix in order to rewrite the winding matrix in form which is left invariant under the action of ρ^* and ρ^{-1}. For example if we take the monodromy

$$\rho(\alpha,\beta) = \begin{pmatrix} a & nb_1 \\ c & d \end{pmatrix}^{\alpha+\beta} \in \mathrm{SL}(2,\mathbb{Z}) \tag{6.12}$$

associated to a supermembrane with central charge n, for particular values of a, b, c, d and n, this includes elliptic, parabolic and hyperbolic monodromies. Then we can take

$$\rho(\alpha,\beta)^* = \begin{pmatrix} a & b_1 \\ nc & d \end{pmatrix} \tag{6.13}$$

and V, U such that

$$V \begin{pmatrix} \lambda_1 & 0 \\ \beta & \lambda_2 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \tag{6.14}$$

Then

$$\rho^*(\alpha,\beta) \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \rho^{-1}(\alpha,\beta) = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \tag{6.15}$$

We then have

$$V^{-1}\rho^* V \begin{pmatrix} \lambda_1 & 0 \\ \beta & \lambda_2 \end{pmatrix} U\rho^{-1}U^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ \beta & \lambda_2 \end{pmatrix} \tag{6.16}$$

that is, the winding matrix is left invariant under the monodromy $\rho(\alpha,\beta)$ provided we consider an associated abelian representation of $\mathrm{SL}(2,\mathbb{Z})$ acting on the homology of the base manifold. Having established the gauge fixing procedure arising from conjugate representations $U\rho(\alpha,\beta)U^{-1}$, we may now ask what is the residual symmetry of the supermembrane on that symplectic torus bundle with monodromy $U\rho(\alpha,\beta)U^{-1}$. The residual symmetry must leave invariant the elements of the coinvariant group of the monodromy. It must act as the identity on the coinvariant group. Consequently it is the same abelian group defining the monodromy. In distinction, a group that commutes with the monodromy group maps the coinvariant group into itself, but it does not need to act as the identity. The latest corresponds to the residual symmetry of a theory when on considers the collection of bundles associated to a given monodromy. The collection procedure occurs when we construct gauged supergravities in 9D from the 11D compactified supermembrane theory on the symplectic torus bundle with the central charge condition.

We may finally express the hamiltonian of the supermembrane with central charges on a symplectic torus bundle with monodromy $\rho(\alpha,\beta)$ in the following way,

$$\begin{aligned} H = &\int_\Sigma T^{2/3}\sqrt{W}\left[\frac{1}{2}\left(\frac{P_m}{\sqrt{W}}\right)^2 + \frac{1}{2}\left(\frac{P\overline{P}}{W}\right) + \frac{T^2}{4}\{X^m, X^n\}^2 + \frac{T^2}{2}\mathcal{D}X^m\overline{\mathcal{D}}X^m + \frac{T^2}{8}\mathcal{F}\overline{\mathcal{F}}\right] \\ &- \int_\Sigma T^{2/3}\sqrt{W}[\overline{\Psi}\Gamma_-\Gamma_m\{X^m,\Psi\} + 1/2\overline{\Psi}\Gamma_-\overline{\Gamma}\{X,\Psi\} + 1/2\overline{\Psi}\Gamma_-\Gamma\{\overline{X},\Psi\}], \end{aligned} \tag{6.17}$$

subject to the first class constraints. We denote

$$\begin{aligned} \mathcal{D}\circ &= D\circ + \{A, \circ\}, \\ \mathcal{F} &= D\overline{A} - \overline{D}A + \{A, \overline{A}\} \end{aligned} \tag{6.18}$$

where

$$D = \frac{\epsilon^{ab}}{\sqrt{W}} 2\pi R(l_r + m_r\tau)\theta^s_r \partial_a \widehat{X}^s \partial_b \tag{6.19}$$

and the matrix θ is given by

$$\theta = (V^{-1}(\rho^*)^{-1}V)^T. \tag{6.20}$$

The matrix θ was derived by the sculpting approach. We have obtained its explicit expression here from the gauge fixing procedure introduced in this section. As mentioned before in this section D and A acquire a phase factor $e^{i\varphi_\rho}$ as a consequence of the monodromy. The hamiltonian is manifestly invariant under this transformation. The moduli R and τ transform as (3.23) where $\Lambda = U\rho U^{-1}$. The factor θ^r_s in the expression of D arises from the transformation of the basis of harmonic one-forms. It can be also interpreted as a transfromation of the winding matrix with components l_r and m_r, $r = 1, 2$. If we take this point of view the winding numbers belong then to an element of the coinvariant group associated to the monodromy $V^{-1}(\rho^*)V$ acting on the cohomology of the base manifold while the KK charges belong to an element of the coinvariant group of monodromy ρ. The mass squared formula remains then invariant under transitions on the symplectic torus bundle provided we interpret the winding numbers and KK charges as equivalence classes of the corresponding coinvariant groups.

7 Gauging of the trombone symmetry on the supermembrane

In the previous section we showed that the supermembrane with central charges may be formulated on a symplectic torus bundle with a nontrivial $\mathrm{SL}(2, Z)$ monodromy. Corresponding to each monodromy we obtain the gauging of an abelian subgroup of $\mathrm{SL}(2, \mathbb{Z})$, the isotopy group of symplectomorphisms preserving the symplectic 2-form introduced in the construction of the supermembrane theory with central charges. The monodromy defined as the homomorphism from $\Pi_1(\Sigma) \to \Pi_0(G) \approx \mathrm{SL}(2, \mathbb{Z})$ was constructed in terms of parabolic, elliptic and hyperbolic $\mathrm{SL}(2, \mathbb{Z})$ matrices. We are going to show that there is also a supermembrane theory with central charges formulated on a symplectic torus bundle with a monodromy corresponding to the gauging of the trombone symmetry introduced in the context of supergravity [31]. See section 2. The first step will be to consider the supermembrane formulated on a symplectic torus bundle with trivial monodromy and obtain the transformation law of the mass squared formula presented in section 3 under the scaling symmetry. We first follow the approach [31] and work out the general compensator in the context of the supermembrane theory. The second step will be to gauge the trombone symmetry in M-theory.

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7.1 The trombone symmetry on the compactified M2 with central charges

Let us obtain the transformation law of the mass squared formula presented in section 3 under the scaling symmetry. Following the lines of [31], we are going to generalize the compensating transformation for arbitrary values of the moduli τ.

The general form of the compensating transformation: we consider a integer lattice of KK charges parametrized by $Q = \binom{p}{q}$. The geometrical interpretation of Q is in terms of the elements of the homology group $H_1(T^2)$ of the fiber, which is a 2-torus. Under the U-duality transformation (3.23) $Q \to \Lambda Q$ with $\Lambda \in \mathrm{SL}(2,\mathbb{Z})$ with the corresponding transformation of the moduli parameters as stated in section 3. We are interested in the most general transformation mapping $Q_i \to Q_j : Q_j = \Lambda_{ij} Q_i$. For a given Q_i we define $\Lambda_i \in \mathrm{SL}(2,\mathbb{Z}) : \Lambda_i Q_0 = Q_i$ where $Q_0 = \binom{1}{0}$. Λ_i is not unique, its most general expression is $\Lambda_i g$ where $g = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ for any integer $m \neq 0$, and $g \in H$ is the Borel group of parabolic $\mathrm{SL}(2,\mathbb{Z})$ matrices. We then have $\Lambda_{ji} = \Lambda_j g \Lambda_i^{-1}$ for any $g \in H$. Under composition we have

$$\Lambda_{kj}\Lambda_{ji} = \Lambda_{ki}. \tag{7.1}$$

For $\Lambda_{ji} \in \mathrm{SL}(2,\mathbb{Z})$ acting on Q_i there is an associated transformation of the moduli parameters as stated in section 3. The mass formula is invariant under the overall transformation. We consider equivalence classes of matrices Λ_{ji}: two elements of the class differ in an element $g \in H$. We denote the class $\widetilde{\Lambda}_{ij}$. We may now introduce the compensator in the approach of [31]. The following result is valid: for each equivalence class $\widetilde{\Lambda}_{ji}$ there exists a unique matrix $H_{ji} \in GL(2,\mathbb{R})$, $H_{ji} = M_{ji}\Lambda_{ji}$ and a unique complex number $h_{ji} \in \mathbb{C}$ such that

(i) $H_{ji}Q_i = Q_j$

(ii) $H_{ji}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = h_{ji}\begin{pmatrix} \tau \\ 1 \end{pmatrix}$

H_{ji} and h_{ji} depend only on the equivalence class, it is independent of $g \in H$. In distinction, the compensator M_{ji} depends explicitly on $g \in H$. Relation ii) is equivalent to the following sequence of transformations:

$$\tau \overset{\Lambda_{ji}}{\to} \tilde{\tau} \overset{M_{ji}}{\to} \tau \tag{7.2}$$

where $\tau \to \widetilde{\tau}$ is the Moebius transformations associated to $\Lambda_{ji} \in \mathrm{SL}(2,\mathbb{Z})$. The general expression of the H_{ji} matrix is,

$$H_{ji} = \begin{pmatrix} -\frac{p_j}{q_j}u + \frac{q_i}{q_j}\mathcal{C} & \frac{p_j}{q_i} + \frac{p_i p_j}{q_i q_j}u - \frac{p_i}{q_j}\mathcal{C} \\ -u & \frac{q_j}{q_i} + \frac{p_i}{q_i}u \end{pmatrix} \tag{7.3}$$

with $u = \frac{(p_j q_i - p_i q_j)}{|p_i - q_i\tau|^2}$, $\mathcal{C} = det M_{ji} = \frac{|p_j - q_j\tau|^2}{|p_i - q_i\tau|^2}$ and $h_{ji} = \frac{p_j - q_j\overline{\tau}}{p_i - q_i\overline{\tau}}$, where $\overline{\tau}$ is the complex conjugate of τ. It then follows that the compensator M_{ji} depends explicitly on $g \in H$ since $M_{ji} = H_{ji}\Lambda_{ji}^{-1}$. Although $H_{ji} \in GL(2,\mathbb{R})$, the non-linear transformation maps integer charges Q_i into integer charges Q_j, as it should in order to satisfy the charge quantization condition. It is straightforward to show that H_{ji} defines a non-linear realization of the $\mathrm{SL}(2,\mathbb{Z})$ group. In fact, if

$$\widetilde{\Lambda}_{21} \to H_{21}, \quad \widetilde{\Lambda}_{32} \to H_{32}, \quad \widetilde{\Lambda}_{31} \to H_{31} \tag{7.4}$$

then $H_{21}Q_1 = Q_2, \quad H_{32}Q_2 = Q_3$ hence $H_{32}H_{21}Q_1 = Q_3$. Analogously,

$$H_{32}H_{21}\begin{pmatrix}\tau\\1\end{pmatrix} = \lambda_{32}\lambda_{21}\begin{pmatrix}\tau\\1\end{pmatrix} = \lambda_{31}\begin{pmatrix}\tau\\1\end{pmatrix} \tag{7.5}$$

The uniqueness of the transformation then implies $H_{31} = H_{32}H_{21}$.

H_{ji} realizes then a nonlinear representation of $\mathrm{SL}(2,\mathbb{Z})$ and it represents the trombone symmetry at the quantum level.

The mass operator transformation under trombone symmetry: having determined the transformation law for the KK charges and the complex moduli τ we may now consider the transformation of the other moduli R, and the winding matrix. From (3.23) we know their transformation law under $\Lambda_{ji} \in \mathrm{SL}(2,\mathbb{Z})$, we may now determine the compensator action on them . We will do so by imposing the condition that the hamiltonian remains invariant under its action. The transformation for the complex moduli τ may be re-written as:

$$\begin{pmatrix}\tau\\1\end{pmatrix} \overset{\frac{\Lambda_{ji}}{l_{ji}}}{\to} \begin{pmatrix}\tau'\\1\end{pmatrix} \overset{\frac{l_{ji}}{h_{ji}}M_{ji}}{\to} \begin{pmatrix}\tau\\1\end{pmatrix} \tag{7.6}$$

where $l_{ji} \equiv c\tau + d$ and $\Lambda \in \mathrm{SL}(2,\mathbb{Z})$, see (3.23) while $\frac{1}{|h_{ji}|}M_{ji} \in \mathrm{SL}(2,\mathbb{R})$ and h_{ji} was defined as in the previous section. The harmonic sector of the supermembrane may be expressed as

$$2\pi R(d\widehat{X}^1, d\widehat{X}^2)\begin{pmatrix}m_1 & l_1\\ m_2 & l_2\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}. \tag{7.7}$$

Under the first transformation in the composition 7.6 the factor $|l_{ji}|^{-1}$ is canceled by the transformation of R:

$$R \overset{|l_{ji}|}{\to} R' = R|l_{ji}|. \tag{7.8}$$

We must then consider

$$R'' = \frac{R'}{|l_{ji}|} \tag{7.9}$$

in order to compensate the factor $|l_{ji}|$ in the second transformation in 7.6. We then have

$$R \to R' \to R. \tag{7.10}$$

Finally, under Λ_{ji} the winding matrix transform as:

$$\begin{pmatrix} m_1 & l_1 \\ m_2 & l_2 \end{pmatrix} \rightarrow \begin{pmatrix} m_1^{'} & l_1^{'} \\ m_2^{'} & l_2^{'} \end{pmatrix} = \begin{pmatrix} m_1 & l_1 \\ m_2 & l_2 \end{pmatrix} \Lambda_{ji}^{-1} \tag{7.11}$$

Consequently, the compensating action must be

$$\begin{pmatrix} m_1^{'} & l_1^{'} \\ m_2^{'} & l_2^{'} \end{pmatrix} \overset{\Lambda_{ji}^{-1}}{\rightarrow} \begin{pmatrix} m_1 & l_1 \\ m_2 & l_2 \end{pmatrix} \tag{7.12}$$

in order to have an invariant hamiltonian under that action. We notice that the harmonic sector is not invariant but its contribution together with the one of its complex conjugate yields an invariant hamiltonian. The winding term in the mass formula also remain invariant while the KK therm varies according to:

$$\frac{|p_i - q_i\tau|}{RIm\tau} \rightarrow \frac{|p_j - q_j\tau|}{RIm\tau} \tag{7.13}$$

7.2 Gauging the trombone

We may finally consider the gauging of the trombone symmetry. The main point in the construction is the geometrical description of the KK charges (p, q) in terms of the elements of the homology group $H_1(T^2)$ of the fiber T^2. The homomorphism $\Pi_1(\Sigma) \rightarrow \Pi_0(G) \approx \mathrm{SL}(2, \mathbb{Z})$ determines a representation $\rho : \Pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbb{Z})$. If we denote $\rho(\alpha, \beta) \in \mathrm{SL}(2, \mathbb{Z})$ the element of $\mathrm{SL}(2, \mathbb{Z})$ associated to $(\alpha, \beta) \in \pi_1(\Sigma)$, its action on $H_1(T^2)$ yields

$$Q_j = \rho(\alpha, \beta)Q_i \tag{7.14}$$

Form section (6.1) we then conclude that $\rho(\alpha, \beta) = \Lambda_{ji}$ and there exists an associated non-linear representation realized in terms of the matrix H_{ji}. The monodromy is then constructed with this non-linear representation of $\mathrm{SL}(2, \mathbb{Z})$. we notice that the $Z[\Pi_1(\sigma)]$-module is the same as the one arising from the linear representation ρ, however its action on τ, R and the winding matrix is different since their transformation is done in terms of H_{ji} matrices. We thus obtain a different global structure for the supermembrane on this symplectic torus bundle. Following the analysis of section 5, the hamiltonian of the supermembrane is well-defined on this symplectic torus bundle. We notice that the (p, q) charges in the KK term of the mass squared formula do not have arbitrary values. In fact the only allowed values are the ones determined from the Z_ρ^2-module. In order to obtain the invariance of the mass squared formula we may consider summation on all the (p, q) values allowed by the Z_ρ^2- module. One arrives to the family of symplectic torus bundle whose monodromy realizes the gauging of the trombone symmetry.

8 T-duality in the supermembrane theory

In this section we introduce the T-duality transformations for the supermembrane theory. This goes beyond the T-duality of superstring theory. In fact, the latter may be directly obtained from the membrane theory by freezing membrane degrees of freedom and quantizing the remaining string states [50]. In this section we present the T-duality of the full degrees of freedom of the supermembrane, when formulated on a dual symplectic torus bundle (i.e. a symplectic torus bundle defined under the T-duality transformation acting on the moduli). It acts on the moduli as well as on the bosonic and fermionic fields. We will see that T-duality become a natural symmetry of the theory that fixes the scale of energy of the supermembrane tension T. The T-duality transformation is a nonlinear map which interchange the winding modes $\mathbb{W}$, previously defined associated to the cohomology of the base manifold with the KK charges, $Q = (p, q)$ associated to the homology of the target torus together with a transformation of the real moduli $R \to \frac{1}{R}$ and complex moduli $\tau \to \widetilde{\tau}$, both in a nontrivial way. In the following all transformed quantities under T-duality are denoted by a tilde, to differenciate from other symmetries. Given a winding matrix $\mathbb{W}$ and KK modes there always exists an equivalent winding matrix $\mathbb{W}' = \begin{pmatrix} l_1' & l_2' \\ m_1' & m_2' \end{pmatrix}$, under the $\mathrm{SL}(2,\mathbb{Z})$ symmetry (3.22) such that for KK charges $Q = \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$,

$$\begin{pmatrix} l_1' \\ m_1' \end{pmatrix} = \Lambda_0 \begin{pmatrix} p \\ q \end{pmatrix} \tag{8.1}$$

where $\Lambda_0 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ with $\alpha = \delta$. This is an intrinsic relation between the equivalence classes of winding matrices and KK modes. In fact, it is preserved under a U-duality transformation (3.22):

$$\begin{aligned} \begin{pmatrix} l_1' \\ m_1' \end{pmatrix} &\longrightarrow \begin{pmatrix} \widehat{l}_1 \\ \widehat{m}_1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} l_1' \\ m_1' \end{pmatrix} \\ \begin{pmatrix} p \\ q \end{pmatrix} &\longrightarrow \begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \end{aligned} \tag{8.2}$$

Hence

$$\begin{pmatrix} \widehat{l}_1 \\ \widehat{m}_1 \end{pmatrix} = \mathbb{M} \begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} \tag{8.3}$$

where

$$\mathbb{M} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \Lambda_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}. \tag{8.4}$$

The matrix $\mathbb{M} \in \mathrm{SL}(2,\mathbb{Z})$ and has equal diagonal terms, provided Λ_0 has $\alpha = \delta$. In order to define the T-duality transformation we introduce the following [50](47) dimensionless variables

$$\mathcal{Z} := TA\widetilde{Y} \quad \widetilde{\mathcal{Z}} := T\widetilde{A}Y \tag{8.5}$$

where T is the supermembrane tension, $A = (2\pi R)^2 Im\tau$ is the area of the target torus and $Y = \frac{RIm\tau}{|q\tau - p|}$. The tilde variables $\widetilde{A}, \widetilde{Y}$ are the transformed quantities under the T-duality.[6] See (8.9) for the explicit value of $\mathcal{Z}$. The T-duality transformation we introduce is given by:

$$\begin{aligned}
&\text{The moduli}: \quad \mathcal{Z}\widetilde{\mathcal{Z}} = 1, \quad \widetilde{\tau} = \frac{\alpha\tau+\beta}{\gamma\tau+\alpha}; \\
&\text{The charges}: \quad \begin{pmatrix} \widetilde{p} \\ \widetilde{q} \end{pmatrix} = \Lambda_0 \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \widetilde{l}_1 & \widetilde{l}_2 \\ \widetilde{m}_1 & \widetilde{m}_2 \end{pmatrix} = \Lambda_0^{-1} \begin{pmatrix} l_1' & l_2' \\ m_1' & m_2' \end{pmatrix}.
\end{aligned} \tag{8.7}$$

We notice that the T-duality transformations for the winding matrix, having Λ_0 equal diagonal terms, becomes of the same form as in (3.22). The main difference is that Λ_0 is determined in terms of the winding and KK modes, defining a nonlinear transformation on the charges of the supermembrane, while (3.22) is a linear transformation on them. With the above definition of T-duality transformation we have

$$\begin{aligned}
\begin{pmatrix} p \\ q \end{pmatrix} &\to \begin{pmatrix} \widetilde{p} \\ \widetilde{q} \end{pmatrix} = \begin{pmatrix} l_1' \\ m_1' \end{pmatrix} \\
\begin{pmatrix} l_1' \\ m_1' \end{pmatrix} &\to \begin{pmatrix} \widetilde{l}_1' \\ \widetilde{m}_1' \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}
\end{aligned} \tag{8.8}$$

See the appendix A for the construction of Λ_0. That is, the KK modes are mapped onto the winding modes and viceversa. The property together with the condition $Z\widetilde{Z} = 1$ ensure that $(\text{T-duality})^2 = \mathbb{I}$, the main property of T-duality. The explicit transformations of the real modulus, obtained from the above T-duality transformation is

$$\widetilde{R} = \frac{|\gamma\tau+\alpha||q\tau-p|^{2/3}}{T^{2/3}(Im\tau)^{4/3}(2\pi)^{4/3}R}, \text{with} \quad \widetilde{\tau} = \frac{\alpha\tau+\beta}{\gamma\tau+\alpha} \quad \text{and} \quad \mathcal{Z}^2 = \frac{TR^3(Im\tau)^2}{|q\tau-p|} \tag{8.9}$$

The winding modes and KK charge contribution in the mass squared formula transform in the following way:

$$\begin{aligned}
Tn^2A^2 &= \frac{n^2}{\widetilde{Y}^2}\mathcal{Z}^2 \\
\frac{m^2}{Y^2} &= T^2m^2\widetilde{A}^2\mathcal{Z}^2
\end{aligned} \tag{8.10}$$

To see how the H_1 (3.17) transforms under T-duality it is important to realize the transformation rules for the fields,

$$\begin{aligned}
&dX^m = ud\widetilde{X}^m, \qquad d\widetilde{X} = ue^{i\varphi}dX, \qquad A = ue^{i\varphi}\widetilde{A} \\
&\text{and} \qquad \Psi = u^{3/2}\widetilde{\Psi}, \qquad \overline{\Psi} = u^{3/2}\widetilde{\overline{\Psi}}
\end{aligned} \tag{8.11}$$

[6]This definition can be more naturally understood in terms of a vector of the 2-torus moduli $\mathcal{V} = (TA, \mathcal{Z}/Y)$ defined in terms of the moduli (R, τ) as

$$\widetilde{\mathcal{V}} = \Omega\mathcal{V} \tag{8.6}$$

being $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Where $u = \mathcal{Z}^2 = \frac{R|\gamma\tau+\alpha|}{\widetilde{R}}$, φ was defined in (3.23) and $dX = dX^1 + idX^2$ and respectively, its dual $d\widetilde{X}$ is

$$d\widetilde{X} = 2\pi\widetilde{R}[(\widetilde{m}_1\widetilde{\tau} + \widetilde{l}_1)d\widehat{X}^1 + (\widetilde{m}_2\widetilde{\tau} + \widetilde{l}_2)d\widehat{X}^2] \tag{8.12}$$

The phase $e^{i\varphi}$ cancels with the h.c. the transformation of the Hamiltonian. The relation between the hamiltonians through a T-dual transformation is

$$H = \frac{1}{\widetilde{\mathcal{Z}}^8}\widetilde{H}, \quad \widetilde{H} = \frac{1}{\mathcal{Z}^8}H. \tag{8.13}$$

We thus obtain for the mass squared formula the following identity,

$$M^2 = T^2n^2A^2 + \frac{m^2}{Y^2} + T^{2/3}H = \frac{1}{\widetilde{\mathcal{Z}}^2}\left(\frac{n^2}{\widetilde{Y}^2} + T^2m^2\widetilde{A}^2\right) + \frac{T^{2/3}}{\widetilde{\mathcal{Z}}^8}\widetilde{H}. \tag{8.14}$$

8.1 T-duality on symplectic bundles

There is bijective relation between the symplectic torus bundles with monodromy $\rho(\alpha,\beta)$ and the elements of the cohomology group $H_2(\Sigma, Z_\rho)$ of the base manifold Σ with coefficients on the module Z^2_ρ, and hence with the elements of the coinvariant group associated to the monodromy group G. That is each equivalence class

$$\{\mathcal{Q} + g\widehat{\mathcal{Q}} - \widehat{\mathcal{Q}}\}, \tag{8.15}$$

for any $g \in G$ and $\widehat{\mathcal{Q}} \in H_1(T^2)$, characterizes one symplectic torus bundle. In the formulation of the supermembrane on that geometrical structure $\mathcal{Q}$ are identified with the KK charges. The action of G, the monodromy group, leaves the equivalence class invariant. G acts as the identity on the coinvariant group. We now consider the duality transformation introduced previously. It interchanges KK modes $\mathcal{Q}$ into components of the winding matrix through the relation (8.1)

$$\begin{pmatrix} l_1 \\ m_1 \end{pmatrix} = \Lambda_0 \begin{pmatrix} p \\ q \end{pmatrix} \tag{8.16}$$

Under the duality transformation the equivalence class transform as

$$\{\Lambda_0\mathcal{Q} + (\Lambda_0 g\Lambda_0^{-1})\Lambda_0\widehat{\mathcal{Q}} - \Lambda_0\widehat{\mathcal{Q}}\}, \tag{8.17}$$

hence for the dual bundle it holds,

$$\left\{\Lambda_0\begin{pmatrix} l_1 \\ m_1 \end{pmatrix} + (\Lambda_0 g\Lambda_0^{-1})\begin{pmatrix} \widehat{l}_1 \\ \widehat{m}_1 \end{pmatrix} - \begin{pmatrix} \widehat{l}_1 \\ \widehat{m}_1 \end{pmatrix}\right\}, \tag{8.18}$$

That is, as an element of the coinvariant group of $\Lambda_0 G\Lambda_0^{-1}$. We then conclude that the duality transformation, in addition to the transformation on the moduli R, τ, also maps the geometrical structure onto an equivalent symplectic torus bundle with monodromy $\Lambda_0 G\Lambda_0^{-1}$. We notice that the transformation depends crucially on the original equivalence class of the coinvariant group. So for a nonequivalent symplectic torus bundle the dual transformations is realized with a different $\mathrm{SL}(2,Z)$ matrix Λ_0. Consequently, this dual transformation between supermembrane on symplectic torus bundles cannot be seen at the level of supergravity theory which only distinguish the monodromy group but not its coinvariant structure.

Now we are in position to determine the T-duality as a natural symmetry for the family of supermembranes with central charges. We take:

$$\widetilde{Z} = Z = 1 \Rightarrow T_0 = \frac{|q\tau - p|}{R^3(Im\tau)^2}. \tag{8.19}$$

It imposes a relation between the energy scale of the tension of the supermembrane and the moduli of the torus fiber and that of its dual. Indeed we can think in two different ways: given the values of the moduli it fixes the allowed tension T_0 or on the other way around, for a fixed tension T_0, the radius, the Teichmuller parameter of the 2-torus, and the KK charges satisfy (8.19). When this T-duality extended to M-theory acts on the stringy states of the supermembrane with central charges wrapping on a $\widetilde{T}^2$ one recovers the standard T-duality relations in string theory [50]. The contribution of the stringy states of the supermembrane with central charges wrapping on a dual $\widetilde{T}^2$ torus was already found in [50]. At the level of supergravity the structure of the fiber bundle base manifold of the supermembrane with central charges is lost and a remanent of it appears as nonvanishing components of the 3-form, which for the supermembrane in the LCG corresponds to C_{-rs} [59]. Following the lines of the noncommutative torus of [58],[7] we can interpret $C_{-rs} = F_{rs}$ in our case the nondegenerate 2-form associated to the central charge condition, then $\int_\Sigma F_{rs} = n$ and at the level the noncommutative structure of the 2-torus in string theory the nonvanishing three form corresponds to the presence of nonvanishing B_{ij} field [60] in the closed string sector. The formulation of the supermembrane in the presence of nonvanishing 3-form has been analyzed in [59]. In our formulation there is a particularity, since the magnetic field on the worldvolume of the supermembrane induced by the monopole contribution is nonconstant and consequently it should be associated to a nonvanishing 4-form flux $G = dC$ in 11D. In [60] the double T-duality is realized for the the closed strings sector and its associated noncommutativity, it would be interesting to see if there is a connection with our results.

9 Discussion and conclusions

We showed that the formulation of the supermembrane in terms of sections of the symplectic torus bundle with a monodromy is a natural way to understand the M-theory origin of the gauging procedures in supergravity theories [64]. Its low energy limit corresponds to the type II $\mathrm{SL}(2,\mathbb{R})$ gauged supergravities in 9D. We have explicitly shown the relation with the type IIB gauged sugras in 9D. The global description is a realization of the sculpting

[7]The work of [58] is mainly done in flat space with Moyal star product in which the noncommutative parameter is given by the 2-form, however as it is signalled in the paper, it can be generalized to curved manifolds, for which the star product is changed to a deformation quantization star product (for example in our case it corresponds to a Fedosov-like product) and then, an additional choice of Poisson structure appears.

mechanism found in [61] and it is associated to the inequivalent classes of symplectic torus bundles with monodromies in $\mathrm{SL}(2,\mathbb{Z})$. The geometrical description of these kind of bundles has been developed in [66]. As already conjectured in [61] we claim that the following diagramme applies:

$$\begin{tikzcd} \text{Compactified M2}(n=0) \text{ in } X_9\times T^2 \arrow[r, "\textit{'Sculpting'}"] \arrow[d, "\text{Low Energies}"'] & \text{M2 with central charges } (n\neq 0) \text{ in } X_9\times T^2 \arrow[d, "\text{Low Energies}"] \\ \text{Type II Maximal Supergrav. 9D} \arrow[r, "\textit{Noether}"] & \text{Type II Gauged Supergravities 9D} \end{tikzcd} \tag{9.1}$$

The supermembrane without any extra topological condition compactified on a 2-torus is a gauge theory on a trivial principle bundle with structure group of the symplectic group homotopic to the identity. The supermembrane with nontrivial central charge is also invariant under the isotopy group of symplectomorphisms, which in the case considered is $\mathrm{SL}(2,\mathbb{Z})$. In this paper we analyze the gauged supermembrane arising from the gauging of the abelian subgroups of this $\mathrm{SL}(2,\mathbb{Z})$ group which has an intrinsic meaning in the theory. The gauging is automatically achieved by formulating the supermembrane with central charges as sections of a symplectic torus bundle with monodromy. The monodromy is also intrinsically defined by considering representations of $\Pi_1(\Sigma)$, the fundamental group of the Riemann base manifold of genus one (Σ), onto $\Pi_0(G)$ the isotopy group of the symplectomorphisms group G. The abelian subgroup of $\mathrm{SL}(2,\mathbb{Z})$ acts naturally on the homology of the target torus (the fiber of the bundle[8]) $H_1(T^2)$. We identify, in our formulation of the supermembrane, the elements of $H_1(T^2)$ with (p,q) KK charges. Besides, the winding numbers are directly related to the cohomology of the base manifold Σ. For a given monodromy there is a one to one correspondence between the symplectic torus bundle with that monodromy and the elements of the coinvariant group of the monodromy [66]. These elements are equivalence clases of KK (p,q) charges which we explicitly described for the elliptic, parabolic, and hyperbolic monodromies. We classified the symplectic torus bundles in terms of the coinvariant group of the monodromy. It turns out that at the level of the supermembrane what is relevant are the elements of the coinvariant group of a given monodromy group. The possible values of the (p,q) charges on a given symplectic torus bundle with that monodromy are restricted to the corresponding equivalence class defining the element of the coinvariant group associated to the bundle. We also analyse the presence of torsion elements in the cohomology of the base of the manifold or equivalently $Z_m\oplus Z_n$ groups as the coinvariant group of the monodromy. We also obtained, using the same geometrical setting, the gauging of the trombone symmetry. It is constructed from a nonlinear representation of $\mathrm{SL}(2,\mathbb{Z})$ and gives rise to a different symplectic torus bundle in comparison to the previous constructions in terms of linear representations.

We showed the existence of a new Z_2 symmetry that plays the role of T-duality in M-theory interchanging the winding and KK charges but leaving the hamiltonian invariant.

[8]The complete fiber corresponds in this set-up to the target space, that in the case considered is $M_9\times T^2$ but the nontrivial topological properties are only associated to the compact sector.

We expect that all monodromies associated to type IIA will arise from the dual symplectic torus bundle obtained from this new T-duality symmetry. Consequently, we expect that the global geometrical formulation of supermembranes we are proposing will provide a unified origin of all type II gauged supergravities in 9D. We may then conjecture that the supermembrane becomes the M-theory origin of all type II nine dimensional supergravities.

From this construction of the supermembrane on symplectic torus bundle one may identify directly corresponding gauged supergravities in 9D. Moreover, a given gauged supergravity can only interact with a corresponding supermembrane on a symplectic torus bundle associated to a coinvariant element of the same monodromy, otherwise, an inconsistency with the transition functions on the bundle will occur. We also obtain the explicit gauge degree of freedom of the theory, discuss a gauge fixing procedure and obtain the residual symmetry once the monodromy has been assumed.

Recently in type II String Phenomenology the role of M2-branes wrapping homological 2-cycles with torsion has been used as a M-theory realization of the so-called discrete gauge symmetries Z_N. These symmetries may have a potential number of bondages from the phenomenological viewpoint as for example to be discrete symmetries that can help to realize proton stability or help to suppress some dangerous operators. It has been conjectured that this M2-branes at low energies would produce Bohm-Aranov particles [72]–[73]. In our constructions many of the M2-branes fiber bundles naturally are wrapped on homological 2-cycles with torsion. It would be interesting to see whether in compactifications down to 4D, it could be a possible connection with our construction.

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A Computation of Λ_0 matrix of the T-dual transformations

We are going to determine Λ_0. Without loss of generality we may assume l_1 and m_1 to be relatively prime integers. We have $\det(\mathbb{W}) = n$ It is important to notice that $\left(\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\right)$ are also relatively prime integers. There always exists $\Lambda_0 \in \mathrm{SL}(2, \mathbb{Z})$ such that

$$\begin{pmatrix} l_1 \\ m_1 \end{pmatrix} = \Lambda_0 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \tag{A.1}$$

We thus have from (5.2):

$$\begin{pmatrix} \widetilde{p}_1 \\ \widetilde{q}_1 \end{pmatrix} = \begin{pmatrix} l_1 \\ m_1 \end{pmatrix}. \tag{A.2}$$

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We now introduce

$$\begin{pmatrix} r_2 \\ r_1 \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} l_2 \\ m_2 \end{pmatrix}. \tag{A.3}$$

Now we define $\mathbb{A} = \left(\begin{smallmatrix} p_1 & r_2 \\ q_1 & r_1 \end{smallmatrix}\right)$ consequently $\mathbb{A} = \Lambda^{-1} \left(\begin{smallmatrix} l_1 & l_2 \\ m_1 & m_2 \end{smallmatrix}\right)$, with $det\mathbb{A} = n$. We notice that $det \left(\begin{smallmatrix} \widetilde{l}_1 & \widetilde{l}_2 \\ \widetilde{m}_1 & \widetilde{m}_2 \end{smallmatrix}\right) = \mathbb{A}$. We thus have a transformation interchanging winding and KK modes. The expression for Λ_0 may be obtained in the following way: There always exists integers (b_2, b_1, d_1, c_1) such that there are $\mathbb{B} = \left(\begin{smallmatrix} p_1 & b_2 \\ q_1 & b_1 \end{smallmatrix}\right)$, and $\mathbb{C} = \left(\begin{smallmatrix} l_1 & d_1 \\ m_1 & c_1 \end{smallmatrix}\right)$, with

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \mathbb{B} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} l_1 \\ m_1 \end{pmatrix} = \mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.4}$$

where $\mathbb{B}, \mathbb{C} \in \mathrm{SL}(2,\mathbb{Z})$. Finally we can determine the transformation matrix Λ_0 . It corresponds to,

$$\Lambda_0 = \begin{pmatrix} l_1 & d_1 \\ m_1 & c_1 \end{pmatrix} \begin{pmatrix} p_1 & b_2 \\ q_1 & b_1 \end{pmatrix}^{-1}. \tag{A.5}$$

and together with the (8.6) condition implies that the T-dual transformation (8.7) $(\text{T-duality})^2 = \mathbb{I}$.

B An example of equivalence between the two monodromy representations considered

In this appendix we are going to prove that the matrix of (6.5) particularized to the values corresponding to $n = 1$, $m > 0$, $\left(\begin{smallmatrix} -2m+1 & 2m+1 \\ -m & m+1 \end{smallmatrix}\right) = T^2ST^{m+1}STS$ is conjugate to $-T^{-3}S(T^{-2}S)^{m-5}$. We denote by $\sim$ two conjugate matrices. We then have

$$\begin{aligned} T^2ST^{m+1}STS &= -T^2ST^mST^{-1} \sim -TST^mS \sim (-1)^mT(STS)^m \\ &\sim T(T^{-1}ST^{-1})^m \sim (T^{-1}ST^{-1})^{m-1}T^{-1}S \sim (T^{-1}ST^{-1})^{m-2}S \\ &\sim (T^{-1}ST^{-1})^{m-4}T^{-1} \sim -T^{-1}ST^{-1}(T^{-1}ST^{-1})^{m-5}T^{-1} \\ &\sim -(T^{-1}ST^{-1})(T^{-1}ST^{-1})^{m-6}T^{-1}ST^{-2} \\ &\sim -T^{-3}ST^{-1}(T^{-1}ST^{-1})^{m-6}T^{-1}S \\ &= -T^{-3}S(T^{-2}S)^{m-5}. \qquad \square \end{aligned} \tag{B.1}$$

Where we have used $STS = -T^{-1}ST^{-1}$.

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